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N. Unnikrishnan Nair

Cochin University of Science and Technology

Silpa Subhash

silpasubhash112@gmail.com

District Office

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Modelling with κ –deformed quantile functions

N. Unnikrishnan Nair¹ and Silpa Subhash²

¹*Department of Statistics, Cochin University of Science and Technology, Cochin-682022, Kerala, India.*

²*Economics and Statistics Department, District Office, Wayanad, Kalpetta-673122, Kerala, India.*

Abstract

In this article, we develop a general framework for constructing probability distributions arising from κ -deformed quantile functions and investigate the impact of the deformation parameter κ on the distributional and tail characteristics of well-known probability models. Motivated by the observation that κ -deformation can substantially modify the tail behaviour of a baseline distribution, we derive several κ -deformed quantile functions from selected classical distributions. The practical usefulness of the proposed class is illustrated through applications to real-life datasets, where parameter estimation and goodness-of-fit analyses are performed and the fitted models are compared with their classical counterparts.

Keywords: κ –deformed distribution, quantile function, survival function, density function, data analysis

1. Introduction

It is an established fact that the standard probability distributions like normal, exponential, gamma, Weibull, etc. fail to provide satisfactory fits to a large number of real data consequent on the divergence in tail behaviour between observed and expected distributions. A classic example of this phenomenon is the attempts to model income distributions over time and space, wherein a unique distribution satisfying income data could not be found, resulting in distributions like the Pareto, gamma, Weibull, Stoppa, Singh-Maddala, Dagum, etc and their generalizations being proposed by different authors. Similar is the case with modelling human mortality. To overcome such difficulties in locating the data-generating mechanisms, several kinds of generalizations and modifications have been proposed in literature to impart sufficient flexible models that can accommodate different shapes in the observed data. One recent attempt in the context is the κ –generalized distributions which stems from application of the κ –deformed mathematical models in physics that makes use of an additional parameter κ to provide more flexibility to the standard distributions.

Email address: unnikrishnannair4@gmail.com, silpasubhash112@gmail.com (N. Unnikrishnan Nair¹ and Silpa Subhash²)

²Corresponding author.

One popular model in the κ -deformed category introduced by [Kaniadakis \(2001\)](#) and [Kaniadakis and Scarfone \(2002\)](#) is the κ -Weibull distribution specified by the survival function

$$\bar{F}_\kappa(x) = \exp_\kappa[-\beta x^\alpha], \quad x > 0; \quad \alpha, \beta > 0, \quad \kappa \in [0, 1), \quad (1.1)$$

where $\exp_\kappa(\cdot)$ is the deformed exponential function defined as

$$\exp_\kappa[x] = (\sqrt{1 + \kappa^2 x^2} + \kappa x)^{\frac{1}{\kappa}}.$$

As a special cases of (1.1), we have the κ -exponential when $\alpha = 1$ and the κ -Rayleigh when $\alpha = 2$. Since $\exp_\kappa[x]$ tends to $\exp[x]$ as κ tends to zero, (1.1) reduces to the usual Weibull distribution as κ tends to zero and hence subsumes all its properties. In addition to several Weibull-like properties, the κ -Weibull model is such that its left tail is of the power distribution type, the right tail resembles that of the Pareto model, and the middle portion is similar to a moderately skewed distribution. These being the inherent nature of an empirical income distribution, several papers were published pointing out the application of (1.1) in income analysis. The κ -Weibull distribution was found to give an excellent fit to income data from the Greece LIS database ([Clementi \(2023\)](#)), household incomes in Germany, Italy, the United Kingdom and Australia ([Clementi and Gallegati \(2005\)](#), [Clementi et al. \(2008\)](#), [Clementi et al. \(2009\)](#), [Clementi et al. \(2010\)](#), [Clementi et al. \(2016\)](#) [Clementi and Gallegati \(2017\)](#)). Similar studies were also held by [Okamoto \(2013\)](#) and [Vallejos et al. \(2019\)](#), concerning income and wealth data. Attention of the deformed family has spread over to other types of distributions like κ -Gaussian([Wada and Suyari \(2006\)](#)), logistic and the generalised gamma ([Kaniadakis \(2021\)](#)) which subsumed the κ -deformed versions of exponential, Erlang, half-normal, gamma, chi-square, Nakagami, Maxwell, Rayleigh, and Weibull models. Along with these developments in distribution theory, the area of application spread to new branches like epidemiology, quantum statistics, astrophysics and cosmology, geophysics, machine learning, nuclear energy and other topics in economics and finance. We refer to [Wada and Suyari \(2006\)](#), [Carvalho et al. \(2008\)](#), [Trivellato \(2013\)](#), [Chen et al. \(2017\)](#), [Lourek and Meuloud \(2017\)](#), [Moretto et al. \(2017\)](#), [da Silva et al. \(2023\)](#), and [Baldi et al. \(2024\)](#). The report by [Hristopulos et al. \(2025\)](#) states that more than 260 papers by around 200 authors were published so far on the seminal work by [Kaniadakis \(2001\)](#) on κ -distributions is sufficient testimony to the growing importance of the topic.

