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Load-Stable Reaction Force Ratios in a Compliant Sphere–Inclined-Plane Contact

Research Article

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Abstract

Constrained contacts may exhibit a nearly load-invariant tangential-to-normal force ratio even while their internal mechanical state continues to evolve. Here, a stainless-steel sphere supported by a compliant square copper rod was loaded quasi-statically against an aluminum inclined plane at six inclinations from 5° to 30° , yielding 76 steady plateaus. At each fixed inclination, the resolved tangential (T) and normal forces (N) were nearly proportional, whereas the mean reaction ratio (T/N) increased from 0.15 ± 0.02 to 1.40 ± 0.02 across the inclination range. A structure–contact model was formulated by combining horizontal support compliance, finite-state Hertz normal indentation, constrained sphere rotation, and a Mindlin tangential elastic reference. Distinguishing the Hertz tangent stiffness from the finite-state secant stiffness leads to an explicit relation for the sphere-center trajectory. The tangential displacement not accounted for by the ideal elastic reference and known rotation is represented by an effective accommodation variable. Elimination of the trajectory angle then yields a compact finite-state relation involving inclination, translational stiffness competition, the local tangential-to-normal contact stiffness ratio, rotational coupling, and normalized tangential accommodation. Across the measured states, the sphere-center trajectory rotates continuously and the accommodation contribution increases with load, while the macroscopic reaction ratio remains confined to a narrow band at each inclination. The results show how a stable external reaction direction can coexist with substantial evolution of contact stiffness, structural deformation, rotation, and internal tangential accommodation.

Keywords: constrained contact; sphere–inclined-plane contact; Hertz–Mindlin contact; structural compliance

1 Introduction

The ratio of tangential to normal force is one of the most frequently reported macroscopic quantities in tribological experiments. In a simple contact, this ratio often provides a compact description of the interfacial response. In a geometrically constrained contact loaded through a compliant structure, however, the measured reactions must simultaneously satisfy structural deformation, local contact deformation, rotational equilibrium, and kinematic compatibility along the loading path. A seemingly simple force ratio can therefore encode both local contact behavior and the mechanical boundary conditions of the test system.

This coupling is closely connected to classical contact mechanics. Hertz theory relates normal load, contact radius, and indentation through a nonlinear finite-state relation [1, 2], whereas Mindlin and Mindlin–Deresiewicz showed that tangentially loaded contacts possess finite tangential compliance and loading-history dependence before gross sliding [3, 4]. Johnson’s experiments and subsequent studies of rough contacts further demonstrated that partial slip, asperity deformation, and rearrangement of the real contact population can modify the macroscopic tangential compliance [5–8]. Consequently, an approximately fixed external reaction direction does not require a fixed contact size, stiffness, or internal displacement partition.

Geometric constraint introduces an additional level of coupling. Halsey and Ertaş showed for a ball-in-groove system that rigid-body equilibrium alone does not uniquely determine the contact reactions; loading history and contact compliance are required to close the problem [9]. Subsequent V-groove and constrained-contact studies likewise showed that contact geometry, external loading stiffness, and local compliance can strongly alter macroscopic force ratios [10, 11]. These results motivate a mechanics-based interpretation in which a stable T/N is treated as a system response selected by the coupled contact and support rather than as evidence of an invariant internal state.

The present study considers a unilateral sphere–inclined-plane contact. A stainless-steel sphere is connected to the external loading structure through a square copper support and pressed against an adjustable 7075 aluminum plane. Compared with a two-sided groove, this configuration contains only one local contact, yet it retains several coupled mechanical features: horizontal support compliance, nonlinear Hertz normal response, and a finite rotational degree of freedom. Experimentally, the tangential and normal reaction forces at the contact remain almost proportional at each fixed inclination, whereas changing the inclination strongly shifts the proportionality level. This combination provides a compact setting in which to examine how a simple macroscopic force ratio emerges from an evolving structural and contact state.

Accordingly, the analysis is organized around the mechanical compatibility of the system rather than around the force ratio alone. Horizontal support deflection and the finite Hertz normal state are first used to reconstruct the sphere-center trajectory. Sphere rotation and an effective tangential accommodation are then introduced to complete the tangential kinematics. Combining these elements produces a finite-state relation that links the measured reaction direction to geometry, structural compliance, local contact stiffness, rotational restraint, and the evolving tangential state. The resulting framework is used to interpret why the external reaction direction can remain nearly load-stable while the internal contact and displacement fields continue to change.

