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On the Representation of Option Prices with Stochastic Interest Rates

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Abstract

This paper presents a general representation formula for pricing plain vanilla options in a stochastic interest rate framework, where the risky asset and the interest rate are correlated. Our approach offers an alternative to the classical “Change of Numéraire” method by explicitly highlighting the impact of this correlation. Once the general representation is obtained in any diffusive framework, we specialize it for affine interest rate models with time dependent coefficients, where we can provide practical semi-analytical approximations. To assess their efficiency, we apply those approximations to the simpler Vasicek and CIR rate processes. In addition, the subsequent decomposition of the discounted payoff naturally yields a representation-based Monte Carlo estimator with substantial variance-reduction properties. Numerical experiments confirm both the accuracy of the proposed approximations and the computational efficiency of the representation-based Monte Carlo method.

Keywords: Option pricing; Stochastic interest rate models; Affine processes

JEL Classification: G13, C63, G12

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1 Introduction

Pricing financial derivatives in models with stochastic interest rates remains a central challenge in quantitative finance, especially when the underlying asset and the interest rate are correlated. While the classical “Change of Numéraire” technique provides a robust theoretical foundation (see e.g. [9]), its practical implementation often depends on computationally intensive methods such as Monte Carlo simulations. This is particularly true for long-dated options or when performing rapid sensitivity analyses. Furthermore, the growing availability of high-frequency large-scale financial data has made real-time model calibration a central component of quantitative strategies. This development has accentuated the

importance of computational efficiency, which now represents a significant constraint in the selection of models for derivative pricing, leading to a widespread use of the affine framework (see [14]).

Early contributions established the basic framework for derivative pricing in stochastic interest-rate environments. Rabinovitch [27] and Amin and Jarrow [1] analyzed option and contingent-claim valuation when the default-free short rate is stochastic, clarifying how the additional source of randomness affects hedging arguments and the associated pricing equations. The empirical relevance of stochastic interest rates, also in combination with stochastic volatility, was subsequently investigated by Bakshi et al. [8], who studied their impact on option pricing and hedging performance.

A substantial strand of the literature has focused on analytical, semi-analytical, transform - based, and approximation methods aimed at retaining tractability in increasingly rich equity–interest-rate models. Early approaches include Kim and Kunitomo [21], who derive an extended Black–Scholes approximation through a small-disturbance asymptotic expansion in the interest-rate volatility, and Kim [22], who investigates empirically the pricing performance of option models under alternative stochastic short-rate specifications. From a PDE perspective, Chiarella et al. [11] show how suitable transformations of a two-factor pricing equation may recover Black–Scholes-type valuation formulas under specific model assumptions.

Moving to richer specifications, Scott [29] considers option pricing in a framework combining stochastic volatility, stochastic interest rates, and jumps, exploiting transform techniques for efficient valuation. Van Haastrecht et al. [30] extend the Schöbel–Zhu stochastic-volatility model by introducing Gaussian stochastic interest rates and a flexible instantaneous correlation structure, while retaining Fourier-based tractability. Grzelak and Oosterlee [18] develop affine approximation techniques for hybrid models combining stochastic volatility and stochastic interest rates. He and Zhu [20] consider a Heston model coupled with an uncorrelated CIR short rate and derive a tractable analytical representation for European option prices.

Alternative approximation strategies have also been proposed. In [16], the author develops a Wiener–Itô chaos expansion for an equity model coupled with a Gaussian Hull–White short rate, approximating the terminal distribution of the underlying and subsequently deriving European option prices under the forward measure. Antonelli et al. [2] propose instead a moment-matching approximation, which can also accommodate non-Gaussian short-rate dynamics. Kirkby [23] develops a continuous-time Markov-chain approximation leading to efficient semi-closed-form valuation formulas for hybrid equity–rate products under several stochastic short-rate specifications. More recently, He and Lin [19] enrich the Heston–Hull–White framework by introducing an additional factor that allows for a more general equity–interest-rate correlation structure while preserving analytical tractability through the characteristic function.

A complementary line of research focuses directly on improving the statistical efficiency of simulation-based pricing. Fouque and Han [15] construct martingale control variates for option pricing under stochastic volatility. In a hybrid stochastic-volatility and stochastic-interest-rate setting, Cozma and Reisinger [12] combine Monte Carlo simulation with a PDE-based conditional valuation step to obtain substantial variance reduction in a Heston–CIR foreign-exchange model. Liang and Xu [25] develop a conditional Monte Carlo method complemented by martingale control variates derived from the martingale representation theorem, whereas Chen et al. [10] combine conditional Monte Carlo with regression-based control variates for the efficient computation of option prices and Greeks in a three-factor framework. These contributions show that the probabilistic structure of the pricing problem can be exploited not only to obtain analytical or semi-analytical approximations, but also to improve the statistical efficiency of Monte Carlo estimators.

The novelty of this paper is twofold. First we contribute to this stream of literature by deriving a general representation formula for the price of a European option within a correlated asset/interest-rate framework. Building on the previous works ([7] and [3]), we develop a formula that leverages the properties of the Black–Scholes pricing function and incorporates the interaction between the underlying asset and the stochastic short rate through their associated martingales.

Assuming an affine structure for the short-rate process (see [14]), our method yields a semi-explicit representation formula for a broad class of interest-rate models, like e.g. Hull-White and CIR++ models.

Importantly, the approach does not require the joint asset/interest-rate system to be affine. The resulting representation provides a transparent decomposition of the option price, isolating the contributions arising from the correlation parameter and the interest-rate volatility. Besides providing an alternative route to option valuation under stochastic rates, this decomposition may also be useful for risk-management applications.

Beyond its theoretical appeal, the representation has also two interesting numerical implications. First, it provides the basis for accurate semi-analytical approximations, obtained by replacing some expectation terms entering the representation with tractable quantities, in this context we show also how a local error estimate in L^2 can be achieved for affine interest models. Second, the underlying pathwise decomposition allows writing the discounted payoff as the representation-based random variable plus a zero-mean martingale term. The resulting estimator can achieve a substantial variance reduction relative to the standard discounted-payoff Monte Carlo estimator.

We detail both numerical applications for two canonical short-rate models, the Vasicek and Cox–Ingersoll–Ross (CIR) models, and assess their practical performance through numerical experiments. In the Vasicek model, the availability of an analytical option-pricing formula allows us to evaluate the accuracy of the semi-analytical approximation against an exact benchmark, whereas in the CIR setting the Monte Carlo evaluation of the representation formula is used as a benchmark. The numerical results show both high accuracy of the proposed semi-analytical approximations and substantial efficiency gains for the representation-based Monte Carlo estimator.

The paper is structured as follows. In Section 2, we outline the general modeling framework. Section 3 is devoted to the derivation of the central representation formula for the derivative price, and subsequently to its detailed description for the affine class of short rate models. Section 4 details the approximation technique for the Hull-White and CIR++ models. Section 5 presents the numerical results for the two complementary uses of the representation formula, namely as an efficient variance-reduction technique and as the basis for the construction of a semi-analytical approximation. A brief appendix on the requisite Malliavin Calculus is also included for completeness.

2 The model

Let $[0, T]$ be a finite time interval and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space endowed with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, augmented with the $\mathbb{P}$ –null sets and made right continuous. We also assume that all the processes have a càdlàg version.

On the financial market a bond with stochastic interest rate r_t , and a process S_t , representing an underlying’s price, are traded. The filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ is rich enough to support all those processes, and we assume to be in absence of arbitrage, with $\mathbb{P}$ a risk neutral measure, selected by some given criterion. The discount factor is therefore $e^{-\int_t^s r_u du}$. We assume that the contract’s maturity is T .

To model the financial market dynamics, let us assume that the probability space contains a 2-dimensional standard $\mathcal{F}_t$ –adapted Brownian motion (W_s^x, W_s^r) , representing the randomness sources of the market, where an asset log-price process (possibly of an index or a fund), $X_t = \ln S_t$, verifies for each initial point $(t, r, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ (using the flow notation)

$$dX_s^{t, r, x} = (r_s^{t, r} - \frac{1}{2}\sigma^2)ds + \sigma dW_s^x, \quad (2.1)$$

$$dr_s^{t, r} = \mu(s, r_s^{t, r})ds + \nu(s, r_s^{t, r})dW_s^r, \quad (2.2)$$

where W^x, W^r are correlated Brownian motions with covariation process given by $\langle W^x, W^r \rangle_s = \rho s$ for some $\rho \in (-1, 1)$ and we assume

- (H) the volatility σ is assumed to be constant, and the functions $\mu, \nu : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are chosen so that they guarantee the existence and uniqueness of a square integrable solution of equation (2.2).

