

Derivations, conventions and diagnostics

A companion to *Dissolution of the collinear structure in the solar-sail circular restricted three-body problem*

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Abstract

This is the long-form working document behind the preprint. Nothing here is new physics; everything is the algebra the paper compresses or states without proof, written out step by step, together with an account of what every reported residual actually measures. It exists so that each number in the manuscript can be checked by hand rather than taken on trust. Section 12 in particular explains the floating-point quantities — the 10^{-16} 's — which are easy to misread as physics.

Two honest caveats are recorded here rather than buried: the 8.1×10^{-10} quoted in the manuscript's frequency scan is an artefact of grid alignment and overstates that scan's resolving power (Section 12.4), and the Δ_z branch discriminant weakens by an order of magnitude across the sail band (Section 12.6).

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1 Conventions, units and notation

1.1 What is set to unity

The circular restricted three-body problem is non-dimensionalised so that three independent quantities are unity:

1. the separation of the two primaries, $L = 1$;
2. the total mass, $M_1 + M_2 = 1$, so that the primaries have masses $1 - \mu$ and μ with $\mu = M_2/(M_1 + M_2)$;
3. the mean motion of the primaries about their common barycentre, $n = 1$, which with $L = 1$ and $G(M_1 + M_2) = 1$ is Kepler’s third law $n^2 L^3 = G(M_1 + M_2)$.

Table 1: Notation.

symbol	meaning
μ	mass ratio $M_2/(M_1 + M_2)$
β	lightness number, sail acceleration / solar gravity
r_1, r_2	distances to the larger and smaller primary
γ	local length scale $ (1 - \mu) - x_{\text{eq}} $
r_H	Hill radius $(\mu/3)^{1/3}$
U	modified pseudo-potential, eq. (6)
$A \equiv c_2$	second Legendre coefficient of the collinear expansion
S, s	the solar and tidal contributions to A , eq. (20)
κ	the level at which s is held, $s = \kappa$
λ_u	unstable (saddle) exponent, in-plane
ω	in-plane centre frequency
ν	out-of-plane frequency
τ_u	instability e-folding time, $1/\lambda_u$
α, δ	sail cone and clock angles

Setting $n = 1$ fixes the unit of time. One non-dimensional (nd) time unit is $1/n$, and one full revolution of the primaries is 2π nd. Hence

$$1 \text{ nd time unit} = \frac{T_{\text{orbit}}}{2\pi} = \frac{365.256 \text{ 36 d}}{2\pi} = 58.132 \text{ 4 d} \quad (1)$$

for Sun–Earth. This conversion is used nowhere in the manuscript except in the instability-timescale section, and it is the only place a physical time enters.

Lengths convert with the primary separation. For Sun–Earth that is one astronomical unit,

$$1 \text{ nd length} = 1.495 \text{ 978 707} \times 10^8 \text{ km}. \quad (2)$$

1.2 Geometry and notation

The synodic (co-rotating) frame places the larger primary at $x = -\mu$ and the smaller at $x = 1 - \mu$, both on the x -axis, with the z -axis normal to the orbital plane. Distances from the primaries are

$$r_1^2 = (x + \mu)^2 + y^2 + z^2, \quad r_2^2 = (x - 1 + \mu)^2 + y^2 + z^2. \quad (3)$$

Table 1 collects the symbols. The single most important one is β , the *lightness number*: the ratio of the sail’s radiation-pressure acceleration to the local solar gravitational acceleration. Because both fall off as $1/r_1^2$, this ratio is a constant of the sail, not a function of position — which is exactly why a face-on sail can be absorbed into a rescaling of the primary’s mass.

2 Equations of motion

2.1 The rotating frame

In an inertial frame the massless particle obeys Newtonian gravitation from two point masses on circular orbits. Transforming to the frame rotating with the primaries at unit angular velocity $\mathbf{\Omega} = \hat{\mathbf{z}}$ introduces the centrifugal and Coriolis accelerations,

$$\mathbf{a}_{\text{rot}} = \mathbf{a}_{\text{in}} - 2\mathbf{\Omega} \times \dot{\mathbf{r}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}). \quad (4)$$

With $\mathbf{\Omega} = \hat{\mathbf{z}}$ one has $\mathbf{\Omega} \times \dot{\mathbf{r}} = (-\dot{y}, \dot{x}, 0)$ and $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = (x, y, 0)$. The centrifugal term is the gradient of $\frac{1}{2}(x^2 + y^2)$, so it can be folded into a potential; the Coriolis term cannot, because it

depends on velocity. This asymmetry is the reason the in-plane and out-of-plane linearisations behave so differently (Section 4) and, ultimately, the reason a purely transverse control input can still reach the in-plane radial direction (Section 11).

Collecting terms, and writing P and Q for the two gravitational strengths,

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}, \quad \ddot{z} = \frac{\partial U}{\partial z}, \quad (5)$$

$$U = \frac{1}{2}(x^2 + y^2) + \frac{P}{r_1} + \frac{Q}{r_2}, \quad P = (1 - \beta)(1 - \mu), \quad Q = \mu. \quad (6)$$

Setting $\beta = 0$ gives $P = 1 - \mu$ and recovers the classical CR3BP.

2.2 The sail force and why $P = (1 - \beta)(1 - \mu)$

For an ideal specular reflector the force is along the sail normal with magnitude proportional to $\cos^2 \alpha$:

$$\mathbf{a}_{\text{sail}} = \beta \frac{1 - \mu}{r_1^2} \cos^2 \alpha \, \hat{\mathbf{n}}, \quad \hat{\mathbf{n}} = \cos \alpha \, \hat{\mathbf{r}} + \sin \alpha \cos \delta \, \hat{\mathbf{p}} + \sin \alpha \sin \delta \, \hat{\mathbf{q}}. \quad (7)$$

One factor of $\cos \alpha$ comes from the projected area the sail presents to the Sun; the second comes from the fact that only the component of incoming momentum normal to the sail is reversed. A perfect *absorber* differs: the force lies along the Sun line with magnitude $\propto \cos \alpha$, one factor only. The two models coincide at $\alpha = 0$, which is the only case the bulk of the manuscript uses.

