

Supplementary Information for Dynamical Asymmetry Reveals Progressive Biodiversity Loss in Mutualistic Ecological Networks

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Supplementary Note 1: General reduction and closure assumptions

In this section, we detail the derivation of the reduced equations (8) and (9) introduced in section 4.1. We consider the macroscopic observables

$$X = \sum_{i=1}^N a_i x_i, \quad Y = \sum_{k=1}^M b_k y_k,$$

with $a_i, b_k \in \mathbb{R}^+$, $\sum_{i=1}^N a_i = 1$ and $\sum_{k=1}^M b_k = 1$.

Our goal is to express the dynamics of X and Y in closed form. We first take the time derivative of X and insert Eq. (6), which leads to

$$\dot{X} = \sum_{i=1}^N a_i \dot{x}_i = \sum_{i=1}^N a_i \left[F_1(x_i) + H_1(h_1(\mathbf{y}), x_i) + \sum_{j=1}^N \beta_{ij} G(x_i, x_j) \right]. \quad (1)$$

By choosing $a = (a_1, a_2, \dots, a_N)$ appropriately, the right-hand side can be expressed solely in terms of X . To achieve this, we now treat the three terms in Eq. (1) separately.

Part 1. Taylor expand around X up to the first order

$$F_1(x_i) = F_1(X) + (x_i - X) F_1'(X) + \mathcal{O}[(x_i - X)^2].$$

Taking the weighted sum with coefficient a_i gives

$$\sum_{i=1}^N a_i F_1(x_i) = F_1(X) + F_1'(X) \sum_{i=1}^N a_i (x_i - X) + \mathcal{O} \left[\sum_{i=1}^N a_i (x_i - X)^2 \right].$$

According to the definition of X , we can get

$$\sum_{i=1}^N a_i F_1(x_i) = F_1(X) + \mathcal{O} \left[\sum_{i=1}^N a_i (x_i - X)^2 \right].$$

Thus, this term does not impose any constraints on the choice of a .

Part 2 We now consider the mutualistic term depending on the y -variables. We assume a concentration condition of the form

$$y = Y\mathbf{1} + \epsilon\eta_y, \quad \|\eta_y\| = O(1), \quad 0 < \epsilon \ll 1,$$

where $Y = b^\top y$ is the average used as the macroscopic coordinate of the y -subsystem. Since $Y = b^\top y$ and $b^\top \mathbf{1} = 1$, this ansatz implies $b^\top \eta_y = 0$. Thus, $\epsilon\eta_y$ represents microscopic deviations from the retained collective coordinate, with zero b -weighted average. This ansatz means that the microscopic variables y_k are weakly heterogeneous around the collective value Y . The choice of b as the principal eigenvector of the induced matrix $M_y = E^\top E$ makes Y the dominant collective coordinate associated with the y -subsystem. Therefore, the closure is expected to be accurate when the components of y orthogonal to this dominant mode remain small. Under this assumption,

$$h_1(y) = h_1(Y\mathbf{1}) + O(\epsilon).$$

We define the scalar function

$$\tilde{h}_1(Y) := h_1(Y\mathbf{1}).$$

Then

$$h_1(y) = \tilde{h}_1(Y) + O(\epsilon).$$

Using the smoothness of H_1 , we obtain

$$\sum_i a_i H_1(h_1(y), x_i) = \sum_i a_i H_1(\tilde{h}_1(Y), x_i) + O(\epsilon).$$

We then perform a first-order Taylor expansion of H_1 with respect to its second argument around X :

$$H_1(\tilde{h}_1(Y), x_i) = H_1(\tilde{h}_1(Y), X) + \partial_2 H_1(\tilde{h}_1(Y), X)(x_i - X) + O((x_i - X)^2).$$

Multiplying by a_i and adding over i gives

$$\sum_i a_i H_1(\tilde{h}_1(Y), x_i) = H_1(\tilde{h}_1(Y), X) + O\left(\sum_i a_i (x_i - X)^2\right).$$

Thus, to leading order, the mutualistic contribution can be written as the generic closed term

$$\mathcal{I}_1(Y, X) := H_1(\tilde{h}_1(Y), X).$$

Part 3. Consider the interaction sum

$$S := \sum_{i,j} a_i \beta_{ij} G(x_i, x_j), \quad \alpha := \sum_{i,j} a_i \beta_{ij}.$$

We introduce normalized weights $w_{ij} = a_i \beta_{ij} / \alpha$, with $\sum_{i,j} w_{ij} = 1$, and define the weighted means of the two arguments of G as

$$\mu^{(1)} := \sum_{i,j} w_{ij} x_i, \quad \mu^{(2)} := \sum_{i,j} w_{ij} x_j.$$

These quantities are the weighted-average values of the first and second arguments of G , with weights given by the coupling coefficients. We parameterize them as $\mu^{(1)} = \theta X$ and $\mu^{(2)} = \phi X$, where θ and ϕ quantify how the coupling weights bias the effective values of x_i and x_j relative to the macroscopic observable X .

