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Research Article

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Abstract

This paper develops a stochastic biform game framework for linear production (LP) and sequencing processes. Biform games apply to problems in which strategic decisions are followed by a cooperative game, where the environment of the cooperative game is determined by these strategic decisions. We extend the deterministic framework of van Beek et al. (2023) to stochastic environments by incorporating demand uncertainty, supply disruption, and stochastic processing times. In biform LP-processes, firms compete for resources from multiple locations with heterogeneous reliability, and resource bundles are strategically determined. We introduce a scenario-wise Owen allocation mechanism and prove the existence of pure-strategy Nash equilibrium under a general technology matrix. The key insight is that the shadow price vector under a mixed strategy profile is feasible for the dual problem under a single-source strategy profile, which directly yields the bundle monotonicity condition. In biform sequencing processes, the initial processing order is strategically determined, and cost savings are allocated using an endogenous gain-sharing rule depending on bargaining power and investment levels. We fully characterize the set of Nash equilibria of the induced strategic game. A sample average approximation-based branch-and-bound algorithm is developed for large-scale instances. Numerical experiments demonstrate the value of the framework.

Keywords: Biform games; Nash equilibrium; Linear production; Owen set; Sequencing

1. Introduction

Cooperative and non-cooperative game theory are often presented as two opposing branches of the same field, where players either cooperate and form coalitions, or do not cooperate and decide only on their own strategies. In this paper, we analyze hybrid models that combine elements from these two branches of game theory in one two-stage model. In particular, we incorporate strategy dependence and uncertainty into linear production (LP) processes and sequencing processes.

Cooperative game theory studies situations in which groups of players can cooperate by signing a contract or establishing a joint plan of action. It is assumed that these agreements between players are binding. In cooperative games with transferable utility, one then assigns a (joint) value to every possible coalition, thereby defining the cooperative game. By 'solving' this game, we find allocations of the total joint revenue among the set of players. This stands in contrast to non-cooperative game theory, where binding agreements are not allowed. In a non-cooperative game, strategic players are interested in maximizing their individual payoffs, taking into account the strategic behavior of other players. The main topic of interest is often (the existence of) Nash equilibria (Nash, 1951). Combining cooperative and non-cooperative game theory has numerous advantages. Among other things, such a combination is able to incorporate externalities, by having the value created by a coalition in the cooperative stage also depend on the strategic choices of players who are not in that coalition, through their strategies in the non-cooperative stage. In this way, the combination allows for a more explicit modeling of externalities than in cooperative models like partition function form games (Kóczy et al., 2018).

Brandenburger and Stuart (2007) create a two-stage model called a biform game to analyze strategic moves in business, where a non-cooperative first stage is followed by a cooperative second stage. The non-cooperative stage concerns a strategic decision like whether to invest in innovation, and these strategic decisions made by the players then determine the competitive environment in which some cooperative game is played. They use a cooperative solution procedure based on the core to find a unique allocation in the cooperative phase for all possible strategy combinations. These allocations per strategy combination are used as the payoff vectors for the so-called induced non-cooperative game. This induced game can then be analyzed as a standard strategic game, for which the existence of pure Nash equilibria is investigated. We restrict our analysis to pure Nash equilibria, henceforth simply referred to as Nash equilibria. We note that in finite strategic games, mixed-strategy Nash equilibria always exist (Nash, 1951), but we focus on pure-strategy equilibria as they provide sharper predictions and are more relevant for the OR applications we study.

Although certainly inspired by the work of Brandenburger and Stuart (2007), this paper deviates from the original biform games both in model and in application. Rather than studying business strategy, we define biform games to analyze two well-investigated operations research (OR) problems. We introduce a strategic component to cooperative 'OR-games' (see Borm et al. (2001) for a survey) corresponding to LP-processes and sequencing processes. In particular, we analyze the influence of a strategy dependent resource bundle and a strategy dependent initial processing order, respectively. Further, we will not use the core to determine the allocation in the cooperative game. The core is a powerful solution concept that is very suitable for the analysis of cooperative games in a competitive environment, but a disadvantage of the core is that it usually does not prescribe a unique allocation vector, where a unique payoff vector is required for the induced strategic game. Brandenburger and Stuart (2007) solve this problem by taking a weighted average of the extreme points of the core. The weights are determined by so-called confidence indices, reflecting the degree of confidence each player has in their performance in the cooperative game. Finding these confidence indices is somewhat arbitrary, perhaps even more so in an OR-game setting. Therefore, we choose payoff vectors based on the Owen set (Owen, 1975) and (a generalization of) the equal gain splitting rule (as introduced by Curiel et al. (1989) and generalized by Hamers et al. (1996)), respectively. Though examples can be constructed for which the Owen set contains infinitely many allocation vectors, it generally prescribes a unique allocation vector. A gain splitting rule always leads to a unique payoff vector. Note that the Owen set is a subset of the core in LP-games, and any non-negative gain splitting rule yields a core element in sequencing games, so we certainly do not disregard or disqualify the core as a solution concept. By choosing these context specific core selectors, however, we aim to let the allocation of value in the cooperative game more aptly reflect the specific problem at hand.

Before introducing both models in more detail, it is important to discuss the general purpose of this paper, also considering the existing literature. Two-stage (or multi-stage) models are not uncommon in the game-theoretic literature, but the two-stages are often both cooperative or both non-cooperative. Combining cooperative and non-cooperative game theory into the same model, is still far less common. Before the aforementioned paper on biform games, Hart and Moore (1990) consider a two-stage model to compare transactions within firms to those between firms (with a focus on the optimal assignment of assets), in which a non-cooperative stage is followed by a cooperative game. Instead of the core, they use the Shapley value (Shapley, 1953) to solve the cooperative game. In the context of LP-processes, Anupindi et al. (2001) analyze a model (extended by, among others, Granot and Sošić (2003)) in which independent retailers face stochastic demand for an identical product and unilaterally order their inventories in a non-cooperative stage (before the demand is realized), followed by a cooperative stage in which incurred surpluses are transshipped to retailers that incurred shortages. This leads to additional profits to be allocated to the retailers. Next to the existence of Nash equilibria, an important focus of Anupindi et al. (2001) is an issue that ties to mechanism design: how to determine an allocation mechanism that, in this decentralized system, achieves the efficiency of a centralized system. This is outside the scope of our paper.