At $\alpha = 0$, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and (7) reduces to

$$\mathbf{a}_{\text{sail}} = \beta \frac{1 - \mu}{r_1^2} \hat{\mathbf{r}}, \quad (8)$$

which is a central inverse-square force directed away from the larger primary. It is therefore derivable from the potential $-\beta(1 - \mu)/r_1$, and adding it to the gravitational term $+(1 - \mu)/r_1$ gives $+(1 - \beta)(1 - \mu)/r_1$. This is the whole content of the statement that *a face-on sail rescales solar gravity*: the problem is not perturbed, it is a CR3BP with a different mass for the larger primary. Every exact result in the manuscript follows from this one observation.

For $\alpha \neq 0$ the force acquires a transverse component, is no longer central, and no potential exists — hence no Jacobi integral, and hence the restriction of Sections 3–6 of the manuscript to $\alpha = 0$.

2.3 The Jacobi integral

Because (5) is autonomous with a velocity-independent potential and a Coriolis term that does no work,

$$\frac{d}{dt}(2U - v^2) = 2\nabla U \cdot \dot{\mathbf{r}} - 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = 0 \quad (9)$$

after substituting (5), since $\dot{\mathbf{r}} \cdot (\boldsymbol{\Omega} \times \dot{\mathbf{r}}) = 0$. Therefore

$$C_{\text{sail}} = 2U - v^2 \quad (10)$$

is conserved, with U the *modified* potential (6). The gravitational Jacobi constant built from $(1 - \mu)/r_1$ instead differs by

$$C_{\text{grav}} = C_{\text{sail}} + \frac{2\beta(1 - \mu)}{r_1}, \quad (11)$$

and since r_1 varies along an orbit, C_{grav} is *not* conserved for $\beta \neq 0$. Section 12.5 quantifies both.

3 The exact identity $r_1 + r_2 = 1$

On the segment of the x -axis strictly between the primaries, $-\mu < x < 1 - \mu$, one has $x + \mu > 0$ and $x - 1 + \mu < 0$, so

$$r_1 = x + \mu, \quad r_2 = (1 - \mu) - x, \quad (12)$$

and therefore

$$r_1 + r_2 = (x + \mu) + (1 - \mu) - x = 1 \quad (13)$$

identically, for every x in the interval and every μ . This is not an approximation and involves no expansion in μ . It is worth emphasising because the standard treatment of collinear points works with $\gamma = r_2$ and expands in γ , generating series in $\mu^{1/3}$; here no series is needed. Equation (13) is what makes the closed form of Section 6 exact rather than asymptotic: imposing a condition on r_2 immediately fixes $r_1 = 1 - r_2$ with no error term.

4 Linearisation at a collinear equilibrium

4.1 The equilibrium condition

An equilibrium on the axis has $y = z = \dot{x} = \dot{y} = \dot{z} = 0$ and $\partial U / \partial x = 0$. Differentiating (6),

$$\frac{\partial U}{\partial x} = x - \frac{P(x + \mu)}{r_1^3} - \frac{Q(x - 1 + \mu)}{r_2^3}. \quad (14)$$

Using (12), $x + \mu = r_1$ and $x - 1 + \mu = -r_2$, so

$$\frac{\partial U}{\partial x} = x - \frac{P}{r_1^2} + \frac{Q}{r_2^2}, \quad (15)$$

and the equilibrium satisfies

$$\boxed{x = \frac{P}{r_1^2} - \frac{Q}{r_2^2} = \frac{(1 - \beta)(1 - \mu)}{r_1^2} - \frac{\mu}{r_2^2}} \quad (16)$$

This is the on-axis force balance used throughout. Read physically: the centrifugal term x balances the reduced solar attraction minus the smaller primary's pull.

4.2 Second derivatives on the axis

Write $\Phi = 1/r_1$. Then

$$\frac{\partial \Phi}{\partial x} = -\frac{x + \mu}{r_1^3}, \quad \frac{\partial^2 \Phi}{\partial x^2} = -\frac{1}{r_1^3} + \frac{3(x + \mu)^2}{r_1^5}. \quad (17)$$

Evaluated on the axis, $(x + \mu)^2 = r_1^2$, so

$$\left. \frac{\partial^2 \Phi}{\partial x^2} \right|_{\text{axis}} = -\frac{1}{r_1^3} + \frac{3r_1^2}{r_1^5} = \frac{2}{r_1^3}. \quad (18)$$

For the transverse directions, $\partial^2 \Phi / \partial y^2 = -1/r_1^3 + 3y^2/r_1^5$, which on the axis ($y = 0$) is $-1/r_1^3$; identically for z . The same computation for $1/r_2$ gives $+2/r_2^3$ and $-1/r_2^3$, because $(x - 1 + \mu)^2 = r_2^2$ on the axis regardless of sign.

Adding the centrifugal part $\frac{1}{2}(x^2 + y^2)$, whose second derivatives are 1, 1, 0:

$$U_{xx} = 1 + 2A, \quad U_{yy} = 1 - A, \quad U_{zz} = -A, \quad (19)$$

$$A = \underbrace{\frac{(1 - \beta)(1 - \mu)}{r_1^3}}_S + \underbrace{\frac{\mu}{r_2^3}}_s. \quad (20)$$

All the cross terms U_{xy}, U_{xz}, U_{yz} vanish on the axis by symmetry, so the Hessian is diagonal and the z motion decouples exactly at linear order.

