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Balasaheb B. Waphare

balasahebwaphare@gmail.com

MAER's MIT Arts Commerce and Science College,Alandi

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The Geometric Product of n Vectors: a Grade-Spectrum Theorem, Exact Second Moments, and a Two-Scale Cut-off for Random Versors

Balasaheb B. Waphare
MAEER's MIT Arts, Commerce and Science College, Alandi (D), Pune 412105, Maharashtra, India
(Affiliated to Savitribai Phule Pune University)
E-mail: balasahebwaphare@gmail.com ORCID: https://orcid.org/0000-0003-0976-0615

Abstract

Let $X = a_1 \cdots a_n$ be a product of nonzero vectors in the Clifford algebra of $\mathbb{R}^d$, and let $\rho_X \in O(d)$ be the orthogonal map it represents. We show that the squared coordinates of X are read off from ρ_X by single determinants, $X_I^2 = |X|^2 \det\big((I_d + \rho_X D_I)/2\big)$, where D_I is the diagonal sign matrix of the blade e_I. The formula holds for all versors, odd ones included. Its grade-wise form, $\sum_s \| \langle X \rangle_s \|^2 \, t^s = |X|^2 \det\Big(\big((1+t)I_d + (1-t)\rho_X\big)/2\Big)$, refines the spin-character identity $\langle X \rangle^2 = \det\big((I_d + \rho_X)/2\big)$ for unit X. For even versors the coordinate formula is equivalent to a consequence of Shirokov's reconstruction of spin group elements from minors, and in dimension three it gives the squared components in Shepperd's quaternion extraction. As consequences, the grade energies are a Krawtchouk transform of the exterior-power traces of ρ_X, and the grade profile is a complete conjugacy invariant with mean $(d - \mathrm{tr}\rho_X)/2$ and variance $(d - \mathrm{tr}\rho_X^2)/4$. The profile is ultra-log-concave on its parity class, and every dominant grade lies within 2 of the mean. For products of independent uniform unit vectors, the mean profile follows the Ehrenfest urn and equilibrates after $\frac{d}{4}\log d$ factors. The law of the profile carries the full distance of the reflection walk on $O(d)$ and equilibrates only after Porod's $\frac{d}{2}\log d$. We give exact second moments of the trace and an explicit second-moment lower bound for the law of the profile. All checkable statements are verified numerically.

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Keywords. Geometric algebra, versor, Pin group, grade decomposition, Krawtchouk polynomials, cut-off phenomenon.

1 Introduction

Clifford's geometric product [3] combines the inner and outer products of vectors: $ab = a \cdot b + a \wedge b$. Products of several vectors, called versors, are the central objects of geometric algebra [8, 9, 15, 16]. They represent compositions of reflections, and through them every orthogonal transformation (the Pin and Spin groups [2, 22, 27]). Their scalar parts are Dirac traces [25].

A versor $X = a_1 \cdots a_n$ has two descriptions. Algebraically, it is a multivector with components $\langle X \rangle_s$ of every grade s. Geometrically, it is the orthogonal map ρ_X that it induces. This paper determines exactly how the two descriptions are related, and then uses that relation to study random versors.

Main results.

(A) *Grade-spectrum and blade-spectrum theorems* (Theorems 4.3 and 4.10). The grade-energy generating function of X is $|X|^2\det\big(((1+t)I_d+(1-t)\rho_X)/2\big)$. More generally, every squared coordinate is a single determinant, $X_I^2=|X|^2\det\big((I_d+\rho_X D_I)/2\big)$, valid for all versors. In $\mathbb{R}^3$ it gives the squared components in Shepperd's quaternion extraction (Corollary 4.12). The profile is ultra-log-concave on its parity class, and its dominant grade lies within 2 of the mean grade (Corollary 4.13). Further consequences follow. The profile is a Poisson-binomial law in the half-angles of ρ_X (Corollary 4.4). It is a Krawtchouk transform of the exterior-power traces $\mathrm{tr}\Lambda^k\rho_X$ (Corollary 4.5). The mean grade is $(d-\mathrm{tr}\rho_X)/2$ and the grade variance is $(d-\mathrm{tr}\rho_X^2)/4$ (Corollary 4.6). The profile is a complete conjugacy invariant (Corollary 4.7).

(B) *Moments of random versors* (Theorems 5.1 and 6.2). For independent isotropic factors, the expected profile is the Ehrenfest urn law, and Theorem 4.3 gives a second, spectral proof of this. For uniform unit factors we give the exact variance of the mean grade and the exact expectation of the grade variance, together with their equilibrium values.

(C) *Averaged versus individual profiles* (Section 7). Averaging the profile over the randomness is a contraction for total variation (Lemma 7.3). The expected profile has the classical $\frac{d}{4}\log d$ cut-off of the Ehrenfest urn. The random profile, being a complete invariant, is exactly as far from equilibrium as the reflection walk on $O(d)$ (Lemma 7.4). It therefore has its cut-off at Porod's $\frac{d}{2}\log d$ [26] (Corollary 7.8). For the lower half, Theorem 7.5 gives an explicit second-moment bound, $\mathrm{TV}\geq 1-8d^{4/d}e^{-2c}-O(\log d/d)$ at $n=\frac{d}{2}(\log d-c)$.

Relation to the literature. Section 3 collects the algebraic structure of $a_1\cdots a_n$ in arbitrary signature. This consists of the partial-matching (Wick) expansion, the Pfaffian scalar part, and the reversion and Gram identities. In substance these are classical [2, 11, 16, 37]. We include them, with short proofs, because the later sections use them.

The case $t=0$ of Theorem 4.3 is classical in another language. When d is even, $\langle X\rangle_0$ is the normalised character of the spin module, and $\langle X\rangle_0^2=\det\big((I+\rho_X)/2\big)$ is the identity $|\chi_S|^2=\sum_k\chi_{\Lambda^k}=\det(I+\rho)$. This identity expresses $S\otimes S^*\cong\Lambda^\bullet V$ [12, Lecture 20], [20, Ch. I] (Remark 4.9). The rotor case in $\mathbb{R}^3$, $\cos^2(\varphi/2)=(1+\mathrm{tr}\rho)/4$, is a familiar special case. Theorem 4.3 refines this identity grade by grade. Each grade energy w_s is a class function on $\mathrm{Pin}(d)$, so it suffices to evaluate it on a maximal torus; Lemmas 4.1 and 4.2 carry out this reduction directly in the Clifford algebra, without the spin module. In canonical form a versor factors into commuting simple factors, a fact known as the invariant decomposition [29]; see also the Cayley-transform (exterior-exponential) parametrisation of Lipschitz, $s=e^{\wedge B}$, discussed in [21, 22]. Expanding such a factorisation gives the grade components in an adapted basis. What Theorem 4.3 adds is a basis-free form: the grade energies are read off from ρ_X alone, through a single determinant. Compound matrices and Pfaffians have also been used to represent geometric algebra itself [28].

For coordinates, the closest work is Shirokov's [32, 33]. He reconstructs an element $S\in\mathrm{Spin}_+(p,q)$ from sums of minors of its pseudo-orthogonal matrix through the identity $2^dS\langle S^{-1}e_F\rangle_0=M_F$ $(d=p+q)$ [33, eq. (18)]. Taking the scalar part of $M_F\widetilde{e_F}$ in that identity yields, for even Euclidean versors, S_F^2 as the signed sum of principal minors in (6). Theorem 4.10 writes these squares as one determinant each, extends them to odd versors (the Pin group), and derives them from the case $t=0$ of Theorem 4.3 by a one-line shift (Remark 4.11). In $\mathbb{R}^3$ it is the classical quaternion-from-rotation-matrix formula of Shepperd [31]. Lasenby, Lasenby and Matsantonis [19] reconstruct a rotor from its action on a frame by means of characteristic multivectors. Related determinant identities appear in mathematical physics in the language of fermionic Gaussian operators. The trace of a product of exponentials of fermion bilinears is,

up to sign, a square root of a determinant of the form $\det(I+R)$ [18, eqs. (6), (37)], the case $t=0$ again. For a fermionic Gaussian state with covariance matrix Γ, the normalised squared Majorana coefficients are the principal minors $\det\Gamma|_{\mathbf{x}}$ divided by $\det(I+\Gamma)$ [4, eq. (6)]. Through the operator–state correspondence these results are likely to contain translated forms of (4) and (6) for even versors; we do not pursue the translation.

Krawtchouk polynomials are also known to be linked with Clifford algebras, in a different form. Staples [36] identified them as traces of blade-conjugation operators, which is an operator-level statement. Corollary 4.5 is an element-level statement: it expresses the grade energies of a single versor through the spectrum of the orthogonal map it induces. Staples [35] also realised the hypercube random walk inside a Clifford algebra by identifying vertices with blades. Our first proof of Theorem 5.1 runs in the opposite direction. There we show that the grade-energy vector of a random product evolves by the Ehrenfest kernel.

The probabilistic inputs are standard. The $\frac{d}{4}\log d$ cut-off of the Ehrenfest urn is classical [6]; we reprove it with explicit constants only for completeness (Theorem 7.1). The constants in Lemma 6.1 are the character ratios of a reflection on Λ^2 and on the traceless part of Sym^2, as in the Fourier analysis of Rosenthal [30] and Porod [26]. The location $\frac{d}{2}\log d$ of the individual cut-off is Porod's theorem, transported to the grade profile by Lemma 7.4.

What we believe is new is the following; we did not find these statements in the literature we consulted.

- The blade formula of Theorem 4.10 for all of $\mathrm{Pin}(d)$, proved from the case $t=0$ of Theorem 4.3 by a one-line shift. For even versors its principal-minor form is equivalent to the scalar part of Shirokov's identity [33, eq. (18)]. What is new is the single-determinant form, the extension to odd versors, and its use as a selection rule (Remark 4.11). Also new are the basis-free grade-spectrum identity (4) and its Corollaries 4.5–4.7 (the Krawtchouk–exterior-power formula, the mean-grade and grade-variance formulas, and complete invariance), and the ultra-log-concavity and dominant-grade bound of Corollary 4.13.

- The observation that the grade profile of a random versor, a quantity defined inside the Clifford algebra, carries the full distance to equilibrium of the reflection walk (Lemma 7.4), while its average mixes at the Ehrenfest rate.

- An explicit second-moment lower bound for the reflection walk (Theorem 7.5) with constant $8d^{4/d}$ (below 16.4 for $d \geq 16$, and tending to 8) and error $O(\log d/d)$, against 16.4 and $O\left(d^{-3/5}\log d\right)$ in [26, Theorem 1.2].

The exact second moments in Theorem 6.2 are a direct consequence of the character ratios of [26, 30]; we record them because the lower bound and the diagnostic of Section 8 use them in closed form.