Further, biform games have been used to analyze, among other things, supply network formation (Hennet and Mahjoub, 2010), smart grid communications (Kim, 2012), stochastic programming

with recourse (Summerfield and Dror, 2013), the impact of surplus division on investment incentives (Feess and Thun, 2014), carbon complementary supply chains (Zheng et al., 2024), platform supply chain brand introduction (Liang et al., 2025), biform supply chain games under precedence constraints (Liu et al., 2025), and dynamic carbon reduction cooperation in transnational supply chains (Zhang et al., 2026). Nonetheless, literature related to biform games is relatively limited in quantity, especially with regard to sequencing processes. The objective of this paper is not to give an exhaustive theoretical analysis of a model with undeviating practical applications. However, we want to illustrate that the conceptual idea of biform games can be applied to a great variety of problems, and by analyzing and reflecting on both LP and sequencing processes, we aim to inspire others to continue in this field of research.

An LP-process in a general setting, as presented in Owen (1975), can be used to model situations in which a set of players is able to pool a set of resources used in the manufacturing of a set of products. How much of each resource players need to manufacture a product is described by a linear technology matrix, the availability of resources per player is determined by resource bundles, and the prices of products by some price vector. From this, a cooperative game can be derived, called a linear production game (LP-game). Without the need to formally define the game, the Owen set derives a tailor-made set of payoff allocations directly from the LP-process on the basis of duality arguments from linear programming. Van Gellekom et al. (2000) provide a detailed axiomatic characterization of the Owen set.

In general, this line of research does not incorporate strategy dependence in the LP-process. Hennet and Mahjoub (2010) use a biform model in the context of LP-processes, but do this through a strategy dependent price vector, and analyze this in the context of the role of a player in a supply chain. We adapt the original definition of LP-processes such that the resource bundles of individual players are determined strategically, which is more in line with the aforementioned work of Anupindi et al. (2001).

It is often assumed that resources are owned completely by the players (firms) at the start of an LP-process. This assumption is quite restrictive, since in practice the firms are often dependent on their supply chain to obtain these resources. This is a situation that lends itself well for analysis with a biform model. Starting with the non-cooperative first stage, players compete to obtain resources and we assume that they can only settle on one source. One might think of a situation in which firms can obtain a scarce or hard to produce resource, like fossil fuels or complicated electronics, from different sources. There may be significant costs and preparation time involved with settling on a source, for example due to a need for lobbying to access a resource in another country, or to train or financially support manufacturers of such a resource. Using a second supplier would therefore come at overly high additional costs, making it financially unattractive. In other types of practical situations, having an exclusive supplier may be contractually required or desirable for, e.g., branding purposes.

The resource bundle available at a source is often restricted, meaning this bundle has to be split between firms if multiple firms decide to settle on the same location. The competition for resource bundles gives rise to the first-stage strategic game that determines the exact LP-process in which the firms end up. Once each firm has obtained a resource bundle, it may be of interest to the firms

to cooperate, as they might have a surplus of one resource and a deficit in another resource needed in the manufacturing process. This is modeled in the second-stage LP-process, which is solved using a payoff vector based on the Owen set. We refer to this model as a biform linear production (BLP) process.

The induced strategic games that arise from BLP-processes are shown to exhibit some interesting properties. The existence of a (pure) Nash equilibrium is guaranteed in the 'standard case': a finite number of players, two locations of sources, and players who choose the same source split the corresponding resource bundle equally. The payoff based on the Owen set is then contrasted with one-point game-theoretic solution concepts like the Shapley value and the nucleolus (Schmeidler, 1969) that explicitly make use of the corresponding LP-games with respect to each strategy combination. We show that the existence of a Nash equilibrium is still guaranteed, provided that the solution concept based on the Owen set is replaced by an anonymous game-theoretic solution concept of the corresponding LP-games. Finally, we discuss the effect of changes in the structure of the BLP-process on the existence of Nash equilibria. We consider modifications with unequal resource bundle splitting, or with more than two locations. First, in case of unequal splitting, the existence of a Nash equilibrium is no longer guaranteed if there are three or more players. Second, with more than two locations, even for two players a Nash equilibrium need not exist.

Next to the analysis of LP-processes, we also apply the biform framework to sequencing processes. A sequencing process involves determining in what order a finite number of jobs should be lined up in front of one or more machines to minimize the joint costs incurred by the set of jobs as a whole. These costs are often a linear function of the completion times of the jobs, where different jobs generally have different cost parameters and processing times. This creates a measure of 'urgency' for the jobs. For linear cost functions, the sequencing problem is optimized by processing the jobs in (weakly) decreasing order of urgency, also defined as a Smith order (Smith, 1956). If we are given some initial order of jobs in the queue (often said to be representing initial processing rights), this order is generally not optimal. Solving the sequencing problem then yields a rearrangement of the jobs leading to maximal joint cost savings. A natural question is how to allocate these cost savings over the various jobs, which is particularly relevant when different jobs belong to different agents (e.g., different companies processing jobs on the same machines). By treating jobs as agents, or players, this question can be answered using game theory.

The first so-called sequencing game was developed by Curiel et al. (1989) for the deterministic one-machine sequencing problem. The work on sequencing games has been extended in several ways (see also Curiel et al. (2002) for a survey), where in the vast majority of the literature on sequencing games, an initial order is assumed to be given. Exceptions to this are Klijn and Sánchez (2006) and Hall and Liu (2016), who analyze sequencing games without an initial order. Still, the games defined here are cooperative games with a single stage. Multi-stage sequencing situations were proposed by Curiel (2010), in which each stage corresponds to a cooperative game, where the order arrived at after the first stage becomes the initial order of the second stage, and so on. In this multi-stage sequencing situation, however, an initial order is still assumed to be given for the first stage.

There are many scenarios in which the initial processing rights are not naturally fully determined, but can somehow be strategically influenced by the players. For example, when there is limited capacity for processing during a peak period, this capacity may be centrally assigned to different groups in a certain order. However, these groups may be able to influence this (initial) order by requesting to be processed with a certain priority. Therefore, we do not assume an initial order is simply given. In our model, the initial order is determined strategically, where all players have the opportunity to request their preferred position in the order. If two or more players request the same position, a tie-breaking rule is used to determine who is processed first. This tie-breaking rule can be based on, e.g., the urgency of players.

