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Disorder-Assisted Reduced-State Quantum Metrology in an Inhomogeneous Kerr-Parametric Cavity under Intrinsic Decoherence

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Abstract

We study an inhomogeneous two-atom Kerr-parametric cavity under Milburn intrinsic decoherence, with particular attention to the distinction between calibrated and uncalibrated static disorder. The Hamiltonian is formulated in the pump rotating frame, uses the standard Kerr interaction $\chi a^{\dagger 2} a^2$, and allows independent emitter detunings and coupling variations. All main calculations use a converged photon cutoff $n_c = 9$. For each known disorder realization we evaluate the reduced atomic QFI and then average it, whereas for unknown disorder we first average the density matrices and evaluate the QFI of the mixed ensemble state. These quantities behave very differently: moderate disorder can increase the calibrated realization-averaged QFI, while the QFI accessible without knowledge of the realization decreases as disorder grows, in agreement with QFI convexity. Negativity and coherence are also analyzed, and convergence with both photon cutoff and number of disorder realizations is reported. The results identify a physically meaningful calibrated-disorder localization effect without claiming that unknown disorder creates metrological information.

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I. INTRODUCTION

Idealized cavity-QED models usually begin with identical emitters: equal transition frequencies, equal coupling strengths and perfectly symmetric preparation. This assumption is extremely useful because it exposes collective physics clearly, but real devices are never perfectly homogeneous. Atomic position, local strain, fabrication spread, field-profile variations and static frequency offsets all break the permutation symmetry that underlies the simplest Tavis-Cummings description. For quantum-information and sensing applications, the practical question is therefore not only how large a quantum resource can become in an ideal model, but how much of that resource survives when the couplings and transition frequencies are slightly different.

The need for such a robustness analysis is especially clear when several quantum indicators are considered together. Entanglement, discord, coherence and quantum Fisher information (QFI) quantify different properties of a state [1–16, 55–65]. Entanglement is often fragile, while discord-type correlations may remain finite after entanglement has been strongly reduced. QFI is operationally different again: it measures sensitivity to an estimated parameter and therefore depends on how that parameter is encoded in the state. A perturbation that reduces entanglement can, in some time windows, rearrange the spectrum and eigenvectors of a reduced state in a way that leaves QFI comparatively robust. Robustness should consequently be evaluated resource by resource rather than inferred from one curve.

The physical platform considered here is a multipartite atom-field system based on the Tavis-Cummings interaction [17–23]. Closely related Hamiltonians appear in atomic cavity QED, circuit QED and hybrid light-matter platforms [38–46]. The cavity provides coherent exchange between the emitters and a common field mode, while the collective structure creates bright and dark combinations of atomic excitations. Perfect symmetry makes these combinations particularly simple. Static inhomogeneity mixes them and changes the frequencies at which amplitudes interfere, so even weak mismatch can alter collapse-revival patterns and the information retained in the atomic subsystem.

We further include two nonlinear optical processes. A Kerr medium adds an intensity-dependent phase through a photon-number-squared term [32–37], whereas a degenerate parametric amplifier drives pair creation and annihilation [25–30]. Together they provide

a controllable nonlinear spectrum. The same nonlinearities that can enhance correlations also make the dynamics more sensitive to detuning and coupling variations because different excitation sectors acquire different phases. This raises a useful design question: can Kerr strength and parametric pumping be chosen so that the useful atomic information becomes less sensitive to static disorder?

Intrinsic decoherence is incorporated through the Milburn model [31], in addition to the static inhomogeneity. Unlike ordinary bath-induced damping, Milburn decoherence is generated by stochasticity in the system's own time evolution and suppresses off-diagonal energy-basis terms according to their energy separation. The combination of static disorder and intrinsic decoherence is therefore nontrivial. Disorder changes the energy gaps and eigenvectors, while the decoherence term acts directly on those modified gaps. A perturbation can thus accelerate the loss of some coherences while shifting other contributions into more slowly dephasing sectors.

Previous work on multipartite atom-field models has examined Kerr nonlinearity, thermal fields, Stark shifts, parametric amplification and intrinsic decoherence in several combinations [67–70]. These studies establish that quantum correlations, coherence, and QFI can respond differently to nonlinear strength, parametric amplification, and decoherence. The next natural step is to abandon exact emitter identity and ask whether the conclusions survive under experimentally plausible mismatch. This extension is more than a numerical complication: it removes a symmetry that strongly constrains the ideal dynamics and therefore tests whether the reported resource enhancement is structurally stable.

We consider two complementary forms of inhomogeneity. First, deterministic mismatch is used to separate the roles of unequal couplings and unequal transition frequencies. We write $g_j = g(1 + \delta_j)$ and $\omega_j = \omega_0 + \Delta_j$ and compare ideal, coupling-mismatched, detuning-mismatched and combined cases. Second, we draw the mismatch parameters from zero-mean distributions and perform ensemble averages. The latter step is important because a single disorder realization can accidentally produce a favorable interference pattern; an experimentally meaningful robustness statement should survive averaging and should be accompanied by a measure of sample-to-sample spread.