2 Preliminaries

Let $V=\mathbb{R}^{p,q}$ have an orthonormal basis $e_1,\dots,e_d$, $d=p+q$, with $e_i^2=\eta_i\in\{\pm1\}$. The Clifford algebra $C\ell_{p,q}$ is generated by V with $uv+vu=2\,u\cdot v$, so $a^2=Q(a)$. For $I=\{i_1<\cdots<i_r\}\subseteq[d]$ put $e_I=e_{i_1}\cdots e_{i_r}$; the 2^d blades e_I form a basis. For $Y=\sum_I Y_I\,e_I$ we write $\langle Y\rangle_r=\sum_{|I|=r}Y_I\,e_I$, $\langle Y\rangle=\langle Y\rangle_0$, and $\|Y\|^2=\sum_I Y_I^2$ for the *coefficient energy*. The outer product $b_1\wedge\dots\wedge b_r$ is the antisymmetrised product. The reversion $\tilde{}$ reverses the order of vector factors, and the grade involution α acts as $(-1)^r$ on grade r. For a vector a and an r-vector B, the left contraction $a\rfloor B=\langle aB\rangle_{r-1}$ is the first sum in Lemma 3.1.

When $q = 0$ (the Euclidean case), the energy is basis-independent, $\| Y \|^2 = \langle \tilde{Y} Y \rangle$, and the grades are mutually orthogonal. For basis blades, $e_I e_i = \theta(I,i)\, e_{I \triangle \{i\}}$ with $\theta(I,i) \in \{\pm 1\}$.

For a non-null a, the map $S_a(v) = -ava^{-1} = v - 2\frac{a \cdot v}{Q(a)} a$ is the reflection in $a^{\perp}$. For $X = a_1 \cdots a_n$ with all $Q(a_i) \neq 0$, the twisted adjoint action is

$$\rho_X(v) = \alpha(X)\, v\, X^{-1} = S_{a_1} \circ \cdots \circ S_{a_n}(v). \qquad (1)$$

It is an orthogonal map with $\det \rho_X = (-1)^n$, and $\rho_{XY} = \rho_X \rho_Y$. In the Euclidean case the unit versors form the group $\mathrm{Pin}(d)$, and $X \mapsto \rho_X$ is a surjective homomorphism $\mathrm{Pin}(d) \to O(d)$ with kernel $\{\pm 1\}$. Surjectivity is the Cartan–Dieudonné theorem [1]; for the kernel see [22, 27].

For Borel probability measures μ, ν on a compact metric space M we use the total variation distance

$$\begin{aligned} \mathrm{TV}(\mu,\nu) &= \sup_A |\mu(A) - \nu(A)| \\ &= 1/2 \sup\left\{ \left| \int f \, d(\mu - \nu) \right| : f \in C(M),\ |f| \leq 1 \right\} \in [0,1], \end{aligned}$$

which is the normalisation used by Porod [26, Definition 2.1].

3 Classical structure of $a_1 \cdots a_n$

Lemma 3.1. For vectors $a, b_1, \ldots, b_r$,

$$a\,(b_1 \wedge \ldots \wedge b_r) = \sum_{j=1}^{r} (-1)^{j-1} \left(a \cdot b_j\right) b_1 \wedge \ldots \wedge \widehat{b_j} \wedge \ldots \wedge b_r + a \wedge b_1 \wedge \ldots \wedge b_r.$$

Proof. Both sides are multilinear and alternating in the b_j, so it suffices to take $a = e_i$ and $b_k = e_{j_k}$ with $j_1 < \cdots < j_r$. If $i \notin J$, then $e_i e_J = e_i \wedge e_J$. If $i = j_m$, then $e_i e_J = (-1)^{m-1} \eta_i\, e_{J \setminus \{i\}}$, which is the m-th term of the sum, while $e_i \wedge e_J = 0$. ▫

A *partial matching* $\mathfrak{m}$ of $[n]$ is a set of disjoint pairs (i,j), each written with $i < j$; its unmatched set is $U(\mathfrak{m})$. Its sign $\varepsilon(\mathfrak{m})$ is the sign of the permutation that lists the pairs (in any order, since exchanging two pairs is an even permutation) followed by $U(\mathfrak{m})$ in increasing order. In (2) the wedge over $U(\mathfrak{m})$ is taken in increasing order.

Theorem 3.2 (partial-matching expansion). In $C\ell_{p,q}$,

$$a_1 a_2 \cdots a_n = \sum_{\mathfrak{m}} \varepsilon(\mathfrak{m}) \prod_{(i,j) \in \mathfrak{m}} \left(a_i \cdot a_j\right) \bigwedge_{u \in U(\mathfrak{m})} a_u. \qquad (2)$$

The sum has $\tau(n)$ terms, where $\tau(n)$ is the number of involutions of $[n]$ [24].

Proof. We use induction on n, writing $a_1 \cdots a_n = a_1 (a_2 \cdots a_n)$. Apply the induction hypothesis to $a_2 \cdots a_n$, and Lemma 3.1 to each term $a_1 \bigwedge_{u \in U(\mathfrak{m}')} a_u$, where $\mathfrak{m}'$ is a partial matching of $\{2, \ldots, n\}$ with paired set P ($|P|$ even) and unmatched indices $u_1 < \cdots < u_r$. The wedge term of Lemma 3.1 corresponds to leaving 1 unmatched: inserting 1 before u_1 in the listing creates $|P|$ inversions, an even number, so the sign is unchanged. The l-th contraction term carries $(-1)^{l-1}\left(a_1 \cdot a_{u_l}\right)$ and corresponds to the matching $\mathfrak{m} = \mathfrak{m}' \cup \{(1, u_l)\}$. Listing $(1, u_l)$ first changes the inversion count by $\#\{p \in P : p < u_l\} - \#\{p \in P : p >$

$u_l\} + (l-1) \equiv |P| + l - 1 \equiv l - 1 \pmod 2$, so $\varepsilon(\mathfrak{m}) = (-1)^{l-1}\varepsilon(\mathfrak{m}')$, which is the sign in Lemma 3.1. Each partial matching of $[n]$ arises exactly once. ▫

Corollary 3.3. Let $X = a_1 \cdots a_n$. Then (a) only the grades $n, n-2, \ldots$ occur; (b) $\langle X\rangle_n = a_1 \wedge \ldots \wedge a_n$; (c) $\langle X\rangle = 0$ for odd n, and $\langle X\rangle = \mathrm{Pf}\big[a_i \cdot a_j\big]_{i<j}$ for even n.

Proof. Part (c) follows because the perfect-matching terms of (2) form the Pfaffian expansion. Parts (a) and (b) are read off from (2) grade by grade. ▫

Example 3.4. In $\mathbb{R}^3$, let $a = e_1 + e_2$, $b = e_2 + e_3$, $c = e_1 + e_3$. Theorem 3.2 gives

$$abc = (a\cdot b)c - (a\cdot c)b + (b\cdot c)a + a\wedge b\wedge c = 2e_1 + 2e_{123}.$$

Theorem 3.5 (reversion and Gram identities). In $C\ell_{p,q}$: (a) $X\tilde{X} = \tilde{X}X = \prod_i Q\,(a_i)$; (b) $\langle (a_1 \wedge \ldots \wedge a_n)^{\sim}(b_1 \wedge \ldots \wedge b_n)\rangle = \det\big[a_i \cdot b_j\big]$. In the Euclidean case the grade-energy identity

$$\sum_s \| \langle X\rangle_s \|^2 = \prod_i |a_i|^2 =: |X|^2 \qquad (3)$$

holds, together with $\| \langle X\rangle_n \|^2 = \det G(a_1, \ldots, a_n)$, the Gram determinant.

Proof. (a) Telescope, using $a_n a_n = Q(a_n)$. (b) Both sides are multilinear and alternating, so it suffices to check them on basis vectors, where both sides equal $\delta_{IJ}\prod_{i\in I}\eta_i$. For (3), use $\| X \|^2 = \langle \tilde{X}X\rangle$ together with (a). The last identity follows from (b) and Corollary 3.3(b). ▫

Remark 3.6. Subtracting $\det G$ from (3) gives Hadamard's inequality $\det G \le \prod |a_i|^2$ [13], and its defect is an explicit sum of squares of the lower-grade parts in (2). For three vectors the defect is $\| (a\cdot b)c - (a\cdot c)b + (b\cdot c)a \|^2$.

4 The grade-spectrum theorem

From now on the signature is Euclidean, $V = \mathbb{R}^d$. Write $w_s(X) = \| \langle X\rangle_s \|^2$ and $G_X(t) = \sum_s w_s\,(X) t^s$.

Lemma 4.1 (conjugation invariance). Let $h \in \mathrm{Pin}(d)$ and let X be a versor. Then $w_s(hXh^{-1}) = w_s(X)$ for every s, and $\rho_{hXh^{-1}} = \rho_h \rho_X \rho_h^{-1}$.

Proof. It suffices to take $h = u$, a unit vector. The automorphism $\mathrm{Ad}_u : Y \mapsto uYu^{-1}$ maps V onto V by the orthogonal map $f = -S_u$. Since $f(e_1), \ldots, f(e_d)$ are orthonormal and hence pairwise anticommuting,

$$\mathrm{Ad}_u(e_I) = f\big(e_{i_1}\big)\cdots f\big(e_{i_r}\big) = f\big(e_{i_1}\big)\wedge \ldots \wedge f\big(e_{i_r}\big).$$

By Theorem 3.5(b) these r-blades are orthonormal for $\langle \tilde{\cdot}\,\cdot\rangle$. So Ad_u maps each grade isometrically onto itself. The second claim follows because ρ is a homomorphism. ▫

Lemma 4.2 (orthogonal factorisation). Let $V = U \oplus W$ be an orthogonal decomposition, and let Y and Z be versors built from vectors of U and of W respectively. Then $G_{YZ}(t) = G_Y(t) G_Z(t)$ and $\rho_{YZ} = \rho_Y|_U \oplus \rho_Z|_W$.

Proof. Choose an orthonormal basis of V that is the union of bases of U and W. For $I \subseteq U$ and $J \subseteq W$ we have $e_I e_J = \pm e_{I\cup J}$, and distinct pairs (I, J) give distinct blades. Hence $(YZ)_{I\cup J} = \pm Y_I Z_J$, and the energies

multiply grade-additively. A vector $w \in W$ anticommutes with every vector of U, so $\alpha(Y)wY^{-1} = w$. Thus ρ_Y is the identity on W, and symmetrically ρ_Z is the identity on U. ▫

Theorem 4.3 (grade-spectrum theorem). Let $X = a_1 \cdots a_n$ with nonzero $a_i \in \mathbb{R}^d$. Then for all t,

$$\sum_{s=0}^{d} \| \langle X \rangle_s \|^2 \, t^s = |X|^2 \det\left(\frac{(1+t)I + (1-t)\rho_X}{2}\right). \qquad (4)$$

Proof. Reduction to unit versors. Write $X = |X|\hat{X}$ with $\hat{X} = \prod_i a_i / |a_i| \in \mathrm{Pin}(d)$. Since $\rho_X = \rho_{\hat{X}}$, we may assume $|X| = 1$.