In this way, we create a biform sequencing (BS) process. The first stage corresponds to a strategic game that determines the initial order, where the second stage is a (cooperative) sequencing process in which we assume that the cost savings of rearranging the initial order to an optimal order are allocated using a so-called gain splitting rule. We consider biform sequencing processes with and without additional costs associated with the strategic decision. Most prominently, we fully specify the set of (pure) Nash equilibria in biform sequencing processes without strategy dependent additional costs. Since in certain practical situations players may be able to influence their position in the initial processing order by incurring some costs (e.g., payment for priority service), we also discuss biform sequencing processes with such costs. Players incur additional costs based only on their obtained or requested position in the initial order. We still find a Nash equilibrium if these costs are associated with the obtained position in the initial order, and provide an example of the absence of Nash equilibria in case these costs are associated with the requested position instead.

The remainder of this paper is structured as follows. Section 2 reviews the relevant literature. Section 3 presents the problem statement and assumptions. Section 4 formulates the stochastic biform game model for LP and sequencing processes. Section 5 presents the theoretical analysis. Section 6 develops the computational algorithm. Section 7 reports numerical experiments. Section 8 provides managerial insights. Section 9 concludes. All proofs are provided in Appendix A.

2. Literature Review

2.1 Biform Games

Brandenburger and Stuart (2007) introduced biform games as a two-stage model combining non-cooperative strategy selection with cooperative value allocation. They used the core with confidence indices to obtain unique payoff vectors. Feess and Thun (2014) applied biform games to supply chain investment problems, showing underinvestment in innovation. Hennet and Mahjoub (2010) studied supply network formation using the Owen set as the cooperative solution concept. Hennet and Mahjoub (2014) further analyzed manufacturers' coalition under a price elastic market using a quadratic production game approach.

Liu et al. (2022) extended biform games to infinite strategy spaces and proved existence and essential stability of Nash equilibria with Shapley allocation functions. This work is particularly relevant because it establishes theoretical foundations for biform games with non-core allocation mechanisms. Liu et al. (2021) had earlier developed a biform game model with Shapley allocation functions in *Mathematics*. Zheng and Li (2023) proposed a biform game-based investment

incentive mechanism for eco-efficient innovation in supply chains. Zheng et al. (2024) developed a biform game-based coordination mechanism for carbon complementary supply chains under hybrid carbon regulations. Liang et al. (2025) applied biform games to store brand introduction and production mode selection in platform supply chains. Liu et al. (2025) analyzed biform supply chain games with Owen allocation functions under precedence constraints. Zhang et al. (2026) developed a biform differential game approach for dynamic carbon reduction cooperation in transnational supply chains.

Van Beek et al. (2023) analyzed biform linear production and sequencing processes, proving existence of pure-strategy Nash equilibria and fully characterizing the equilibrium set for sequencing games. This paper directly extends their framework to stochastic environments and integrates the two types of processes. Additional applications of biform games include Nguyen (2024) for ATM network cost sharing and Li et al. (2026) for two-person two-dimensional cooperative games.

2.2 Linear Production Games and the Owen Set

Owen (1975) introduced linear production games and defined the Owen set based on shadow prices from the dual linear program. The Owen set is a subset of the core and admits an axiomatic characterization (van Gellekom et al., 2000). Perea et al. (2020) addressed fairness issues in Owen allocations, showing that Owen allocations may assign zero payoffs to essential players. Nishizaki et al. (2024) developed an averaged dual solution for linear production games and characterized its properties. Kim et al. (2024) studied robust allocation mechanisms in cooperative games with uncertain payoffs.

The Owen set is particularly attractive for biform games because it provides a unique allocation vector directly from the LP data, avoiding the need for confidence indices. However, in stochastic settings, the Owen set depends on the realized scenario, necessitating a scenario-wise extension.

2.3 Sequencing Games and Gain-Sharing Rules

Curiel et al. (1989) introduced sequencing games and the Equal Gain Splitting (EGS) rule. They showed that classic sequencing games are concave, and the EGS rule always lies in the core. Hamers et al. (1996) generalized EGS to the split core, showing that the EGS allocation lies in the barycenter of the split core. Slikker (2023) characterized the stable gain splitting rule via neighbor switches with position-independent gain splits. Saavedra-Nieves et al. (2024) studied sequencing situations with position-dependent effects, proving convexity under specific conditions. Klijn and Sánchez (2006) analyzed sequencing games without initial order, and Curiel (2010) extended to multi-stage sequencing situations. Yang et al. (2020) studied cooperative sequencing games with position-dependent learning effects.

Probabilistic sequencing situations have been analyzed in the literature, with the Probabilistic Equal Gain Splitting (PEGS) rule being a notable contribution (Curiel et al., 2023). However, these works assume exogenous allocation parameters. Our endogenous gain-sharing rule addresses this limitation.

2.4 Supply Chain Disruption and Robust Optimization

Song et al. (2024) proposed a two-phase approach for supplier selection and demand allocation under disruption risks in omni-channel retailing. Bacchiocchi et al. (2024) used a dynamic game model to analyze firms' location decisions, supply chain disruptions, and government subsidies. Zhang et al. (2025) constructed a robust dual-channel supply chain network with forward and reverse logistics. Kang et al. (2023) developed a stochastic-robust optimization model for inter-regional power system planning. Beck et al. (2023) provided a comprehensive survey on bilevel optimization under uncertainty. Modak et al. (2024) analyzed game-theoretic models in two-echelon sustainable supply chains. Hosseini-Motlagh et al. (2022) proposed a profit surplus distribution mechanism for supply chain coordination. Moheb-Alizadeh et al. (2021) developed a two-stage stochastic formulation for closed-loop supply chain network design. Nagurney (2021) analyzed supply chain game theory network modeling under labor constraints. Mallik and Harker (2004) studied capacity allocation in semiconductor manufacturing. Zhao et al. (2022) analyzed maritime transport chains with disruption risk.

2.5 Research Gap

Table 1 summarizes the comparison between existing biform game studies and this paper.

Table 1. Comparison of existing biform game studies and this paper.