Expanding G to the first order around $(\mu^{(1)}, \mu^{(2)})$ gives

$$G(x_i, x_j) \approx G(\theta X, \phi X) + G_1(\theta X, \phi X)(x_i - \theta X) + G_2(\theta X, \phi X)(x_j - \phi X).$$

Substituting into S yields

$$\begin{aligned} S &\approx \alpha G(\theta X, \phi X) + G_1(\theta X, \phi X) \sum_{i,j} a_i \beta_{ij} (x_i - \theta X) \\ &\quad + G_2(\theta X, \phi X) \sum_{i,j} a_i \beta_{ij} (x_j - \phi X). \end{aligned}$$

Recall that $M_x = [\beta_{ij}]_{i,j=1}^N$ is a nonnegative matrix, i.e., $\beta_{ij} \geq 0$. Since β_{ij} denotes the strength of the interaction from x_j to x_i , the sum of the rows $k_i^{\text{in}} = \sum_{j=1}^N \beta_{ij}$ represents the total input going into node i . We therefore define the diagonal matrix K with entries $K_{ii} = k_i^{\text{in}} = \sum_{j=1}^N \beta_{ij}$. Then

$$\sum_{i,j} a_i \beta_{ij} x_i = x^\top K a, \quad \sum_{i,j} a_i \beta_{ij} x_j = x^\top M_x^\top a.$$

The exact cancellation of the linear terms for all microscopic states x requires

$$K a = \alpha \theta a, \quad M_x^\top a = \alpha \phi a.$$

In other words, a would need to be a common eigenvector of K and $M_x^\top$, which is generally impossible. If a is chosen as the Perron eigenvector of $M_x^\top$ and normalized by $\mathbf{1}^\top a = 1$, then $\phi = 1$ and α coincides with the corresponding eigenvalue.

When $K a \neq \alpha \theta a$, the reduction leaves a residual term

$$G_1(\theta X, X) x^\top (K a - \alpha \theta a),$$

which must be small for the closure to remain accurate. This requires either a weak heterogeneity of k_i or a strong concentration of x along the direction a .

We choose θ by minimizing the mean-squared error (MSE)

$$\theta^* = \arg \min_{\theta} \|Ka - \theta \alpha a\|^2,$$

where the symbol $\|\cdot\|$ denotes the standard Euclidean norm.

$$\|Ka - \theta \alpha a\|^2 = (Ka - \theta \alpha a)^\top (Ka - \theta \alpha a) \quad (2)$$

$$= (Ka)^\top Ka - 2\theta \alpha a^\top Ka + \theta^2 \alpha^2 (a^\top a). \quad (3)$$

Taking the derivative with respect to θ gives

$$\frac{\partial}{\partial \theta} \|Ka - \theta \alpha a\|^2 = -2\alpha a^\top Ka + 2\theta \alpha^2 (a^\top a).$$

Setting the derivative to zero,

$$-2\alpha a^\top Ka + 2\theta \alpha^2 (a^\top a) = 0,$$

we get

$$\theta^* = \frac{a^\top Ka}{\alpha a^\top a}.$$

Define $c_i = \frac{a_i^2}{\sum_i a_i^2}$, i.e., the normalized squared entries of a . Then

$$\theta^* = \frac{c^\top k_{\text{in}}}{a^\top k_{\text{in}}}.$$

Therefore, θ^* is close to unity when the a -weighted and a^2 -weighted averages of k^{in} are close, i.e., when $c^\top k^{\text{in}} \approx a^\top k^{\text{in}}$. This condition is satisfied exactly for regular networks and approximately when the Perron vector a is weakly localized. The parameter θ measures the residual degree-weighted bias in the first argument of G , arising from the mismatch between Ka and αa . For a regular network with constant degree k , we have $K = kI$. Since the Perron eigenvalue of M_x is then $\alpha = k$, we obtain $\theta = 1$ exactly. For heterogeneous networks, θ^* may deviate from unity, but it remains close to 1 when the Perron vector is weakly localized. This approximation is supported by the numerical tests reported below.

Combining the three contributions, we get

$$\dot{X} = F_1(X) + \mathcal{I}_1(Y, X) + \alpha G(\theta X, X),$$

where

$$\theta = \frac{1}{\alpha} \frac{a^\top Ka}{a^\top a}.$$

Applying the induced Gram-matrix closure to the y -layer gives

$$\dot{Y} = F_2(Y) + \mathcal{I}_2(X, Y),$$

where $\mathcal{I}_2(X, Y)$ is the corresponding closed interaction term for the y -equation.