2 Experimental Method and Mechanical Formulation

2.1 Experimental setup and step loading

The experimental configuration is shown in Fig. 1. A stainless-steel sphere of radius $R = 4$ mm is supported by a square copper rod with a $5 \text{ mm} \times 5 \text{ mm}$ cross-section and an effective length of 30 mm. The rod axis is normal to the analyzed x – z plane; within this plane, the rod primarily provides lateral bending compliance and also supplies a finite torsional constraint on sphere rotation. The plane inclination γ is measured from the horizontal and was set to 5° , 10° , 15° , 20° , 25° , and 30° .

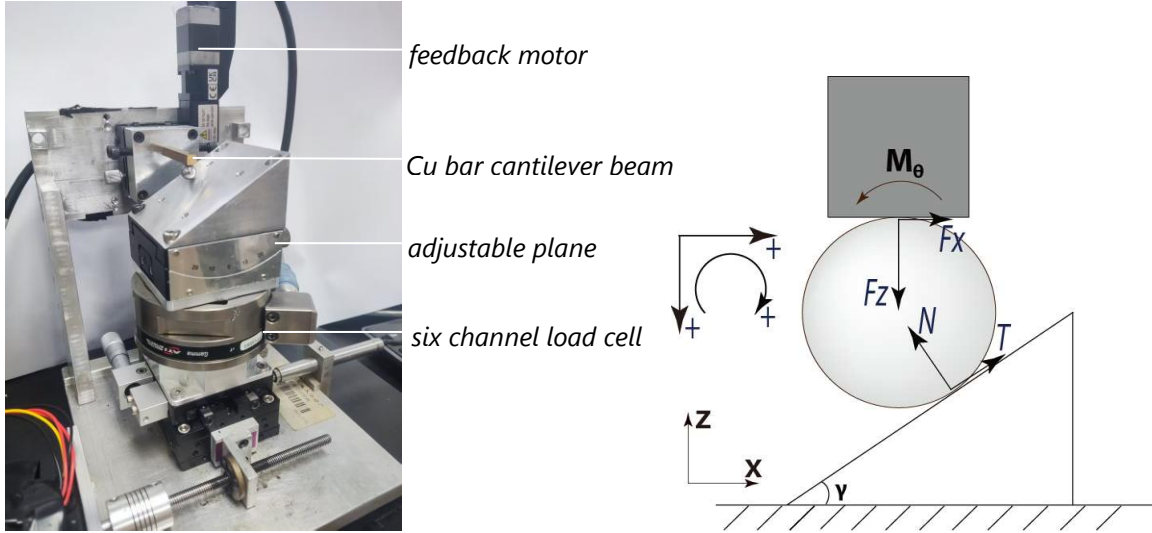


Fig. 1 Experimental setup and force definitions for the sphere–inclined-plane step-loading tests. The load cell measures the laboratory-frame reactions F_x and F_z ; T and N denote the tangential and normal reactions in the local plane coordinate system

At each inclination, the load was increased in discrete steps. After each increment, the loading stage was held until the measured reactions reached a steady plateau; the next increment was then applied. A time-averaged value from the stable portion of each plateau was treated as an independent quasi-static state. The six inclination groups contained 12, 12, 12, 13, 13, and 14 valid plateaus, respectively, for a total of 76 states.

The normal and tangential forces were simultaneously measured using a high-precision six channel force sensor (model: Net F/T SI-32-2.5, manufacturer: ATI Industrial Automation). The sensor was configured to measure the normal force F_z and tangential force F_x during the inclined-plane loading and unloading process. The measurement ranges for the normal and tangential channels were 100 N and 32 N, respectively, with resolutions of $1/80$ N and $1/160$ N. The tangential and normal reaction forces at the contact interface in the local plane frame are denoted by T and N , respectively, and are related by the coordinate transformation

$$F_x = T \cos \gamma - N \sin \gamma. \quad (1a)$$

$$F_z = T \sin \gamma + N \cos \gamma. \quad (1b)$$

The macroscopic tangential-to-normal reaction ratio is defined as

$$q \equiv \frac{T}{N}. \quad (2)$$

2.2 Horizontal structural compliance and the finite Hertz normal state

The first-order bending stiffness of the square copper support in the horizontal x direction is denoted by K_x . For the present geometry and material properties, $K_x = 636.57 \text{ N mm}^{-1}$. The horizontal displacement of the sphere center is therefore written as