The QFI is evaluated with respect to the initial atomic angle using the same parameter-independent dynamical map for neighboring values of that angle. This guarantees the expected data-processing behavior and avoids interpreting numerical differentiation artifacts

as metrological enhancement [8–16]. Negativity and l_1 coherence are calculated from the reduced two-atom state [55–59]. Late-time averages are used when comparing disorder strengths because they distinguish persistent robustness from isolated transient peaks.

A key possibility explored in this work is disorder-assisted reduced-state metrology. This phrase is used deliberately and conservatively. We do not claim that disorder creates information globally or that increasing disorder is generally beneficial. Rather, after interaction with the cavity and subsequent tracing over the field, a moderate symmetry-breaking perturbation can redistribute information between field and atoms so that the atomic subsystem retains a larger time-averaged QFI over a finite interval than the perfectly symmetric reference. Such an effect must disappear or reverse when disorder becomes too strong, and it must be checked against entanglement, coherence and ensemble fluctuations before it can be regarded as useful.

The numerical results are organized around this robustness question. We first examine deterministic symmetry breaking in the time domain, then average over disorder realizations, and finally construct a Kerr-parametric map showing where the reduced atomic QFI is most resilient. QFI, negativity and coherence are discussed separately so that the physical role of coupling mismatch, detuning mismatch, nonlinear compensation and correlation redistribution remains clear. Each figure is discussed under its own heading.

The remainder of the paper is arranged as follows. Section 2 defines the inhomogeneous Kerr-parametric Hamiltonian and Milburn dynamics. Section 3 introduces the quantum-resource and robustness measures. Section 4 presents a figure-by-figure discussion of deterministic disorder, ensemble-averaged behavior, and the nonlinear robustness map. Section 5 places the results in relation to previous work and states the novelty and limitations. Section 6 gives the main conclusions and outlines extensions to larger ensembles, cavity loss and experimentally calibrated disorder distributions.

A further distinction is essential for interpreting disorder in a metrological setting. If a static disorder realization is known from calibration, one may compute the QFI of each realization and then average performance over devices. If the realization is unknown, however, the physically accessible state is the ensemble-averaged density operator and the relevant QFI must be evaluated only after this averaging. Because QFI is convex, these two procedures are not interchangeable. Both quantities are therefore reported, and their difference separates calibrated disorder-assisted localization from apparent enhancement caused by

averaging QFI values that are not simultaneously accessible.

II. INHOMOGENEOUS KERR-PARAMETRIC MODEL

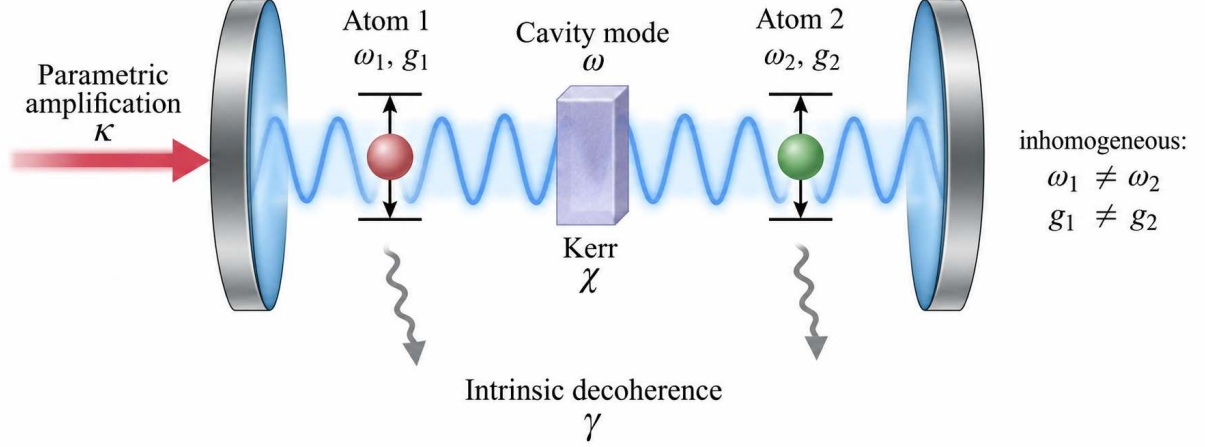


Figure 1. Inhomogeneous Kerr-parametric cavity in the pump rotating frame. The atoms can have different detunings Δ_j and couplings g_j , and the cavity contains a Kerr medium and a degenerate parametric amplifier. Intrinsic decoherence acts on each static realization.

A. Rotating-frame Hamiltonian

The ideal Tavis-Cummings interaction [18, 19] is generalized by emitter-dependent parameters. The nonlinear cavity terms follow standard Kerr and parametric-amplifier models [22, 26], and the intrinsic decoherence follows Milburn [31]. In the frame rotating at half the pump frequency, the static Hamiltonian of one disorder realization r is

$$H_r = \Delta_c a^\dagger a + \sum_{j=1}^N \frac{\Delta_j^{(r)}}{2} \sigma_z^{(j)} + \sum_{j=1}^N g_j^{(r)} \left(a \sigma_+^{(j)} + a^\dagger \sigma_-^{(j)} \right) + \chi a^{\dagger 2} a^2 + \frac{i\kappa}{2} (a^{\dagger 2} - a^2). \quad (1)$$

Here Δ_c is the cavity detuning from half the pump frequency, Δ_j is the corresponding atomic detuning, g_j is the atom-cavity coupling, and κ is the constant two-photon pump amplitude. We use $\Delta_c=0$ in the numerical results. The Kerr term is $\chi a \hat{A}^2 a \hat{A}^2 = \chi n(n-1)$, which avoids hiding a linear χn shift inside the cavity frequency.