For the Holling-II mutualistic model used in the main text, the mutualistic terms are separable in the abundance of the focal species. In particular, they can be written in the form

$$H_1(h, x) = x g_1(h), \quad H_2(h, y) = y g_2(h).$$

In this case,

$$\mathcal{I}_1(Y, X) = X H_1(Y), \quad \mathcal{I}_2(X, Y) = Y H_2(X),$$

after absorbing the scalar field functions $g_1(\tilde{h}_1(Y))$ and $g_2(\tilde{h}_2(X))$ into the notation $H_1(Y)$ and $H_2(X)$, respectively. Therefore, the reduced equations used in the main text become

$$\dot{X} = F_1(X) + X H_1(Y) + \alpha G(\theta X, X),$$

and

$$\dot{Y} = F_2(Y) + Y H_2(X).$$

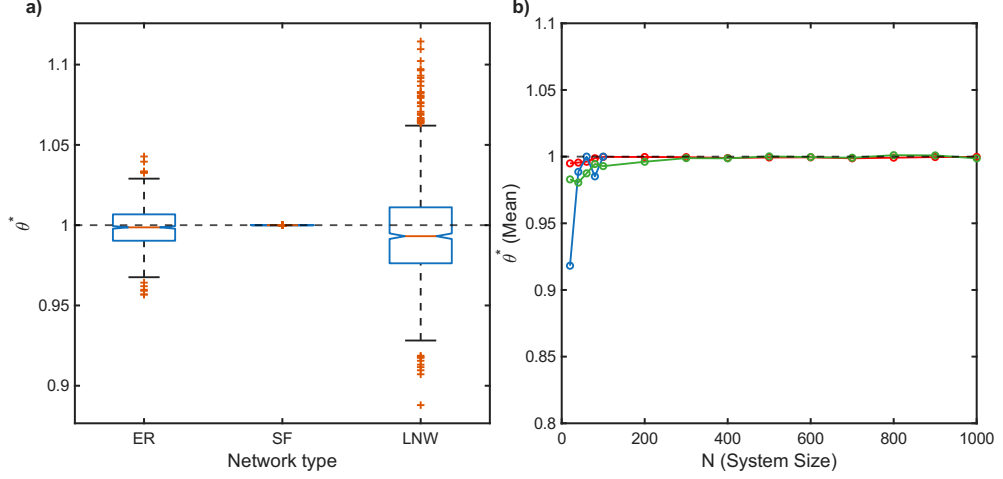
Numerical verification of the approximation $\theta^* \simeq 1$ We compute θ^* directly from simulated interaction networks. Considering three classes of network ensembles: Erdős–Rényi (ER), scale-free, and weighted lognormal networks. For each realization, we compute the input-degree vector

$$k_i^{\text{in}} = \sum_j \beta_{ij},$$

construct the diagonal matrix K , and evaluate θ^* .

Figure 1 summarizes the behavior of θ^* across different network ensembles and system sizes. Panel 1a shows boxplots of θ^* obtained from different network classes. In all cases, the distributions remain strongly concentrated around $\theta^* = 1$. ER and scale-free networks exhibit particularly narrow distributions, while weighted lognormal networks display slightly broader fluctuations and occasional outliers due to stronger weight heterogeneity. Panel 1b examines the dependence of the mean value of θ^* on the size of the system N . For all network ensembles considered, θ^* rapidly approaches unity as N increases. Finite-size fluctuations are most visible in small scale-free networks but become negligible for larger systems.

Taken together, these results support the leading-order approximation $\theta^* \simeq 1$ used in the reduced dynamics. The results further suggest that the Perron projection efficiently absorbs residual degree heterogeneity, even in strongly heterogeneous networks.



Supplementary Figure 1 Numerical verification of the approximation $\theta^* \simeq 1$. **a.** Boxplots of θ^* for three network ensembles: Erdős-Rényi (ER), scale-free (SF), and lognormal-weight (LNW) networks. **b.** Mean value of θ^* as a function of the system size N . Across all network ensembles and system sizes, θ^* remains close to unity, supporting the leading-order approximation used in the reduced dynamics.

Supplementary Note 2: Resilience function of the asymmetric plant–pollinator model with dilution

The dynamics of the reduced model are governed by

$$\dot{X} = X(\alpha^{(x)} - \beta X) + X \cdot \frac{\gamma_0^{(x)} Y}{1 + h\gamma_0^{(x)} Y},$$

$$\dot{Y} = Y(\alpha^{(y)} - \kappa) + Y \cdot \frac{\gamma_0^{(y)} X}{1 + h\gamma_0^{(y)} X + \eta Y}.$$

A key difference from the classical symmetric formulation is the role of the dilution term ηY in the plant equation. In the asymmetric formulation, the intrinsic plant contribution $Y(\alpha^{(y)} - \kappa)$ is purely linear. Therefore, without the density-dependent dilution term in the mutualistic response, plant growth would not be bounded by any self-regulating mechanism. For a feasible equilibrium to exist, the linear part must be negative in the absence of mutualistic input. Hence, we require $c = \kappa - \alpha^{(y)} > 0$. This condition means that plant persistence must be maintained by mutualistic feedback, while the dilution term ηY prevents unbounded growth at high plant abundance. Let

$$f_x(X, Y) = \frac{\gamma_0^{(x)} Y}{1 + h\gamma_0^{(x)} Y}, \quad f_y(X, Y) = \frac{\gamma_0^{(y)} X}{1 + h\gamma_0^{(y)} X + \eta Y}.$$