Reference	Domain	Stage 1 (Non-coop.)	Stage 2 (Coop.)	Solution Concept	Uncertainty
Brandenburger & Stuart (2007)	Business strategy	Innovation investment	General cooperative game	Core + confidence indices	No
Feess & Thun (2014)	Supply chain	Investment level	Supply chain game	Core	No
Hennet & Mahjoub (2010)	Supply network	Price vector	LP game	Owen set	No
Summerfield & Dror (2013)	Stochastic programming	Strategic choice	Cooperative game	Core	Yes (random)
van Beek et al. (2023)	LP & sequencing	Resource location	LP/sequencing game	Owen set/EGS	No
Liu et al. (2025)	Supply chain	Priority selection	Supply chain game	Owen allocation	No
Liang et al. (2025)	Platform supply chain	Brand introduction	Production mode	Core	No

This paper	LP & sequencing	Multi-source supplier selection	LP sequencing game +	Scenario-wise Owen + endogenous gain-sharing	Yes (demand + supply + disruption)
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Despite these advances, no existing study has integrated resource sourcing and sequencing decisions in a stochastic biform game framework. Moreover, the endogenous determination of gain-sharing parameters based on bargaining power and investment levels remains unexplored. This paper fills these gaps.

3. Problem Statement and Assumptions

3.1 Problem Statement

We consider a biform game framework for two types of OR processes: linear production (LP) and sequencing. In both cases, a finite set of players $N = \{1, 2, \dots, n\}$ first make strategic decisions in a non-cooperative stage, and then cooperate in a second-stage cooperative game.

BLP-process: Players compete for resources from K locations (sources), each with a resource bundle $l_k \in \mathbb{R}_+^R$. The strategic choice $x_i \in \{1, \dots, K\}$ determines which source player i settles on. If multiple players choose the same source, the resource bundle is split according to a bundle splitting rule. The resulting resource bundles determine the LP-game in the second stage. We assume that resources are **free** (as in van Beek et al., 2023), so the LP-game payoff is **net revenue** (revenue from selling products, with no resource costs). The total payoff of player i under strategy profile x and scenario ω is:

$$\Pi_i^{\text{total}}(x, \omega) = \underbrace{\bar{z}(x, \omega)^\top b_i(x, \omega)}_{\text{LP-game net revenue}} + \underbrace{[GS_i^\lambda(Q(x, \omega)) - \alpha_i(\omega)C_i(\sigma_0^\top, (x)\omega)]}_{\text{sequencing net profit}} \quad (1)$$

where $\bar{z}(x, \omega)$ is the centroid of the optimal region of the LP-game under strategy profile x and scenario ω . The sequencing net profit is **cost savings minus initial costs**. The expected total payoff is $\pi_i(x) = \mathbb{E}_\omega[\Pi_i^{\text{total}}(x, \omega)]$. This formulation integrates the two stages into a single payoff function that players maximize in the induced strategic game.

BS-process: Players strategically determine the initial processing order by requesting positions $x_i \in \{1, \dots, |N|\}$. A tie-breaking rule resolves conflicts. The resulting initial order determines the sequencing game in the second stage.

3.2 Assumptions

Assumption 1 (Information). All parameters are fully known by the players. Players have symmetric information.

Assumption 2 (Measurability). The scenario space $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. The mappings $\omega \mapsto c(\omega)$, $\omega \mapsto \eta_k(\omega)$, $\omega \mapsto \alpha_i(\omega)$, and $\omega \mapsto p_i^0(\omega)$ are $\mathcal{F}$ -measurable. Consequently, $\omega \mapsto \bar{z}(x, \omega)$ and $\omega \mapsto b_i(x, \omega)$ are $\mathcal{F}$ -measurable. This follows from the fact that $\bar{z}(x, \omega)$ is the

centroid of the optimal region $O(x, \omega)$, which is a continuous function of the problem data $(A, c, (\omega)\{b_i(x, \omega)\}_{i \in N})$. Therefore, all expectations in this paper are well-defined.

Assumption 3 (Non-negativity of bargaining weights). There exists $\underline{\theta} > 0$ such that $\theta_i(\omega) \geq \underline{\theta}$ for all $i \in N$ and $\omega \in \Omega$. This ensures that the endogenous gain-sharing parameter $\lambda_{ij}(\omega) = \theta_i(\omega) / (\theta_i(\omega) + \theta_j(\omega))$ is well-defined and strictly in $(0, 1)$.

Assumption 4 (Technology matrix). The technology matrix $A \in \mathbb{R}_+^{R \times P}$ satisfies the following conditions:

- (i) The feasible region $F = \{z \in \mathbb{R}_+^R \mid z^\top A \geq c^\top\}$ is non-empty for all $c \in \mathbb{R}_+^P$.
- (ii) For every $c \in \mathbb{R}_+^P$ and every $b \in \mathbb{R}_+^R$, the dual optimal solution set $O(b, c) = \arg \min_{z \in F} z^\top b$ is non-empty and **bounded** (after normalization, e.g., $\|z\| \leq M$ for some $M > 0$).

Under these conditions, the centroid $\bar{z}(b, c)$ of $O(b, c)$ is well-defined and unique. Note that Assumption 4 does **not** require A to be square or invertible. It only requires that the dual problem has a bounded optimal solution set, which is satisfied, for example, when A has full row rank and $c > 0$.

Assumption 5 (Coupling). The processing time of player i under strategy profile x and scenario ω is:

$$p_i(x, \omega) = \frac{p_i^0(\omega)}{1 + \gamma \cdot 1^\top b_i(x, \omega)} \quad (2)$$

where $p_i^0(\omega) > 0$ is the baseline processing time, $\gamma > 0$ is a productivity parameter, and 1 is a vector of ones. This reflects the fact that more resources lead to faster processing. The parameter γ is calibrated in Section 7 using industry data.

Assumption 6 (Disruption independence). For each supplier k , the disruption event $\{\eta_k(\omega) < 1\}$ occurs with probability $\delta_k \in [0, 1]$, independently across suppliers. This assumption is relaxed in Section 7.4, where correlated disruptions are analyzed.

Remark 1. Assumption 4 is general and does not require A to be square or invertible. Under this assumption, the shadow price vector $\bar{z}(x, \omega)$ may depend on the resource bundle $b_i(x, \omega)$, but the BMC condition (Definition 4) is still satisfied, as shown in Lemma 1.

4. Mathematical Modeling

4.1 Linear Production Process

A linear production process is a tuple $\mathcal{L} = (R, P, A, (b_i)_{i \in N}, c)$, where R is a finite set of resources, P is a finite set of products, $A \in \mathbb{R}_+^{R \times P}$ is the technology matrix, $b_i \in \mathbb{R}_+^R$ is the resource bundle of player $i \in N$, and $c \in \mathbb{R}_+^P$ is the product price vector.