B. Controlled and random inhomogeneity

$$g_j = g(1 + \delta_j), \quad \Delta_j = \bar{\Delta} + \xi_j. \quad (2)$$

For deterministic comparisons we use moderate symmetry breaking: $g_1/g=1.10$, $g_2/g=0.90$ and/or $\Delta_1/g=+0.15$, $\Delta_2/g=-0.15$. Random disorder is sampled independently in coupling and detuning. To avoid unbounded Gaussian tails and sign reversals of g_j , the normal distributions are truncated at $|\delta_j| \leq 0.45$ and $|\xi_j/g| \leq 0.45$:

$$\delta_j \sim \mathcal{N}(0, \sigma_g^2) \text{ (truncated)}, \quad \xi_j/g \sim \mathcal{N}(0, \sigma_\Delta^2) \text{ (truncated)}. \quad (3)$$

When a single horizontal disorder coordinate σ is shown for compactness, $\sigma_g = \sigma_\Delta = \sigma$ is used; the text treats this only as a one-dimensional benchmark, not as a claim that coupling and frequency disorder have the same microscopic origin.

C. Initial state and Milburn solution

$$\rho_A(0) = (1-p)|\psi_\theta\rangle\langle\psi_\theta| + p|gg\rangle\langle gg|, \quad |\psi_\theta\rangle = \cos\theta|gg\rangle + \sin\theta|ee\rangle. \quad (4)$$

$$\rho(0) = \rho_A(0) \otimes |0\rangle\langle 0|. \quad (5)$$

For each static realization, the Milburn equation is

$$\dot{\rho}_r = -i[H_r, \rho_r] - \frac{\gamma}{2}[H_r, [H_r, \rho_r]]. \quad (6)$$

If $H_r|E_m\rangle = E_m|E_m\rangle$, the evolution is exact in that energy basis:

$$(\rho_r)_{mn}(t) = (\rho_r)_{mn}(0) \exp \left[-i(E_m - E_n)t - \frac{\gamma t}{2}(E_m - E_n)^2 \right]. \quad (7)$$

The reduced atomic state is $\rho_A(t) = \text{Tr}_F \rho_r(t)$. The derivative $\partial_t \rho$ is propagated by the same exact map using the analytic derivative of the initial state.

III. METROLOGICAL AND CORRELATION MEASURES

A. QFI for calibrated and uncalibrated disorder

$$F_\theta[\rho] = 2 \sum_{m,n: \lambda_m + \lambda_n > 0} \frac{|\langle m | \partial_\theta \rho | n \rangle|^2}{\lambda_m + \lambda_n}. \quad (8)$$

For calibrated static disorder, the realization r is assumed to be known or independently characterized. The device-averaged performance is then

$$\langle F_\theta \rangle_{\text{cal}} = \sum_r p_r F_\theta[\rho_A^{(r)}]. \quad (9)$$

If the realization is unknown, the accessible state is the ensemble mixture

$$\bar{\rho}_A = \sum_r p_r \rho_A^{(r)}, \quad F_\theta^{\text{uncal}} = F_\theta[\bar{\rho}_A]. \quad (10)$$

QFI convexity gives the inequality

$$F_\theta \left[\sum_r p_r \rho_A^{(r)} \right] \leq \sum_r p_r F_\theta[\rho_A^{(r)}]. \quad (11)$$

This convexity inequality is central to the interpretation below. Equality can occur in special cases, so no strict ordering is assumed. A rise in $\langle F_\theta \rangle_{\text{cal}}$ does not imply that unknown disorder improves accessible metrology.

B. Negativity, coherence, and averaging

$$\mathcal{N} = \frac{\|\rho_A^{T_2}\|_1 - 1}{2}, \quad C_{l_1} = \sum_{i \neq j} |(\rho_A)_{ij}|. \quad (12)$$

Late-time averages are taken over $15 \leq gt \leq 30$. In the Milburn equation used here, H has frequency units and γ has units of inverse frequency (time); therefore the dimensionless intrinsic-decoherence parameter is $\gamma g = 0.025$, not γ/g . For the principal one-dimensional disorder scan, $R = 500$ realizations are used at the seven displayed disorder values $\sigma = 0, 0.075, \dots, 0.45$. The disorder grid and ensemble size are independent numerical controls in the reproduction code. Error bars on the calibrated mean QFI are 95% confidence intervals

estimated as 1.96 times the standard error of the mean, $\text{SEM} = s_F / \sqrt{R}$, rather than the realization-to-realization standard deviation. This separates uncertainty in the estimated ensemble mean from the physical spread of individual devices. A separate R-convergence calculation is reported for $R=100, 250$ and 500 . The main Fock cutoff is $n_c=9$, after checking $n_c=3, 5, 7, 9$.

IV. RESULTS AND DISCUSSION

A. Discussion of Figure 2: Deterministic QFI

Moderate deterministic symmetry breaking shifts the timing and amplitude of QFI revivals without allowing the reduced atomic QFI to exceed its initial preparation value. The combined mismatch can temporarily localize a larger fraction of the initial information in the atoms than the perfectly symmetric case at selected times. This is a reduced-state redistribution effect caused by modified bright/dark-state mixing and altered atom-field exchange.