Characterization of a probability distribution can be facilitated by either the distribution function or by the quantile function, and all the functions carried out by one of them can also be performed by the other. However, there are theoretical and practical situations where analysis using quantile functions becomes more desirable and therefore it is important of investigate the methodology based on quantile functions in all areas where distribution-based tools are relevant. Moreover, in all fields in which κ -deformed distributions have a role, the corresponding deformed quantile functions can provide alternative methodology and results that the distribution function approach may fail to deliver. This motivates an investigation of the κ -deformed quantile functions in the present work, which

could eventually lead to a more robust inference framework, easy generation of random samples for simulation experiments, more flexible models involving simple functional forms and powerful graphical methods. Our procedure in finding κ -deformed quantile functions consists of prescribing a method of construction based on a given distribution that may be existing or new. When generated from an existing distribution, we get a generalization that includes the baseline model. The basic properties of the models so generated are expressed in terms of the quantile function and hence provide alternative methodology in practical problems.

The paper consists of six sections. In the next section some mathematical results relating to deformed functions that are used in the later discussions are presented. The κ -deformed quantile functions are derived in Section 3. Further, in Section 4, several properties of the quantile-based deformed distributions are proposed. Finally, some applications of the models to real data are considered in Section 5.

2. Preliminaries

In this section, we present some mathematical results that are used in the sequel concerning deformed functions. A detailed exposition of the theoretical foundations is available in [Kaniadakis \(2013\)](#). The κ -deformed exponential function is defined for real x as

$$\exp_{\kappa}[x] = \left(\sqrt{1 + \kappa^2 x^2} + \kappa x \right)^{\frac{1}{\kappa}}, \quad 0 \leq \kappa \leq 1. \quad (2.1)$$

The function (2.1) reduces to the usual exponential function $\exp[x]$ as $\kappa \rightarrow 0$. Recalling the definition of the hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$\sinh^{-1} x = \log \left(\sqrt{1 + x^2} + x \right),$$

it is noted that

$$\exp_{\kappa}[x] = \exp \left[\frac{1}{\kappa} \sinh^{-1} \kappa x \right].$$

Further

$$\exp_{\kappa}[x] = \exp_{-\kappa}[x],$$

$\exp_{\kappa}[-\infty] = 0^+$, $\exp_{\kappa}[0] = 1$, $\exp_{\kappa}[\infty] = \infty$, $\exp_{\kappa}[x] \exp_{\kappa}[-x] = 1$, and

$$\{\exp_{\kappa}[x]\}^{\gamma} = \exp_{\frac{\kappa}{\gamma}}[\gamma x].$$

The κ -logarithmic function corresponding to $e_{\kappa}(\cdot)$ is defined as

$$\log_{\kappa}[\exp_{\kappa}[x]] = \exp_{\kappa}[\log_{\kappa} x] = x$$

from which we can find

$$\log_{\kappa}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} = \frac{\sinh(\kappa \log x)}{\kappa}.$$

We have,

$$\log_0(x) = \log x,$$

$\log_{-\kappa}(x) = \log_{\kappa}(x)$, $\log_{\kappa} 0 = -\infty$, $\log_{\kappa} 1 = 0$ and $\log_{\kappa} \infty = \infty$ together with $\log_{\kappa}(\frac{1}{x}) = -\log_{\kappa}(x)$ and $\log_{\kappa} x^{\gamma} = \gamma \log_{\kappa}(x)$. For the laws of composition of $x, y \in \mathcal{R}$ their sum is

$$x \oplus y = x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2}, \quad -1 < \kappa < 1,$$

forming an abelian group $(\mathcal{R}, \oplus)$ with identity element zero and inverse of x is $-x$ satisfying $x \oplus (-x) = -x \oplus x = 0$. Similarly the κ -product of x and y is

$$x \otimes y = \frac{1}{\kappa} \sinh \left(\frac{1}{\kappa} \sinh^{-1}(\kappa x) \cdot \sinh^{-1}(\kappa y) \right).$$

As before $(\mathcal{R}, \otimes)$ is an abelian group with identity l satisfying

$$x \oplus l = l \otimes x = x, \quad l = \kappa^{-1} \sinh x$$

and inverse element

$$x^{-1} = \kappa^{-1} \sinh \left(\frac{\kappa^2}{\sinh^{-1} \kappa x} \right).$$

The κ -derivative of a function $a(x)$ is

$$\begin{aligned} \frac{da(x)}{d_{\kappa}x} &= \lim_{z \rightarrow x} \frac{a(z) - a(x)}{z \ominus x}, \quad z \ominus x = z \oplus (-x) \\ &= \sqrt{1 + \kappa^2 x^2} \frac{da(x)}{d_{\kappa}x}. \end{aligned}$$

Finally, the κ -integral of $a(x)$ is

$$\int a(x) d_{\kappa}x = \int \frac{a(x) dx}{\sqrt{1 + \kappa^2 x^2}}.$$

Both the κ -derivative and κ -integral reduce to the usual ones as $\kappa \rightarrow 0$ and are governed by the rules of conventional differentiation and integration.