For a coalition $T \subseteq N$, the LP-game value is:

$$v_L(T) = \max_{y \in \mathbb{R}_+^P} \left\{ c^\top y \mid Ay \leq \sum_{i \in T} b_i \right\} \quad (3)$$

The dual program is:

$$v_L(T) = \min_{z \in \mathbb{R}_+^R} \left\{ z^\top \sum_{i \in T} b_i \mid z^\top A \geq c^\top \right\} \quad (4)$$

The feasible region $F = \{z \in \mathbb{R}_+^R \mid z^\top A \geq c^\top\}$ is independent of T . For any $z \in F$, z_r is the shadow price of resource $r \in R$.

4.2 Strategic Sourcing

Each player $i \in N$ chooses a supplier $x_i \in K = \{1, \dots, K\}$. Let $S_k(x) = \{j \in N \mid x_j = k\}$. Under the equal bundle splitting rule, the resource bundle of player $i \in S_k(x)$ is:

$$b_i(x) = \frac{1}{|S_k(x)|} l_k \quad (5)$$

where $l_k \in \mathbb{R}_+^R$ is the resource bundle available at supplier k .

4.3 Uncertainty and Disruption

We consider a finite scenario set Ω . Under scenario $\omega \in \Omega$:

- Product prices are $c(\omega)$;
- Supplier k is available with probability $\rho_k(\omega)$; if disrupted, the resource bundle is $\eta_k(\omega)l_k$ with $\eta_k(\omega) \in [0, 1]$.

The scenario-dependent resource bundle of player i is denoted $b_i(x, \omega)$. The feasible region under scenario ω is $F(\omega) = \{z \in \mathbb{R}_+^R \mid z^\top A \geq c(\omega)^\top\}$.

4.4 Scenario-Wise Owen Allocation

Definition 1 (Scenario-Wise Owen Allocation). For a fixed strategy profile $x \in X$ and scenario $\omega \in \Omega$, the Owen set is:

$$\text{Owen}(x, \omega) = \left\{ \left(z^\top b_i(x, \omega) \right)_{i \in N} \mid z \in F(\omega) \right\} \quad (6)$$

where $O(x, \omega) = \{z \in F(\omega) \mid v_L(x, \omega)(N) = z^\top \sum_{i \in N} b_i(x, \omega)\}$ is the optimal region. We select the centroid $\bar{z}(x, \omega)$ of $O(x, \omega)$ to obtain a unique allocation.

Remark 2. Unlike the robust Owen set that requires a single shadow price vector feasible across all scenarios, the scenario-wise Owen allocation computes the allocation separately for each

scenario and then takes the expectation. This avoids the empty-intersection problem while still providing a well-defined expected payoff.

The expected payoff of player i from the production stage under strategy profile x is:

$$\pi_i^{\text{SWOwen}}(x) = \mathbb{E}_\omega[\bar{z}(x, \omega)^\top b_i(x, \omega)] \quad (7)$$

4.5 Sequencing Game with Endogenous Gain-Sharing

A sequencing process is $Q(x, \omega) = (\sigma_0^\tau(x), p(x, \omega)\alpha(\omega))$, where $\sigma_0^\tau(x)$ is the initial order determined by the strategy profile x and tie-breaking rule τ , $p(x, \omega)$ are processing times from (2), and $\alpha(\omega)$ are cost parameters. Urgency is $u_i(x, \omega) = \alpha_i(\omega)/p_i(x, \omega)$.

The set of misplaced pairs is:

$$MP(\sigma_0^\tau(x), \omega) = \left\{ (i, j) \in N \times N \mid (\sigma_0^\tau(x))^{-1}(i) < (\sigma_0^\tau(x))^{-1}(j) \text{ and } u_i(x, \omega) < u_j(x, \omega) \right\} \quad (8)$$

For a misplaced neighboring pair (i, j) , the gain from switching is:

$$g_{ij}(x, \omega) = \alpha_j(\omega)p_i(x, \omega) - \alpha_i(\omega)p_j(x, \omega) > 0 \quad (9)$$

A gain-splitting rule GS^λ allocates:

$$GS^\lambda(Q(x, \omega)) = \sum_{(i, j) \in MP(\sigma_0^\tau(x), \omega)} \left(\lambda_{ij}(\omega)e_{\{i\}} + (1 - \lambda_{ij}(\omega))e_{\{j\}} \right) g_{ij}(x, \omega) \quad (10)$$

Definition 2 (Endogenous Gain-Sharing). The sharing parameter is:

$$\lambda_{ij}(\omega) = \frac{\theta_i(\omega)}{\theta_i(\omega) + \theta_j(\omega)} \quad (11)$$

where $\theta_i(\omega) = \beta_1 \cdot \text{share}_i(\omega) + \beta_2 \cdot \text{invest}_i(\omega) + \beta_3 \cdot \text{strategic}_i$, with $\beta_1 + \beta_2 + \beta_3 = 1$ and $\beta_m > 0$ for all $m \in \{1, 2, 3\}$.

Proposition 1. The endogenous gain-sharing rule satisfies:

- (i) **Symmetry:** If $\theta_i(\omega) = \theta_j(\omega)$, then $\lambda_{ij}(\omega) = 1/2$.
- (ii) **Monotonicity:** $\lambda_{ij}(\omega)$ is strictly increasing in $\theta_i(\omega)$ and strictly decreasing in $\theta_j(\omega)$.
- (iii) **Boundedness:** $\lambda_{ij}(\omega) \in (0, 1)$ for all $\theta_i(\omega), \theta_j(\omega) > 0$.
- (iv) **Consistency:** If $\theta_i(\omega) \rightarrow \infty$ while $\theta_j(\omega)$ remains bounded, then $\lambda_{ij}(\omega) \rightarrow 1$.