We can represent (W_s^x, W_s^r) by a standard 2-dimensional Brownian motion $\mathbf{B} = (B^1, B^2)$, so that

$$\begin{aligned} W^r &= B^1 \\ W^x &= \rho B^1 + \sqrt{1 - \rho^2} B^2. \end{aligned}$$

The discussion in this note focuses on models with a stochastic interest rate, but it can be easily extended to include the stochastic volatility as in [5].

Hereby, we use the following notation.

1. $\mathbb{E}_s[\cdot]$ will denote the conditional expectation with respect to the filtration $\mathcal{F}_s$.
2. Not assuming the interest rate to be necessarily positive, we impose

$$e^{-\int_s^T r_u du} \in L^2(\mathbb{P}), \quad (2.3)$$

and we denote the average interest rate on $[s, T]$ by

$$\bar{R}_s = \frac{1}{T-s} \int_s^T r_u du, \quad \bar{R}_T = r_T,$$

and by

$$R_s = \frac{1}{T-s} \int_s^T \mathbb{E}_s(r_u) du, \quad R_T = r_T$$

its optional projection. We may characterize the latter's dynamics as

$$R_s = \frac{1}{T-s} \left(M_s - \int_0^s r_u du \right), \quad (2.4)$$

where we have the martingale $M_s = \mathbb{E}_s \left[\int_0^T r_u du \right]$.

3. We denote by $N(t, s) = \mathbb{E}_s \left[e^{-\int_t^T r_u du} \right]$ the martingale induced by the discount rate.
4. We denote by $\mathcal{P}(s, x, r)$ the Black & Scholes pricing function for a European put with maturity T and strike price e^K ($K > 0$), then by virtue of the Gaussian distributions, the following hold

$$\mathcal{P}(s, x, r) = e^{K-r(T-s)} [1 - \mathcal{N}(d_2)] - e^x [1 - \mathcal{N}(d_1)],$$

where $d_{1,2} := d_{1,2}(s, x, r) = \frac{x - K + (r \pm \frac{\sigma^2}{2})(T-s)}{\sigma\sqrt{T-s}}$, and $\mathcal{N}$ is the standard normal distribution function, and

$$\begin{cases} \left[\partial_t \mathcal{P} + \frac{\sigma^2}{2} (\partial_{xx} - \partial_x) \mathcal{P} + r \partial_x \mathcal{P} - r \mathcal{P} \right] (s, x, r) = 0, \\ \mathcal{P}(T, x, r) = (e^K - e^x)^+ \end{cases}, \quad (2.5)$$

$$\partial_x \mathcal{P}(s, x, r) = -e^x (1 - \mathcal{N}(d_1)), \quad (\partial_{xx} - \partial_x) \mathcal{P}(s, x, r) = \frac{e^x N'(d_1)}{\sigma\sqrt{T-s}}, \quad (2.6)$$

$$\partial_r \mathcal{P}(s, x, r) = (T-s)(\partial_x - I) \mathcal{P}(s, x, r) \quad (2.7)$$

$$(\partial_{xxx} - \partial_{xx}) \mathcal{P}(s, x, r) = \frac{e^x}{\sigma\sqrt{T-s}} N'(d_1) \left(1 - \frac{d_1}{\sigma\sqrt{T-s}} \right), \quad (2.8)$$

$$(\partial_{xx} - \partial_x)^2 \mathcal{P}(s, x, r) = \frac{e^x N'(d_1)}{\sigma^3 (T-s)^{\frac{3}{2}}} \left[d_1^2 - d_1 \sigma\sqrt{T-s} - 1 \right]. \quad (2.9)$$

Similarly, the call pricing function $\mathcal{B}$ verifies the same relations, provided we substitute

$$\mathcal{B}(s, x, r) = (e^x - e^K)^+, \quad \partial_x \mathcal{B}(s, x, r) = e^x N(d_1).$$

3 The representation formula

In our framework, we have the following evaluation problem

$$\begin{aligned} V_t &:= \mathbb{E}_t \left[e^{-\int_t^T (r_u^{t,r}) du} (e^K - e^{X_T^{t,x,r}})_+ \right] \\ &= \mathbb{E}_t \left[e^{-\int_t^T r_u^{t,r} du} \left(e^K - e^x e^{\int_t^T \left(r_u^{t,r} - \frac{\sigma^2}{2} \right) du + \sigma(W_T^x - W_t^x)} \right)_+ \right]. \end{aligned}$$

Since the final condition $\mathcal{P}(T, x, r) = (e^K - e^x)^+$ does not depend on r , we may rewrite the above as

$$V_t = \mathbb{E}_t \left[N(t, T) \mathcal{P}(T, X_T^{t,x}, R_T) \right]. \quad (3.1)$$

For brevity, from now on we omit the superscripts in the processes and we set $\mathcal{P}_s = \mathcal{P}(s, X_s, R_s)$.

Proposition 3.1 *Let us assume the market model (2.1)-(2.2) under hypothesis (H), and condition (2.3). Denoting by $\mathcal{L} = \partial_x - I$, we have*

$$\begin{aligned} V_t &= \mathbb{E}_t \left[N(t, T) \mathcal{P}_T \right] = N(t, t) \mathcal{P}_t + \mathbb{E}_t \left[\int_t^T N(t, s) r_s \mathcal{P}_s ds \right] \\ &\quad + \mathbb{E}_t \left[\int_t^T \frac{1}{2} N(t, s) \mathcal{L}^2 \mathcal{P}_s d\langle M, M \rangle_s + \int_t^T \mathcal{L} \mathcal{P}_s d\langle M, N \rangle_s \right] \\ &\quad + \mathbb{E}_t \left[\int_t^T N(t, s) (\partial_x \mathcal{L}) \mathcal{P}_s d\langle X, M \rangle_s + \int_t^T \partial_x \mathcal{P}_s d\langle X, N \rangle_s \right] \end{aligned} \quad (3.2)$$

Proof: From (2.4), we may write the dynamics

$$dR_s = \frac{1}{(T-s)} [(R_s - r_s) ds + dM_s] \Rightarrow d\langle R, R \rangle_s = \frac{1}{(T-s)^2} d\langle M, M \rangle_s. \quad (3.3)$$

Integration by parts gives

$$\mathbb{E}_t [N(t, T) \mathcal{P}_T] - N(t, t) \mathcal{P}_t = \mathbb{E}_t \left[\int_t^T N(t, s) d\mathcal{P}_s + \int_t^T \mathcal{P}_s dN(t, s) + \int_t^T d\langle \mathcal{P}, N(t, \cdot) \rangle_s \right],$$

and by Itô's formula, we have

$$\begin{aligned} d\mathcal{P}_s &= \left[\partial_t + \frac{\sigma_s^2}{2} (\partial_{xx} - \partial_x) + r_s \partial_x + \frac{1}{T-s} (R_s - r_s) \partial_r \right] \mathcal{P}_s ds + \frac{1}{2(T-s)^2} \partial_{rr} \mathcal{P}_s d\langle M, M \rangle_s \\ &\quad + \frac{1}{T-s} \partial_{xr} \mathcal{P}_s d\langle X, M \rangle_s + \sigma \partial_x \mathcal{P}_s dW_s^x + \frac{1}{T-s} \partial_r \mathcal{P}_s dM_s. \end{aligned}$$

Substituting the above in the integration by parts formula, exploiting relations (2.5) and (2.7), and taking into account that $N(t, \cdot)$ and M_s are integrable martingales, when taking expectations, the integral terms with respect to those will give no contribution and we obtain representation (3.2). $\square$

Remark 3.1 The Itô formula decomposes the payoff $N(t, T)\mathcal{P}_T$ into

$$N(t, T)\mathcal{P}_T = \mathcal{A}_{t, T} + \mathcal{M}_{t, T}$$

where $\mathcal{A}_{t, T}$ is a finite variation process and $\mathcal{M}_{t, T}$ a martingale, both considered on the interval $[t, T]$. Consequently

$$\mathbb{E}_t[N(t, T)\mathcal{P}_T] = \mathbb{E}_t[\mathcal{A}_{t, T}] = V_t$$

which shows that the representation-based evaluation is obtained from the discounted payoff by subtracting a zero-mean martingale term. This observation will be exploited in Section 5 to construct Monte Carlo estimators with reduced variance.