From the steady-state condition $dY = 0$, we can write

$$f_y = \frac{\gamma_0^{(y)} X}{1 + h\gamma_0^{(y)} X + \eta Y} = c,$$

which implies

$$Y(X) = \frac{\gamma_0^{(y)}(1 - ch)}{c\eta} X - \frac{1}{\eta}.$$

This is a linear relation between Y and X , which can be written as

$$Y(X) = uX + v,$$

where the coefficients are given by

$$u = \frac{\gamma_0^{(y)}(1 - ch)}{c\eta}, \quad v = -\frac{1}{\eta}.$$

For $dX = 0$, we have

$$\alpha^{(x)} - \beta X + f_x = 0,$$

where

$$f_x = \frac{\gamma_0^{(x)} Y}{1 + h\gamma_0^{(x)} Y}.$$

Substituting $Y = uX + v$, we have

$$\alpha^{(x)} - \beta X + \frac{\gamma_0^{(x)}(uX + v)}{1 + h\gamma_0^{(x)}(uX + v)} = 0.$$

Multiplying by the denominator $1 + h\gamma_0^{(x)}(uX + v)$ leads to the quadratic equation

$$C_2 X^2 + C_1 X + C_0 = 0, \tag{4}$$

where

$$C_2 = -\beta h\gamma_0^{(x)} u, \quad C_1 = \alpha^{(x)} h\gamma_0^{(x)} u - \beta(1 + h\gamma_0^{(x)} v) + \gamma_0^{(x)} u, \quad C_0 = \alpha^{(x)}(1 + h\gamma_0^{(x)} v) + \gamma_0^{(x)} v.$$

The local stability of a steady state (X^*, Y^*) is determined by the Jacobian matrix J :

$$J = \begin{pmatrix} -\beta X^* & \frac{\gamma_0^{(x)} X^*}{(1 + h\gamma_0^{(x)} Y^*)^2} \\ \frac{\gamma_0^{(y)} Y^* (1 + \eta Y^*)}{(1 + h\gamma_0^{(y)} X^* + \eta Y^*)^2} & -\frac{\eta \gamma_0^{(y)} X^* Y^*}{(1 + h\gamma_0^{(y)} X^* + \eta Y^*)^2} \end{pmatrix}. \tag{5}$$

A steady state is locally asymptotically stable if $\text{tr}(J) < 0$ and $\det(J) > 0$. Since all parameters and equilibrium values are positive, $\text{tr}(J) < 0$ is automatically satisfied.

The critical value of the effective competition strength can now be obtained from the saddle-node conditions. At a positive equilibrium, the plant equation gives

$$X = \frac{c(1 + \eta Y)}{\gamma_0^{(y)}(1 - ch)}, \quad (6)$$

provided that $c > 0$ and $1 - ch > 0$. The saddle-node condition is $\det(J) = 0$. Using the Jacobian above and the positive-equilibrium condition, this condition becomes

$$\beta = \frac{\gamma_0^{(x)}(1 + \eta Y)}{\eta X \left(1 + h\gamma_0^{(x)}Y\right)^2}. \quad (7)$$

Substituting Eq. (6) into Eq. (7), and then imposing the pollinator equilibrium condition, yields

$$\alpha^{(x)}\eta \left(1 + h\gamma_0^{(x)}Y\right)^2 + \gamma_0^{(x)} \left(\eta h\gamma_0^{(x)}Y^2 - 1\right) = 0. \quad (8)$$

The positive solution of Eq. (8) is

$$Y_c = \frac{\chi - \alpha^{(x)}\eta h}{\eta h\gamma_0^{(x)}(1 + h\alpha^{(x)})}, \quad \chi = \sqrt{\eta h \left[\gamma_0^{(x)}(1 + h\alpha^{(x)}) - \alpha^{(x)}\eta\right]}. \quad (9)$$