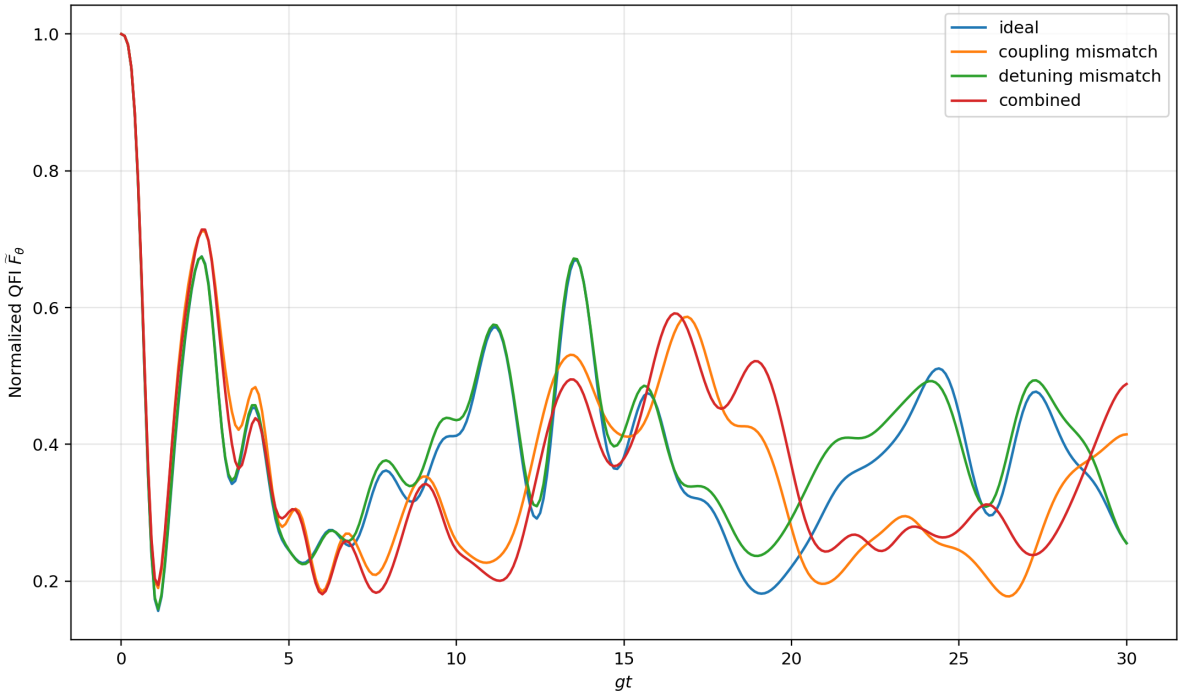


Figure 2. Atomic QFI for ideal coupling, coupling mismatch, detuning mismatch, and combined moderate mismatch.

B. Discussion of Figure 3: Entanglement under mismatch

Negativity is more sensitive to mismatch than QFI. Detuning and coupling asymmetry deform entanglement revivals and can shorten the intervals of nonzero negativity. A finite QFI in a low-negativity interval is not contradictory because the parameter may remain encoded in atomic coherences that do not produce a negative partial-transpose eigenvalue.