Reduction to canonical form. By the real normal form of orthogonal matrices, there is $g \in O(d)$ such that $g\rho_X g^{-1} = D$, where

$$D = R(\theta_1) \oplus \ldots \oplus R(\theta_m) \oplus (-I_\ell) \oplus I_r,$$

$R(\theta)$ is the plane rotation by $\theta \in (0, \pi)$, and $2m + \ell + r = d$. Lift g to $h \in \mathrm{Pin}(d)$. By Lemma 4.1, both sides of (4) are unchanged when X is replaced by hXh^{-1}, and $\rho_{hXh^{-1}} = D$. Let $Y = R_1 \cdots R_m \, e_{2m+1} \cdots e_{2m+\ell}$ with $R_k = \cos\frac{\theta_k}{2} - \sin\frac{\theta_k}{2} \, e_{2k-1}e_{2k}$; with the convention $R(\theta)e_1 = \cos\theta \, e_1 + \sin\theta \, e_2$, a direct computation gives $\rho_{R_k} = R(\theta_k)$ on its plane. Then $\rho_Y = D$, so $hXh^{-1} = \pm Y$ because the kernel of ρ is $\{\pm 1\}$. The sign changes neither side of (4).

Factors. By Lemma 4.2, both sides of (4) for Y factor over the blocks of D. For a plane rotor R_k the left side is $\cos^2\frac{\theta_k}{2} + t^2\sin^2\frac{\theta_k}{2}$. On the right side, with $\alpha = \frac{1+t}{2}$ and $\beta = \frac{1-t}{2}$,

$$\begin{aligned} \det\big(\alpha I + \beta R(\theta_k)\big) &= \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta_k = (1 + t/2)^2 + (1 - t/2)^2 + 1 - t^2/2\cos\theta_k \\ &= \cos^2\theta_k/2 + t^2\sin^2\theta_k/2\,. \end{aligned}$$

For a unit vector e_j both sides equal t, since $\rho_{e_j} = -1$ on $\mathbb{R}e_j$ and $\alpha - \beta = t$. On the fixed subspace both sides equal 1. ▫

Corollary 4.4 (Poisson-binomial form). Let X be a unit versor. Suppose ρ_X has rotation angles $\theta_1, \ldots, \theta_m \in (0, \pi)$ and eigenvalue -1 with multiplicity ℓ (outside the rotation blocks). Then

$$\sum_s w_s(X) t^s = t^\ell \prod_{k=1}^{m} (\cos^2\theta_k/2 + t^2\sin^2\theta_k/2).$$

Thus $w_s(X) = \Pr(\ell + 2\sum_k \xi_k = s)$ for independent Bernoulli variables ξ_k with $\Pr(\xi_k = 1) = \sin^2(\theta_k/2)$.

Corollary 4.5 (Krawtchouk–exterior-power formula). Let

$$K_s(k) = \sum_i (-1)^i \binom{k}{i}\binom{d-k}{s-i}$$

be the Krawtchouk polynomials [23]. Then

$$\| \langle X \rangle_s \|^2 = \frac{|X|^2}{2^d} \sum_{k=0}^{d} K_s(k) \, \mathrm{tr}\Lambda^k \rho_X.$$

Proof. We have $\det(\alpha I + \beta\rho) = \sum_k \alpha^{d-k}\,\beta^k \mathrm{tr}\Lambda^k\rho$ and $(1+t)^{d-k}(1-t)^k = \sum_s K_s\,(k)t^s$. Compare the coefficients of t^s. ▫

Corollary 4.6 (mean, variance, extreme grades). For a unit versor X, let $M(X) = \sum_s s\ w_s$ and $\varsigma^2(X) = \sum_s s^2\,w_s - M(X)^2$ (the grade variance). Then

$$M(X) = \frac{d - \mathrm{tr}\rho_X}{2}, \qquad \varsigma^2(X) = \frac{d - \mathrm{tr}\rho_X^2}{4},$$

$$\langle X\rangle^2 = \det\frac{I+\rho_X}{2}, \qquad \| \langle X\rangle_d \|^2 = \det\frac{I-\rho_X}{2}.$$

Proof. Write $\log G_X(t) = \sum_j \log\frac{(1+t)+(1-t)\lambda_j}{2}$ over the eigenvalues λ_j of ρ_X. Then $M = (\log G)'(1) = \sum_j \frac{1-\lambda_j}{2}$ and $\varsigma^2 = (\log G)''(1) + (\log G)'(1) = \sum_j \frac{1-\lambda_j^2}{4}$. The last two identities are $G_X(0)$ and the leading coefficient of G_X. ▫

Corollary 4.7 (complete invariant). For unit versors X and Y in $\mathbb{R}^d$ the following are equivalent: (i) $w_s(X) = w_s(Y)$ for all s; (ii) ρ_X and ρ_Y are conjugate in $O(d)$; (iii) $Y = \pm hXh^{-1}$ for some $h \in \mathrm{Pin}(d)$.

Proof. With $t = \frac{1-x}{1+x}$, (4) becomes $(1+x)^d G_X\left(\frac{1-x}{1+x}\right) = \det(I + x\rho_X)$. So the profile determines the characteristic polynomial of ρ_X. Orthogonal matrices with the same characteristic polynomial are conjugate in $O(d)$ by the real normal form, so (i) implies (ii). Lifting through $\mathrm{Pin}(d) \to O(d)$ gives (ii)⇒(iii), and Lemma 4.1 gives (iii)⇒(i). ▫

Example 4.8. For $X = abc$ in Example 3.4, $|X|^2 = 8$ and the normalised profile is $\left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$. So $G_X(t) = 8 \cdot t\,\frac{1+t^2}{2} = 4t + 4t^3$. By Corollary 4.4, ρ_X is a rotoreflection whose rotation angle φ satisfies $\sin^2(\varphi/2) = \frac{1}{2}$, so $\varphi = \pi/2$. Equivalently, $\mathrm{tr}\rho_X = d - 2M = -1$. In $\mathbb{R}^3$, an even unit versor (a rotor) with rotation angle φ has $M = 2\sin^2(\varphi/2)$, which recovers $\langle R\rangle^2 = \cos^2(\varphi/2)$.

Remark 4.9 (the case $t = 0$ and spin characters). Let $d = 2m$ be even. The complexification $C\ell_{d,0} \otimes \mathbb{C}$ is isomorphic to $\mathrm{End}(S)$ for the spin module S of dimension 2^m. Every blade e_I with $I \neq \emptyset$ is traceless on S: it anticommutes with some e_j, namely $j \notin I$ when $|I|$ is odd and $j \in I$ when $|I|$ is even. Hence $\langle Y\rangle = 2^{-m}\mathrm{tr}_S(Y)$ for all Y. For $X \in \mathrm{Spin}(d)$ whose image ρ_X has angles $\theta_1, \ldots, \theta_m$ (with $\theta_k \in \{0, \pi\}$ allowed), the weights of S give $\mathrm{tr}_S(X) = \pm\prod_k 2\cos(\theta_k/2)$. (Here $\mathrm{tr}_S(hYh^{-1}) = \mathrm{tr}_S(Y)$ for every $h \in \mathrm{Pin}(d)$, since S is a module for the whole algebra, so we may conjugate X to canonical form.) So $\langle X\rangle^2 = \prod_k \cos^2(\theta_k/2) = \det\big((I+\rho_X)/2\big)$, which is (4) at $t = 0$. In representation-theoretic form this is $|\chi_S|^2 = \sum_k \chi_{\Lambda^k} = \det(I+\rho)$, coming from $S \otimes S^* \cong \Lambda^\bullet V_{\mathbb{C}}$ [12, Lecture 20], [20, Ch. I]. For odd X both sides vanish, since ρ_X has eigenvalue -1. For odd d, embed $\mathbb{R}^d \subset \mathbb{R}^{d+1}$: $\langle X\rangle$ is unchanged and ρ_X becomes $\rho_X \oplus 1$, so $\det\big((I+\rho_X)/2\big)$ is unchanged.

Fig. 1(a) shows (4) for a random versor in $\mathbb{R}^{10}$.

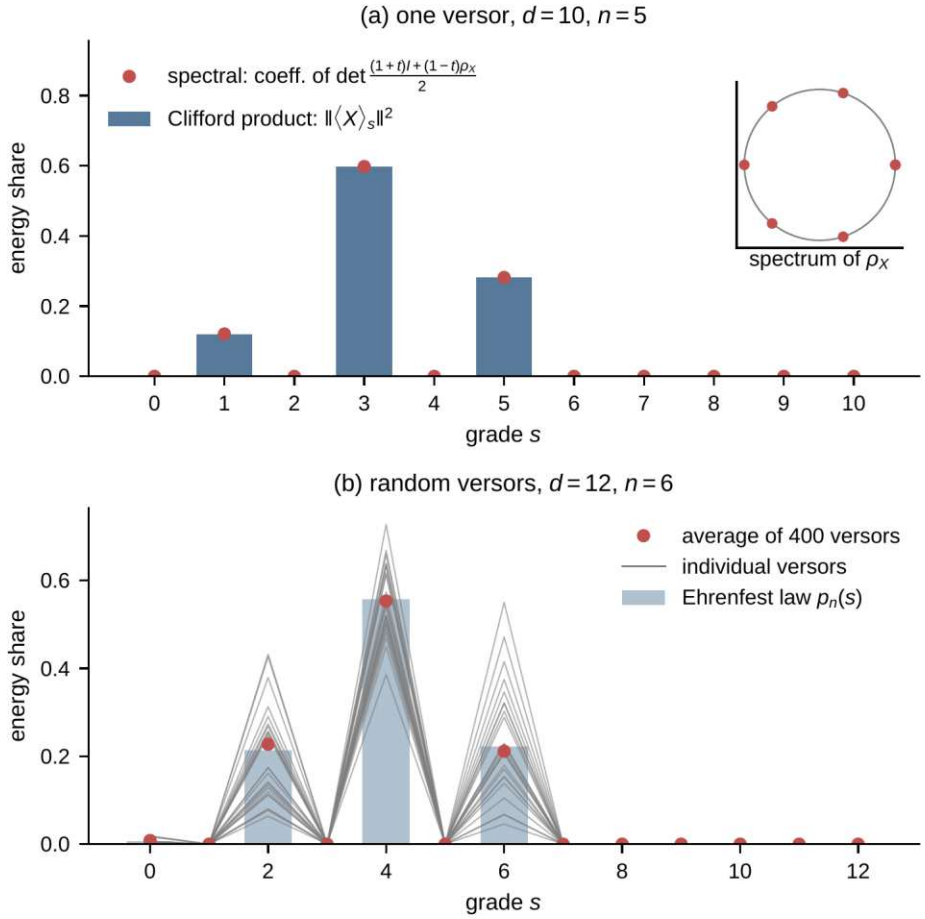


Fig. 1. *(a) Grade energies of a random product of* 5 *unit vectors in* $\mathbb{R}^{10}$ *(bars), and the coefficients of* $det\left(((1+t)I+(1-t)\rho_X)/2\right)$ *(dots). The inset shows the spectrum of* ρ_X*. (b) Profiles of individual random versors (*$d = 12$*,* $n = 6$*; thin lines) and their average over* 400 *samples (dots), compared with the Ehrenfest law of Theorem 5.1 (bars).*

4.1 Blade energies

The grade energies are sums of squared coordinates X_I^2 over blades of one grade. A shift by a coordinate blade turns the case $t = 0$ of Theorem 4.3 into a formula for each coordinate separately. For $I \subseteq [d]$ let D_I be the diagonal matrix with entries -1 at positions $i \in I$ and $+1$ elsewhere; in this subsection we write I_d for the identity matrix.