Consequently, the theoretical saddle-node threshold is

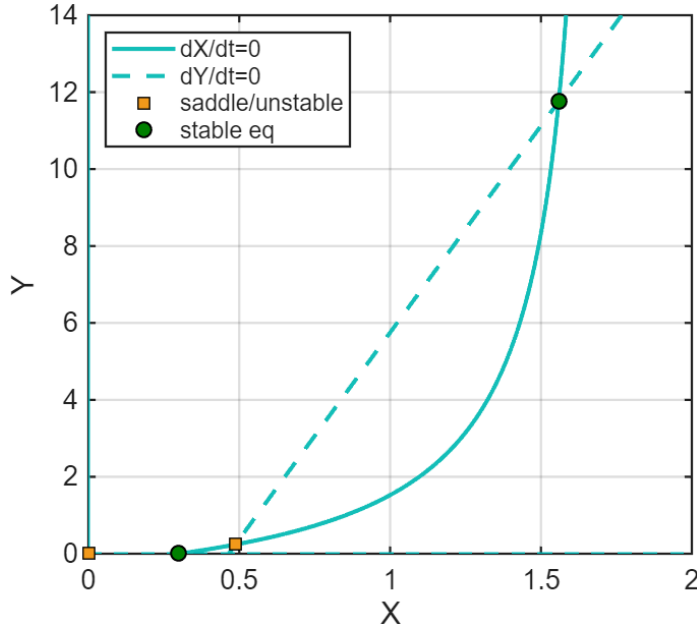
$$\beta_c = \frac{\gamma_0^{(x)}\gamma_0^{(y)}\eta(1 - ch)(1 + h\alpha^{(x)})^2}{c(\eta + \chi)^2}, \quad \chi = \sqrt{\eta h \left[\gamma_0^{(x)}(1 + h\alpha^{(x)}) - \alpha^{(x)}\eta\right]}. \quad (10)$$

Equivalently,

$$\beta_c = \frac{\gamma_0^{(x)}\gamma_0^{(y)}\eta \left[1 - h(\kappa - \alpha^{(y)})\right] \left[1 + h\alpha^{(x)}\right]^2}{(\kappa - \alpha^{(y)}) \left[\eta + \sqrt{\eta h \left\{\gamma_0^{(x)} \left[1 + h\alpha^{(x)}\right] - \alpha^{(x)}\eta\right\}}\right]^2}. \quad (11)$$

This expression applies when the resulting critical equilibrium is positive and the saddle-node is nondegenerate. As β increases through β_c , the high-abundance stable branch and the intermediate unstable branch coalesce and disappear, leaving the low-abundance regime as the remaining attractor of the reduced system.

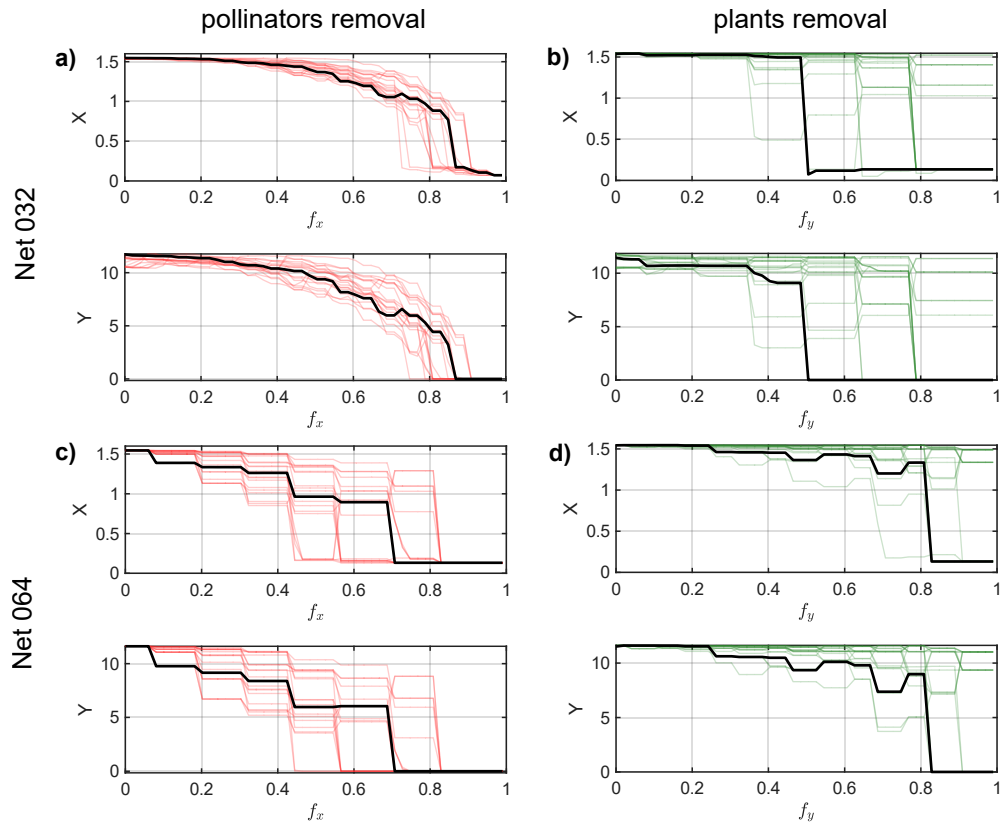
The nullclines provide a graphical representation of these equilibrium branches. The solid curve is the solution of $dX/dt = 0$, whereas the dashed curve represents $dY/dt = 0$; their intersections are therefore equilibria of the reduced model. In the parameter regime shown below, four intersections are present, two of which are stable: the high-abundance equilibrium corresponds to the survival state, whereas the low-abundance equilibrium corresponds to the extinction state.