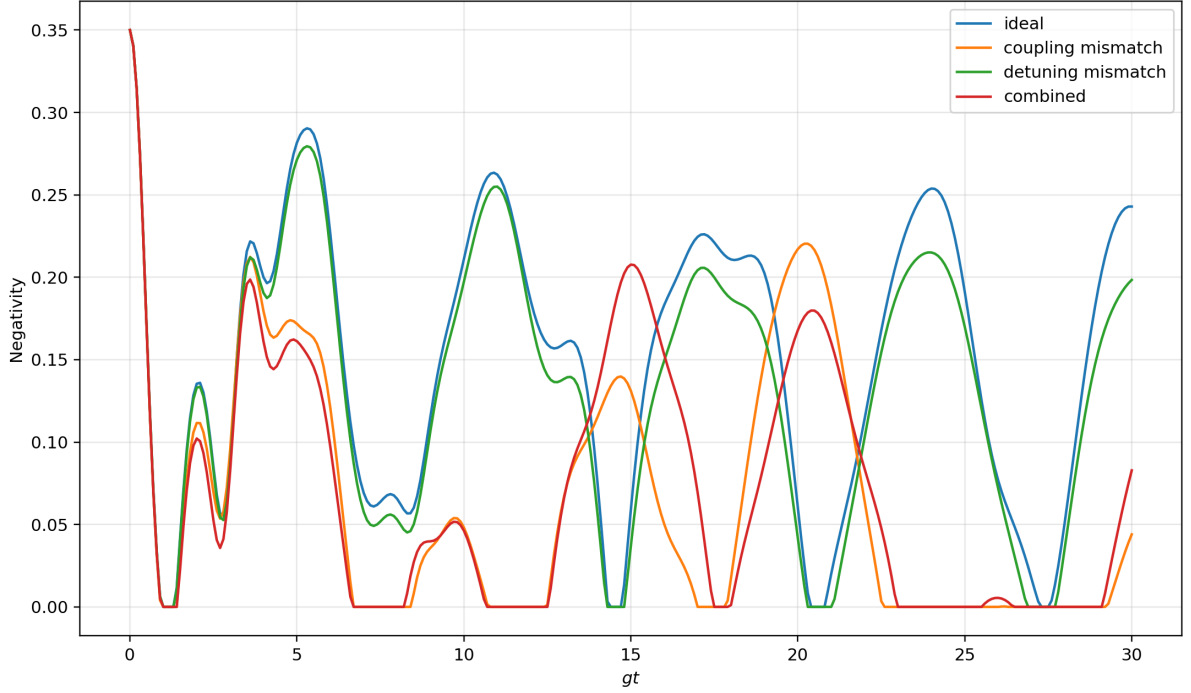


Figure 3. Two-atom negativity for the four deterministic symmetry cases.

C. Discussion of Figure 4: Atomic coherence

The coherence curves show that inhomogeneity first modifies the phase structure of the atomic state rather than simply destroying all off-diagonal elements. The comparison with Figs. 2 and 3 separates three resource notions: metrological sensitivity, entanglement, and basis coherence.

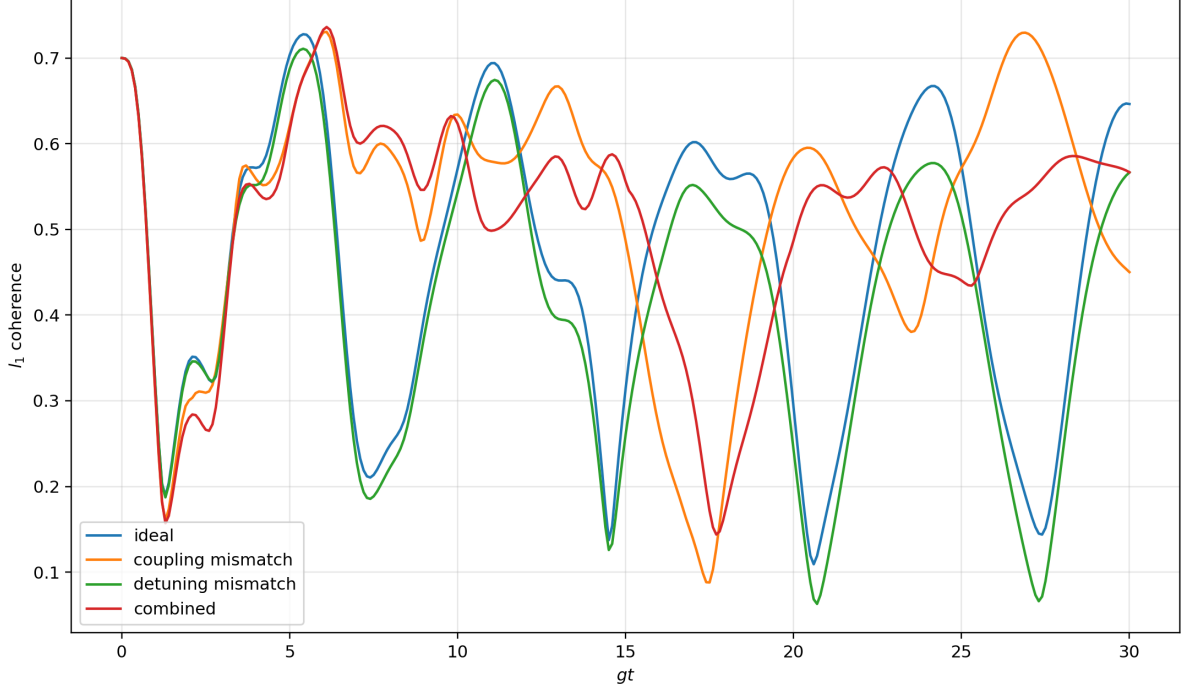


Figure 4. l_1 coherence of the reduced atomic state under deterministic mismatch.

D. Discussion of Figure 5: Calibrated versus uncalibrated disorder

This figure sharpens the interpretation of the disorder effect. With $R=500$, the clean calibrated mean is 0.8300. The calibrated realization-averaged QFI first decreases and then rises above the clean reference, reaching 0.8788 at $\sigma=0.375$ with an approximate 95% confidence interval of ± 0.0097 (1.96 SEM). The enhancement relative to the clean value is therefore larger than the uncertainty of the estimated ensemble mean for this benchmark. In contrast, the uncalibrated QFI falls strongly with σ because averaging unknown static realizations removes parameter-sensitive phase information. The two trends are fully consistent with QFI convexity. Consequently, disorder-assisted metrology is justified only for calibrated or realization-resolved devices; unknown disorder is detrimental in the parameter range studied.