$$x = \frac{F_x}{K_x}. \quad (3)$$

The local normal approach of the sphere–plane contact is described by the finite-state Hertz relation. The contact radius is [1, 2]

$$a = \left(\frac{3NR}{4E^*} \right)^{1/3}, \quad (4)$$

and the normal indentation is

$$u_n = \left(\frac{3N}{4E^*\sqrt{R}} \right)^{2/3}. \quad (5)$$

A finite-state normal secant stiffness is defined as

$$\bar{c}_n \equiv \frac{N}{u_n} = \frac{4}{3} E^* a. \quad (6)$$

The secant stiffness $\bar{c}_n$ must be distinguished from the Hertz normal tangent stiffness $c_n^{\tan} = 2E^*a$. The former represents the current finite state through N/u_n , whereas the latter is the incremental slope about that state. All finite-displacement closures used below employ $\bar{c}_n$.

2.3 Finite-state relation for the sphere-center trajectory

Let the sphere center undergo a resultant displacement s , whose direction makes an angle β with the inclined plane. The geometry gives

$$u_n = s \sin \beta, \quad x = s \cos(\beta + \gamma). \quad (7)$$

Combining Eqs. (1a)–(1b) and Eqs. (2)–(7) gives the explicit finite-state trajectory relation (Appendix A)

$$\cot \beta = \tan \gamma + \rho(q - \tan \gamma), \quad (8)$$

where

$$\rho \equiv \frac{\bar{c}_n}{K_x}. \quad (9)$$

ρ measures the current finite-state normal contact stiffness relative to the horizontal structural stiffness. Because $a \propto N^{1/3}$, it follows that $\rho \propto N^{1/3}$. Thus, even when q is nearly constant at a fixed inclination, the trajectory angle β can evolve systematically with load.

2.4 Sphere rotation and effective tangential accommodation

Figure 2 defines the kinematic variables used in the remaining formulation. The sphere center moves from O_0 to O_1 ; the geometric tangential component is $s \cos \beta$, and the normal approach is $s \sin \beta = u_n$. Sphere rotation contributes a tangential displacement $R\theta$ at the contact, while the remaining tangential displacement not explicitly represented by the ideal elastic response and the known rotation is denoted by d .

2.5 Finite-state coupled formulation

Using Eqs. (13)–(15) together with $N = \bar{c}_n u_n$, the reaction ratio can be written as

$$q = \kappa \left(\cot\beta - \eta - \frac{R\theta}{u_n} \right). \quad (16)$$

Equation (11), together with Eqs. (1a) and (2), is then used to eliminate θ . For compactness, define

$$D_\gamma \equiv 1 - \chi + \chi \cos\gamma. \quad (17a)$$

The rotational balance can then be written as

$$q D_\gamma = \kappa (\cot\beta - \eta) + \chi \sin\gamma. \quad (17b)$$

For a specified sphere-center trajectory and internal accommodation state, the reaction ratio becomes

$$q = \frac{\kappa (\cot\beta - \eta) + \chi \sin\gamma}{D_\gamma}. \quad (18)$$

Finally, substituting Eq. (8) eliminates β and yields the finite-state relation (Appendix B)

$$q = \frac{\kappa [(1 - \rho) \tan\gamma - \eta] + \chi \sin\gamma}{1 - \chi(1 - \cos\gamma) - \kappa \rho}. \quad (19)$$

Equation (19) combines the principal mechanical ingredients of the constrained contact in a single finite-state relation. Its terms represent the plane inclination γ , the translational stiffness ratio ρ , the local contact-stiffness ratio κ , the rotational coupling χ , and the normalized tangential accommodation η . The relation therefore connects the measured reaction direction to quantities associated with geometry, external structural compliance, local contact mechanics, sphere rotation, and loading-dependent tangential accommodation.

With the Hertz–Mindlin reference,

$$\kappa = \frac{c_t}{\bar{c}_n} = \frac{6G^*}{E^*}, \quad (20)$$

so that κ is independent of load for a fixed material pair, whereas

$$\rho \propto N^{1/3}, \quad \chi \propto N^{1/3}. \quad (21)$$

Equation (19) is a finite-state relation, not a fully closed forward constitutive law. It identifies the mechanical variables that participate in the current state, but the internal accommodation η must be supplied by an independent evolution law or displacement measurement before the equation can predict the reaction ratio from load and material parameters alone. For this reason, the results below use the reconstructed variables to characterize state evolution and avoid treating algebraic back-substitution of quantities derived from the same force measurements as an independent validation.