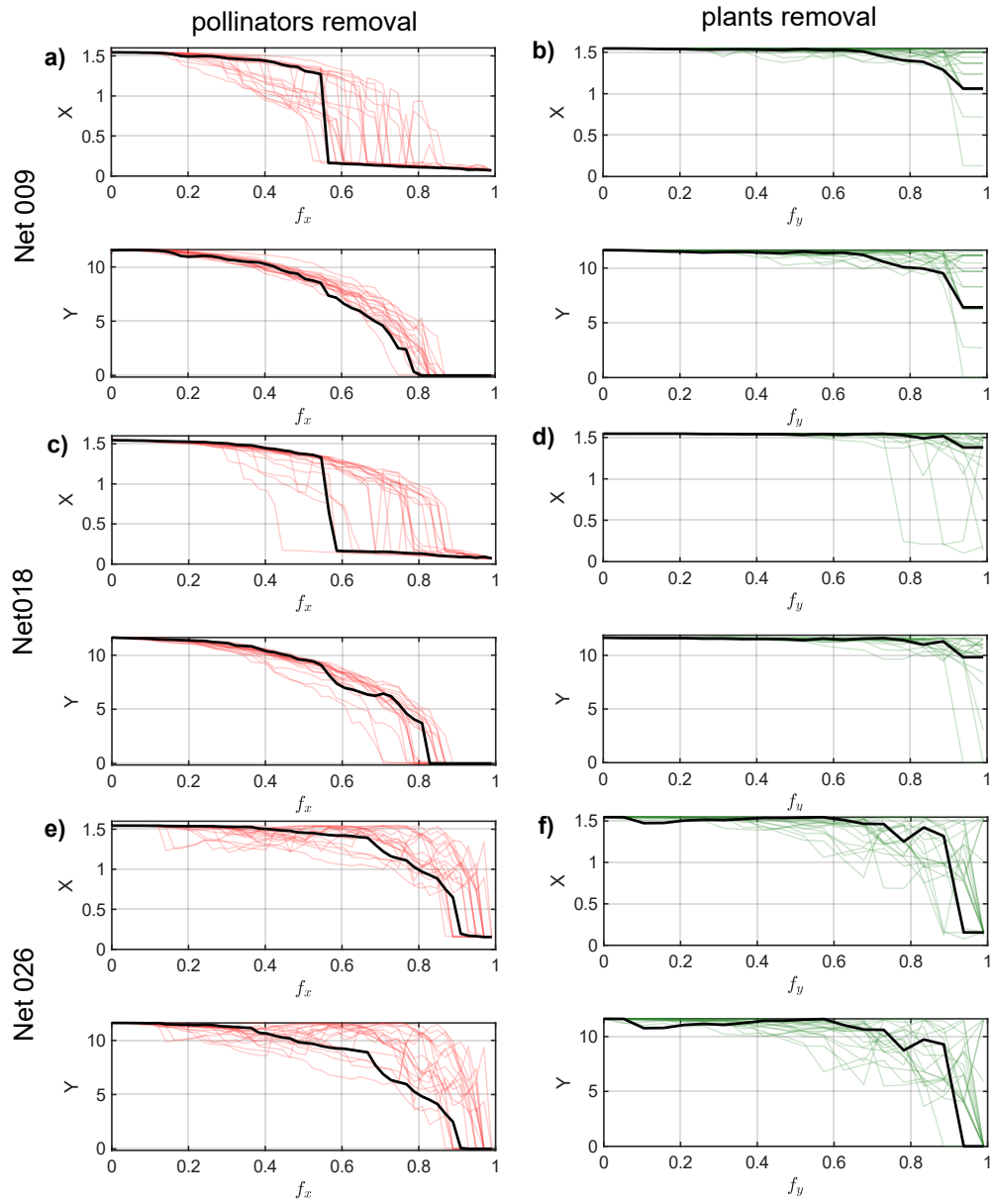
Supplementary Figure 2 Nullclines and equilibria of the reduced system. Nullclines are shown for parameter values $\alpha^{(x)} = \alpha^{(y)} = 0.3$, $\beta = 1$, $\kappa = 0.5$, $h = 0.7$, $\gamma_0^{(x)} = 0.7$, $\gamma_0^{(y)} = 0.5$, and $\eta = 0.2$. The solid and dashed curves represent the pollinator and plant nullclines, respectively. Squares indicate saddle equilibria, whereas green circles indicate stable equilibria.