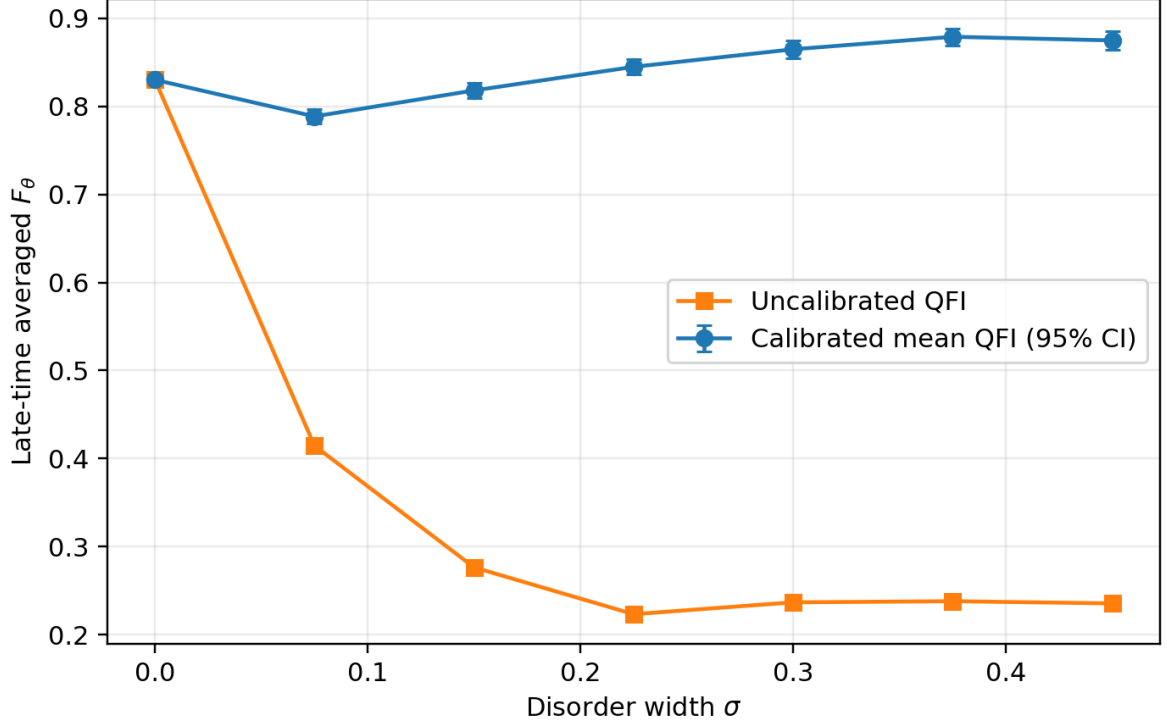


Figure 5. Late-time calibrated mean QFI with approximate 95% confidence intervals (± 1.96 SEM) and uncalibrated QFI of the ensemble-averaged state versus disorder width σ . The calibrated statistics use $R=500$ realizations. The clean calibrated mean is 0.8300 , while the maximum shown is 0.8788 ± 0.0097 at $\sigma=0.375$.

E. Discussion of Figure 6: Kerr-parametric robustness map

The map uses the physically stricter uncalibrated QFI, not the mean of realization-by-realization QFIs. It is recomputed for $\sigma_g = \sigma \Delta = 0.30$ using truncated Gaussian disorder, $R=200$ common disorder realizations at every grid point, $n_c=9$, $\gamma g=0.025$, $p=0.30$, $\theta=\pi/4$, $\Delta_c=0$, and the late-time window $15 \leq t \leq 30$. It therefore identifies nonlinear parameter regions that retain sensitivity even when the exact disorder realization is not available. The dominant trend in the map is a smooth robustness gradient across the Kerr-parametric plane, with smaller local structure superimposed by the competition between Kerr-induced number-dependent phases and parametric pair creation. We therefore interpret Fig. 6 as a robustness landscape rather than claiming a generic non-monotonic response.