Theorem 4.10 (blade-spectrum theorem). Let $X = a_1 \cdots a_n$ with nonzero $a_i \in \mathbb{R}^d$, and let $T = \mathrm{diag}(t_1, \ldots, t_d)$, with ρ_X written as a matrix in the basis $e_1, \ldots, e_d$ that defines the coordinates X_I. Then

$$\sum_{I \subseteq [d]} X_I^2 \prod_{i \in I} t_i = |X|^2 \det\left(\frac{I_d + \rho_X}{2} + \frac{I_d - \rho_X}{2}\, T\right). \qquad (5)$$

Equivalently, for every $I \subseteq [d]$,

$$X_I^2 = |X|^2 \det \frac{I_d + \rho_X D_I}{2} = \frac{|X|^2}{2^d} \sum_{A \subseteq [d]} (-1)^{|A \cap I|} \det(\rho_X)_{AA}, \qquad (6)$$

where $(\rho_X)_{AA}$ is the principal submatrix on A (and $\det(\rho_X)_{\emptyset\emptyset} = 1$).

Proof. For orthonormal basis blades, $\langle e_J \widetilde{e_I} \rangle = \delta_{IJ}$, so $\langle X \widetilde{e_I} \rangle = X_I$. The element $Y = X\widetilde{e_I}$ is a versor, a product of the a_i and of coordinate vectors, with $|Y| = |X|$ and $\rho_Y = \rho_X \rho_{\widetilde{e_I}} = \rho_X D_I$. Theorem 4.3 at $t = 0$ gives $\langle Y \rangle^2 = |Y|^2 \det\big((I_d + \rho_Y)/2\big)$, which is the first equality in (6). The j-th column of $(I_d + \rho_X D_I)/2$ is the j-th column of $(I_d - \rho_X)/2$ if $j \in I$ and of $(I_d + \rho_X)/2$ otherwise. Since the determinant is multilinear in the columns, and right multiplication by T scales the j-th column by t_j, expanding the right side of (5) gives $\sum_I \prod_{i \in I} t_i \det\big((I_d + \rho_X D_I)/2\big)$, which proves (5). Finally, $\det(I_d + M) = \sum_A \det M_{AA}$ and $\det(\rho_X D_I)_{AA} = (-1)^{|A \cap I|} \det(\rho_X)_{AA}$ give the principal-minor form. ▫

Remark 4.11. Setting $T = tI_d$ in (5) recovers Theorem 4.3. Summing (6) over $|I| = s$ recovers Corollary 4.5, because $\sum_{|I|=s} (-1)^{|A \cap I|} = K_s(|A|)$ and $\sum_{|A|=k} \det (\rho_X)_{AA} = \mathrm{tr} \Lambda^k \rho_X$. So the whole of this section follows from the scalar case $\langle X \rangle^2 = |X|^2 \det\big((I_d + \rho_X)/2\big)$, which is the spin-character identity of Remark 4.9, by the shift $X \mapsto X\widetilde{e_I}$.

For even unit versors and $|I|$ even, (6) also follows from Shirokov's identity $2^d X \langle X^{-1} e_I \rangle_0 = M_I$ [33, eq. (18)]. On the left, $\langle M_I \widetilde{e_I} \rangle = 2^d X_I^2$; on the right, only the terms with $A = B$ survive, with sign $(-1)^{|A \cap I|}$, which gives the principal-minor form. (For $|I|$ odd both sides of (6) vanish.) Our formula serves as a selection rule for his reconstruction. Choosing I with the largest determinant in (6) gives $|M_I| = 2^d |X_I| \geq 2^{(d+1)/2}$ for a unit even versor, because its 2^{d-1} even coordinates have squares summing to 1; this is the d-dimensional analogue of Shepperd's rule of choosing the largest quaternion component.

Setting $T = I_d$ in (5), whose expansion uses only multilinearity, gives $\sum_I \det (I_d + \rho D_I) = 2^d$ for every square matrix ρ. For orthogonal ρ each term is nonnegative, and (6) identifies it as 2^d times a squared coordinate of either lift $\pm X \in \mathrm{Pin}(d)$.

Corollary 4.12 (quaternions from rotation matrices). Let $d = 3$ and let $q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$ be the unit quaternion of a rotation $R \in SO(3)$, with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ identified with $-e_{23}, -e_{31}, -e_{12}$ (any choice of signs gives the same squares). Then $q_0^2 = \frac{1}{4}(1 + \mathrm{tr} R)$ and $q_i^2 = \frac{1}{4}(1 + 2R_{ii} - \mathrm{tr} R)$ for $i = 1,2,3$.

Proof. For $Q \in SO(3)$ with angle φ, $\det\big((I_3 + Q)/2\big) = \frac{1+\cos\varphi}{2} = \frac{1+\mathrm{tr} Q}{4}$. Apply (6) with $I = \emptyset$, and with $I = \{j, k\} = [3] \backslash \{i\}$, where $D_I \in SO(3)$ and $\mathrm{tr}(R D_I) = R_{ii} - R_{jj} - R_{kk} = 2R_{ii} - \mathrm{tr} R$. ▫

Corollary 4.13 (unimodality and dominant grade). Let X be a unit versor, with ℓ and $\theta_1, \ldots, \theta_m$ as in Corollary 4.4, and put $u_j = w_{\ell+2j}(X)$ for $0 \leq j \leq m$. Then:

(a) $u_j^2 \geq \left(1 + \frac{1}{j}\right)\left(1 + \frac{1}{m-j}\right) u_{j-1} u_{j+1}$ for $1 \leq j \leq m-1$, that is, $\big(u_j\big)_{0 \leq j \leq m}$ is ultra-log-concave of order m; in particular it is log-concave and unimodal;

(b) every grade s^* of maximal energy satisfies $|s^* - M(X)| < 2$, where $M(X) = (d - \mathrm{tr} \rho_X)/2$.

Proof. By Corollary 4.4, $\sum_j u_j x^j = \prod_{k=1}^m \left(\cos^2 \frac{\theta_k}{2} + x \sin^2 \frac{\theta_k}{2}\right)$, a polynomial with positive coefficients and only real roots, since $\theta_k \in (0, \pi)$. Newton's inequalities [14, 34] give (a). For (b), u is the law of a Poisson-binomial variable $B = \sum_k \xi_k$ with mean $\mu_B = \sum_k \sin^2 (\theta_k/2)$. By Darroch's theorem [5], every mode j^* of B satisfies $|j^* - \mu_B| < 1$. Since $s^* = \ell + 2j^*$ and $M(X) = \ell + 2\mu_B$, (b) follows. The constant

2 is best possible: if $\ell = 0$ and $\sin^2(\theta_k/2) = \frac{1}{m+1} - \varepsilon$ for all k, the unique dominant grade is 0, while $M(X) = \frac{2m}{m+1} - 2m\varepsilon$. ▫

5 Random products: first moments

A random vector $a \in \mathbb{R}^d$ is *isotropic with scale* σ^2 if $\mathbb{E}[aa^{\mathsf{T}}] = \frac{\sigma^2}{d} I$. The Ehrenfest chain [10, 17] on $\{0, \dots, d\}$ moves $r \to r-1$ with probability r/d and $r \to r+1$ with probability $(d-r)/d$. Let p_n be its law at time n, starting from 0.

Theorem 5.1 (Ehrenfest law). Let $a_1, \dots, a_n$ be independent, with a_k isotropic of scale σ_k^2. Then

$$\mathbb{E} \parallel \langle a_1 \cdots a_n \rangle_s \parallel^2 = \left(\prod_k \sigma_k^2 \right) p_n(s), \qquad p_n(s) = \frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} K_s(k) \left(1 - \frac{2k}{d}\right)^n .$$

The first identity holds in any signature $\mathbb{R}^{p,q}$ for the coefficient energy, with isotropy understood in the chosen orthonormal basis.

Proof. First proof (any signature). The coefficient of e_J in Ya is $\sum_i \theta \; a_i Y_{J \triangle \{i\}}$ with $\theta = \pm 1$. By independence and isotropy,

$$\mathbb{E}(Ya)_J^2 = \frac{\sigma^2}{d} \sum_i \mathbb{E}\, Y_{J \triangle \{i\}}^2 .$$

A set K with $|K| = s-1$ arises as $J \triangle \{i\}$ from $d-s+1$ sets J of size s, and a set with $|K| = s+1$ from $s+1$ of them. Summing over $|J| = s$ therefore gives

$$\mathbb{E} w_s(Ya) = \frac{\sigma^2}{d} \left[(d-s+1)\, \mathbb{E} w_{s-1}(Y) + (s+1)\, \mathbb{E} w_{s+1}(Y) \right],$$

so the normalised grade-energy vector evolves by the transposed Ehrenfest kernel. The closed form is the Fourier inversion for the walk on $\{0,1\}^d$ that flips one coordinate per step [6].

Second proof (Euclidean, via Corollary 4.5, assuming $\Pr(a_k = 0) = 0$*).* Let H_a be the matrix of S_a. On $\Lambda^k V$ we have $|a|^2 \Lambda^k(H_a) B = |a|^2 B - 2\, a \wedge (a \lrcorner B)$, and $\mathbb{E}[a \wedge (a \lrcorner B)] = \frac{\sigma^2}{d} \sum_i e_i \wedge (e_i \lrcorner B) = \frac{\sigma^2 k}{d} B$. So $\mathbb{E}[|a|^2 \Lambda^k(H_a)] = \sigma^2 \left(1 - \frac{2k}{d}\right) I$. Since Λ^k is multiplicative and the factors are independent,

$$\mathbb{E}[|X|^2 \operatorname{tr} \Lambda^k \rho_X] = \binom{d}{k} \left(\prod_j \sigma_j^2 \right) \left(1 - \frac{2k}{d}\right)^n .$$

Substituting into Corollary 4.5 gives the closed form. ▫

Corollary 5.2. Let the factors be unit-scale ($\sigma_k^2 = 1$), and set $\kappa_1 = \left(1 - \frac{2}{d}\right)^n$ and $\kappa_2 = \left(1 - \frac{4}{d}\right)^n$. Then: (a) $p_n(s) = 0$ unless $s \equiv n \pmod 2$; (b) $\mathbb{E} \det G(a_1, \dots, a_n) = \prod_{j<n} \left(1 - \frac{j}{d}\right)$; (c) the Ehrenfest mean is $\mu_n = \frac{d}{2}(1 - \kappa_1)$ and the Ehrenfest variance is $v_n = \frac{d}{4}[1 + (d-1)\kappa_2 - d\kappa_1^2]$. Here $\kappa_1 = \kappa_1(n)$ and $\kappa_2 = \kappa_2(n)$ depend on n; this notation is used throughout.