3 Results

3.1 Load stability and angle dependence of the reaction ratio

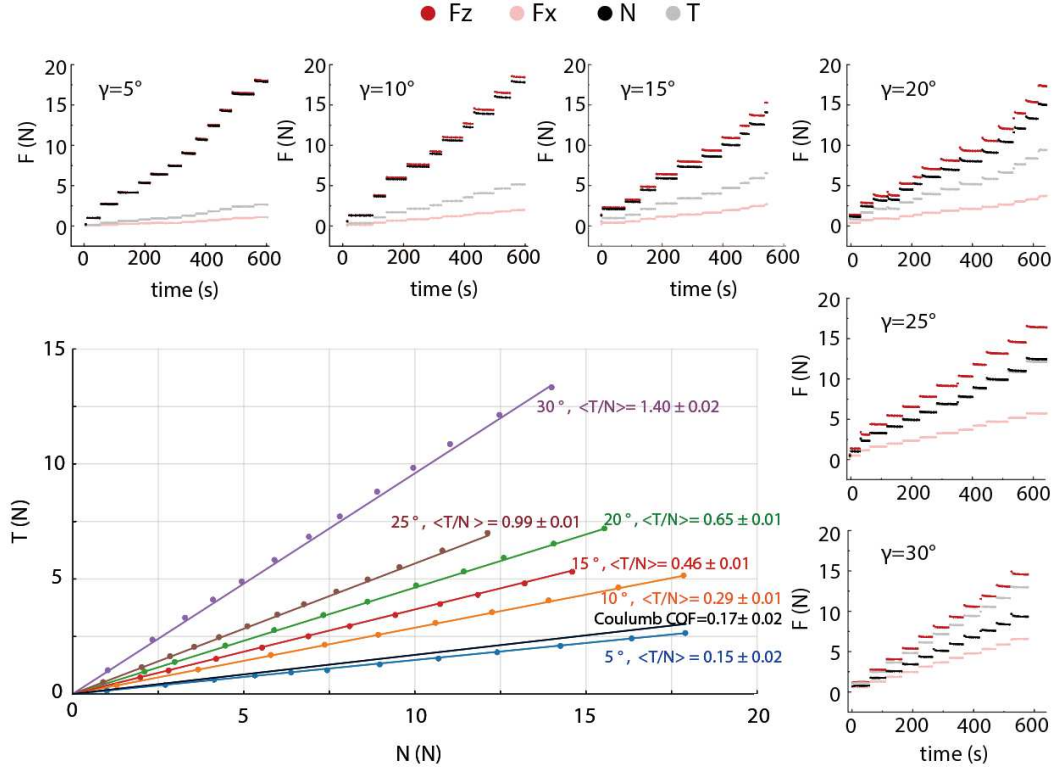


Fig. 3 Raw step-loading reactions and steady T - N relationships for the six plane inclinations. The upper and right panels show the stepwise F_z , F_x , N , and T signals; the main panel shows the steady T - N states and the mean T/N level for each inclination. A black line shows the Coulomb coefficient of friction using gross-sliding experiment at $\gamma = 0$.

Figure 3 shows the raw stepwise responses and the corresponding steady T - N states. The relationship between T and N is highly linear for all six inclinations, with regression R^2 values of 0.99956, 0.99990, 0.99989, 0.99981, 0.99982, and 0.99998. The plateau-wise q values are plotted against N in Fig. 4. At each fixed inclination, they form a narrow band, with means of 0.147, 0.289, 0.467, 0.643, 0.986, and 1.397 and corresponding standard deviations of approximately 0.002, 0.003, 0.004, 0.030, 0.012, and 0.024.

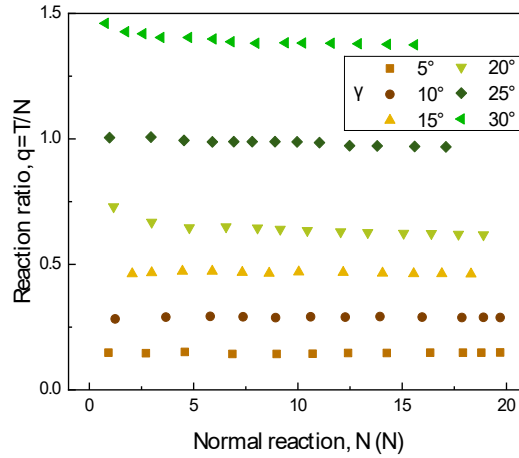


Fig. 4 Tangential-to-normal reaction ratio $q = T/N$ as a function of normal reaction N . At each fixed inclination, q remains within a narrow band, whereas its level changes strongly with inclination

These measurements establish two macroscopic features that the mechanical description must accommodate simultaneously: the resultant reaction ratio is nearly insensitive to load at fixed geometry, while a change in plane inclination produces a pronounced shift in T/N .