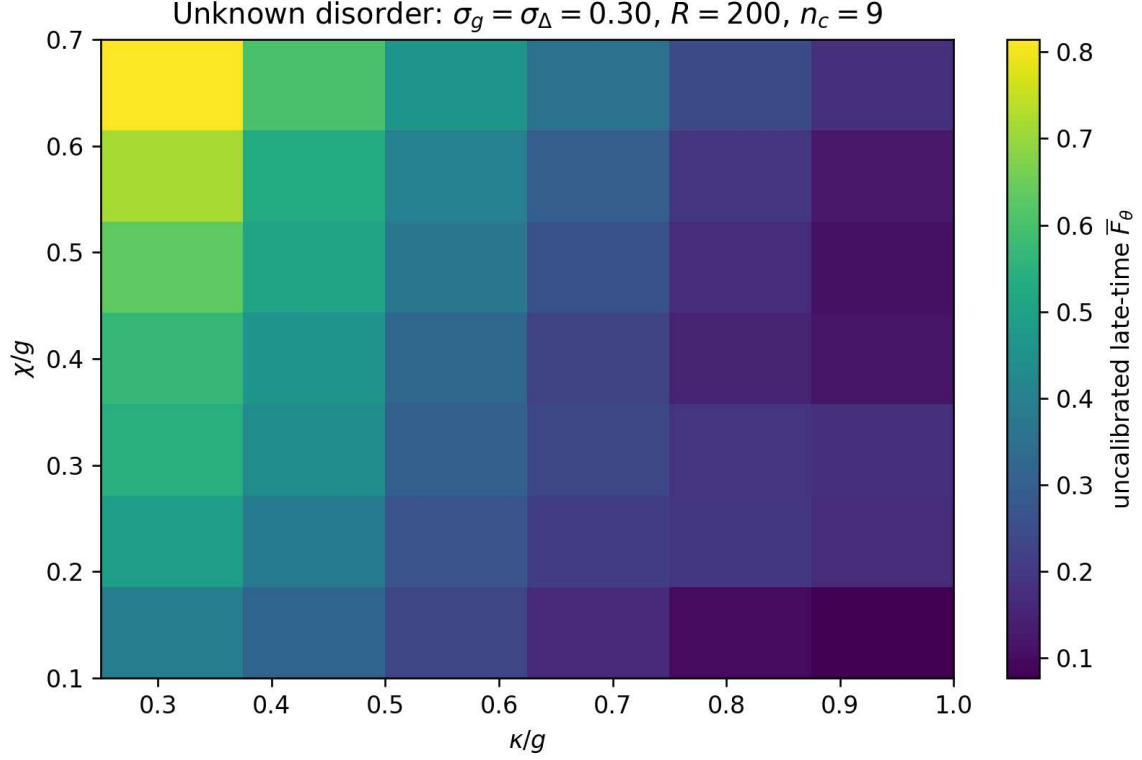


Figure 6. Late-time uncalibrated QFI of the ensemble-averaged state in the Kerr-parametric plane for $\sigma_g = \sigma_\Delta = 0.30$ (truncated at 0.45), $R=200$ common disorder realizations, $n_c=9$, $\gamma g=0.025$, $p=0.30$, $\theta=\pi/4$, $\Delta_c=0$, and averaging window $15 \leq g t \leq 30$.

F. Discussion of Figure 7: Photon-cutoff convergence

For this convergence test we plot the normalized QFI $\tilde{F}_\theta(t) = F_\theta(t)/F_\theta(0)$, so every cutoff curve starts exactly from unity. The $n_c=3$ curve differs visibly from the larger-basis results, demonstrating that a very small field basis is inadequate when a two-photon pump is present. The $n_c=7$ and $n_c=9$ results are much closer, so $n_c=9$ is used throughout the principal calculations. The cutoff is therefore treated solely as a numerical convergence parameter.

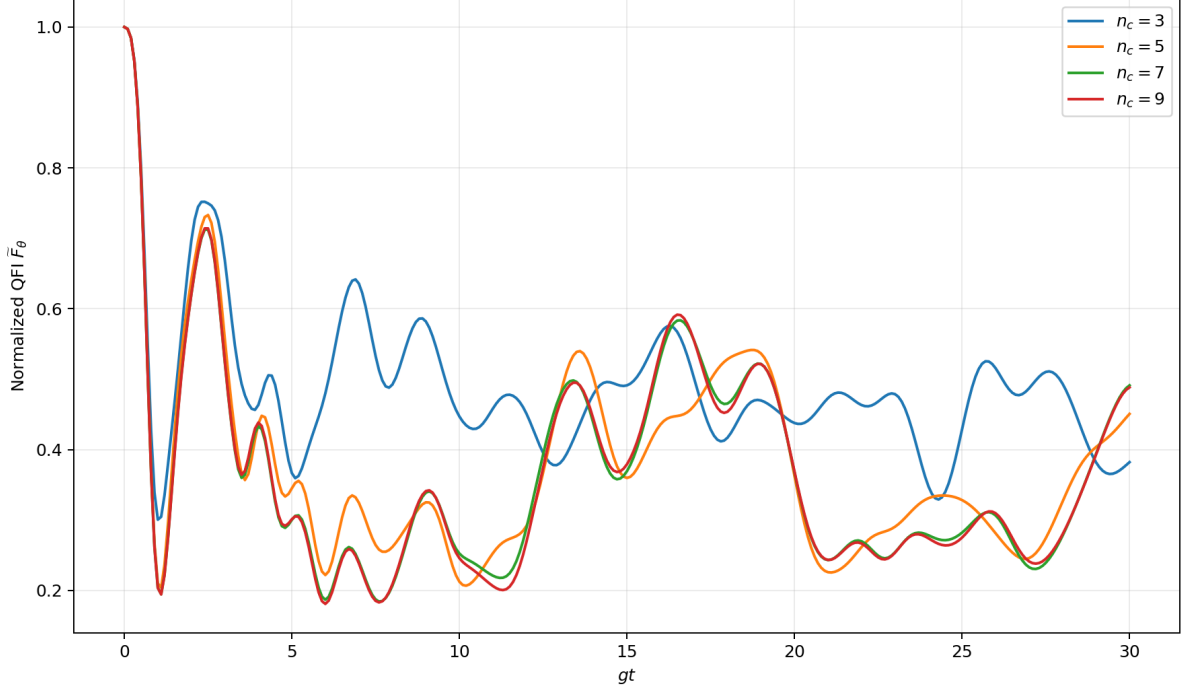


Figure 7. Normalized QFI convergence for the combined deterministic mismatch at $n_c=3,5,7,9$. All curves satisfy $\tilde{F}_\theta(0) = 1$.

G. Discussion of Figure 8: Disorder-sample convergence

Both calibrated and uncalibrated quantities are recomputed as R is increased from 100 to 500. The calibrated average changes only weakly, whereas the uncalibrated QFI is more sensitive because it is evaluated after density-matrix averaging. At the benchmark $\sigma=0.30$, the calculated late-time values approach a stable trend over $R=100, 250$ and 500 . The principal disorder scan therefore uses $R=500$. This convergence test reduces the risk that a finite-sample fluctuation is misidentified as a disorder-assisted effect.