Proof. (a) Each step changes the grade by ± 1. (b) $\| \langle X \rangle_n \|^2 = \det G$ by Theorem 3.5, and reaching grade n in n steps forces only upward moves. (Part (b) is also classical: by the Cauchy–Binet formula, $\mathbb{E}\det G = n! \binom{d}{n} d^{-n}$.) (c) Write the grade as a sum of hypercube coordinates and use $\mathbb{E}(-1)^{x_i} = \kappa_1$ and $\mathbb{E}(-1)^{x_i + x_k} = \kappa_2$ for $i \neq k$. ▫

Table 1 lists p_n for $d = 3$, and Fig. 2 shows p_n for $d = 8$ together with Monte Carlo averages over uniform unit factors.

Table 1. *Exact $p_n(s)$ for $d = 3$.*

n	$s = 0$	$s = 1$	$s = 2$	$s = 3$
1	0	1	0	0
2	1/3	0	2/3	0
3	0	7/9	0	2/9
4	7/27	0	20/27	0
5	0	61/81	0	20/81
6	61/243	0	182/243	0

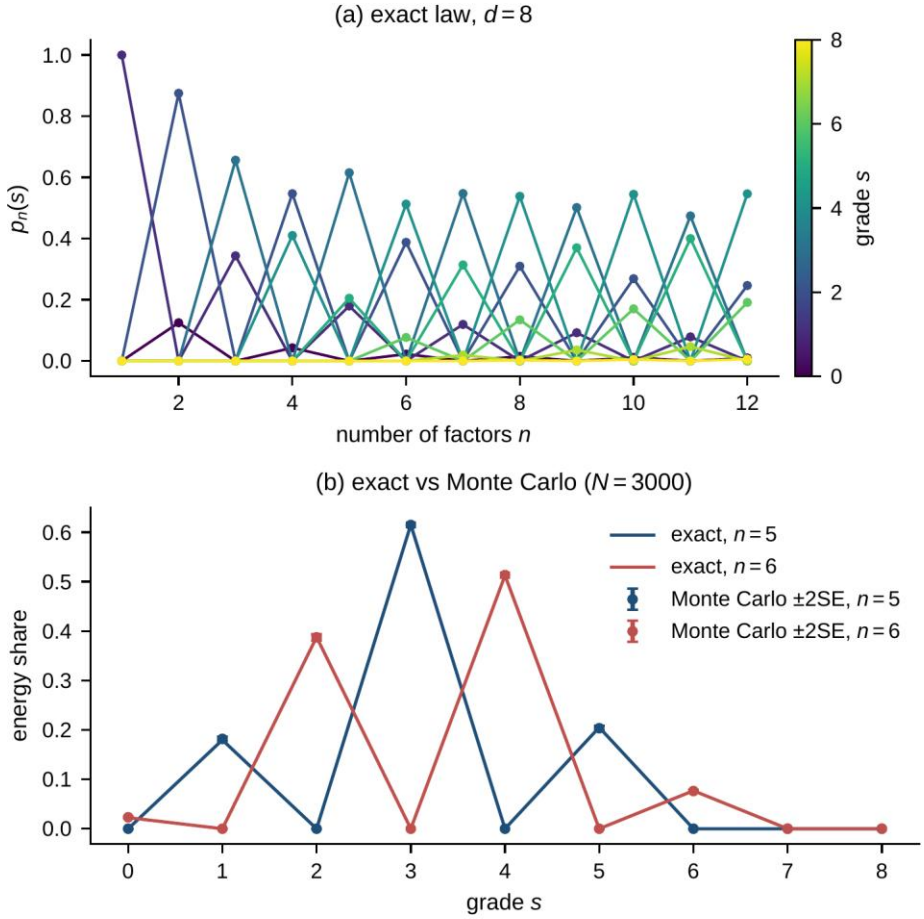


Fig. 2. *(a) Exact Ehrenfest law $p_n(s)$ for $d = 8$. (b) Exact p_5 and p_6 against Monte Carlo averages over uniform unit vectors ($N = 3000$, ± 2 standard errors).*

6 Exact second moments

From now on, $a_1, a_2, \ldots$ are independent and uniform on the unit sphere S^{d-1}. Let $\rho_n = \rho_{a_1 \cdots a_n} = H_1 \cdots H_n$, where $H_i = I - 2a_i a_i^{\mathsf{T}}$ is the matrix of S_{a_i}, and set $T_n = \mathrm{tr}\rho_n$. By Corollary 4.6, the mean grade is $M_n = (d - T_n)/2$ and the grade variance is $\varsigma_n^2 = (d - \mathrm{tr}\rho_n^2)/4$.

Lemma 6.1. Let u be uniform on S^{d-1} and $H = I - 2uu^{\mathsf{T}}$. On $V \otimes V$ the operator $\mathbb{E}[H \otimes H]$ acts as 1 on $\Omega = \sum_i e_i \otimes e_i$; as $c_A = \frac{d-4}{d}$ on antisymmetric tensors; and as $c_S = \frac{d-2}{d+2}$ on traceless symmetric tensors.

Proof. Write $H \otimes H = I - 2P \otimes I - 2I \otimes P + 4P \otimes P$ with $P = uu^{\mathsf{T}}$, so that $\mathbb{E}P = \frac{1}{d} I$. If g is a standard Gaussian vector, then $u = g/|g|$ is independent of $|g|$, and $\mathbb{E}g_i g_j g_k g_l = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. Dividing by $\mathbb{E}|g|^4 = d(d+2)$ gives

$$\mathbb{E}[P \otimes P](x \otimes y) = \frac{(x \cdot y)\,\Omega + x \otimes y + y \otimes x}{d(d+2)}.$$

This vanishes on $x \otimes y - y \otimes x$, maps Ω to Ω/d, and multiplies a traceless symmetric tensor by $\frac{2}{d(d+2)}$. Collecting terms gives the three eigenvalues. ▫

Theorem 6.2 (exact second moments). For $n \geq 0$,

$$\mathbb{E}T_n = d\kappa_1, \qquad \mathbb{E}T_n^2 = 1 + \binom{d}{2} c_A^n + \frac{(d-1)(d+2)}{2} c_S^n,$$
$$\mathbb{E}\mathrm{tr}\rho_n^2 = 1 + \frac{(d-1)(d+2)}{2} c_S^n - \binom{d}{2} c_A^n.$$

Consequently $\mathbb{E}M_n = \mu_n$, $\mathrm{Var}M_n = \frac{1}{4}\mathrm{Var}T_n$ and $\mathbb{E}\varsigma_n^2 = \frac{1}{4}(d - \mathbb{E}\mathrm{tr}\rho_n^2)$, and the law of total variance $v_n = \mathbb{E}\varsigma_n^2 + \mathrm{Var}M_n$ holds exactly.

Proof. By independence, $\mathbb{E}\left[\rho_n^{\otimes 2}\right] = \mathbb{E}[H \otimes H]^n$. We have $T_n^2 = \mathrm{tr}\left(\rho_n^{\otimes 2}\right)$ and $\mathrm{tr}\rho_n^2 = \mathrm{tr}\left(\rho_n^{\otimes 2}\,\mathrm{SW}\right)$, where SW is the swap, which acts as $+1$ on symmetric and -1 on antisymmetric tensors. Lemma 6.1 and the dimensions 1, $\binom{d}{2}$ and $\frac{(d-1)(d+2)}{2}$ of the three eigenspaces give the formulas. The first moment follows from $\mathbb{E}H = \left(1 - \frac{2}{d}\right) I$. For the decomposition, note that the energy-weighted law of the grade, averaged over X, is p_n. ▫

Proposition 6.3 (equilibrium). Let $d \geq 3$, and let g be Haar-distributed on $SO(d)$ or on the coset $O^-(d) = \{\det = -1\}$. Then $\mathbb{E}\mathrm{tr}g = 0$ and $\mathbb{E}(\mathrm{tr}g)^2 = \mathbb{E}\mathrm{tr}g^2 = 1$. Consequently, at equilibrium the mean grade has expectation $d/2$ and variance $1/4$, and the grade variance of a typical versor has expectation $(d-1)/4$. These are the limits of Theorem 6.2 as $n \to \infty$.

Proof. Left invariance gives $\mathbb{E}g = h\,\mathbb{E}g$ for every $h \in SO(d)$. So every column of $\mathbb{E}g$ is fixed by $SO(d)$, and hence $\mathbb{E}g = 0$. The operator $\mathbb{E}_{SO}[g \otimes g]$ is the orthogonal projection onto the $SO(d)$-invariant tensors, which for $d \geq 3$ form the line $\mathbb{R}\Omega$. Hence $\mathbb{E}(\mathrm{tr}g)^2 = 1$ and $\mathbb{E}\mathrm{tr}g^2 = \langle \Omega, \mathrm{SW}\,\Omega\rangle/|\Omega|^2 = 1$. On the coset, write $g = g_0 h$ with $h \in SO(d)$ and note that $(g_0 \otimes g_0)\Omega = \Omega$. The limits follow as $n \to \infty$, since $|c_A|, |c_S| < 1$ for $d \geq 3$. For $d = 2$, $c_A = -1$, so the hypothesis $d \geq 3$ is sharp. ▫

Remark 6.4. The averaged equilibrium profile $\pi_n(s) \propto \binom{d}{s}$ (Section 7) has spread $\sqrt{d}/2$. A typical versor has a profile of almost the same spread, $\sqrt{d-1}/2$, but its centre $M = (d-T)/2$ lies within $O(1)$ of $d/2$ (standard deviation $\frac{1}{2}$). For Haar measure on the whole of $O(d)$, $\mathrm{tr}g$ is asymptotically standard normal [7]; only the exact moments of Proposition 6.3 are used below. The slow statistic $\mathrm{tr}\rho_n$ is invisible in p_n because it is drowned by the $\sqrt{d}$ spread. This is the mechanism behind Section 7.