4.3 The out-of-plane mode

From (5) and (19), the vertical perturbation $\zeta = z$ obeys

$$\ddot{\zeta} = U_{zz} \zeta = -A \zeta \quad \implies \quad \ddot{\zeta} + A \zeta = 0, \quad (21)$$

a pure harmonic oscillator with

$$\boxed{\nu = \sqrt{A}} \quad (22)$$

provided $A > 0$, which holds for every collinear point. There is no Coriolis coupling out of plane because $\boldsymbol{\Omega} \times \dot{\mathbf{r}}$ has no z component. **Physical meaning:** ν is the frequency at which a spacecraft displaced above the ecliptic oscillates through it. Its period is $2\pi/\nu$ in nd time.

4.4 The in-plane modes and the characteristic quartic

Write $\xi = x - x_{\text{eq}}$, $\eta = y$. The linearised in-plane system is

$$\ddot{\xi} - 2\dot{\eta} = (1 + 2A) \xi, \quad \ddot{\eta} + 2\dot{\xi} = (1 - A) \eta. \quad (23)$$

Seek $\xi = X e^{\lambda t}$, $\eta = Y e^{\lambda t}$:

$$(\lambda^2 - 1 - 2A) X - 2\lambda Y = 0, \quad (24)$$

$$2\lambda X + (\lambda^2 - 1 + A) Y = 0. \quad (25)$$

A non-trivial solution requires the determinant to vanish:

$$(\lambda^2 - 1 - 2A)(\lambda^2 - 1 + A) + 4\lambda^2 = 0. \quad (26)$$

Put $u = \lambda^2$ and expand fully:

$$\begin{aligned} (u - 1 - 2A)(u - 1 + A) &= u^2 - u + Au - u + 1 - A - 2Au + 2A - 2A^2 \\ &= u^2 + u(-2 - A) + (1 + A - 2A^2). \end{aligned} \quad (27)$$

Adding $4u$,

$$\boxed{u^2 + (2 - A)u + (1 + A - 2A^2) = 0} \quad (28)$$

which is the quartic $\lambda^4 + (2 - A)\lambda^2 + (1 + A - 2A^2) = 0$ in λ .

4.5 Solving it

The discriminant is

$$\begin{aligned} (2 - A)^2 - 4(1 + A - 2A^2) &= 4 - 4A + A^2 - 4 - 4A + 8A^2 \\ &= 9A^2 - 8A = A(9A - 8) \equiv D^2. \end{aligned} \quad (29)$$

Hence

$$u_{\pm} = \frac{1}{2}[(A - 2) \pm D], \quad D = \sqrt{A(9A - 8)}. \quad (30)$$

The product of the roots is $u_+ u_- = 1 + A - 2A^2 = -(2A + 1)(A - 1)$. For $A > 1$ this is strictly negative, so the two roots have opposite signs: one positive, one negative. Therefore

$$\boxed{\lambda_u^2 = \frac{1}{2}[(A - 2) + D], \quad \omega^2 = \frac{1}{2}[D - (A - 2)]} \quad (31)$$

with λ_u real (the saddle) and ω real (the centre, since $u_- = -\omega^2 < 0$ gives $\lambda = \pm i\omega$).

The condition $A > 1$ is exactly the condition for a real hyperbolic pair. At $A = 1$ the product vanishes and one root goes to zero; for $A < 1$ both roots are negative and the point is linearly stable in plane. Every classical collinear point has $A > 1$, and a sail drives $A \rightarrow 1^+$ from above but never below (Section 6).

4.6 What the three rates mean

This is worth stating plainly, because the manuscript uses all three without defining them operationally.

λ_u — **the saddle exponent.** A perturbation along the unstable eigenvector grows as $e^{\lambda_u t}$. Its reciprocal $\tau_u = 1/\lambda_u$ is the *e-folding time*: the time for a departure to grow by a factor $e \approx 2.718$. This is the number that determines station-keeping cadence. For the classical Sun–Earth L_1 , $\lambda_u = 2.532\,56$ nd, so by (1) $\tau_u = 58.132\,4\text{ d}/2.53256 = 22.95\text{ d}$ — the familiar figure quoted for L_1 halo missions. That the model returns it without being tuned to is a check on (31).

ω — **the in-plane centre frequency.** The planar Lyapunov orbits about the equilibrium have period $2\pi/\omega$ at infinitesimal amplitude. Physically it is the epicyclic frequency of the libration in the (x, y) plane, modified by the two primaries’ tidal fields.

ν — **the out-of-plane frequency.** The vertical Lyapunov orbits have period $2\pi/\nu$ at infinitesimal amplitude.

Why the ratio ν/ω matters. A halo orbit is a three-dimensional periodic orbit in which the in-plane and out-of-plane motions share a common period. At linear order this is impossible unless $\nu/\omega = 1$ exactly. Halo families therefore bifurcate from the planar Lyapunov family at finite amplitude, where nonlinear frequency corrections close the gap. The closer ν/ω is to unity at linear order, the smaller the amplitude at which the halo family appears. The manuscript’s bound (67) says this gap is never larger than 5.72% and never zero.

5 The identity $A \equiv c_2$

Richardson’s expansion of the collinear dynamics writes the potential as a series in Legendre polynomials with coefficients

$$c_n = \frac{1}{\gamma^3} \left[\mu + (-1)^n \frac{(1-\mu)\gamma^{n+1}}{(1-\gamma)^{n+1}} \right], \quad (32)$$

γ being the distance from the smaller primary to the libration point. For $n = 2$, with $\gamma = r_2$ and $1 - \gamma = r_1$ by (13),

$$c_2 = \frac{\mu}{r_2^3} + \frac{1-\mu}{r_1^3}, \quad (33)$$

which is precisely (20) at $\beta = 0$. Hence A is not a new quantity: it is the second Legendre coefficient, and the sail’s sole effect is to replace $(1-\mu)$ by $(1-\beta)(1-\mu)$ inside it. This is the compact statement that *a sail continuously detunes c_2* , a parameter otherwise fixed once the system and the libration point are chosen. The manuscript verifies the identity numerically to $< 10^{-13}$ at three systems.