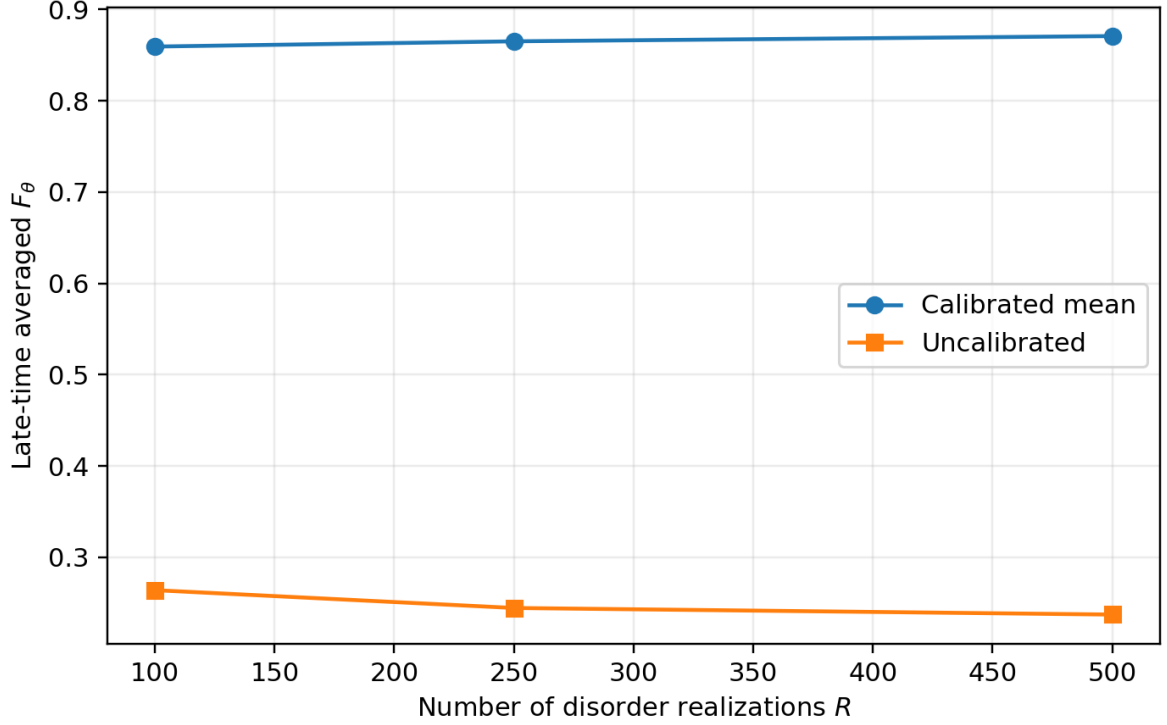


Figure 8. Convergence of calibrated and uncalibrated late-time QFI with disorder-sample size $R=100, 250$ and 500 at the representative benchmark disorder.

V. PHYSICAL INTERPRETATION AND LIMITATIONS

The central result is a distinction between information localization in a calibrated device and information loss under unknown static disorder. A known mismatch can alter collective atom-field pathways and, after tracing over the cavity, leave more of the originally encoded preparation information in the atoms at selected times. No global information is created, and the reduced QFI remains bounded by its initial value. When the mismatch is not known and density matrices must be averaged first, parameter-sensitive phases are washed out and the accessible QFI decreases.

The model remains phenomenological with respect to intrinsic decoherence and omits ordinary cavity leakage, spontaneous emission, thermal photons, and time-dependent technical noise. The one-dimensional σ scan also sets $\sigma_g = \sigma_\Delta$ only for compactness. A more device-specific study should map independent σ_g and σ_Δ values using experimentally measured distributions.

VI. CONCLUSIONS

The inhomogeneous Kerr-parametric model is formulated in the pump rotating frame with the standard Kerr interaction $\chi \in \hat{A}^2 \otimes \hat{A}^2$, moderate deterministic mismatch, truncated random distributions, a converged photon basis, and an explicit distinction between calibrated and uncalibrated disorder. Using $R=500$ realizations, the calibrated mean increases from the clean value 0.8300 to 0.8788 ± 0.0097 at $\sigma=0.375$, while the QFI of the unknown-disorder ensemble state decreases strongly. The result therefore identifies a calibrated-device disorder-assisted reduced-state localization effect rather than a generic claim that uncontrolled disorder improves metrology. The field-basis and disorder-sampling convergence checks make this distinction reproducible.

Declarations

Author contributions. S. Jamal Anwar: conceptualization, methodology, investigation, numerical analysis, visualization, writing—original draft, and writing—review and editing. S. Abdel-Khalek: methodology, validation, interpretation, supervision, and writing—review and editing. Both authors read and approved the final manuscript.

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Data availability. No external data sets were generated or analyzed in this study. The results reported in this article are based on the theoretical model and numerical calculations described in the manuscript.

Code availability. The MATLAB codes used for the numerical calculations are available from the corresponding author upon reasonable request.

Competing interests. The authors declare that they have no competing interests or conflicts of interest.

Ethics approval. Not applicable.

Consent to participate. Not applicable.

Consent for publication. Not applicable.

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