7 Averaged versus individual profiles: a two-scale cut-off

Let $\pi_n(s) = \binom{d}{s}2^{1-d}$ for $s \equiv n \pmod 2$ and 0 otherwise. Let L_n be the law of the random profile $w(X_n) = (w_0, \dots, w_d)$ of $X_n = a_1 \cdots a_n$, a probability measure on the simplex Δ_d. Let Π_n be the law of $w(X)$ when ρ_X is Haar-distributed on the coset $O^{(n)} = \{g \in O(d): \det g = (-1)^n\}$. As before, $\kappa_1 = \left(1 - \frac{2}{d}\right)^n$ and $\kappa_2 = \left(1 - \frac{4}{d}\right)^n$. Two quantities are compared below: the distance $\mathrm{TV}(p_n, \pi_n)$ of the *mean profile*, a distance between two probability vectors on $\{0, \dots, d\}$; and the distance $\mathrm{TV}(L_n, \Pi_n)$ of the *law of the profile*, a distance between two measures on Δ_d.

Theorem 7.1 (expected profile; classical). Let $d \geq 4$ and $c > 0$. (i) If $n \geq \frac{d}{4}(\log d + c)$, then $\mathrm{TV}(p_n, \pi_n) \leq \frac{1}{2}\sqrt{e^{e^{-c}} - 1}$. (ii) If $0 \leq n \leq \frac{d}{4}(\log d - c)$, then $\mathrm{TV}(p_n, \pi_n) \geq 1 - 8d^{2/d}e^{-c}$.

The cut-off itself is classical [6]; a short proof with these explicit constants is given in Appendix A.

Lemma 7.2 (equilibrium mean profile). For $d \geq 2$ and every n, $\mathbb{E}_{\Pi_n}[w] = \pi_n$.

Proof. Let g be Haar on $O^{(n)}$. For $0 < k < d$ the $SO(d)$-module $\Lambda^k V$ has no nonzero invariant vector: for $|J| = k$ choose $i \in J$ and $j \notin J$; the element of $SO(d)$ acting as -1 on e_i and e_j and as 1 elsewhere sends e_J to $-e_J$ and fixes or negates every other basis blade, so an invariant vector has zero e_J-coefficient. (This covers $d = 2$ and the middle degree $k = d/2$, where $\Lambda^k V$ is reducible.) Hence $\mathbb{E}[\Lambda^k h] = 0$ for h Haar on $SO(d)$, since $\mathbb{E}[\Lambda^k h]$ is the orthogonal projection onto the invariants. Writing $g = g_0 h$ gives $\mathbb{E}\mathrm{tr}\Lambda^k g = \mathrm{tr}\left(\Lambda^k g_0\, \mathbb{E}\Lambda^k h\right) = 0$ for $0 < k < d$. Also $\mathrm{tr}\Lambda^0 g = 1$ and $\mathrm{tr}\Lambda^d g = \det g = (-1)^n$. By Corollary 4.5 and $K_s(0) = \binom{d}{s}$, $K_s(d) = (-1)^s\binom{d}{s}$,

$$\mathbb{E}w_s = 2^{-d}\left(K_s(0) + (-1)^n K_s(d)\right) = 2^{-d}\binom{d}{s}\left(1 + (-1)^{n+s}\right) = \pi_n(s).$$

▫

Lemma 7.3 (averaging is a contraction). For every n, $\mathrm{TV}(p_n, \pi_n) \leq \mathrm{TV}(L_n, \Pi_n)$.

This is the data-processing inequality for the Markov kernel that sends a profile $w \in \Delta_d$ to the law w on $\{0, \dots, d\}$; we include the two-line proof.

Proof. By Theorem 5.1 with unit factors and Lemma 7.2, $p_n = \int w\, dL_n$ and $\pi_n = \int w\, d\Pi_n$. With $\mu = L_n - \Pi_n$ and $|\mu|$ its total variation measure,

$$2\mathrm{TV}(p_n, \pi_n) = \sum_s \left|\int w_s\, d\mu\right| \leq \int \sum_s w_s\; d|\mu| = |\mu|(\Delta_d) = 2\mathrm{TV}(L_n, \Pi_n),$$

since $w \geq 0$ and $\sum_s w_s = 1$ on Δ_d. ▫

Lemma 7.4 (the profile carries the full distance). For every n,

$$\mathrm{TV}(L_n, \Pi_n) = \mathrm{TV}\big(\mathcal{L}(\rho_n), \mathrm{Haar}_{O^{(n)}}\big).$$

Proof. By Theorem 4.3, the profile is a continuous function $w = W(\rho_X)$ of ρ_X (the coefficients of a determinant). By Corollary 4.7, the fibres of W on $O^{(n)}$ are the $O(d)$-conjugacy classes. Since L_n and Π_n are the images of the two measures under W, the inequality $\leq$ holds. For $\geq$, let $\mu = \mathcal{L}(\rho_n) - \mathrm{Haar}_{O^{(n)}}$. This signed measure is invariant under conjugation by $O(d)$, because the step law is: $gH_ug^{-1} = H_{gu}$. On the compact space $O^{(n)}$ we have $2\mathrm{TV} = \sup\{\left|\int f\, d\mu\right| : f \text{ continuous}, |f| \leq 1\}$. Given such an f, let $\bar{f}(x) = \int_{O(d)} f\,(gxg^{-1})\, dg$. Then $\bar{f}$ is a continuous class function with $|\bar{f}| \leq 1$ and $\int \bar{f}\, d\mu = \int f\, d\mu$. Because $\bar{f}$ is constant on the fibres of W, we can write $\bar{f} = \varphi \circ W$. The function φ is continuous on the compact image, since a continuous surjection between compact Hausdorff spaces is a quotient map. Hence $\left|\int f\, d\mu\right| = \left|\int \varphi\, d(L_n - \Pi_n)\right| \leq 2\mathrm{TV}(L_n, \Pi_n)$. ▫

Theorem 7.5 (individual profiles: lower bound). Let $d \geq 3$ and $n \geq 0$.

(a) (second-moment form) $\mathrm{TV}(L_n, \Pi_n) \geq 1 - \frac{4\,(1+\mathrm{Var}T_n)}{(d\kappa_1)^2}$, with $\mathrm{Var}T_n$ given exactly by Theorem 6.2.

(b) (explicit form) Let $d \geq 4$, $c > 0$ and $n \leq \frac{d}{2}(\log d - c)$. Then

$$\mathrm{TV}(L_n, \Pi_n) \geq 1 - 8d^{4/d}e^{-2c} - 4\varepsilon_{d,n}, \qquad \varepsilon_{d,n} = \frac{1 + 1/d}{2}\left(e^{4n/(d^2-4)} - 1\right).$$

In this range $\varepsilon_{d,n} \leq \frac{1+1/d}{2}\left(d^{2d/(d^2-4)} - 1\right) \sim \frac{\log d}{d}$.

Proof. The statistic $T = \mathrm{tr}\rho = d - 2M$ is a function of the profile (Corollary 4.6). Under Π_n, Proposition 6.3 gives $\mathbb{E}T = 0$ and $\mathbb{E}T^2 = 1$. Under L_n, Theorem 6.2 gives $\mathbb{E}T_n = d\kappa_1 > 0$. Let $A = \{T > d\kappa_1/2\}$. Chebyshev's inequality gives $\Pi_n(A) \leq \frac{4}{(d\kappa_1)^2}$ and $L_n(A^c) \leq \frac{4\mathrm{Var}T_n}{(d\kappa_1)^2}$, and $\mathrm{TV} \geq L_n(A) - \Pi_n(A)$ gives (a).

For (b), Theorem 6.2 gives $\mathrm{Var}T_n = 1 + \binom{d}{2}c_A^n + \frac{(d-1)(d+2)}{2}c_S^n - d^2\kappa_1^2$. Since $c_A^n = \kappa_2 \leq \kappa_1^2$ and $c_S = \left(1 - \frac{2}{d}\right)^2\left(1 + \frac{4}{d^2-4}\right)$, we get $c_S^n \leq \kappa_1^2 e^{4n/(d^2-4)}$. Together with $\binom{d}{2} + \frac{(d-1)(d+2)}{2} = d^2 - 1$ and $(d-1)(d+2) \leq d^2\left(1 + \frac{1}{d}\right)$, this gives $\mathrm{Var}T_n \leq 1 + (d\kappa_1)^2\varepsilon_{d,n}$. Finally, $\log\kappa_1 \geq -\frac{2n}{d}\left(1 + \frac{2}{d}\right) \geq -(\log d - c)\left(1 + \frac{2}{d}\right)$, so $d\kappa_1 \geq e^c d^{-2/d}$ and $\frac{8}{(d\kappa_1)^2} \leq 8d^{4/d}e^{-2c}$. ▫

The explicit form (b) is informative only for moderately large d; for example, it is negative at $d = 16$, $c = 1$. The second-moment form (a) is sharper; Table 2 lists it at $n = \lfloor\frac{d}{2}(\log d - c)\rfloor$.

Remark 7.6 (comparison with Porod's lower bound). Porod [26, Theorem 1.2(b)] shows that for $N \geq 2$, $c > 0$ and $k = \frac{1}{2}N\log N - cN$ even, the walk at time $k - 1$ is at distance at least $1 - 16.4e^{-4c} - 46.3\,N^{-3/5}\log N$ from Haar measure on $O^-(N)$ (and part (a) gives the analogous statement at time k). In the parametrisation of Theorem 7.5, $n = \frac{d}{2}(\log d - 2c)$, part (b) gives $1 - 8d^{4/d}e^{-4c} - 4\varepsilon_{d,n}$ with $4\varepsilon_{d,n} \sim 4\log d/d$. So the leading constant improves from 16.4 to $8d^{4/d}$ for $d \geq 16$ (and $8d^{4/d} \to 8$), and the error term from $O\left(d^{-3/5}\log d\right)$ to $O(\log d/d)$. Both bounds rest on the same character ratios (Lemma 6.1).