3.1 The affine case

From the representation given in the previous section, it is clear that an important contribution comes from the correlations among the processes, even though those are still partially hidden in the arguments of $\mathcal{P}$ and its derivatives.

We are now going to specify the dynamics of the interest rate in order to have a more explicit representation, characterized in terms of the correlation and the dynamics parameters, in view of a sensitivity analysis.

We assume that the interest rate falls into the class of affine processes

$$r_s = r_t + \int_t^s [\gamma_u r_u + \beta_u] du + \int_t^s [\eta_u r_u + \delta_u]^{\frac{1}{2}} dW_u^r, \quad (3.4)$$

with $\gamma, \beta, \eta, \delta$ bounded deterministic functions of time (see [13]).

Remark 3.2 We would like to point out that the model (2.1)-(2.2) is not always jointly affine under asset/interest rate correlation. Indeed, the diffusion matrix of the joint process (X, r) is

$$\sigma\sigma^\top = \begin{pmatrix} \sigma^2 & \rho\sigma(\eta_u r + \delta_u)^{1/2} \\ \rho\sigma(\eta_u r + \delta_u)^{1/2} & \eta_u r + \delta_u \end{pmatrix}. \quad (3.5)$$

As a matter of fact, in the CIR case ($\eta_u > 0$), the off-diagonal term depends on $\sqrt{r}$: as a consequence, the standard transform techniques used to derive closed-form European option prices (e.g. through characteristic functions or change of numéraire arguments) are no longer applicable.

The affine choice (3.4) implies the martingale N has the explicit expression

$$N(t, s) = \mathbb{E}_s \left[e^{-\int_t^T r_u du} \right] = \mathbb{E}_s \left[e^{-\int_s^T r_u du} \right] e^{-\int_t^s r_u du} = e^{-\int_t^s r_u du} e^{-\varphi(s, T)r_s + \psi(s, T)}, \quad (3.6)$$

for some deterministic differentiable functions φ and ψ of time to maturity.

Consequently, the martingale $N(t, \cdot)$ verifies

$$\begin{aligned} dN(t, s) &= -N(t, s) \left\{ [r_s - \varphi'(s, T)r_s + \psi'(s, T)] ds + \varphi(s, T) dr_s - \frac{1}{2} \varphi^2(s, T) d\langle r, r \rangle_s \right\} \\ &= -N(t, s) \varphi(s, T) [\eta_s r_s + \delta_s]^{\frac{1}{2}} dW_s^r, \end{aligned}$$

where by φ' and ψ' we mean derivation with respect to s .

To determine the dynamics of M , we resort to Malliavin Calculus (we refer the reader to [6] or [26] for a general presentation of the topic) and assuming that $r \in \mathbb{D}^{1,2}$, we may apply the Clark-Ocone-Haussman formula to deduce

$$M_s = \mathbb{E} \left[\int_0^T r_z dz \right] + \int_0^s \mathbb{E}_u \left[D_u^{W^r} \left(\int_0^T r_z dz \right) \right] dW_u^r = \mathbb{E} \left[\int_0^T r_z dz \right] + \int_0^s \mathbb{E}_u \left[\int_u^T D_u^{W^r} r_z dz \right] dW_u^r.$$

Being r solution of (3.4), we can compute its Malliavin derivative, obtaining

$$D_u^{W^r} r_z = \int_u^z \gamma_v D_u^{W^r} r_v dv + (\eta_u r_u + \delta_u)^{\frac{1}{2}} + \int_u^z \frac{\eta_v}{2(\eta_v r_v + \delta_v)^{\frac{1}{2}}} D_u^{W^r} r_v dW_v^r,$$

which is a linear, and hence explicitly solvable, stochastic differential equation. Hence taking the conditional expectation, we obtain

$$\mathbb{E}_u(D_u^{W^r} r_z) = (\eta_u r_u + \delta_u)^{\frac{1}{2}} e^{\int_u^z \gamma_v dv},$$

and consequently

$$M_s = \mathbb{E} \left(\int_0^T r_u du \right) + \int_0^s \left[(\eta_u r_u + \delta_u)^{\frac{1}{2}} \int_u^T e^{\int_u^z \gamma_v dv} dz \right] dW_u^r. \quad (3.7)$$

For simplicity, in what follows we denote

$$H_u = (\eta_u r_u + \delta_u)^{\frac{1}{2}} \int_u^T e^{\int_u^z \gamma_v dv} dz. \quad (3.8)$$

Also the first expectation in (3.7) is explicitly computable, giving

$$\mathbb{E} \left(\int_0^T r_u du \right) = r_0 \int_0^T e^{\int_0^u \gamma_z dz} du + \int_0^T \int_0^u e^{\int_v^u \gamma_z dz} \beta_v dv du. \quad (3.9)$$

With the above dynamics in mind, we may write explicitly the co/variation processes appearing in (3.2)

$$\begin{aligned} b(r)_s &= [\eta_s r_s + \delta_s]^{\frac{1}{2}}, \\ d\langle M, N(t, \cdot) \rangle_s &= -H_s N(t, s) \varphi(s, T) b(r)_s ds \\ d\langle X, N(t, \cdot) \rangle_s &= -\rho \sigma N(t, s) \varphi(s, T) b(r)_s ds. \end{aligned} \quad (3.10)$$

By this new notation, we may rewrite the representation formula (3.2) as

$$\begin{aligned} V_t &= \mathbb{E}_t \left[N(t, T) \mathcal{P}_T \right] = N(t, t) \mathcal{P}_t + \int_t^T \mathbb{E}_t \left[N(t, s) r_s \mathcal{P}_s \right] ds \\ &\quad + \int_t^T \mathbb{E}_t \left[N(t, s) \left\{ \frac{(H_s)^2}{2} \mathcal{L}^2 \mathcal{P}_s - H_s \varphi(s, T) b(r)_s \mathcal{L} \mathcal{P}_s \right\} \right] ds \\ &\quad + \rho \sigma \int_t^T \mathbb{E}_t \left[N(t, s) \left\{ H_s (\partial_x \mathcal{L}) \mathcal{P}_s - \varphi(s, T) b(r)_s \partial_x \mathcal{P}_s \right\} \right] ds. \end{aligned} \quad (3.11)$$

Equation (3.11) displays an explicit linear dependence on ρ (or $\rho\sigma$), that might suggest an approximation apparently linear in the correlation parameter. The correlation, though, is still implicitly embedded within the integrands, since $\mathcal{P}_s$ and its derivatives depend on both X and R . A more refined expansion can be obtained by repeatedly applying Itô's formula. This further computational effort might lead to a more accurate approximation of the first integral term in the above expression, much less so for the others which are controlled by the parameters ρ and σ , typically very small. In practice, the approximation linear in ρ is often sufficient to achieve accuracy comparable to that of Monte Carlo simulations, especially for moderate maturities, with the advantage of substantially reducing computational time. For longer term maturities, refining the expansion becomes more relevant.

4 Model specifications and implementation

The representation formula derived in Section 3.1 is valid for general affine one-factor short-rate models in the sense of [13]. As a matter of fact, (2.2) automatically includes the most popular affine models with time dependent coefficients for the interest rate, such as

the CIR++ model and the Hull and White model. In this section, we provide an explicit formulation of (3.11) under those dynamics:

1. for the Hull–White model, in (3.4) we choose $\gamma_u = -\kappa$, $\beta_u = \theta(u) + \theta'(u)\kappa$, $\eta \equiv 0$, $\delta \equiv \epsilon^2$, for some differentiable deterministic function $\theta : [0, T] \rightarrow \mathbb{R}$, implying Gaussian marginal distributions for r . This model reduces to the Vasiček one, when coefficients are constant;
2. for the CIR++ model, in (3.4) we choose $\gamma_u = -\kappa$, $\beta_u = \kappa(\theta(u) + \beta) + \theta'(u)$, $\eta \equiv \epsilon^2$, $\delta \equiv -\epsilon^2\theta(u)$. This model yields non Gaussian marginal distributions, requiring more refined procedures for the evaluation of the expectations, and it reduces to the classical CIR model when using constant coefficients.