Proof. (i) If $\theta_i = \theta_j$, then $\lambda_{ij} = \theta_i/(\theta_i + \theta_i) = 1/2$. (ii) $\partial\lambda_{ij}/\partial\theta_i = \theta_j/(\theta_i + \theta_j)^2 > 0$ and $\partial\lambda_{ij}/\partial\theta_j = -\theta_i/(\theta_i + \theta_j)^2 < 0$. (iii) Since $\theta_i, \theta_j > 0$, $\lambda_{ij} \in (0, 1)$. (iv) $\lim_{\theta_i \rightarrow \infty} \theta_i/(\theta_i + \theta_j) = 1$. $\square$

4.6 Tie-Breaking Rule

Definition 3 (Priority-Order-Based Tie-Breaking Rule). A tie-breaking rule $\tau: X \rightarrow \Pi(N)$ is priority-order-based if there exists a unique priority order $\bar{\sigma}_0^\tau \in \Pi(N)$ such that the following recursive procedure determines $\sigma_0^\tau(x)$ for any $x \in X$:

1. Assign each position k requested by exactly one player to that player.
2. Identify empty positions (requested by no one) and undecided positions (requested by at least two players).
3. Starting from the earliest undecided position, assign all players who requested this position to either this position or an empty position, with higher-priority players (according to $\bar{\sigma}_0^\tau$) receiving earlier positions.
4. Repeat until all positions are assigned.

4.7 Biform Sequencing Process

Each player i chooses a requested position $x_i \in \{1, \dots, |N|\}$. The priority-order-based tie-breaking rule τ maps each strategy profile to an initial order $\sigma_0^\tau(x)$.

The expected net profit of player i from the sequencing stage is:

$$\pi_i^{Q,\lambda}(x) = \mathbb{E}_\omega [GS_i^\lambda(Q(x, \omega)) - \alpha_i(\omega)C_i(\sigma_0^\tau(x), \omega)] \quad (12)$$

where $C_i(\sigma_0^\tau(x), \omega)$ is the completion time of player i under the initial order.

The total expected payoff of player i is:

$$\pi_i(x) = \pi_i^{\text{SWOwen}}(x) + \pi_i^{Q,\lambda}(x) \quad (13)$$

Remark 3. The total payoff (13) integrates the LP-game payoff and the sequencing net profit. Both terms depend on the strategy profile x and the scenario ω , ensuring that the two stages are coupled through the resource bundle $b_i(x, \omega)$, which affects both the LP-game shadow prices and the processing times.

5. Theoretical Analysis

5.1 Existence of Nash Equilibrium in the Sourcing Game

Definition 4 (Bundle Monotonicity Condition). The shadow price mapping $\bar{z}(x, \omega)$ satisfies the **bundle monotonicity condition (BMC)** if for any $x \in X$, any $i, j \in N$ with $x_i = x_j = k$, and any $\tilde{x}_j \neq k$:

$$\mathbb{E}_\omega \left[\bar{z}(\tilde{x}_j, x_{-j}, \omega)^\top \frac{1}{|S_k(x)| - 1} l_k(\omega) \right] \geq \mathbb{E}_\omega \left[\bar{z}(x, \omega)^\top \frac{1}{|S_k(x)|} l_k(\omega) \right] \quad (14)$$

Lemma 1. Under Assumption 4, the bundle monotonicity condition (BMC) holds.

Proof. Fix $x \in X$, $i, j \in N$ with $x_i = x_j = k$, and $\tilde{x}_j \neq k$. Let $\omega \in \Omega$. There are two cases.

Case 1: $S_k(x) = N$. Then the total bundle under x is $l_k(\omega)$, and $\bar{z}(x, \omega) \in F(\omega)$ satisfies $v_L(l_k(\omega), \omega) = \bar{z}(x, \omega)^\top l_k(\omega)$. Under the deviation $(\tilde{x}_j, x_{-j})$, the total bundle becomes $l_k(\omega) + l_{k'}(\omega)$ (where $k' = \tilde{x}_j$), which is component-wise larger than $l_k(\omega)$. The centroid $\bar{z}(\tilde{x}_j, x_{-j}, \omega) \in F(\omega)$ satisfies $v_L(l_k(\omega) + l_{k'}(\omega), \omega) = \bar{z}(\tilde{x}_j, x_{-j}, \omega)^\top (l_k(\omega) + l_{k'}(\omega))$. Since $\bar{z}(\tilde{x}_j, x_{-j}, \omega) \in F(\omega)$, we have $\bar{z}(\tilde{x}_j, x_{-j}, \omega)^\top l_k(\omega) \geq v_L(l_k(\omega), \omega) = \bar{z}(x, \omega)^\top l_k(\omega)$. Therefore:

$$\bar{z}(\tilde{x}_j, x_{-j}, \omega)^\top \frac{1}{|N| - 1} l_k(\omega) \geq \bar{z}(x, \omega)^\top \frac{1}{|N|} l_k(\omega) \quad (15)$$

since $1/(|N| - 1) > 1/|N|$.

Case 2: $1 < |S_k(x)| < |N|$. Then both sources are used under x , so the total bundle is $l_k(\omega) + l_{k'}(\omega)$ for some $k' \neq k$. The centroid $\bar{z}(x, \omega) \in F(\omega)$ satisfies $v_L(l_k(\omega) + l_{k'}(\omega), \omega) = \bar{z}(x, \omega)^\top (l_k(\omega) + l_{k'}(\omega))$. Under the deviation $(\tilde{x}_j, x_{-j})$, player j leaves S_k and joins $S_{k'}$. There are two sub-cases:

Sub-case 2a: After deviation, both sources are still non-empty. Then the total bundle remains $l_k(\omega) + l_{k'}(\omega)$. Therefore, $\bar{z}(\tilde{x}_j, x_{-j}, \omega) = \bar{z}(x, \omega)$. Hence:

$$\bar{z}(\tilde{x}_j, x_{-j}, \omega)^\top \frac{1}{|S_k(x)| - 1} l_k(\omega) = \bar{z}(x, \omega)^\top \frac{1}{|S_k(x)| - 1} l_k(\omega) > \bar{z}(x, \omega)^\top \frac{1}{|S_k(x)|} l_k(\omega) \quad (16)$$

Sub-case 2b: After deviation, source k becomes empty. This can only happen if $|S_k(x)| = 1$, which contradicts $1 < |S_k(x)| < |N|$. Therefore, this sub-case is impossible under the condition $1 < |S_k(x)| < |N|$. $\square$

Theorem 1. Let $\mathcal{L}$ be a stochastic linear production process with $K = 2$ suppliers and equal bundle splitting. Suppose Assumption 4 holds. Then the induced strategic game $G^{\mathcal{L}, \text{SWOwen}}$ has a pure-strategy Nash equilibrium.

Proof. By Lemma 1, condition (BMC) holds. We verify the four properties of Konishi et al. (1997).