Table 2. *Lower bound (a) of Theorem 7.5 for* $TV(L_n, \Pi_n)$ *at* $n = \lfloor \frac{d}{2}(logd - c) \rfloor$*, and the exact* $TV(p_n, \pi_n)$ *at the same* n*, computed from the Ehrenfest chain in 50-digit arithmetic for* $d \leq 256$ *and in double precision otherwise. As* d *grows,* $n/(dlogd)$ *tends to* $\frac{1}{2}$*; the mean profile is then past its cut-off at* $\frac{1}{4}$*, while the bound for the law of the profile stays near* $1 - 8e^{-2c}$*.*

d	c	n	$\frac{n}{d\log d}$	$\mathrm{Var} T_n$	bound (a) for $\mathrm{TV}(L_n, \Pi_n)$	$\mathrm{TV}(p_n, \pi_n)$
64	2	69	0.259	1.0531	0.8397	0.3534
64	3	37	0.139	0.9379	0.9802	0.8190
256	2	453	0.319	1.0141	0.8501	0.1820
256	3	325	0.229	1.0423	0.9796	0.4702
1024	2	2524	0.356	1.0020	0.8524	0.0916
1024	3	2012	0.283	1.0085	0.9800	0.2462
4096	2	12938	0.380	1.0002	0.8531	0.0460
4096	3	10890	0.320	1.0011	0.9801	0.1246

Theorem 7.7 (individual profiles: Porod's upper bound, transported). Let c_0 be the universal constant of [26, Theorem 1.1]. If $d \geq 16$, $c \geq c_0$ and $n \geq \frac{d}{2}\log d + cd + 1$, then

$$\mathrm{TV}(L_n, \Pi_n) \leq 10.6\, e^{-c/9}.$$

Proof. Porod's walk has step law uniform on the reflections, that is, the law of H_u with u uniform on S^{d-1}, which is ours. It compares the walk at even times with normalised Haar measure on $SO(d)$, and at odd times with normalised Haar measure on $O^-(d)$, in the total variation distance of Section 2 [26, Section 1 and Definition 2.1]. These two measures are $\mathrm{Haar}_{O(n)}$. Let $k = n$ if n is even and $k = n - 1$ otherwise, and put $c' = \left(k - \frac{d}{2}\log d\right)/d \geq c \geq c_0$. Then $k = \frac{d}{2}\log d + c'd$ is even. Porod's Theorem 1.1(a) (for $n = k$) or (b) (for $n = k + 1$) gives $\mathrm{TV}\left(\mathcal{L}(\rho_n), \mathrm{Haar}_{O(n)}\right) \leq 10.6\, e^{-c'/9} \leq 10.6\, e^{-c/9}$. Lemma 7.4 transfers this to the profile. ▫

Corollary 7.8 (two-scale cut-off). Let $d \to \infty$, and let $c_d \to \infty$ with $c_d \leq \log d$; in (b) take d large enough that $d \geq 16$ and $c_d \geq c_0$.

(a) Mean profile: $\mathrm{TV}(p_n, \pi_n) \to 0$ for $n = \lceil \frac{d}{4}(\log d + c_d) \rceil$, and $\mathrm{TV}(p_n, \pi_n) \to 1$ for $n = \lfloor \frac{d}{4}(\log d - c_d) \rfloor$.

(b) Law of the profile: $\mathrm{TV}(L_n, \Pi_n) \to 1$ for $n = \lfloor \frac{d}{2}(\log d - c_d) \rfloor$, and $\mathrm{TV}(L_n, \Pi_n) \to 0$ for $n = \lceil \frac{d}{2}\log d + c_d d \rceil + 1$.

In words: the mean grade profile equilibrates after about $\frac{d}{4}\log d$ reflections, while the profile of an individual product needs about $\frac{d}{2}\log d$, twice as many. With c fixed instead of $c_d \to \infty$, Theorems 7.1, 7.5 and 7.7 show that both windows are $O(d)$.

Proof. (a) is Theorem 7.1 with $c = c_d$. (b) follows from Theorem 7.5(b) with $c = c_d$, since $\varepsilon_{d,n} \to 0$, and from Theorem 7.7 with $c = c_d$. ▫

Remark 7.9. Between $\frac{d}{4}\log d$ and $\frac{d}{2}\log d$ reflections, the mean profile is already close to equilibrium while the law of an individual profile is still far from Π_n. Part (a) of Corollary 7.8 is the classical Ehrenfest cut-off, and by Lemma 7.4 part (b) is equivalent to Porod's theorem. What the grade profile adds is a reading of this distance inside the Clifford algebra, through the slow statistic $M = (d - \mathrm{tr}\rho)/2$. Fig. 3 illustrates this: in panel (b) the shift of the mean grade, in units of its fluctuation, falls below 1 at $n/(d\log d) = \frac{1}{4}$ for the averaged profile ($\sqrt{d}\,\kappa_1$) but only at $\frac{1}{2}$ for an individual versor ($d\kappa_1$). The Monte Carlo means in panel (c), -4.18, -0.92, -0.18, agree with $-d\kappa_1 = -4.19$, -0.90, -0.19 to within 2.4 standard errors (standard error ≈ 0.009).

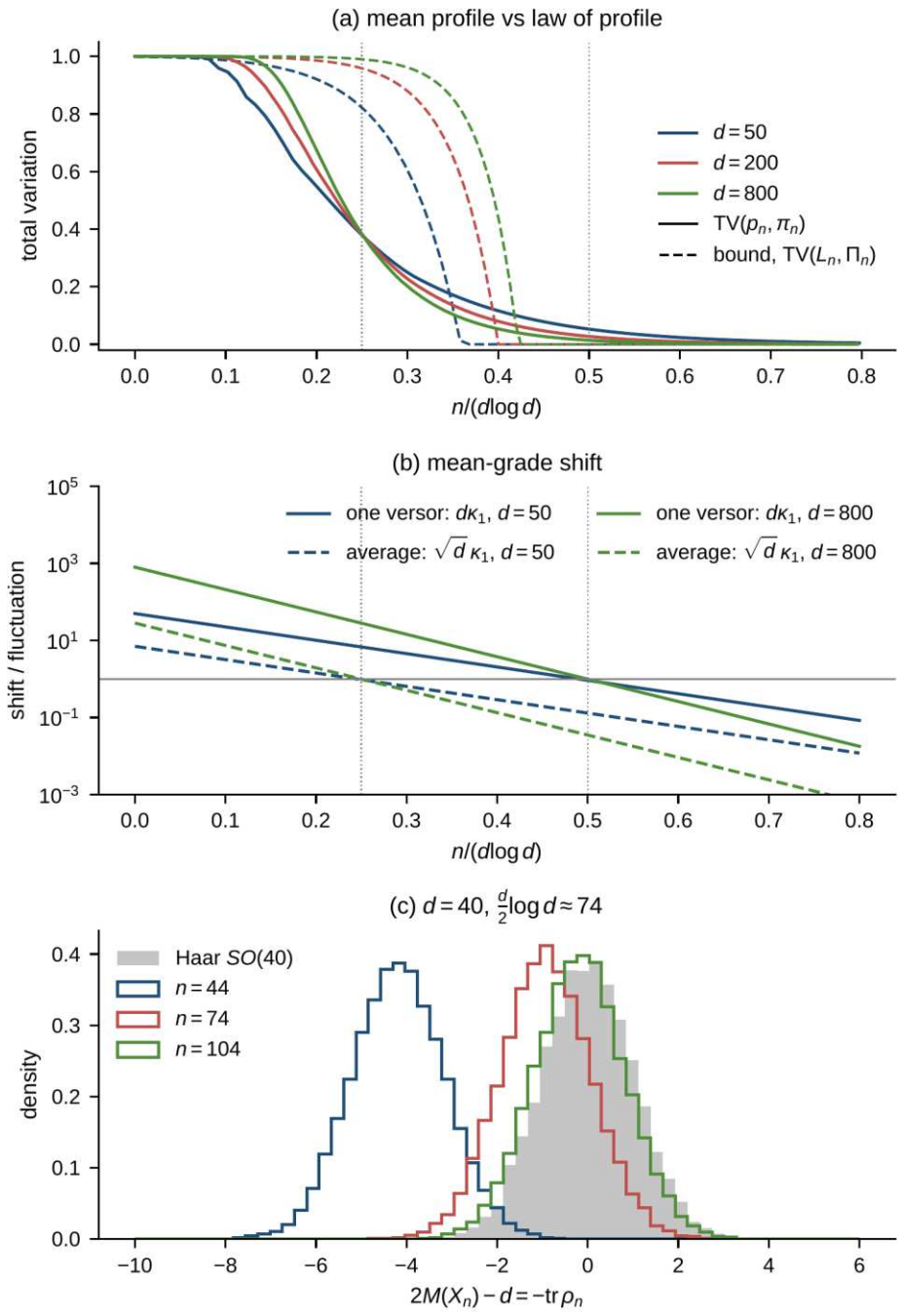


Fig. 3. *The two-scale cut-off. (a) Exact* $TV(p_n, \pi_n)$ *(solid) and the lower bound of Theorem 7.5(a) for* $TV(L_n, \Pi_n)$ *(dashed, clipped at* 0*) for* $d = 50,\ 200,\ 800$*; as* $d \to \infty$ *the cut-offs are at* $n/(d\log d) = \frac{1}{4}$ *and* $\frac{1}{2}$*. (b) Shift of the mean grade from* $d/2$ *in units of its fluctuation:* $d\kappa_1$ *for an individual versor (solid) and* $\sqrt{d}\,\kappa_1$ *for the averaged profile (dashed). (c) Monte Carlo samples of* $2M(X_n) - d = -tr\rho_n$ *in* $d = 40$ *for* $n = 44,\ 74,\ 104$ *(*$12{,}000$ *samples each), against Haar measure on* $SO(40)$*.*

8 Applications

Dirac traces. Let γ_μ satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}I_4$, so that $e_\mu \mapsto \gamma_\mu$ represents $C\ell_{1,3}$. The trace kills every blade except the scalar. (The argument of Remark 4.9 applies in any signature, since each e_j is invertible.) Hence, by Corollary 3.3(c), $\mathrm{tr}(a/_1 \cdots a/_n) = 4\langle a_1 \cdots a_n\rangle$ equals 0 for odd n and $4\mathrm{Pf}\big[a_i \cdot a_j\big]_{i<j}$ for even n. This recovers the trace theorems of quantum electrodynamics [25].

Volumes of random designs. By Corollary 5.2(b), n independent isotropic columns in $\mathbb{R}^d$ of unit scale have expected Gram determinant (D-criterion) $\prod_{j<n}(1 - j/d)$. Only second moments of the columns enter; the formula is classical (Cauchy–Binet), and here it appears as the top grade of the Ehrenfest law.

A diagnostic for random rotations. Products of random reflections have been proposed for generating random orthogonal matrices; see the discussion in [26, Remark 1.6]. Corollary 7.8 gives a practical warning. The averaged grade profile is close in total variation to equilibrium after about $\frac{d}{4}\log d$ reflections, but each output remains detectably non-uniform until $\frac{d}{2}\log d$. The statistic that detects this is the trace $\mathrm{tr}\rho_n = d - 2M(X_n)$, which costs $O(d)$ per sample once ρ_n is available as a matrix. Its exact mean $d(1 - 2/d)^n$ and variance (Theorem 6.2) give a calibrated test. In $\mathbb{R}^3$, Theorem 6.2 ($\mathbb{E}T_n = 3^{1-n}$) and $\mathrm{tr}\rho = 1 + 2\cos\varphi$ give $\mathbb{E}\cos\varphi = -\frac{1}{2} + \frac{1}{2\cdot 3^{n-1}}$ for the rotation angle after an even number n of uniform reflections.

Rotors from rotation matrices. Theorem 4.10 gives the magnitude of every coordinate of a rotor or versor directly from its matrix, at the cost of one $d \times d$ determinant per coordinate. In particular it identifies a coordinate with $X_I^2 \geq 2^{1-d}$ (Remark 4.11), which is a well-conditioned choice of index for Shirokov's reconstruction with signs [33]; for an odd versor, apply this to Xe_1. In $\mathbb{R}^3$, this selection followed by Shirokov's formula is Shepperd's algorithm [31]. Corollaries 4.4, 4.6 and 4.13 likewise give the grade content of a rotor, such as its mean grade, its dominant grade and the squared magnitude $\det\frac{I-\rho}{2}$ of its pseudoscalar part, without Clifford arithmetic. Rotors can also be reconstructed from frames by characteristic multivectors [19].