3.2 Contact stiffening and continuous rotation of the sphere-center trajectory

The evolution of the sphere-center trajectory reconstructed from Eq. (8) is presented in Fig. 5. In contrast to the nearly load-independent reaction ratio q , the trajectory angle β exhibits a continuous decrease with increasing normal reaction for all six inclination angles. At $\gamma = 5^\circ$, β decreases from 70.49° at the lowest load to 49.89° at the highest load. Similar reductions are observed for $\gamma = 10^\circ$ and 15° , where β decreases from 55.17° to 32.88° and from 35.67° to 21.10° , respectively. For the higher inclination groups, the terminal β values reach 16.22° , 8.96° , and 5.93° for $\gamma = 20^\circ$, 25° , and 30° , respectively. The group-averaged trajectory angles are 56.96° , 38.61° , 25.82° , 18.95° , 11.39° , and 8.00° for the six inclination angles. The continuous reduction of β indicates that the sphere-center displacement progressively rotates toward the plane-tangential direction during loading. This behavior is further illustrated by the reconstructed trajectories in the fixed x - z coordinate system (Fig. 5b). Although all trajectories originate from the same initial reference position, their evolution with increasing load deviates from a single straight-line path. At each inclination angle, the trajectory gradually changes orientation, demonstrating that the sphere-center motion is not governed by a constant geometric direction but evolves with the changing contact state.

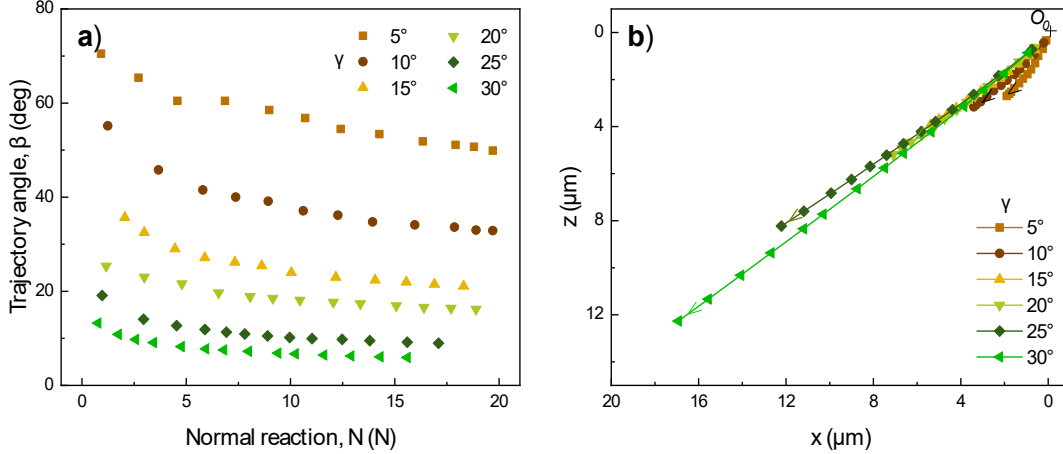


Fig. 5 Evolution of the sphere-center trajectory angle β and reconstructed sphere-center trajectories in the fixed x - z coordinate system. (a) The trajectory angle β decreases continuously with increasing normal reaction N for all inclination angles, indicating progressive rotation of the displacement direction toward the plane-tangential direction. (b) The reconstructed trajectories show that the sphere-center motion evolves continuously during loading rather than following a fixed displacement direction.

3.3 Evolution of the translational stiffness ratio ρ , the rotational coupling χ , and the normalized tangential accommodation η .