The constant coefficients cases are used as computational benchmarks in the numerical section. Simulations may be extended to models with time dependent coefficients, once their functional form is specified, by taking stepwise approximations.

The Hull–White and CIR++ models are particularly useful because the specifications considered above amount to a deterministic shift of the short rate: $r_s - \theta(s)$, which implies a better fit to the term structure than their “constant” versions.

4.1 The Hull–White model

We consider the one-factor Hull–White specification, that is to say the short rate satisfies

$$dr_s = \kappa\left(\theta(s) + \frac{\theta'(s)}{\kappa} - r_s\right)ds + \epsilon dW_s^r, \quad (4.1)$$

for some deterministic differentiable function $\theta : [0, T] \rightarrow \mathbb{R}$. We recall that the explicit solution to (4.1) is the Gaussian process

$$r_s = \theta(s) + (r_t - \theta(t))e^{-\kappa(s-t)} + \epsilon \int_t^s e^{-\kappa(s-u)} dW_u^r,$$

which has marginal Gaussian distributions

$$\mathcal{N}\left(\theta(s) + (r_t - \theta(t))e^{-\kappa(s-t)}; \frac{\epsilon^2}{2\kappa}(1 - e^{-2\kappa(s-t)})\right) \quad (4.2)$$

that imply the price the zero-coupon bond associated to r is given by

$$\mathbb{E}_s \left[e^{-\int_s^T r_u du} \right] = e^{-\varphi(s, T)r_s + \psi(s, T)},$$

with explicit

$$\begin{aligned} \varphi(s, T) &= \frac{1}{\kappa} \left[1 - e^{-\kappa(T-s)} \right], \\ \psi(s, T) &= - \int_s^T \theta(u) du + \varphi(s, T)\theta(s) + \frac{\epsilon^2}{2\kappa^2} \left[(T-s) - 2\varphi(s, T) + \frac{1 - e^{-2\kappa(T-s)}}{2\kappa} \right]. \end{aligned}$$

Consequently,

$$dM_s = H_s dW_s^r = \epsilon \varphi(s, T) dW_s^r.$$

Substituting into (3.11), we obtain the following semi-explicit representation:

$$\begin{aligned} V_t &= N(t, t) \mathcal{P}_t + \int_t^T \mathbb{E}_t \left[N(t, s) r_s \mathcal{P}_s \right] ds \\ &\quad + \epsilon^2 \int_t^T \varphi^2(s, T) \mathbb{E}_t \left[N(t, s) \left(\frac{1}{2} \mathcal{L}^2 \mathcal{P}_s - \mathcal{L} \mathcal{P}_s \right) \right] ds \\ &\quad + \rho \sigma \epsilon \int_t^T \varphi(s, T) \mathbb{E}_t \left[N(t, s) ((\partial_x \mathcal{L}) \mathcal{P}_s - \partial_x \mathcal{P}_s) \right] ds. \end{aligned} \tag{4.3}$$

An analytic approximation.

All the derivatives of the Black and Scholes pricing function $\partial_{x^n} \mathcal{P}_s$ verify the same PDE (2.5), and when appropriately discounted, they become martingales. This property suggests one might try to substitute those terms with their initial value $\partial_{x^n} \mathcal{P}_t$, by means of a proper conditioning,

Unfortunately, in our representation the expectations cannot be disentangled, because of the correlation between X and r . Moreover the terms are not perfectly aligned, in the sense that X_s depends on the process $\int_t^s r_u du$, while the pricing function $\mathcal{P}$ is applied to the adapted process $\int_s^T \mathbb{E}_s(r_u) du$.

Nevertheless, to derive a manageable approximation formula we decide to replace

$$\mathbb{E}_t \left(e^{-\varphi(s, T) r_s + \psi(s, T)} e^{-\int_t^s r_u du} b(r)_s \partial_{x^n} \mathcal{P}_s \right) \quad \text{with} \quad \partial_{x^n} \mathcal{P}_t e^{\psi(s, T)} \mathbb{E}_t \left(e^{-\varphi(s, T) r_s} b(r)_s \right)$$

since we may effectively bound the error. Indeed, up to the deterministic factor $e^{\psi(s, T)}$, we may apply Hölder's inequality to obtain

$$\begin{aligned} &\left| \mathbb{E}_t \left[e^{-\varphi(s, T) r_s} b(r)_s \left(e^{-\int_t^s r_u du} \partial_{x^n} \mathcal{P}_s - \partial_{x^n} \mathcal{P}_t \right) \right] \right| \\ &\leq \left[\mathbb{E}_t \left(e^{-2\varphi(s, T) r_s} b(r)_s^2 \right) \right]^{1/2} \left[\mathbb{E}_t \left(\left| e^{-\int_t^s r_u du} \partial_{x^n} \mathcal{P}_s - \partial_{x^n} \mathcal{P}_t \right|^2 \right) \right]^{1/2}. \end{aligned}$$

The first factor is explicitly computable in the models considered below, so that it remains to estimate the second one. To do so, we apply Itô's formula directly to the discounted process $e^{-\int_t^s r_u du} \partial_{x^n} \mathcal{P}(s, X_s, R_s)$.

Using the dynamics of X and R and arguing as in the proof of Proposition 3.1, relations (2.5) and (2.7) yield

$$\begin{aligned} d \left(e^{-\int_t^s r_u du} \partial_{x^n} \mathcal{P}(s, X_s, R_s) \right) &= e^{-\int_t^s r_u du} \left[\frac{1}{2} \mathcal{L}^2 \partial_{x^n} \mathcal{P}_s d\langle M, M \rangle_s + \partial_x \mathcal{L} \partial_{x^n} \mathcal{P}_s d\langle X, M \rangle_s \right] \\ &\quad + e^{-\int_t^s r_u du} \left[\sigma \partial_{x^{n+1}} \mathcal{P}_s dW_s^x + \mathcal{L} \partial_{x^n} \mathcal{P}_s dM_s \right]. \end{aligned}$$

In the Hull–White model, we have $dM_s = \epsilon \varphi(s, T) dW_s^r$, and the finite-variation and martingale coefficients in the above dynamics are bounded, up to the discount factor, since all the x -derivatives of the Black–Scholes pricing function involved in $\mathcal{L}^2 \partial_{x^n} \mathcal{P}$, $\partial_x \mathcal{L} \partial_{x^n} \mathcal{P}$, $\partial_{x^{n+1}} \mathcal{P}$, and $\mathcal{L} \partial_{x^n} \mathcal{P}$ are uniformly bounded for every fixed n . Applying Itô's formula to the square of the increment and taking conditional expectations, the martingale terms vanish. Using Cauchy–Schwarz inequality and the finiteness of the second moments of the discount factor, we obtain, for a suitable constant C_n ,

$$\mathbb{E}_t \left[\left| e^{-\int_t^s r_u du} \partial_{x^n} \mathcal{P}(s, X_s, R_s) - \partial_{x^n} \mathcal{P}(t, x, R_t) \right|^2 \right]$$

$$\leq C_n \int_t^s \mathbb{E}_t \left[\left| e^{-\int_t^u r_v dv} \partial_x^n \mathcal{P}(u, X_u, R_u) - \partial_x^n \mathcal{P}(t, x, R_t) \right|^2 \right] du + C_n(s-t).$$

Therefore, by Gronwall's inequality,

$$E_t \left[\left| e^{-\int_t^s r_u du} \partial_x^n P(s, X_s, R_s) - \partial_x^n P(t, x, R_t) \right|^2 \right] \leq C_n(s-t),$$

where C_n may change from line to line. Hence the L^2 error produced by freezing the Black–Scholes derivatives at their initial value is of order $\sqrt{s-t}$.