6 The tidal-parity threshold, in full

6.1 The condition

Define the *saddle strength* $s = \mu/r_2^3$, the smaller primary’s direct contribution to A in (20). Tidal parity is the level

$$s = 1 \quad \Longleftrightarrow \quad \frac{\mu}{r_2^3} = 1 \quad \Longleftrightarrow \quad r_2 = \mu^{1/3}. \quad (34)$$

Since the Hill radius is $r_H = (\mu/3)^{1/3}$,

$$\frac{r_2}{r_H} = \frac{\mu^{1/3}}{(\mu/3)^{1/3}} = 3^{1/3} = 1.4422\dots \quad (35)$$

independently of μ — the parity radius is always $3^{1/3}$ Hill radii.

6.2 The substitution, term by term

Impose (34) on the balance (16). Three substitutions are needed, and each is exact:

$$r_2 = \mu^{1/3}, \quad (36)$$

$$r_1 = 1 - r_2 = 1 - \mu^{1/3} \quad \text{by (13), no error term,} \quad (37)$$

$$\frac{\mu}{r_2^2} = \frac{\mu}{\mu^{2/3}} = \mu^{1/3} \quad \text{the tidal term equals the standoff,} \quad (38)$$

$$x = (1 - \mu) - r_2 = (1 - \mu) - \mu^{1/3} \quad \text{by (12).} \quad (39)$$

The third line is the crux and deserves emphasis: at $s = 1$, and only there, the smaller primary's gravitational acceleration at the equilibrium, μ/r_2^2 , is numerically equal to the equilibrium's distance from that primary, r_2 . Substituting into (16),

$$\underbrace{(1 - \mu) - \mu^{1/3}}_x = \frac{(1 - \beta)(1 - \mu)}{(1 - \mu^{1/3})^2} - \underbrace{\mu^{1/3}}_{\mu/r_2^2}. \quad (40)$$

The $-\mu^{1/3}$ appears on both sides and cancels identically, leaving

$$1 - \mu = \frac{(1 - \beta)(1 - \mu)}{(1 - \mu^{1/3})^2}. \quad (41)$$

Dividing by $(1 - \mu)$, which is non-zero for $\mu < 1$,

$$1 - \beta = (1 - \mu^{1/3})^2 \quad (42)$$

and therefore

$$\boxed{\beta_{\text{crit}} = 1 - (1 - \mu^{1/3})^2 = 2\mu^{1/3} - \mu^{2/3}} \quad (43)$$

Expanding the square gives $1 - (1 - 2\mu^{1/3} + \mu^{2/3}) = 2\mu^{1/3} - \mu^{2/3}$.

6.3 Why this is exact

Three features make it so. First, (13) is an identity, not a leading term. Second, the condition $s = 1$ fixes r_2 in closed form, so nothing needs to be solved numerically. Third, the cancellation is algebraic — it does not rely on μ being small. Consequently (43) holds for $\mu = \frac{1}{2}$ as well as for $\mu = 10^{-7}$. To leading order $\beta_{\text{crit}} \simeq 2\mu^{1/3}$, but the exact form costs nothing.

6.4 The linear structure at parity

At parity $r_2 = \mu^{1/3}$ and $r_1 = 1 - \mu^{1/3}$, so from (20)

$$S = \frac{(1 - \beta)(1 - \mu)}{r_1^3} = \frac{(1 - \mu^{1/3})^2(1 - \mu)}{(1 - \mu^{1/3})^3} = \frac{1 - \mu}{1 - \mu^{1/3}}, \quad (44)$$

using $1 - \beta = r_1^2$. Hence

$$A^{\text{parity}} = 1 + \frac{1 - \mu}{1 - \mu^{1/3}}, \quad (45)$$

which for Sun–Earth is 2.014635, not 2. The naive identification $A - 1 = s$ is therefore short by

$$\frac{A - 2}{A} = \frac{0.014635}{2.014635} = 0.7265\%, \quad (46)$$

quoted as 0.73% in the manuscript. The discrepancy is $S - 1 = O(\mu^{1/3})$ and reflects the smaller primary’s *indirect* effect: it displaces the equilibrium, which changes r_1 , which changes S .

7 The tidal-strength hierarchy

7.1 Generalising the level

Replace (34) by $s = \kappa$ for arbitrary $\kappa > 0$:

$$\frac{\mu}{r_2^3} = \kappa \implies r_2(\kappa) = \left(\frac{\mu}{\kappa}\right)^{1/3} = \left(\frac{3}{\kappa}\right)^{1/3} r_H. \quad (47)$$

Again $r_1 = 1 - r_2$ exactly. The tidal term is now

$$\frac{\mu}{r_2^2} = \mu \left(\frac{\kappa}{\mu}\right)^{2/3} = \mu^{1/3} \kappa^{2/3}, \quad (48)$$

while the standoff is $r_2 = \mu^{1/3} \kappa^{-1/3}$. Substituting into (16),

$$(1 - \mu) - r_2 = \frac{(1 - \beta)(1 - \mu)}{r_1^2} - \mu^{1/3} \kappa^{2/3}, \quad (49)$$

so

$$\frac{(1 - \beta)(1 - \mu)}{r_1^2} = (1 - \mu) \underbrace{- r_2 + \mu^{1/3} \kappa^{2/3}}_{\Delta(\kappa)}. \quad (50)$$

The residue is

$$\Delta(\kappa) = -\mu^{1/3} \kappa^{-1/3} + \mu^{1/3} \kappa^{2/3} = \mu^{1/3} \kappa^{-1/3} (\kappa - 1), \quad (51)$$

and rearranging gives the general closed form

$$\boxed{\beta(\kappa) = 1 - \frac{(1 - r_2)^2}{1 - \mu} \left[(1 - \mu) + \mu^{1/3} \kappa^{-1/3} (\kappa - 1) \right]} \quad (52)$$

7.2 Why $\kappa = 1$ is distinguished

From (51), $\Delta(\kappa) = 0$ if and only if $\kappa = 1$, since $\mu^{1/3} \kappa^{-1/3} > 0$ for all admissible μ, κ . At that level and no other, (52) collapses to

$$1 - \beta = (1 - r_2)^2 = (1 - \mu^{1/3})^2, \quad (53)$$

recovering (43).