The dimensionless state variables involved in the finite-state reaction relation are summarized in Fig. 6. Across the 76 steady states, ρ increases continuously from approximately 4.16 to 12.27 with increasing normal reaction. This indicates that the Hertzian contact progressively stiffens relative to the external support during loading. Consequently, the sphere-center displacement becomes increasingly governed by the constraint imposed by the contact geometry rather than by the deformation of the supporting structure alone, resulting in the continuous trajectory rotation observed in Fig. 5. Similar to ρ , χ increases monotonically with normal reaction, ranging from approximately 0.40 to 1.18. This evolution reflects the increasing role of tangential contact elasticity in transmitting displacement through the rotational degree of freedom. Importantly, both ρ and χ are determined by the evolving contact state rather than prescribed constants, because the Hertz–Mindlin stiffnesses increase with contact size. The normalized tangential accommodation $\eta = d/u_n$ characterizes the remaining tangential displacement required to satisfy kinematic compatibility after accounting for the ideal elastic reference and the explicitly modeled sphere rotation. Unlike ρ and χ , which are primarily controlled by contact stiffening, η exhibits a pronounced dependence on inclination angle. At larger inclinations, the smaller normal approach $u_n = s \sin \beta$ amplifies the normalized value of the accommodation variable. Therefore, the increase of η does not imply a proportionally larger absolute tangential displacement alone, but rather reflects the changing balance between tangential accommodation and normal approach.

Together, the three dimensionless quantities reveal that the apparently simple proportionality between T and N occurs while multiple internal mechanical states evolve simultaneously. The reaction direction is therefore not associated with a fixed contact configuration, but emerges from the coupled evolution of structural compliance, contact stiffness, rotational constraint, and tangential accommodation.

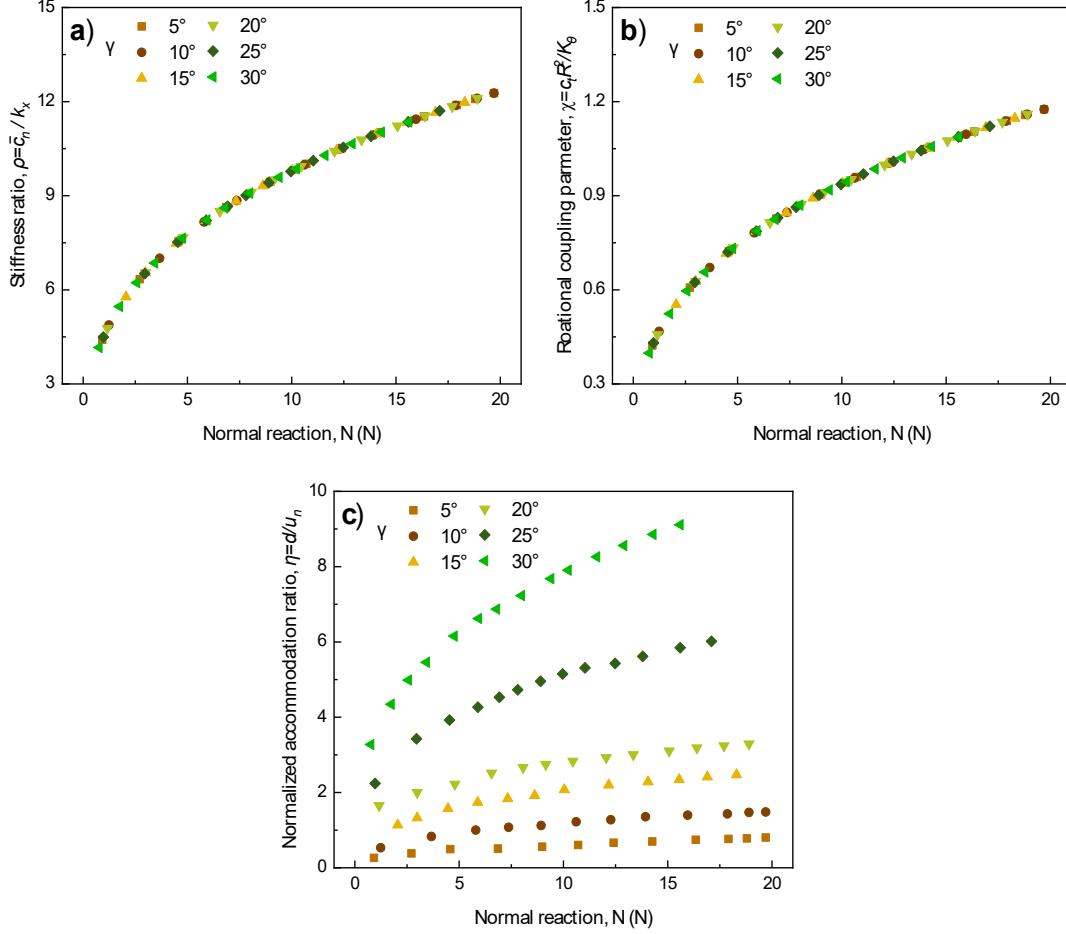


Fig. 6 Dimensionless translational stiffness ratio ρ , the rotational coupling χ , and the normalized tangential accommodation η as a function of normal reaction N