Supplementary Note 3: Extended numerical validation

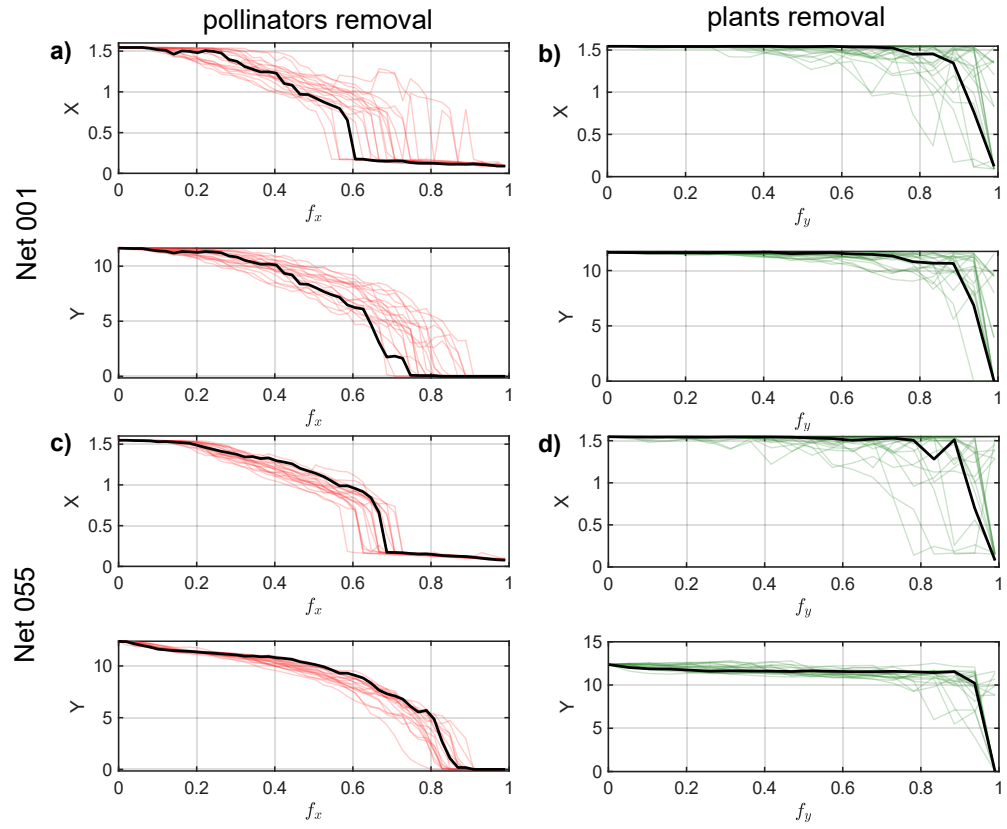
This section provides the extended network-level validation that supports the Results section. The panels are grouped by network size and show the responses to pollinator and plant removal across the ten empirical networks. These displays document the breadth of the tested ecological systems and complement the reduced-space organization presented in the main text.



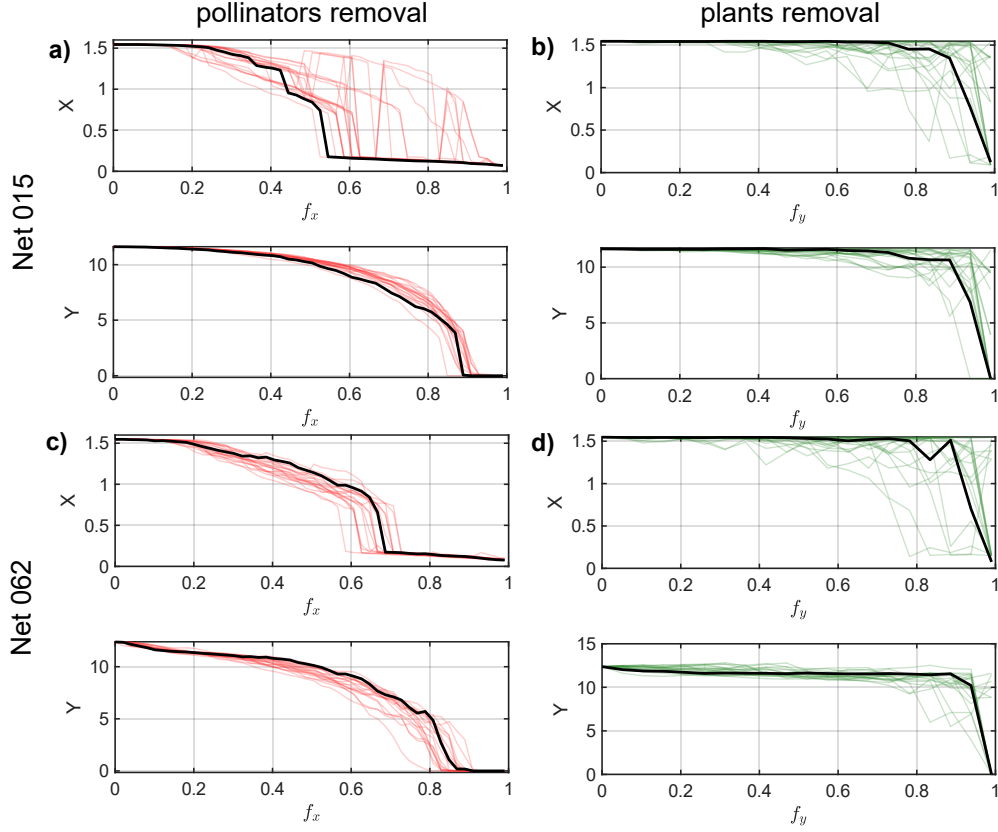
Supplementary Figure 3 Results obtained for two small networks.