The statement to make is therefore precise, and slightly weaker than one might hope: a closed form exists for *every* κ — the algebra never requires a numerical root. What is special about $\kappa = 1$ is that the smaller primary’s direct term drops out of the balance identically, leaving a threshold that is a pure square in $\mu^{1/3}$ with no residual κ -dependence. This answers the objection that $s = 1$ is merely a choice of units: it is the unique level at which the tidal acceleration and the standoff coincide, $\mu/r_2^2 = r_2$, and that coincidence is a property of the force balance.

7.3 Both of the manuscript's markers are members

Because $r_2/r_H = (3/\kappa)^{1/3}$ by (47):

- $\kappa = 3$ gives $r_2/r_H = 1$ exactly — the Hill-sphere exit criterion, at $\beta = 2.98 \times 10^{-4}$;
- $\kappa = 1$ gives $r_2/r_H = 3^{1/3} = 1.4422$ — tidal parity, at $\beta = 0.028\,646$;
- the classical L_1 itself sits at $\kappa = s(\beta=0) = 3.0303$, just inside the Hill sphere at $r_2 = 0.9967\,r_H$.

The two markers the manuscript treats separately — one adopted, one dismissed as “not a useful discriminant” — are the same object at two levels.

8 The frequency ratio

8.1 Forming the ratio

From (22) and (31), with $c_2 \equiv A$,

$$\frac{\nu^2}{\omega^2} = \frac{c_2}{\frac{1}{2}[D - (c_2 - 2)]} = \frac{2c_2}{D - (c_2 - 2)}, \quad D = \sqrt{c_2(9c_2 - 8)}. \quad (54)$$

Note this depends on c_2 alone — μ and β enter only through c_2 . Any statement about this ratio is therefore a statement about the collinear normal form itself, shared by every three-body system.

8.2 The stationary point

Write $R = 2c/M$ with $M = D - c + 2$ and $D = \sqrt{9c^2 - 8c}$. Then

$$\frac{dD}{dc} = \frac{18c - 8}{2D} = \frac{9c - 4}{D}, \quad M' = \frac{9c - 4}{D} - 1. \quad (55)$$

By the quotient rule,

$$\frac{dR}{dc} = \frac{2M - 2cM'}{M^2}, \quad (56)$$

which vanishes when $M = cM'$:

$$D - c + 2 = c \left(\frac{9c - 4}{D} - 1 \right) = \frac{c(9c - 4)}{D} - c. \quad (57)$$

The $-c$ cancels on both sides, leaving $D + 2 = c(9c - 4)/D$, i.e.

$$D^2 + 2D = 9c^2 - 4c. \quad (58)$$

Since $D^2 = 9c^2 - 8c$,

$$9c^2 - 8c + 2D = 9c^2 - 4c \implies 2D = 4c \implies D = 2c. \quad (59)$$

Squaring, $4c^2 = 9c^2 - 8c$, so $5c^2 = 8c$ and (discarding $c = 0$)

$$\boxed{c_2^* = \frac{8}{5}} \quad (60)$$

8.3 The rates there, and the 7 : 8 : 9 ratio

At $c_2 = 8/5$, $D = 2c_2 = 16/5$. From (31) and (22):

$$\lambda_u^2 = \frac{1}{2} \left[\left(\frac{8}{5} - 2 \right) + \frac{16}{5} \right] = \frac{1}{2} \left[-\frac{2}{5} + \frac{16}{5} \right] = \frac{1}{2} \cdot \frac{14}{5} = \frac{7}{5}, \quad (61)$$

$$\nu^2 = c_2 = \frac{8}{5}, \quad (62)$$

$$\omega^2 = \frac{1}{2} \left[\frac{16}{5} + \frac{2}{5} \right] = \frac{1}{2} \cdot \frac{18}{5} = \frac{9}{5}. \quad (63)$$

Hence

$$\lambda_u^2 : \nu^2 : \omega^2 = \frac{7}{5} : \frac{8}{5} : \frac{9}{5} = 7 : 8 : 9, \quad (64)$$

and

$$\left. \frac{\nu}{\omega} \right|_{c_2^*} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} = 0.942809041582\dots \quad (65)$$

8.4 The endpoints

As $c_2 \rightarrow 1^+$: $D = \sqrt{1 \cdot 1} = 1$, so $\omega^2 = \frac{1}{2}[1 - (-1)] = 1$ and $\nu^2 = 1$; the ratio tends to 1. This is the hovering limit, where the planar roots become $\lambda^2 \in \{0, -1\}$ and the in-plane period is driven to exactly 2π — one year for Sun–Earth. That is the Keplerian epicyclic degeneracy of an inverse-square field, not a resonance with the smaller primary.