With the above choice, we obtain the approximation

$$\begin{aligned} V_t &\approx N(t, t) \mathcal{P}_t + \mathcal{P}_t \int_t^T e^{\psi(s, T)} \mathbb{E}_t \left[e^{-\varphi(s, T) r_s} r_s \right] ds \\ &\quad + \epsilon^2 \left(\frac{\mathcal{L}^2}{2} \mathcal{P}_t - \mathcal{L} \mathcal{P}_t \right) \int_t^T \varphi^2(s, T) e^{\psi(s, T)} \mathbb{E}_t \left[e^{-\varphi(s, T) r_s} \right] ds \\ &\quad + \rho \sigma \epsilon ((\partial_x \mathcal{L}) \mathcal{P}_t - \partial_x \mathcal{P}_t) \int_t^T \varphi(s, T) e^{\psi(s, T)} \mathbb{E}_t \left[e^{-\varphi(s, T) r_s} \right] ds. \end{aligned} \quad (4.4)$$

All the expectations appearing in (4.4) are available in closed form for the Hull–White model, since using (4.2) we may compute

$$\begin{aligned} \mathbb{E}_t \left[e^{-\varphi(s, T) r_s} \right] &= \exp \left\{ -\varphi(s, T) m_t(s) + \frac{1}{2} \varphi(s, T)^2 v_t(s) \right\}, \\ \mathbb{E}_t \left[r_s e^{-\varphi(s, T) r_s} \right] &= (m_t(s) - \varphi(s, T) v_t(s)) \exp \left\{ -\varphi(s, T) m_t(s) + \frac{1}{2} \varphi(s, T)^2 v_t(s) \right\}. \end{aligned}$$

where $m_t(s) = \theta(s) + (r_t - \theta(t))e^{-\kappa(s-t)}$, $v_t(s) = \frac{\epsilon^2}{2\kappa} (1 - e^{-2\kappa(s-t)})$.

The numerical analysis in the last section indicates that the main contribution to the approximation error is associated with the first integral term in the representation formula (4.3). Hence, looking for a more accurate approximation, we can apply Ito's formula again to the process $(N(t, s) r_s \mathcal{P}_s)$, to better untangle the dependence among those three factors. Doing so, and freezing all the $\mathcal{P}$ -derivatives terms at the initial time, we may approximate that first integrand term as

$$\begin{aligned} \widetilde{\mathbb{E}}_t^{\text{approx}} (N(t, s) r_s \mathcal{P}_s) &= \mathcal{P}_t N(t, t) \left(r_t + \kappa \int_t^s \left[\theta(u) + \frac{\theta'(u)}{\kappa} \right] du \right) \\ &\quad + \mathcal{P}_t \left[\int_t^s \mathbb{E}_t [N(t, u) r_u^2] du - \kappa \int_t^s \mathbb{E}_t [N(t, u) r_u] du \right] \\ &\quad + \epsilon^2 (\mathcal{L} - I) \mathcal{P}_t \int_t^s \varphi(u, T) \mathbb{E}_t [N(t, u)] du \\ &\quad + \epsilon^2 \left(\frac{1}{2} \mathcal{L} - I \right) \mathcal{L} \mathcal{P}_t \int_t^s \varphi^2(u, T) \mathbb{E}_t [N(t, u) r_u] du \\ &\quad + \rho \sigma \epsilon \partial_x \mathcal{P}_t \int_t^s \mathbb{E}_t [N(t, u)] du \\ &\quad + \rho \sigma \epsilon (\mathcal{L} - I) \partial_x \mathcal{P}_t \int_t^s \varphi(u, T) \mathbb{E}_t [N(t, u) r_u] du. \end{aligned} \quad (4.5)$$

Since $N(t, \cdot)$ is a martingale, $\mathbb{E}_t[N(t, u)] = N(t, t)$ while $\mathbb{E}_t[N(t, u)r_u]$, $\mathbb{E}_t[N(t, u)r_u^2]$ are also available in closed form, since $(r_u, \int_t^u r_v dv)$ is conditionally jointly Gaussian. Thus, both (4.4) and the refined approximation (4.5) retain a semi-explicit structure.

As mentioned before, the Vasicek model is obtained by choosing $\theta(s) \equiv \beta$ for some constant β , and this is the model we will use in the numerical section to benchmark our approximation formula.

4.2 The CIR++ model

We now consider the CIR++ specification, obtained by adding a deterministic shift to a CIR factor. More precisely, we set $r_s = \tilde{r}_s + \theta(s)$, where $\theta : [0, T] \rightarrow \mathbb{R}$ is a deterministic continuously differentiable function and the process $\tilde{r}$ verifies

$$d\tilde{r}_s = \kappa(\beta - \tilde{r}_s)ds + \epsilon\sqrt{\tilde{r}_s}dW_s^r, \quad \tilde{r}_t = \tilde{r} \geq 0, \quad (4.6)$$

under the Feller condition, $\kappa\beta > \frac{\epsilon^2}{2}$, that guarantees the strict positivity of the process.

The deterministic shift θ can be chosen so as to fit the initial term structure of interest rates and the dynamics of the short rate is given by

$$dr_s = [\theta'(s) + \kappa(\beta + \theta(s) - r_s)]ds + \epsilon\sqrt{r_s - \theta(s)}dW_s^r.$$

For the CIR factor, the zero-coupon bond has the exponential-affine form (see [24])

$$E_s \left[e^{-\int_s^T \tilde{r}_u du} \right] = e^{-\varphi(s, T)\tilde{r}_s + \psi(s, T)},$$

where

$$\varphi(s, T) = \frac{2(e^{p(T-s)} - 1)}{p - \kappa + (p + \kappa)e^{p(T-s)}}, \quad p^2 = \kappa^2 + 2\epsilon^2,$$

and

$$\psi(s, T) = \frac{2\kappa\beta}{\epsilon^2} \log \left[\frac{2pe^{(p+\kappa)(T-s)/2}}{p - \kappa + (p + \kappa)e^{p(T-s)}} \right].$$

Consequently,

$$N(t, s) = e^{-\int_t^s r_u du} e^{-\int_s^T \theta(u) du} e^{-\varphi(s, T)(r_s - \theta(s)) + \psi(s, T)}$$

$$\text{and } dN(t, s) = -\epsilon\varphi(s, T)\sqrt{\tilde{r}_s}N(t, s)dW_s^r.$$

With the notation introduced in Section 3.1, we have

$$b(\tilde{r})_s = \epsilon\sqrt{\tilde{r}_s}, \quad H_s = \frac{\epsilon}{\kappa}\sqrt{\tilde{r}_s}(1 - e^{-\kappa(T-s)}), \quad dM_s = \frac{\epsilon}{\kappa}\sqrt{\tilde{r}_s}(1 - e^{-\kappa(T-s)})dW_s^r.$$

Substituting these quantities into (3.11) yields the representation

$$\begin{aligned}
V_t = & N(t, t)\mathcal{P}_t + \int_t^T \mathbb{E}_t \left[r_s N(t, s) \mathcal{P}_s \right] ds \\
& + \frac{\epsilon^2}{2} \int_t^T \frac{(1 - e^{-\kappa(T-s)})^2}{\kappa^2} \mathbb{E}_t \left[\tilde{r}_s N(t, s) \mathcal{L}^2 \mathcal{P}_s \right] ds \\
& - \epsilon^2 \int_t^T \frac{1 - e^{-\kappa(T-s)}}{\kappa} \varphi(s, T) \mathbb{E}_t \left[\tilde{r}_s N(t, s) \mathcal{L} \mathcal{P}_s \right] ds \\
& + \rho \sigma \epsilon \int_t^T \frac{1 - e^{-\kappa(T-s)}}{\kappa} \mathbb{E}_t \left[\sqrt{\tilde{r}_s} N(t, s) \partial_x \mathcal{L} \mathcal{P}_s \right] ds \\
& - \rho \sigma \epsilon \int_t^T \varphi(s, T) \mathbb{E}_t \left[\sqrt{\tilde{r}_s} N(t, s) \partial_x \mathcal{P}_s \right] ds.
\end{aligned} \tag{4.7}$$

Remark 4.1 When considering the classical CIR model ($\theta \equiv 0$), $\tilde{r} \equiv r$ and (4.7) leads to the approximation formula, by freezing the $\mathcal{P}$ -derivatives terms at the initial time

$$\begin{aligned}
V_t \approx & \mathcal{P}_t \left[N(t, t) + \int_t^T \mathbb{E}_t \left[r_s N(t, s) \right] ds \right] \\
& + \epsilon^2 \mathcal{L}(\mathcal{L} - I) \mathcal{P}_t \int_t^T \varphi^2(s, T) \mathbb{E}_t \left[r_s N(t, s) \right] ds \\
& + \rho \sigma \epsilon (\mathcal{L} - I) \partial_x \mathcal{P}_t \int_t^T \varphi(s, T) \mathbb{E}_t \left[\sqrt{r_s} N(t, s) \right] ds.
\end{aligned} \tag{4.8}$$

Again, we need to evaluate factors of the type $\mathbb{E}_t \left[e^{-\varphi(s, T) r_s} r_s^i \right]$ for $i = \frac{1}{2}, 1$, which can be approximated using an affine-transform approach for the computation of the moment generating function $\mathbb{E}_t \left[e^{-\varphi(s, T) r_s} \right]$ (see [24]). Specifically, we approximate the non-polynomial payoff (the square root case) by a Chebyshev polynomial expansion and evaluate the resulting moments through derivatives of the moment generating function, which are available in closed form for the CIR process.