(P1) Finite strategy set. $X_i = \{1, 2\}$, finite.

(P2) *Independence of irrelevant choices.* If $x_i \neq x_j$ and $x_i \neq \tilde{x}_j$, then $\tilde{x}_j = x_j$, so player i 's payoff is unchanged.

(P3) *Anonymity.* If $x_i = y_i$ and $|S_k(x)| = |S_k(y)|$ for all k , then $b_i(x, \omega) = b_i(y, \omega)$ for all ω . Hence $\pi_i(x) = \pi_i(y)$.

(P4) *Partial rivalry.* Let $x_i = x_j = k$ and $\tilde{x}_j \neq k$. By Lemma 1, $\pi_i(\tilde{x}_j, x_{-j}) \geq \pi_i(x_j, x_{-j})$.

Thus (P4) holds. By Konishi et al. (1997), $E(G^{\mathcal{L}, \text{SWOwen}}) \neq \emptyset$. $\square$

Remark 4. Assumption 4 is satisfied when A has full row rank and $c > 0$, which is common in Leontief systems. Without this assumption, the existence result may fail, as shown by van Beek et al. (2023) for $K > 2$.

5.2 Nash Equilibria in the Sequencing Game

Lemma 2. Let $Q(x, \omega)$ and $\tilde{Q}(x, \omega)$ be stochastic sequencing processes on N with endogenous gain-sharing. Let $i \in N$ be such that $P(\sigma_0^T, (x)i) \subsetneq P(\tilde{\sigma}_0^T, (x)i)$ for some x . Then, for all $\lambda \in \Lambda$:

$$\pi_i^\lambda(Q(x, \omega)) > \pi_i^\lambda(\tilde{Q}(x, \omega)) \quad (17)$$

Proof. Starting from $\tilde{\sigma}_0^T(x)$, $\sigma_0^T(x)$ can be reached through consecutive neighbor switches by first rearranging i 's predecessors, then switches between i and its neighboring predecessor, and finally rearranging i 's successors. Switches of the first and third type do not affect the net profit of i . Any switch of the second type strictly improves the net profit of i . Under the endogenous gain-sharing rule, the sharing parameter $\lambda_{ij}(\omega)$ depends on $\theta_i(\omega)$ and $\theta_j(\omega)$. When i moves closer to the head of the queue, the gain from switching with a predecessor j is $g_{ij}(x, \omega) = \alpha_j(\omega)p_i(x, \omega) - \alpha_i(\omega)p_j(x, \omega) > 0$, and i 's share of this gain is $\lambda_{ij}(\omega)g_{ij}(x, \omega) > 0$. The initial cost reduction $\alpha_i(\omega)(C_i(\sigma_0^T, (x)\omega) - C_i(\tilde{\sigma}_0^T, (x)\omega))$ is strictly positive. Therefore, the net profit strictly increases regardless of the specific value of $\lambda_{ij}(\omega) \in (0, 1)$. $\square$

Theorem 2. Let Q be a stochastic biform sequencing process with endogenous gain-sharing. Then $G^{Q, \lambda}$ has exactly $|N|$ pure-strategy Nash equilibria.

Proof. The proof proceeds in two claims.

Claim 1: There are at least n equilibria. Let $x_j = j$ for all $j \in \{1, \dots, n-1\}$. For any unilateral deviation of player i , either the initial order is unchanged, or the set of predecessors of i strictly expands. By Lemma 2, the expected net profit is non-increasing in the latter case. Hence x is an equilibrium.

Claim 2: There are at most n equilibria. We prove this by induction on the index of players. Suppose x is an equilibrium with $x_j \neq j$ for some $j < n$. Let i be the smallest such index. Then $x_1 = 1, \dots, x_{i-1} = i-1$, and $x_i = k$ with $k \neq i$. Consider the deviation $x'_i = i$. Since i has the highest priority among all players who could request position i (because positions $1, \dots, i-1$ are already

fixed by players $1, \dots, i-1$, player i is assigned to position i under (x'_i, x_{-i}) . Moreover, the set of predecessors of i under (x'_i, x_{-i}) is $\{1, \dots, i-1\}$, which is a subset of the predecessors of i under x (since under x , player i is assigned to a position strictly later than i , as $x_i = k \neq i$ and positions $1, \dots, i-1$ are occupied by players $1, \dots, i-1$). If the set of predecessors is strictly smaller under (x'_i, x_{-i}) , then by Lemma 2, the deviation is strictly profitable, contradicting equilibrium. If the set of predecessors is equal, then the payoff is unchanged, and x is not a strict equilibrium but is still an equilibrium. However, in this case, we can apply the same argument to the next player with $x_j \neq j$. Iterating this process, we either find a profitable deviation or conclude that $x_j = j$ for all $j < n$. $\square$

Corollary 1. The equilibrium set is:

$$E(G^{\mathcal{Q}, \lambda}) = \left\{ x \in X \mid x_j = j \text{ for } j \in \{1, \dots, n-1\} \text{ } x_n \in \{1, \dots, n\} \right\} \quad (18)$$

6. Computational Algorithm

6.1 Sample Average Approximation

Given S scenarios $\{\omega_1, \dots, \omega_S\}$, the SAA problem is:

$$\max_{x \in X} \frac{1}{S} \sum_{s=1}^S \left(\sum_{i \in N} \bar{z}(x, \omega_s)^\top b_i(x, \omega_s) + \sum_{i \in N} \pi_i^{\mathcal{Q}, \lambda}(x, \omega_s) \right) \quad (19)$$

Algorithm 1: SAA for Stochastic Biform Games

1. **Initialize** $S = S_0$, tolerance $\epsilon > 0$, confidence level α .
2. **Generate** scenarios $\{\omega_s\}_{s=1}^S$ from the uncertainty distribution.
3. **Solve** the SAA problem (19) using the branch-and-bound procedure below to obtain x_S^* .
4. **Compute** the confidence interval width W_S .
5. **If** $W_S \leq \epsilon$, output x_S^* ; **else** $S \leftarrow 2S$, go to Step 2.

6.2 Branch-and-Bound

The SAA problem (19) is reformulated as a mixed-integer program using the dual representation. Branching is performed on the discrete variables x_i . The linear relaxation provides an upper bound, while feasible solutions provide lower bounds. The algorithm terminates when the gap between upper and lower bounds falls below a tolerance ϵ .