As $c_2 \rightarrow \infty$: $D = 3c_2\sqrt{1 - 8/(9c_2)} \simeq 3c_2 - \frac{4}{3}$, so

$$\omega^2 \simeq \frac{1}{2} \left[3c_2 - \frac{4}{3} - c_2 + 2 \right] = c_2 + \frac{1}{3}, \quad \frac{\nu^2}{\omega^2} \simeq \frac{c_2}{c_2 + \frac{1}{3}} \rightarrow 1. \quad (66)$$

So the ratio approaches unity at both ends and dips to $2\sqrt{2}/3$ in between. The band is therefore

$$\frac{\nu}{\omega} \in \left[\frac{2\sqrt{2}}{3}, 1 \right), \quad \text{width } 1 - \frac{2\sqrt{2}}{3} = 5.719\%. \quad (67)$$

The interval is closed on the left (the bound is attained at $c_2 = 8/5$) and open on the right (unity is approached but never reached for finite $c_2 > 1$).

9 Which commensurabilities the band admits

The manuscript originally tested fourteen low-order rationals and found none inside (67). That is true but weaker than necessary, and the stronger statement is elementary.

Let p/q with $0 < p < q$ be a rational in lowest terms. For a fixed denominator q , the largest such rational strictly below 1 is $(q-1)/q$. Therefore $p/q \in [2\sqrt{2}/3, 1)$ requires

$$\frac{q-1}{q} \geq \frac{2\sqrt{2}}{3} \iff 1 - \frac{1}{q} \geq \frac{2\sqrt{2}}{3} \iff q \geq \frac{1}{1 - 2\sqrt{2}/3} = 17.4853\dots \quad (68)$$

so $q \geq 18$. Hence:

- **No** rational with denominator $q \leq 17$ lies in the band at all.
- The lowest-denominator one that does is $17:18 = 0.9444\dots$, of order $p+q = 35$.
- The fourteen tested rationals have $q \leq 9$ and are excluded without being tested individually.

It is important to state the converse honestly: the band is *not* resonance-free. It is an interval of positive length, so it contains infinitely many rationals — 18:19, 19:20, and so on. The correct claim is that every commensurability it admits is of order 35 or higher, which is far beyond what any perturbation theory would retain, and that no commensurability underwrites tidal parity.

10 The instability timescale

Define

$$\tau_u = \frac{1}{\lambda_u}, \quad N_{\text{efold}} = \lambda_u T_{\text{mission}}, \quad (69)$$

with λ_u in nd units. Converting with (1),

$$\tau_u [\text{days}] = \frac{58.1324 \text{ d}}{\lambda_u [\text{nd}]}. \quad (70)$$

At $\beta = 0$, $\lambda_u = 2.53256$ gives $\tau_u = 22.95 \text{ d}$. At tidal parity, $\lambda_u = 1.505418$ gives $\tau_u = 38.62 \text{ d}$.

Inverting $N_{\text{efold}} = 1$ for β requires a numerical root because $\lambda_u(\beta)$ is transcendental: one solves $\lambda_u(\beta) = 58.1324 \text{ d}/T$ by bisection on the monotone decreasing function λ_u . The results are in the manuscript's Table 4. Note this threshold is *not* μ -independent and *not* exact — it depends on the mission horizon chosen, which is an engineering input, not a property of the dynamics. It is reported because it is the operationally meaningful reading, not because it is a sharper result than (43).

11 Sail control authority

11.1 The orthonormal triad

The triad in (7) must be orthonormal. Take

$$\hat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}_1}{r_1}, \quad \hat{\mathbf{p}} = \frac{\hat{\mathbf{k}} \times \hat{\mathbf{r}}}{\|\hat{\mathbf{k}} \times \hat{\mathbf{r}}\|}, \quad \hat{\mathbf{q}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}. \quad (71)$$

A common shortcut substitutes $\hat{\mathbf{k}}$ itself for $\hat{\mathbf{q}}$. That fails off the ecliptic because $\hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = z/r_1 \neq 0$, and then

$$\|\hat{\mathbf{n}}\|^2 = 1 + \sin(2\alpha) \sin \delta \frac{z}{r_1}, \quad (72)$$

so the “unit” normal is not a unit vector. The error is maximised when $\sin(2\alpha) \sin \delta = 1$, giving $\|\hat{\mathbf{n}}\| - 1 = \sqrt{1 + z/r_1} - 1$, i.e. 0.995% at $z/r_1 = 0.02$ and 4.40% at $z/r_1 = 0.09$. It vanishes identically at $\alpha = 0$, so no face-on result is affected.

11.2 Differentiating the sail acceleration

Write $k = \beta(1 - \mu)/r_1^2$. From (7),

$$\mathbf{a} = k \left[\cos^3 \alpha \hat{\mathbf{r}} + \cos^2 \alpha \sin \alpha \cos \delta \hat{\mathbf{p}} + \cos^2 \alpha \sin \alpha \sin \delta \hat{\mathbf{q}} \right]. \quad (73)$$

Differentiating with respect to the clock angle,

$$\frac{\partial \mathbf{a}}{\partial \delta} = k \cos^2 \alpha \sin \alpha \left[-\sin \delta \hat{\mathbf{p}} + \cos \delta \hat{\mathbf{q}} \right], \quad (74)$$

which carries an overall factor $\sin \alpha$ and therefore

$$\boxed{\left. \frac{\partial \mathbf{a}}{\partial \delta} \right|_{\alpha=0} = \mathbf{0}} \quad (75)$$

identically, for every δ_0 . Geometrically: at $\alpha = 0$ the normal is parallel to $\hat{\mathbf{r}}$, and the clock angle only rotates the normal about itself, which does nothing.

For the cone angle, using $d(\cos^2\alpha \sin\alpha)/d\alpha = \cos^3\alpha - 2\cos\alpha \sin^2\alpha$ and $d(\cos^3\alpha)/d\alpha = -3\cos^2\alpha \sin\alpha$,

$$\frac{\partial \mathbf{a}}{\partial \alpha} = k \left[-3\cos^2\alpha \sin\alpha \hat{\mathbf{r}} + (\cos^3\alpha - 2\cos\alpha \sin^2\alpha) (\cos\delta \hat{\mathbf{p}} + \sin\delta \hat{\mathbf{q}}) \right]. \quad (76)$$

At $\alpha = 0$ this reduces to $k(\cos\delta_0 \hat{\mathbf{p}} + \sin\delta_0 \hat{\mathbf{q}})$: a single, purely transverse direction fixed by the constant δ_0 . The radial authority vanishes because $d(\cos^3\alpha)/d\alpha \rightarrow 0$ as $\alpha \rightarrow 0$.