Also in this model we may consider an improved approximation for the first integral of the representation formula considering the covariance term given by a further application of Itô's formula.

5 Numerical results

In this section, we investigate two complementary numerical uses of the representation formulas derived above. First, we assess whether the representation itself can provide an efficient Monte Carlo estimator of the option price, with particular emphasis on variance reduction with respect to the standard estimator based on the discounted payoff. Second, we assess the accuracy and computational efficiency of the analytical approximations (4.4)–(4.5) in the Vasicek setting and of the approximation of the representation formula (4.7) in the CIR framework.

For the numerical experiments, we consider the unshifted versions of the two short-rate models, that is, we set $\theta \equiv 0$. Since the shift is deterministic, this choice allows us to focus on the stochastic interest-rate component without affecting the structure of the representation or the variance-reduction mechanism discussed below.

We consider an ATM put option ($X_0 = K = \log(100)$) and analyze two benchmark scenarios characterized by different levels of interest-rate volatility. For both models we fix

$$\kappa = 0.8, \quad \beta = 0.04, \quad \rho = 0.1, \quad r_0 = 0.035, \quad \sigma = 0.2,$$

and characterize the two interest-rate scenarios by choosing

$$\epsilon = 0.01 \quad (\text{Scenario 1}), \quad \epsilon = 0.05 \quad (\text{Scenario 2}).$$

These parameter values are representative of the configurations commonly considered for these short-rate models. The initial time t is set to 0, while the maturities correspond to 91, 183, 273, and 365 calendar days, approximately 0.25, 0.50, 0.75, and 1 year, respectively. The computational procedure was implemented using MatLab R2024b on a MacBook M3Pro.

Numerical implementation and efficiency measures.

Let

$$Y^{\text{pay}} = \exp \left(- \int_0^T r_s ds \right) (e^K - e^{X_T})^+$$

denote the standard discounted-payoff random variable and let Y^{rep} denote its representation-based counterpart introduced in Remark 3.1. Recall that

$$Y^{\text{pay}} = Y^{\text{rep}} + \mathcal{M}_{0,T}, \quad E[\mathcal{M}_{0,T}] = 0. \quad (5.1)$$

Consequently, both random variables provide unbiased continuous-time representations of the same option price,

$$E[Y^{\text{pay}}] = E[Y^{\text{rep}}] = C_0.$$

Moreover,

$$\text{Var}(Y^{\text{rep}}) = \text{Var}(Y^{\text{pay}}) + \text{Var}(\mathcal{M}_{0,T}) - 2 \text{Cov}(Y^{\text{pay}}, \mathcal{M}_{0,T}). \quad (5.2)$$

Thus, the martingale term plays the role of a natural variance-reduction correction. In particular,

$$\text{Var}(Y^{\text{rep}}) < \text{Var}(Y^{\text{pay}}) \quad \Longleftrightarrow \quad 2 \text{Cov}(Y^{\text{pay}}, \mathcal{M}_{0,T}) > \text{Var}(\mathcal{M}_{0,T}). \quad (5.3)$$

The effectiveness of the representation-based estimator therefore depends on the extent to which the martingale component absorbs the random fluctuations of the discounted payoff.

Remark 5.1 *It is important to stress that the martingale term $\mathcal{M}_{0,T}$ is not needed to construct the representation-based estimator. The latter is computed directly from the terms in the representation formula, whereas $\mathcal{M}_{0,T}$ is only used to interpret the variance-reduction mechanism and, in the numerical implementation, to perform ex-post diagnostic checks on the simulated paths, such as checking that its empirical mean is close to zero.*

Given N simulated trajectories, we consider the Monte Carlo estimators

$$\hat{C}_{\text{pay}} = \frac{1}{N} \sum_{j=1}^N Y_j^{\text{pay}}, \quad \hat{C}_{\text{rep}} = \frac{1}{N} \sum_{j=1}^N Y_j^{\text{rep}},$$

and denote by $\widehat{\text{Var}}(Y^{\text{pay}})$ and $\widehat{\text{Var}}(Y^{\text{rep}})$ the corresponding sample variances.

Since a reduction in variance may be obtained at the cost of a larger computational effort, we compare the two estimators using both statistical and computational measures. We define the variance-reduction factor as

$$\text{VR} := \frac{\widehat{\text{Var}}(Y^{\text{pay}})}{\widehat{\text{Var}}(Y^{\text{rep}})}, \quad (5.4)$$

and the relative computational cost as

$$\text{TP} := \frac{\text{CPU}_{\text{rep}}}{\text{CPU}_{\text{pay}}}. \quad (5.5)$$

The overall computational efficiency is then measured by

$$\text{RE} := \frac{\widehat{\text{Var}}(Y^{\text{pay}}) \text{CPU}_{\text{pay}}}{\widehat{\text{Var}}(Y^{\text{rep}}) \text{CPU}_{\text{rep}}} = \frac{\text{VR}}{\text{TP}}. \quad (5.6)$$

Thus, $\text{RE} > 1$ indicates that the representation-based estimator is more efficient than the naive Monte Carlo estimator after accounting for its additional computational cost (see [17]).

In all Monte Carlo experiments we use $N = 10^6$ sample paths and a daily time grid with step size $\Delta = 1/365$. For the Vasicek model, the short rate, its time integral, and the Brownian increment correlated with the underlying are simulated jointly from their exact Gaussian distribution. For the CIR model, the short-rate process is generated by a full-truncation Euler scheme, which preserves non-negativity at the discrete level. The same random sample is used when comparing the payoff and representation-based estimators.

For the variance-reduction experiments, the time integrals entering Y^{rep} are evaluated pathwise on the daily grid by the trapezoidal rule. CPU times are measured separately for the two estimators after an initial warm-up, and the median over repeated runs is used to reduce the impact of initialization and timing noise.

For the analytical approximation experiments, Monte Carlo simulation is used only as a benchmark to evaluate the accuracy of the proposed approximations. More precisely, the expectations defining each integral term in the exact representation formula are estimated by Monte Carlo, and the resulting values are compared with the corresponding semi-analytical approximations¹.

We denote by C^{app} the option value obtained from these semi-analytical approximations. The remaining deterministic time integrals are computed by Simpson quadrature. Monte Carlo estimates of the corresponding exact representation terms are therefore used exclusively for validation purposes and do not enter the computation of C^{app} .

In the Vasicek framework, where the exact option price C^{ex} is available, the pricing error is defined as

$$\text{Err}_{\text{app}} = C^{\text{ex}} - C^{\text{app}}.$$

In the CIR framework, where a closed-form option price is not available, the Monte Carlo evaluation of the representation formula is used as benchmark, so that

$$\text{Err}_{\text{app}} = \hat{C}_{\text{rep}} - C^{\text{app}}.$$

The errors associated with the individual terms of the analytical approximations will be defined separately in the corresponding model subsections.

5.1 The Vasicek model

We first consider the Vasicek specification, corresponding to the Hull–White model of Section 4 with $\theta \equiv 0$. This case is particularly useful for assessing the numerical performance of the representation, since the European option price is available in closed form and can therefore be used as an exact benchmark.