3.4 Mechanical origin of the load-stable reaction direction

The nearly constant reaction ratio at a fixed inclination is a consequence of mechanical compatibility among several evolving components rather than the invariance of an individual state variable. Equation (19) provides a compact description of this coupling by linking the reaction direction to inclination geometry, structural stiffness competition, contact stiffness evolution, rotational constraint, and tangential accommodation.

The inclination angle directly modifies the geometric projection terms in Eq. (19), thereby determining the overall level of the reaction ratio for different configurations. This explains why changing the plane inclination produces a substantial shift in the mean T/N , while the ratio remains relatively stable during loading at each fixed inclination. In contrast, increasing normal reaction changes the internal state variables in a coordinated manner. The growth of ρ rotates the sphere-center trajectory, the increase of χ modifies the contribution of rotational compatibility, and the evolution of η adjusts the remaining tangential displacement required by the coupled system. These changes occur simultaneously and compensate through the constraint relation, allowing the resultant reaction direction to remain within a narrow range.

Therefore, the observed load-stable T/N should not be interpreted as evidence of an unchanged contact state. Instead, it represents an emergent property of the constrained contact system, where structural deformation, contact stiffening,

rotation, and tangential accommodation continuously redistribute while maintaining a nearly fixed external force direction.

4 Discussion

4.1 Macroscopic force stability with an evolving contact state

The nearly linear relationship between tangential and normal reactions at each fixed inclination provides a simple macroscopic description of the constrained contact response. However, the underlying mechanics are more complex than the measured force ratio alone suggests. The stable reaction direction emerges while the contact system continuously changes through load-dependent contact stiffness, structural deformation, sphere rotation, and tangential accommodation. A key implication is that a load-stable reaction ratio should be regarded as a property of the constrained loading path rather than a direct indication of an unchanged contact condition. During loading, the contact radius, finite-state stiffnesses, sphere-center trajectory, and internal displacement partition evolve simultaneously. The measured proportionality between T and N therefore reflects the coupled mechanical balance established by the complete contact–structure system.

The effects of load and inclination are mechanically distinct. For a fixed material pair and support configuration, the local stiffness ratio κ remains essentially unchanged, whereas increasing load modifies the finite-state variables through contact evolution. In contrast, changing inclination directly alters the geometric contribution to the reaction state and modifies the compatibility requirements of the system. This difference explains why inclination produces a pronounced shift in the reaction ratio, while loading at a fixed inclination produces only limited variation.

4.2 Physical meaning and limits of effective tangential accommodation

The effective tangential accommodation d was introduced to represent the portion of geometric tangential displacement that cannot be explained by the Mindlin elastic reference and the explicitly modeled sphere rotation. It is therefore an internal compatibility variable rather than a direct measurement of local microslip distance.

The normalized quantity $\eta = d/u_n$ provides a convenient measure of the relative importance of this accommodation compared with the normal indentation scale. The increase of η with load, particularly at larger inclinations, indicates that the unresolved tangential contribution becomes increasingly significant in the overall kinematic balance. This evolution is consistent with the fact that the normal approach u_n becomes relatively smaller as the trajectory rotates toward the plane-tangential direction. The present analysis does not assign a unique microscopic origin to d . Contributions may include partial-slip-related compliance, asperity deformation and rearrangement, local interfacial adjustments, and additional structural effects not explicitly included in the first-order model.

4.3 Relation to earlier constrained-contact frameworks

The present unilateral contact is mechanically related to, but distinct from, the ball-in-groove and V-groove problems that motivated earlier constrained-contact analyses [9–11]. In a groove, the distribution of reactions among multiple contacts is statically underdetermined until contact compliance and loading history are specified. Removing the second wall eliminates that multi-contact force indeterminacy, but it does not eliminate path dependence. Instead, the closure problem shifts to compatibility among the direction-dependent support compliance, the nonlinear normal contact state, tangential contact elasticity, and the finite rotational degree of freedom of the supported sphere. The present framework therefore extends constrained-contact analysis from force partitioning among multiple contacts to compatibility among multiple deformation modes. The variables ρ , χ , and η describe different mechanical roles: ρ links the finite normal contact state with external structural compliance, χ describes the coupling between tangential contact stiffness and rotational restraint, and η represents the remaining tangential accommodation required by compatibility. More generally, the results show that macroscopic force ratios in compliant tribological systems can conceal substantial internal evolution. Similar considerations may apply to flexure-mounted probes, constrained contacts, and component-level tribometers where contact and fixture compliances are comparable.