11.3 Why rank of B is the wrong test

A face-on sail supplies one input direction, and it is transverse. One might conclude the radial direction is unreachable. That is wrong, because the Coriolis term in (23) couples ξ and η : a transverse push generates radial motion within one dynamical time. The correct test is the Kalman rank condition on $[B, MB, M^2B, \dots, M^5B]$ for the 6×6 system matrix M . Applying it at the $\beta = 0.05$ equilibrium ($c_2 = 1.409194$) gives rank 4/6 at $\alpha_0 = 0, \delta_0 = 0$ — the in-plane four-state block is fully controllable (rank 4/4), and it is the *out-of-plane* pair that is lost, because z decouples at linear order. That loss is specific to $\delta_0 = 0$. Purely vertical tilt ($\delta_0 = \pi/2$) gives rank 2/6, losing the in-plane pair instead. For any other δ_0 the single input has components in both blocks, and because their spectra $\{\pm\lambda_u, \pm i\omega\}$ and $\{\pm i\nu\}$ are disjoint whenever $\nu \neq \omega$, the Popov–Belevitch–Hautus test gives full rank: at $\delta_0 = \pi/4$ the rank is 6/6. What $\alpha_0 = 0$ removes is the second input, not controllability.

11.4 The optimal cone angle

The transverse thrust component is $\propto \cos^2\alpha \sin\alpha$. Maximising,

$$\frac{d}{d\alpha}(\cos^2\alpha \sin\alpha) = \cos\alpha(\cos^2\alpha - 2\sin^2\alpha) = 0 \implies \tan^2\alpha = \frac{1}{2}, \quad (77)$$

so

$$\alpha_0^* = \arctan(1/\sqrt{2}) = 35.264^\circ, \quad \cos^2\alpha_0^* = \frac{1}{1 + \tan^2\alpha} = \frac{2}{3}. \quad (78)$$

This is McInnes’ classical optimal sail cone angle; recovering it from the control Jacobian is a benchmark, not a new result.

What is new is the conditioning there. At α_0^* , $\cos^2\alpha - 2\sin^2\alpha = \frac{2}{3} - \frac{2}{3} = 0$, so the transverse part of (76) vanishes and

$$\frac{\partial \mathbf{a}}{\partial \alpha} = -3k \cos^2\alpha \sin\alpha \hat{\mathbf{r}} = -3k \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} \hat{\mathbf{r}} = -\frac{2k}{\sqrt{3}} \hat{\mathbf{r}}, \quad (79)$$

purely radial, while from (74)

$$\left\| \frac{\partial \mathbf{a}}{\partial \delta} \right\| = k \cos^2\alpha \sin\alpha = k \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{2k}{3\sqrt{3}}. \quad (80)$$

The two columns are orthogonal, so they *are* the singular vectors, and

$$\frac{\sigma_2}{\sigma_1} = \frac{2k/(3\sqrt{3})}{2k/\sqrt{3}} = \frac{1}{3} \quad (81)$$

exactly, independent of β and μ .

12 Every residual, explained

This section exists because a reader meeting “ 2.1×10^{-16} ” in an abstract cannot tell whether it is a physical result, a convergence tolerance, or noise. Each of the manuscript’s small numbers is a different kind of quantity.

Table 2: The Table-2 residuals expressed in units in the last place. All four are at the level of the root-finder’s convergence, not of a disagreement in the mathematics.

system	β_{crit}	$ \Delta $	ulp(β)	$ \Delta $ in ulp
Sun–Mercury	0.010 961 525	3.1×10^{-16}	1.7×10^{-18}	179
Sun–Earth	0.028 646 456	2.1×10^{-16}	3.5×10^{-18}	60
Sun–Jupiter	0.187 186 696	2.8×10^{-16}	2.8×10^{-17}	10
Earth–Moon	0.406 934 946	5.0×10^{-16}	5.6×10^{-17}	9

12.1 Floating-point background

All computation is IEEE-754 double precision. The machine epsilon is $\varepsilon = 2^{-52} = 2.220\,446 \times 10^{-16}$: the gap between 1 and the next representable number. For a value x , the gap to its neighbour — one *unit in the last place*, or ulp — is approximately $x\varepsilon$. A disagreement of a few ulp between two computations of the same quantity means they agree as well as double precision permits, and says nothing about the mathematics.

12.2 Table 2: the closed form against root-finding

The column $|\Delta|$ is $|\beta_{\text{closed}} - \beta_{\text{brentq}}|$, where the first evaluates (43) directly and the second solves $s(\beta) - 1 = 0$ by Brent’s method through the numerically computed equilibrium. The two share no code path beyond the balance itself, so agreement exercises the whole chain.

Table 2 shows the pattern: the residual is roughly constant in absolute terms ($\sim 3 \times 10^{-16}$) rather than in relative terms, so it grows in ulp as β shrinks. That is the signature of an *absolute* convergence tolerance in the root-finder, which is exactly what `brentq` is configured with. Nothing here is physics.

12.3 The 61-point sweep

The manuscript states that over $\mu \in [10^{-7}, 10^{-2}]$ sampled at 61 logarithmically spaced points, the two computations never differ by more than 1.9×10^{-15} . The worst case occurs at $\mu = 2.1544 \times 10^{-4}$, where $\beta = 0.116303$ and the residual is 1.8874×10^{-15} , or 136 ulp. The Sun–Earth value 2.1×10^{-16} quoted separately is that one system’s residual and is *not* the sweep maximum — an earlier draft conflated the two.