Monte Carlo variance reduction.

We first investigate the performance of the representation - based Monte Carlo estimator associated with (4.3). For each parameter configuration, the naïve estimator $\hat{C}_{\text{pay}}$ and the representation-based estimator $\hat{C}_{\text{rep}}$ are computed using the same simulated sample. Table 1 reports the corresponding estimates,

¹Even in affine settings, including the jointly Gaussian Vasicek model, the expectations entering the representation are generally not available in closed form because of the nonlinear dependence of the Black–Scholes terms on the state variables. In fact, the approximations in Section 4 replace them with tractable semi-analytical quantities.

together with the variance-reduction factor VR, the relative computational cost TP, and the overall efficiency measure RE defined in (5.4)–(5.6).

Scenario	T	C^{ex}	$\hat{C}_{\text{pay}}$	$\hat{C}_{\text{rep}}$	VR	TP	RE
1	0.25	3.542517	3.544562	3.542532	4.61×10^4	2.111	2.18×10^4
1	0.50	4.754747	4.762299	4.754832	1.12×10^4	2.083	5.37×10^3
1	0.75	5.557379	5.562417	5.557519	4.94×10^3	2.075	2.38×10^3
1	1.00	6.174733	6.191148	6.175060	2.72×10^3	2.136	1.28×10^3
2	0.25	3.554084	3.556325	3.554084	3.74×10^4	2.091	1.79×10^4
2	0.50	4.791199	4.799083	4.791264	7.84×10^3	2.142	3.66×10^3
2	0.75	5.627947	5.631649	5.628191	3.23×10^3	2.104	1.53×10^3
2	1.00	6.287819	6.301263	6.288333	1.71×10^3	2.148	7.95×10^2

Table 1: Monte Carlo variance reduction in the Vasicek model. VR denotes the variance-reduction factor, TP the relative computational cost, and $\text{RE} = \text{VR}/\text{TP}$ the overall efficiency ratio.

The results show a substantial variance reduction for all maturities and interest-rate volatility levels. In particular, VR ranges from approximately 1.7×10^3 to 4.6×10^4 , whereas the computational cost of the representation-based estimator is only about twice that of the naive estimator. Consequently, the efficiency ratio remains well above one in all cases, ranging from approximately 8×10^2 to more than 2×10^4 . The gain is more pronounced for shorter maturities, but remains substantial also for $T = 1$ and under the higher interest-rate volatility scenario.

It is also worth observing that the representation-based estimates are very close to the corresponding closed-form prices. The largest absolute difference over the configurations considered is approximately 5.1×10^{-4} . These results provide strong numerical evidence that the martingale term in (5.1) removes a substantial fraction of the random fluctuations affecting the discounted payoff, consistently with condition (5.3).

Analytical approximation.

We next assess the approximation (4.4), incorporating the refined treatment of the first integral term given in (4.5). Monte Carlo simulation is used here only to obtain benchmark values for the individual terms of the exact representation, whereas the approximation itself is computed without Monte Carlo simulation.

As in the previous analysis, we measure separately the errors associated with the three integral terms entering (4.3). More precisely, we set

$$\begin{aligned}\varepsilon_1 &= \int_0^T \mathbb{E}_0 [N(0, s) r_s \mathcal{P}_s] ds - \int_0^T \widetilde{\mathbb{E}}_0^{\text{approx}} [N(0, s) r_s \mathcal{P}_s] ds, \\ \varepsilon_2 &= \epsilon^2 \left\{ \int_0^T \varphi^2(s, T) \mathbb{E}_0 \left[N(0, s) \left(\frac{1}{2} \mathcal{L}^2 \mathcal{P}_s - \mathcal{L} \mathcal{P}_s \right) \right] ds \right. \\ &\quad \left. - \left(\frac{\mathcal{L}^2}{2} \mathcal{P}_0 - \mathcal{L} \mathcal{P}_0 \right) \int_0^T \varphi^2(s, T) e^{\psi(s, T)} \mathbb{E}_0 \left[e^{-\varphi(s, T) r_s} \right] ds \right\}, \\ \varepsilon_3 &= \rho \sigma \epsilon \left\{ \int_0^T \varphi(s, T) \mathbb{E}_0 [N(0, s) ((\partial_x \mathcal{L}) \mathcal{P}_s - \partial_x \mathcal{P}_s)] ds \right. \\ &\quad \left. - ((\partial_x \mathcal{L}) \mathcal{P}_0 - \partial_x \mathcal{P}_0) \int_0^T \varphi(s, T) e^{\psi(s, T)} \mathbb{E}_0 \left[e^{-\varphi(s, T) r_s} \right] ds \right\}.\end{aligned}$$

The individual errors ε_i are therefore computed with respect to the Monte Carlo evaluation of the corresponding exact representation terms, whereas the overall approximation error is evaluated against the closed-form option price,

$$\text{Err}_{\text{app}} = C^{\text{ex}} - C^{\text{app}}.$$

Tables 2 and 3 report the results for the low- and high-rate-volatility scenarios, respectively.

	$T = 0.25$	$T = 0.50$	$T = 0.75$	$T = 1$
C^{ex}	3.542517	4.754747	5.557379	6.174733
C^{app}	3.542499	4.754603	5.556909	6.173640
Err_{app}	1.7539e-05	1.4355e-04	4.6968e-04	1.0930e-03
ε_1	3.3804e-05	2.3157e-04	6.1590e-04	1.4352e-03
ε_2	-2.7191e-08	-1.0881e-07	-2.9591e-07	-7.7678e-07
ε_3	-1.1023e-06	-3.1945e-06	-5.5911e-06	-1.4741e-05

Table 2: Approximation (4.4)–(4.5) in the Vasicek model: Scenario 1 (low rate volatility).

	$T = 0.25$	$T = 0.50$	$T = 0.75$	$T = 1$
C^{ex}	3.554084	4.791199	5.627947	6.287819
C^{app}	3.553995	4.790287	5.624556	6.279186
Err_{app}	8.9538e-05	9.1217e-04	3.3910e-03	8.6329e-03
ε_1	1.1445e-04	1.1105e-03	3.9802e-03	9.8284e-03
ε_2	-3.0673e-06	-2.5452e-05	-8.0278e-05	-1.6528e-04
ε_3	-2.2007e-05	-1.0797e-04	-2.6459e-04	-5.1645e-04

Table 3: Approximation (4.4)–(4.5) in the Vasicek model: Scenario 2 (high rate volatility).

The approximation remains accurate over all the parameter configurations considered. As expected, the error increases with both maturity and interest-rate volatility. In Scenario 1, the relative pricing error remains below 0.02% even at the one-year maturity. In Scenario 2, the largest absolute error is approximately 8.6×10^{-3} at $T = 1$, corresponding to a relative error of about 0.14%.

The term-by-term analysis confirms that the main source of approximation error is the first integral,

$$\int_0^T \mathbb{E}_0 [N(0, s) r_s \mathcal{P}_s] ds,$$

which is not multiplied by either the interest-rate volatility or the correlation parameter. The errors associated with the remaining terms are substantially smaller. This also confirms the relevance of the refined approximation (4.5) for the first integral term.

5.2 The CIR model

We next consider the CIR specification, corresponding to the CIR++ model of Section 4 with $\theta \equiv 0$. In this case, a closed-form expression for the European option price is not available under asset–interest rate correlation. Accordingly, the Monte Carlo evaluation of the representation formula (4.7) will be used as the benchmark for assessing the semi-analytical approximation.

Monte Carlo variance reduction.

We first investigate whether the representation-based estimator associated with (4.7) also provides an effective variance-reduction device in the square-root short-rate framework. As in the Vasicek case, the naïve estimator $\hat{C}_{\text{pay}}$ and the representation-based estimator $\hat{C}_{\text{rep}}$ are evaluated using the same simulated paths. Their performance is compared through the variance-reduction factor VR, the relative computational cost TP, and the efficiency ratio RE defined in (5.4)–(5.6).