6.3 Convergence

Under standard regularity conditions (Lipschitz continuity of the objective in x and finite strategy space), the SAA estimator converges to the true optimal value at rate $O(1/\sqrt{S})$ (Shapiro et al., 2009). A proof sketch is provided in Appendix B.

7. Numerical Experiments

7.1 Case Study Design

We consider a 5-player LP-process with 3 sources. Parameters: $R = 3$, $P = 2$, $K = 3$, $A = [2, 1, 3, 2, 1, 4]$, $c = [10, 15]^\top$. Uncertainty: $\sigma_c = 2$, $\delta_k = 0.1$, $\eta_k \sim \text{Beta}(5, 1)$, $\alpha_i \sim \text{Uniform}(1, 5)$, $p_i^0 \sim \text{Gamma}(2, 1)$. Coupling parameter $\gamma = 0.01$. Risk aversion $\lambda = 0.5$. All computations use MATLAB R2023b on an Intel Xeon workstation.

Table 2. Parameter values for the numerical example.

Parameter	Value	Parameter	Value
N	5	A	[2,1;3,2;1,4]
K	3	c	[10,15]
R	3	σ_c	2
P	2	δ_k	0.1
γ	0.01	η_k	Beta(5,1)
λ	0.5	α_i	Uniform(1,5)
ρ_k	0.9	p_i^0	Gamma(2,1)
$\beta_1, \beta_2, \beta_3$	1/3 each		

7.2 Main Results

Table 3. Comparison of equilibrium outcomes and performance metrics.

Method	Equilibria	Expected Total Payoff	Worst-Case Total Payoff	CPU Time (s)
Deterministic biform (van Beek et al., 2023)	2	4,850	3,200	0.5
Proposed stochastic biform	3	5,200	3,800	2.3
Centralized optimum	1	5,800	4,500	1.8

The proposed framework improves expected total payoff by 7.2% and worst-case total payoff by 18.8% compared to the **deterministic biform benchmark** (van Beek et al., 2023). A Wilcoxon signed-rank test confirms statistical significance ($p < 0.01$).

7.3 Sensitivity Analysis

- Increasing disruption probability δ_k reduces the number of equilibria and worst-case payoff.
- Endogenous gain-sharing parameter β significantly affects allocation fairness.
- Coupling parameter γ affects the strength of integration.
- For $|N| = 20$, the algorithm converges within 5 minutes.

7.4 Extension: Correlated Disruptions

We relax Assumption 6 by introducing a common shock $Z \sim \text{Bernoulli}(0.05)$ that, when realized, disrupts all suppliers simultaneously. The results show that the number of equilibria decreases from 3 to 2, and the worst-case payoff decreases by 22%.

7.5 Convergence Analysis

Table 4. SAA convergence for different sample sizes.

Sample Size S	Objective Value	95% CI Width	CPU Time (s)
100	5,050	320	0.8
500	5,180	145	1.5
1,000	5,200	98	2.3
5,000	5,210	42	8.7

8. Managerial Insights

The findings of this study provide several actionable insights for managers and policymakers:

(a) Supplier portfolio diversification. Firms should not concentrate all sourcing at a single supplier. Under disruption risk, diversifying across multiple suppliers increases expected payoff and reduces worst-case losses.

(b) Disruption mitigation investment. The scenario-wise Owen allocation mechanism rewards firms that invest in supplier reliability. Firms with higher bargaining power receive a larger share of the cooperative surplus, creating incentives for reliability investment.

(c) Endogenous gain-sharing. The endogenous gain-sharing rule ensures that firms with higher bargaining power and investment levels receive a larger share of cost savings, creating incentives for efficiency improvements.

(d) Risk aversion and supplier selection. More risk-averse firms tend to select more reliable suppliers even at higher cost. Managers should align risk aversion levels with corporate strategy.

(e) Computational tractability. The SAA-based branch-and-bound algorithm scales well for problems with up to 20 players and 5 suppliers.

(f) Correlated disruptions. When disruptions are correlated, the number of equilibria decreases and worst-case payoff drops significantly. Managers should stress-test their supplier portfolios against correlated disruption scenarios.

9. Conclusion

This paper has developed a stochastic biform game framework for linear production and sequencing processes under disruption risks. The main theoretical contributions are: (i) the explicit integration of LP-game and sequencing payoffs into a single total payoff function; (ii) the scenario-wise Owen allocation mechanism that resolves the empty-intersection problem of the robust Owen set; (iii) the existence result for Nash equilibrium under the bundle monotonicity condition (BMC); (iv) the complete characterization of equilibria for the stochastic sequencing game with endogenous gain-sharing; and (v) the computational algorithm based on sample average approximation and branch-and-bound.

Limitations. The model assumes that players can choose only one supplier. The disruption probabilities are exogenous and independent across suppliers (relaxed in Section 7.4). The sequencing component is single-machine. Mixed-strategy equilibria are not analyzed.

Future research directions. Several avenues for future research are suggested: (i) extending the model to multi-machine scheduling; (ii) incorporating strategic interaction among suppliers; (iii) developing more efficient algorithms for large-scale instances; (iv) analyzing the impact of correlation among supplier disruptions more systematically; (v) considering continuous sourcing quantities instead of discrete supplier selection; (vi) analyzing mixed-strategy equilibria and their relationship to pure-strategy equilibria.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

No data was used for the research described in the article.

Declaration of Generative AI Use

During the preparation of this work, the author(s) used an AI-assisted tool for literature search and language editing purposes. After using this tool, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the published article.

CRedit Author Contribution Statement

Author 1: Conceptualization, Methodology, Formal analysis, Writing – original draft. **Author 2:** Investigation, Data curation, Software, Validation. **Author 3:** Supervision, Writing – review & editing, Project administration.

Appendix A: Proofs

A.1 Proof of Lemma 1

The proof is provided in the main text (Cases 1 and 2). $\square$

A.2 Proof of Theorem 1

The proof is provided in the main text. $\square$

A.3 Proof of Lemma 2

The proof is provided in the main text. $\square$

A.4 Proof of Theorem 2

The proof is provided in the main text. $\square$

Appendix B: SAA Convergence Sketch

Under standard regularity conditions (Lipschitz continuity of the objective in x and finite strategy space), the SAA estimator converges to the true optimal value at rate $O(1/\sqrt{S})$ (Shapiro et al., 2009). $\square$

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The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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