4.4 Model scope and information required for forward prediction

The Hertz–Mindlin model is used here as a first-order elastic reference without additional empirical fitting functions. Hertz theory gives maximum contact pressures of approximately 0.32–0.93 GPa over the 76 plateaus, so local elastoplasticity, surface roughness, and the real contact structure may influence the absolute values at higher loads. The structural form of Eq. (19) does not require the Hertz analytical stiffness specifically: more accurate finite-state normal

stiffness $\bar{c}_n(N)$ and effective tangential stiffness $c_t(N)$, obtained for example from finite-element analysis, indentation, or direct displacement measurements, could replace the present expressions without altering the geometric-rotational closure.

The present force data provide a state decomposition but do not, by themselves, constitute a forward predictor when the internal accommodation is unknown. A predictive formulation requires independent information for η . Direct measurement of the sphere-center x - z motion would test the trajectory predicted by Eq. (8); optical or laser measurement of sphere rotation would test Eqs. (10)–(11); and local displacement-field or contact-area measurements could separate interfacial partial slip from structural compliance. Establishing an independent evolution law for $\eta(N, \gamma)$ would then convert Eq. (19) from a state description into a forward model of the reaction-ratio response.

5 Conclusions

A constrained sphere–inclined-plane contact was examined over 76 quasi-static steady plateaus at six plane inclinations. The results establish a finite-state picture in which a simple macroscopic reaction direction coexists with substantial evolution of contact and structural state. The principal findings are:

1. At each fixed inclination, T and N are highly linear, with the mean $q = T/N$ increasing from 0.147 at 5° to 1.397 at 30° . Thus, the macroscopic reaction direction is weakly load dependent within a given geometry but strongly sensitive to inclination.
2. The sphere-center trajectory is governed jointly by horizontal structural compliance and the finite normal contact state. The relation $\cot \beta = \tan \gamma + \rho(q - \tan \gamma)$ shows that an approximately constant q does not imply a constant β ; as ρ increases with load, the sphere-center path rotates progressively toward the plane-tangential direction.
3. Beyond sphere rotation and the Mindlin tangential elastic reference, an effective tangential accommodation d that increases with load is required to satisfy overall kinematic compatibility. The quantity d should be interpreted as an internal state variable rather than as a directly measured microslip distance.
4. The finite-state relation in Eq. (19), expressed through $\gamma, \rho, \kappa, \chi$, and η , shows that the macroscopic reaction ratio emerges from the combined effects of geometry, structural compliance, local contact stiffness, constrained rotation, and internal tangential accommodation. Its load stability at fixed inclination reflects coordinated evolution of these state variables rather than a frozen contact configuration.

Appendix A Derivation of the sphere-center trajectory relation

From Eq. (7),

$$\frac{x}{u_n} = \frac{\cos(\beta + \gamma)}{\sin \beta} = \cot \beta \cos \gamma - \sin \gamma. \quad (\text{A1})$$

On the other hand, using Eqs. (1a), (2)–(3), and $u_n = N/\bar{c}_n$,

$$\frac{x}{u_n} = \frac{\bar{c}_n F_x}{K_x N} = \rho(q \cos \gamma - \sin \gamma). \quad (\text{A2})$$

Equating Eqs. (A1) and (A2) and dividing by $\cos \gamma$ gives Eq. (8) in the main text.

Appendix B Derivation of the reaction-ratio state relation

From Eqs. (10)–(11),

$$\frac{R\theta}{u_n} = \frac{\bar{c}_n R^2}{K_\theta} [q(\cos \gamma - 1) - \sin \gamma]. \quad (\text{B1})$$

Substituting into Eq. (16) and using $\kappa \bar{c}_n = c_t$ and $\chi = c_t R^2/K_\theta$ gives Eq. (17b) after rearrangement. Equation (8) is then written as

$$\cot \beta = \rho q + (1 - \rho) \tan \gamma, \quad (\text{B2})$$

Substitution into Eq. (17b), followed by collection of all terms containing q using Eq. (17a), gives Eq. (19).

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