9 Verification and methods

The checks use a direct implementation of the geometric product on bitmask-indexed blades, written independently of the proofs (it never uses (2) or (4) to compute a product). Algebraic identities are checked in exact rational arithmetic (`fractions`, `sympy`) or at 50 significant digits (`mpmath`). Probabilistic statements are checked by exact enumeration, by 50-digit evaluation, or by Monte Carlo (MC), as stated in Table 3. Lemmas 7.3 and 7.4 and Theorem 7.7 are not numerically checkable. The script `gp_verify.py` (checks V1–V39) is supplied, and all checks pass. The figures are produced by `gp_figures.py`.

Use of a large language model. In preparing this work the author used Claude (Anthropic), a large language model, to help draft and typeset the manuscript, to write the verification and figure scripts, to search the literature and to check references. The author directed the research, checked the mathematical content and the numerical results, and takes full responsibility for the content of the paper. No AI tool is listed as an author.

Table 3. *Verification checks. "Exact" means rational arithmetic; MC means Monte Carlo in double precision, with the largest $|z|$-score reported.*

Check	Statement and scope
V1–V2	Expansion (2) against the direct product in 140 random integer cases, $n \le 7$, five signatures (exact); term count $\tau(n)$ for $n \le 10$
V3–V6	Pfaffian scalar part; $X\tilde{X} = \prod Q(a_i)$; Gram identity; grade-energy identity (3) and $\| \langle X \rangle_n \|^2 = \det G$ (exact)
V7–V8	Three-vector Hadamard defect (Remark 3.6); Example 3.4 (exact)
V9	Krawtchouk closed form of p_n against the Ehrenfest chain, $d \le 10$, $n \le 12$ (exact)
V10–V11	Theorem 5.1 by full enumeration with the coordinate step law (a uniform on $\{\pm e_i\}$, which is isotropic), $d \le 4$, $n \le 4$; indefinite signatures $C\ell_{1,3}$, $C\ell_{2,1}$ (exact)
V12	Theorem 5.1 for factors uniform on S^7 ($d = 8$, $n = 5$, $N = 20{,}000$), MC: max $\|z\| = 0.27$
V13–V14	$\mathbb{E}\det G$, μ_n, v_n and parity (Corollary 5.2) by enumeration with the coordinate step law (exact)
V15–V16	Dirac traces, $n \le 6$; reflection formula and $\langle R \rangle^2 = (1 + \mathrm{tr}\rho)/4$ (50 digits, error $\le 7 \times 10^{-49}$)
V17	Table 1 (exact)
V18	Rotation-angle law in $\mathbb{R}^3$: algebraic rewriting (exact) and the law itself (MC, max $\|z\| = 0.60$)
V19	Grade-spectrum identity (4), 70 random integer versors, $d = 2, \dots, 6$, $n \le 7$ (`sympy`, exact)
V20–V22	Corollaries 4.5, 4.6 and 4.7, including conjugation invariance and recovery of $\det(I + x\rho_X)$ from the profile (exact)
V23	Eigenvalue and Poisson-binomial forms, $d = 4, \dots, 7$, $n \in \{d, d+1\}$ (50 digits, error 1.8×10^{-50})

Table 3 (continued)

Check	Statement and scope
V24–V25	$\mathbb{E}T_n^2$ and $\mathbb{E}\mathrm{tr}\rho_n^2$ against $\mathrm{tr}(\mathbb{E}[H \otimes H]^n)$ from the exact fourth-moment tensor, $d = 3, \dots, 6$, $n \le 8$; law of total variance (exact)
V26	$d = 8$, $n = 6$: Clifford-side M equals $(d - \mathrm{tr}\rho)/2$ sample by sample to 6×10^{-15}; moments of T_n (MC, max $\|z\| = 2.02$)
V27	Haar on $SO(d)$ and $O^-(d)$, $d = 6{,}11$, $N = 2 \times 10^5$: $\mathbb{E}T = 0$, $\mathbb{E}T^2 = \mathbb{E}\mathrm{tr}g^2 = 1$ (MC, max $\|z\| = 1.61$)
V28	Both bounds of Theorem 7.1 against $\mathrm{TV}(p_n, \pi_n)$, $d = 16{,}64{,}256$, $c = 1{,}2{,}3$, cases with $n \ge 0$ (50 digits)
V29	Inequality chain of Theorem 7.5(b), and (a) $\ge$ (b), for $d = 2^2, \dots, 2^{10}$ and $c \in \{\frac{1}{2}, 1{,}2{,}3\}$ (50 digits)
V30	Eigenvalues and multiplicities of $\mathbb{E}[H \otimes H]$ (Lemma 6.1), $d = 3, \dots, 6$ (exact)
V31	$\mathbb{E}\mathrm{tr}\Lambda^k\rho_n = \binom{d}{k}\left(1 - \frac{2k}{d}\right)^n$: coordinate step law, $d = 4$, $n \le 3$ (exact); factors on S^5 (MC, max $\|z\| = 1.56$)

Check	Statement and scope
V32	Lemma 7.2: Krawtchouk identity (exact); $\mathbb{E}\mathrm{tr}\Lambda^k g = 0$ for $0 < k < d$ under Haar, $d = 6{,}7$, both cosets (MC, max $\lvert z\rvert = 1.77$)
V33	Remark 4.9: $\langle X\rangle = 2^{-m}\mathrm{tr}_S(X)$ and $\langle X\rangle^2 = \det\frac{I+\rho_X}{2}$, $d = 4{,}6$ (50 digits, error 2×10^{-51})
V34	Table 2: bound (a) at 50 digits; $\mathrm{TV}(p_n, \pi_n)$ by the Ehrenfest chain (50 digits for $d \le 256$, double precision otherwise)
V35	Illustration only: $d = 40$, $n = 44$, binned empirical TV of $\mathrm{tr}\rho_n$ against a Haar sample ($N = 12{,}000$) is 0.959, above bound (a) $= 0.532$ (MC)
V36	Theorem 4.10, identity (5) in all $t_1, \dots, t_d$: 45 random integer versors (26 odd), $d = 2, \dots, 6$, $n \le 7$ (`sympy`, exact)
V37	Determinant and principal-minor forms (6) for every blade I of the same versors (exact)
V38	Corollary 4.12: Shepperd's formulas for 40 random integer-vector rotors in $\mathbb{R}^3$ (exact)
V39	Corollary 4.13: Newton inequalities and $\lvert s^* - M\rvert < 2$, 300 random integer versors, $d \le 8$, $n \le 9$ (exact)

10 Conclusions and open problems

The grade-spectrum theorem turns questions about the grade structure of products of vectors into questions about the spectrum of orthogonal matrices. At $t = 0$ it reduces to the spin-character identity $\langle X\rangle^2 = \det\big((I + \rho_X)/2\big)$, that is, $|\chi_S|^2 = \det(I + \rho)$ for even d. Applied to random products, it shows that the Ehrenfest law describes only the averaged profile, which is a contraction of the law of the profile. Individual profiles carry the full mixing information of the reflection walk and equilibrate on a time scale twice as long. Several questions remain open:

- a proof of the upper bound in Theorem 7.7 by the Krawtchouk structure of Corollary 4.5, independent of the representation theory of $O(d)$;
- the limiting shape of $\mathrm{TV}(L_n, \Pi_n)$ inside the cut-off window, which is plausibly governed by $T_n \Rightarrow N(e^{-c}, 1)$ at $n = \frac{d}{2}(\log d + c)$ (this is consistent with Theorem 6.2);
- an analogue of Theorem 4.3 in indefinite signature, where coefficient energy and quadratic form differ;
- random versors with anisotropic or heavy-tailed factors.

Appendix A. Proof of Theorem 7.1

Proof of Theorem 7.1. (i) Let x_n be the walk on $\{0{,}1\}^d$, started at 0, that flips one uniformly chosen coordinate per step (the walk in the proof of Theorem 5.1), so that p_n is the law of $|x_n|$, and let U_n be the uniform law on the parity class $\{x: |x| \equiv n \pmod 2\}$, whose image is π_n. Both laws are invariant under coordinate permutations, so $\mathrm{TV}(p_n, \pi_n) = \mathrm{TV}(x_n, U_n)$. Both are supported on a set of size 2^{d-1}. The Fourier coefficients of $\mathcal{L}(x_n) - U_n$ are $\left(1 - \frac{2|t|}{d}\right)^n$ for $t \notin \{0, (1, \dots, 1)\}$ and vanish otherwise. Cauchy–Schwarz and Parseval give

$$\begin{aligned}\mathrm{TV}(x_n, U_n)^2 &\le \frac{2^{d-1}}{4}\cdot 2^{-d}\sum_{j=1}^{d-1}\binom{d}{j}\Big(1-\frac{2j}{d}\Big)^{2n}\\ &\le \frac{1}{4}\sum_{1\le j<d/2}\frac{\big(de^{-4n/d}\big)^j}{j!} \le \frac{1}{4}\big(e^{e^{-c}}-1\big).\end{aligned}$$

Here we used $\left(1-\frac{2j}{d}\right)^{2n} \le e^{-4jn/d}$, the symmetry $j \leftrightarrow d-j$, the vanishing of the $j = d/2$ term, and $\binom{d}{j} \le d^j/j!$.

(ii) Under π_n the grade has mean $d/2$ and variance $d/4$ (since $d \ge 3$). Under p_n it has mean $\mu_n = \frac{d}{2}(1-\kappa_1)$ and variance $v_n \le \frac{d}{4}$, because $\kappa_2 \le \kappa_1^2$ when $d \ge 4$. Let $A = \{s < \frac{d}{2} - \frac{d}{4}\kappa_1\}$. Chebyshev's inequality gives $\pi_n(A) \le \frac{4}{d\kappa_1^2}$ and $p_n(A^c) \le \frac{4}{d\kappa_1^2}$, so $\mathrm{TV}(p_n, \pi_n) \ge 1 - \frac{8}{d\kappa_1^2}$. Since $\log(1-x) \ge -x - x^2$ for $0 \le x \le \frac{1}{2}$, we have $\log\kappa_1 \ge -\frac{2n}{d}\left(1+\frac{2}{d}\right)$. For $n \le \frac{d}{4}(\log d - c)$ this gives $d\kappa_1^2 \ge d\,e^{-(\log d - c)(1+2/d)} \ge e^c d^{-2/d}$. ▫

Declarations

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Data availability. No datasets were generated or analysed during the current study. The verification script `gp_verify.py` and the figure script `gp_figures.py`, which reproduce every numerical statement and figure, are supplied with the manuscript as supplementary material.

Use of AI tools. The use of a large language model is documented in Section 9.

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