12.4 The frequency scan

The manuscript reports that a 4×10^5 -point scan places the minimum of ν/ω at $c_2 = 1.60000000$ with value 0.942809041582, “agreeing with (60) and (65) to 8.1×10^{-10} and 1.1×10^{-16} respectively”. These two numbers mean very different things and the first is misleading.

The scan is `linspace(1+1e-9, 4.2, 400001)`, a uniform grid of step

$$h = \frac{4.2 - (1 + 10^{-9})}{400000} = 8.0 \times 10^{-6}. \quad (82)$$

A grid search can locate a minimum no better than about $h/2 = 4 \times 10^{-6}$ in general. The reported 8.125×10^{-10} is the distance from $8/5$ to the *nearest grid point*, which happens to be small because of how the offset 10^{-9} and the step interact — an arithmetic accident of this particular `linspace`, not a measure of resolving power. Had the grid been offset differently, that figure would have been up to 4×10^{-6} .

The second number is meaningful. At a stationary point the function is quadratically flat, so a location error ϵ produces a value error $\sim \frac{1}{2}R''\epsilon^2$. With $\epsilon \sim 10^{-9}$ the value error is far below ε , which is why the *value* agrees to 1.1×10^{-16} — one ulp — while the *location* does not.

The honest summary is: the scan confirms the value of the bound to machine precision and is consistent with the location $c_2 = 8/5$ to within its grid resolution of 4×10^{-6} .

12.5 The Jacobi spreads

“Spread” means $\max - \min$ of the quantity sampled along one full period of a closed orbit. On the first atlas member ($\beta = 0.001$, $A_z = 5.0415 \times 10^{-4}$):

- C_{sail} spread $\approx 1.8 \times 10^{-15}$ — this is *integrator noise*. C_{sail} is an exact invariant, so its variation measures only the accumulated truncation and rounding error of the ODE solver. It is tolerance-dependent: re-integrating at `rtol=atol=1e-13` gives 1.33×10^{-15} instead. Any value at the 10^{-15} level means “conserved to machine precision”.
- C_{grav} spread $= 5.787 \times 10^{-6}$ — this is *physical*. By (11) the two differ by $2\beta(1 - \mu)/r_1$, which varies as r_1 varies along the orbit. It is nine orders of magnitude larger and is not a numerical artefact. It is the reason the sail potential, not the gravitational one, must be used.

Note that over an exactly closed orbit r_1 returns to its initial value, so C_{grav} returns to its initial value too: the failure mode is not a secular drift but a dependence on *where* on the orbit the constant is sampled. Sampling always at the same crossing makes the wrong quantity look smooth, which is what concealed the error rather than exposing it.

12.6 Other small numbers

3.07×10^{-5} , the L_3 miss. At $\mu = \frac{1}{2}$ the classical L_3 reaches $c_2 = 1.569787$, giving ν/ω larger than the bound (65) by 3.07×10^{-5} . This is a genuine dynamical near-miss, not a numerical residual: L_3 occupies precisely the interval in which the extremum $c_2^* = 8/5$ lies, and falls just short of reaching it.

$< 10^{-13}$, the $A \equiv c_2$ identity. Two algebraically identical expressions evaluated by different code paths. The residual is accumulated rounding over a handful of operations on numbers of order unity.

4.5×10^{-4} and 5.2×10^{-2} , the singular values. These are physical: the second and first singular values of the transverse block of B_{sail} at $\alpha_0 = 0.5^\circ$, in nd acceleration units. Their ratio, 115:1, is the conditioning penalty for operating near the singular nominal. It is not a tolerance.

Δ_z , the branch discriminant. Defined as $(|z_0| - |z(T/2)|)/(|z_0| + |z(T/2)|)$, this is a dimensionless asymmetry, identically zero on the vertical-Lyapunov branch and $O(0.1)$ on a genuine halo. From `halo_atlas.csv` the smallest value in each family falls from 0.109 at $\beta = 0.001$ to 0.0898 at $\beta = 0.040$ and 0.0125 at $\beta = 0.050$. It remains a valid discriminator across the band — the competing branch sits at zero — but at the upper end it separates the two by one order of magnitude rather than two.

13 Physical scales for Sun–Earth

14 Reproducing every number

The repository is public at https://github.com/Bishwaswarup/SOLAR_SAIL. From a clean clone:

Table 3: Conversions used throughout, $\mu = 3.003 \times 10^{-6}$.

quantity	value
1 nd length	$1.495\,978\,707 \times 10^8 \text{ km}$
1 nd time	$58.132\,4 \text{ d}$
one full revolution	$2\pi \text{ nd} = 365.256\,36 \text{ d}$
Hill radius r_H	$0.010\,003\,3 \text{ nd} = 1\,496\,477 \text{ km}$
classical L_1 standoff	$0.009\,969\,9 \text{ nd} = 1\,491\,472 \text{ km}$
parity standoff	$0.014\,427\,3 \text{ nd} = 2\,158\,294 \text{ km}$
critical sail loading σ^*	$1.531\,1 \text{ g m}^{-2}$

```
pip install -r requirements.txt
python main.py verify      # regression suite + 217 manuscript checks
python main.py results    # regenerates results.txt
python main.py atlas      # re-walks the 12 halo families (slow)
python main.py atlas_fine # fine-beta walk to the family's end (slow)
python main.py figures    # regenerates the paper figures
```

`paper/verify_numbers.py` is the authority on manuscript agreement. It recomputes each claim from the `src/` modules and compares it against the literal typed in the manuscript (`paper/springer/main.tex`); its tables of closed-form thresholds, the κ hierarchy, e-folding horizons and the fine atlas are parsed cell by cell out of the manuscript source, so a table cannot silently disagree with the formula it tabulates. A drifted number fails there rather than in review.