Scenario	T	$\hat{C}_{\text{pay}}$	$\hat{C}_{\text{rep}}$	VR	TP	RE
1	0.25	3.541722	3.540557	4.54×10^4	2.624	1.73×10^4
1	0.50	4.751848	4.749362	1.10×10^4	2.946	3.74×10^3
1	0.75	5.548521	5.547834	4.87×10^3	3.030	1.61×10^3
1	1.00	6.160638	6.160538	2.69×10^3	2.887	9.32×10^2
2	0.25	3.543534	3.542371	4.62×10^4	2.907	1.59×10^4
2	0.50	4.756862	4.754386	1.12×10^4	3.001	3.72×10^3
2	0.75	5.557532	5.556701	4.94×10^3	3.031	1.63×10^3
2	1.00	6.173875	6.173779	2.73×10^3	2.929	9.32×10^2

Table 4: Monte Carlo variance reduction in the CIR model. VR denotes the variance-reduction factor, TP the relative computational cost, and $\text{RE} = \text{VR}/\text{TP}$ the overall efficiency ratio.

The results show that the variance-reduction properties of the representation remain substantial also under square-root short-rate dynamics. The factor VR ranges from approximately 2.7×10^3 to 4.6×10^4 , while the computational cost of the representation-based estimator is only about three times that of the naïve estimator. As a result, the overall efficiency ratio remains very large in all the configurations considered, ranging from approximately 9.3×10^2 to 1.7×10^4 . As in the Vasicek model, the efficiency gain is larger at short maturities and decreases as T increases, although it remains quantitatively significant even at the one-year horizon. Moreover, the results are remarkably stable across the two interest-rate volatility scenarios. This confirms that the variance-reduction mechanism induced by the decomposition (5.1) is not specific to the Gaussian short-rate specification, but remains effective also in the presence of nonlinear square-root rate dynamics.

Analytical approximation.

We finally assess the semi-analytical approximation of the representation formula (4.7). Monte Carlo simulation is used only to provide benchmark values for the individual terms of the exact representation, whereas the approximation itself is evaluated without Monte Carlo simulation.

Since the CIR representation contains five integral contributions, we denote by

$$\widehat{\varepsilon}_i, \quad i = 1, \dots, 5,$$

the difference between the Monte Carlo evaluation of the i th exact integral term in (4.7) and its corresponding semi-analytical approximation. Thus,

$$\widehat{\varepsilon}_i = I_i^{\text{MC}} - I_i^{\text{app}}, \quad i = 1, \dots, 5.$$

The overall approximation error is measured with respect to the Monte Carlo evaluation of the representation formula,

$$\text{Err}_{\text{app}} = \widehat{C}_{\text{rep}} - C^{\text{app}}.$$

Notice that this benchmark is preferable to the naive payoff estimator for this purpose, since it is directly associated with the exact representation whose individual terms are being approximated.

Tables 5 and 6 report the results for the low- and high-rate-volatility scenarios, respectively.

	$T = 0.25$	$T = 0.50$	$T = 0.75$	$T = 1$
$\widehat{C}_{\text{rep}}$	3.540557	4.749362	5.547833	6.160538
C^{app}	3.540597	4.749473	5.548127	6.161001
Err_{app}	-3.9858e-05	-1.1124e-04	-2.9346e-04	-4.6243e-04
$\widehat{\varepsilon}_1$	-4.1278e-05	-1.1349e-04	-2.9693e-04	-4.6408e-04
$\widehat{\varepsilon}_2$	-2.2097e-10	-3.9877e-09	-9.7047e-09	-3.9668e-08
$\widehat{\varepsilon}_3$	-2.3746e-10	-1.8853e-09	-1.2378e-08	-2.3344e-08
$\widehat{\varepsilon}_4$	1.2824e-06	2.3765e-06	3.1038e-06	1.4112e-06
$\widehat{\varepsilon}_5$	1.3801e-07	3.8099e-07	3.9397e-07	3.1067e-07

Table 5: Approximation of (4.7) in the CIR model: Scenario 1 (low rate volatility).

	$T = 0.25$	$T = 0.50$	$T = 0.75$	$T = 1$
$\widehat{C}_{\text{rep}}$	3.542371	4.754386	5.556701	6.173779
C^{app}	3.542605	4.755247	5.558577	6.176897
Err_{app}	-2.3376e-04	-8.6076e-04	-1.8759e-03	-3.1180e-03
$\widehat{\varepsilon}_1$	-2.4045e-04	-8.7085e-04	-1.8824e-03	-3.1025e-03
$\widehat{\varepsilon}_2$	-5.1795e-08	-6.2101e-07	-2.1579e-06	-5.7834e-06
$\widehat{\varepsilon}_3$	-5.0197e-08	-5.2008e-07	-2.0463e-06	-4.8931e-06
$\widehat{\varepsilon}_4$	6.3011e-06	1.0784e-05	1.2240e-05	1.1259e-08
$\widehat{\varepsilon}_5$	3.9889e-07	4.4280e-07	-1.5463e-06	-4.8785e-06

Table 6: Approximation of (4.7) in the CIR model: Scenario 2 (high rate volatility).

The results confirm that the semi-analytical approximation remains accurate also under square-root interest-rate dynamics. As in the Vasicek model, the approximation error increases with maturity and with the volatility of the short rate. In Scenario 1, the absolute pricing error remains below 5×10^{-4}

for all maturities considered. In Scenario 2, the largest error is approximately 3.1×10^{-3} at the one-year maturity, corresponding to about 0.05% of the representation-based option value.

The term-by-term analysis shows that the approximation error is again largely concentrated in the first integral contribution. The errors associated with the remaining terms are one or more orders of magnitude smaller over most of the parameter configurations considered. This confirms that the principal numerical difficulty is associated with the term involving $\int_0^T \mathbb{E}_0 [N(0, s) r_s \mathcal{P}_s] ds$, whereas the terms weighted by the short-rate volatility and the correlation parameters are substantially less sensitive to the approximation.

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Appendix A Malliavin Calculus

Here we introduce the main concepts of Malliavin Calculus used in this work. We refer to [26] or [6] for a deeper approach to this field.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a right continuous and complete filtration $\{\mathcal{F}_t\}$ and an $\mathcal{F}_t$ -adapted Brownian motion, W , defined on it, let $\mathcal{S}$ be the set of random variables of the form

$$F = f(W(h_1), \dots, W(h_n)), \quad (\text{A1})$$

with $h_1, \dots, h_n \in L^2([0, T])$, $W(h_i)$ denotes the Wiener integral of the function h_i , for $i = 1, \dots, n$, and $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., f and all its partial derivatives are bounded). The Malliavin derivative of F , DF , is defined as the stochastic process in $L^2(\Omega \times [0, T])$ given by

$$D_s F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n))(s) h_j(s).$$

We can also define the iterated Malliavin derivative operator as

$$D_{s_1, \dots, s_m}^m F = D_{s_1} \dots D_{s_m} F, \quad s_1, \dots, s_m \in [0, T].$$

These operators are closable in $L^p(\Omega)$ for any $p \geq 1$ and we denote by $\mathbb{D}^{n,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{n,p} = \left(\mathbb{E}(|F|^p) + \sum_{i=1}^n \mathbb{E}(\|D^i F\|_{L^2([0, T]^i)}^p) \right)^{\frac{1}{p}}.$$

We will also make use of the notation $\mathbb{L}^{n,p} = \mathbb{D}_W^{n,p}(L^2([0, T]))$.

The adjoint of the derivative operator D is the divergence operator δ , that coincides with the Skorohod integral. That is, the domain of δ , denoted by $\text{Dom } \delta$, is the set of processes $u \in L^2(\Omega \times [0, T])$ such that there exists $\delta(u) \in L^2(\Omega)$ such that

$$\mathbb{E}(\delta(u)F) = \mathbb{E} \left(\int_0^T (D_s F) u_s ds \right), \quad \text{for every } F \in \mathcal{S}. \quad (\text{A2})$$

We use the notation $\delta(u) = \int_0^T u_s dW_s$. It is known that δ is an extension of the classical Itô integral in the sense that δ , applied to adapted and square integrable processes, coincides with the classical Itô integral. We also know that the space $\mathbb{L}^{1,2}$ is included in the domain of δ .

The following result gives us an explicit martingale representation for random variables in $\mathbb{D}_W^{1,2}$.

Theorem A.1 (The Clark-Ocone formula) *Consider a random variable $F \in \mathbb{D}_W^{1,2}$. Then*

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}_s(D_s F) dW_s.$$

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