Supplementary Figure 4 Results obtained for three medium networks.



Supplementary Figure 5 Results obtained for two large networks.



Supplementary Figure 6 Results obtained for two extra large networks.

Supplementary Note 4: Robustness to alternative diagonal-reinforcement rules

The diagonal-reinforcement rule used in the main text is a modeling assumption that maps the loss of bipartite connectivity to an increase in effective self-limitation. To examine whether the shared reduced-space organization depends on this particular functional form, we repeated the analysis under alternative reinforcement sensitivities.

Let

$$r(f) = \frac{\langle K \rangle_f}{\langle K \rangle_0}$$

be the connectivity ratio after deletion. In the main text, reinforcement strength is defined as

$$\rho(f) = 2[1 - r(f)]^2.$$

We compare this baseline rule with two alternative sensitivity regimes,

$$\rho(f) = 2[1 - r(f)]^{0.5}, \quad \rho(f) = 2[1 - r(f)]^4,$$

corresponding to high and low sensitivity reinforcement, respectively. For each rule, the effective competition matrix is reconstructed as

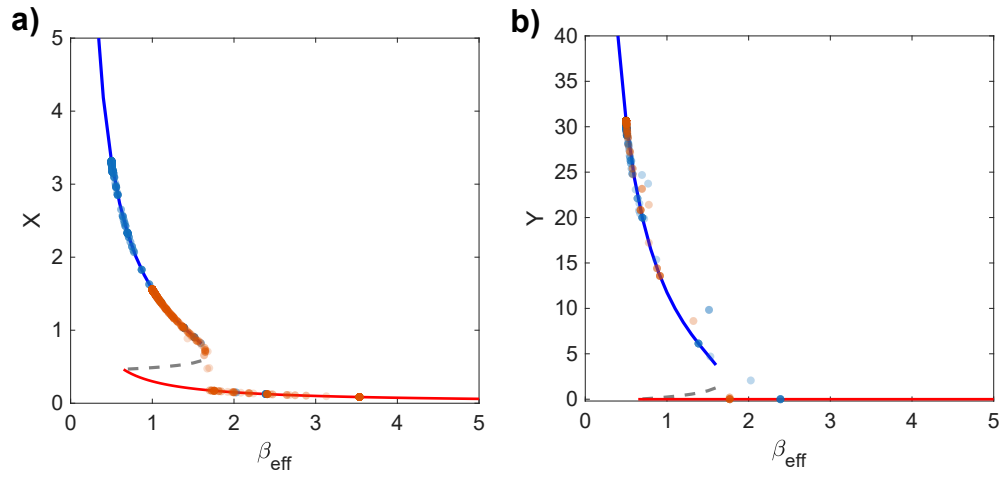
$$M_x^{(f)} = M_{x,0}^{(f)} + \rho(f) \text{diag} \left(d^{(f)} + \bar{d}^{(f)} \mathbf{1} \right).$$

Using the same empirical network and species-removal protocol as in the main text, we repeated both pollinator removal and plant removal experiments under the two alternative sensitivity regimes. For each deletion fraction, 20 independent realizations were performed. The resulting high-dimensional trajectories were then projected onto the bifurcation diagram of the reduced model and compared with the theoretical resilience curves.

Although alternative reinforcement rules produce different relationships between connectivity loss and diagonal enhancement, the projected trajectories remain in excellent agreement with the theoretical bifurcation structure (Fig. 7). The agreement is observed for both state variables of the bipartite system and persists under both pollinator removal and plant removal perturbations.

These results show that the reduced-space organization persists under substantial changes in the sensitivity of diagonal reinforcement. The result therefore does not depend, within the tested parameter ranges, on the particular choice of the baseline exponent $\gamma_p = 2$.

Since the diagonal-reinforcement rule is a modeling assumption that maps deletion fraction to effective competition strength, we tested alternative mappings to assess whether the observed collapse depends on this specific choice. These tests show that alternative reinforcement rules can change the detailed trajectories in deletion space, while the organization of the data in β_{eff} -space remains similar for the parameter ranges tested (Supplementary Fig. 7).



Supplementary Figure 7 Projection of high-dimensional simulations onto the reduced-model bifurcation diagram under different diagonal-reinforcement sensitivities. a, b Projections for the two state variables of the bipartite system, respectively. Across different reinforcement sensitivities, the projected trajectories closely follow the theoretical resilience curves predicted by the reduced model.