3. Deformed quantile functions

Consider a probability space $(\mathcal{X}, \mathbb{B}, P)$ in which $X : \mathcal{X} \rightarrow \mathbb{R}_+$ is a non-negative continuous random variable with survival function $\bar{F}$, distribution function F and probability density function f . Let $g_{\kappa}(x) = G(\kappa x)$, where G is a strictly increasing function and $\kappa \in [0, 1)$ is a parameter. Then the κ -deformed random variable X_{κ} is defined to have realisations

$$x_\kappa = \frac{1}{\kappa} \log \left(\sqrt{1 + g_\kappa^2(x)} + g_\kappa(x) \right) \quad (3.1)$$

where $g_\kappa(x)$ is called the generator of the deformation with the survival function of X_κ has the exponential version

$$\bar{F}_\kappa(x) = \left[\sqrt{1 + g_\kappa^2(x)} - g_\kappa(x) \right]^{\frac{1}{\kappa}} = \exp_\kappa[-g_\kappa(x)]. \quad (3.2)$$

As $\kappa \rightarrow 0$ we have

$$\bar{F}(x) = \exp[-g(x)]. \quad (3.3)$$

By giving specific functional forms for $g_\kappa(x) = G(\kappa x)$ we have different members of the κ -generalized family of distributions, which contains the distribution $\bar{F}(x)$ as $\kappa \rightarrow 0$. Thus when $g_\kappa(x) = \left(\frac{\kappa x}{\alpha}\right)^\beta$, we have κ -Weibull distribution which is a reparametrized version of (1.1)

$$\bar{F}_{\kappa W}(x) = \exp_\kappa \left[- \left(\frac{x}{\alpha} \right)^\beta \right] = \left[\sqrt{1 + \kappa^2 \left(\frac{x}{\alpha} \right)^{2\beta}} - \kappa \left(\frac{x}{\alpha} \right)^\beta \right]^{\frac{1}{\kappa}}. \quad (3.4)$$

As clear from the above, we made use of the κ -exponential function in (2.1), and so as $\kappa \rightarrow 0$ we have the usual Weibull law.

From the construction of κ -distributions as in (3.2), we note that $g_\kappa(x) \rightarrow -\log_\kappa \bar{F}_\kappa(x)$ where $\log_\kappa \bar{F}_\kappa(x)$ is the κ -logarithmic function mentioned in Section 2, satisfying

$$\exp_\kappa(\log_\kappa g_\kappa(x)) = \log_\kappa(\exp_\kappa[g_\kappa(x)]) = g_\kappa(x).$$

This means that

$$\left[\sqrt{1 + \kappa^2 \log_\kappa g_\kappa^2(x)} + \kappa \log_\kappa g_\kappa(x) \right]^{\frac{1}{\kappa}} = g_\kappa(x)$$

from the definition of $\exp_\kappa[x]$.

Solving

$$\log_\kappa g_\kappa(x) = \frac{g_\kappa^\kappa(x) - g_\kappa^{-\kappa}(x)}{2\kappa} \quad (3.5)$$

expressing the formula for converting κ -logarithm to ordinary variables.

The quantile function of the deformed variables X_κ is

$$Q_\kappa(u) = \inf[x | F_\kappa(x) \geq u], \quad 0 < u < 1,$$

which reduces to

$$Q_\kappa(u) = g_\kappa^{-1}(-\log_\kappa(1 - u)) \quad (3.6)$$

from (3.2) since $\bar{F}_\kappa(x)$ is strictly decreasing and continuous. Using (3.5)

$$\log_\kappa g_\kappa(x) = \frac{\exp(\kappa \log g_\kappa(x)) - \exp(-\kappa \log g_\kappa(x))}{2\kappa}$$

and the definition of $\sinh x$, we write

$$\log_{\kappa} g_{\kappa}(x) = \kappa^{-1} \sinh(\kappa \log g_{\kappa}(x)).$$

Thus the κ -quantile function of X admits the representation

$$Q_{\kappa}(u) = g_{\kappa}^{-1} \left(-\frac{1}{\kappa} \sinh(\kappa \log(1-u)) \right) \quad (3.7)$$

$$= g_{\kappa}^{-1} \left(\frac{(1-u)^{-\kappa} - (1-u)^{\kappa}}{2\kappa} \right). \quad (3.8)$$

Based on the above general formulas, some examples of κ -distributions in quantile form of a few well known baseline line models are discussed below. The method is to write the survival function in these cases in the form $\bar{F}(x) = \exp[-g(x)]$, evaluate $g^{-1}(x)$ and then derive the quantile function as $Q_{\kappa}(u) = g_{\kappa}^{-1}(-\log(1-u))$.

1. Weibull distribution

$$\bar{F}(x) = \exp[-\beta x^{\alpha}], x > 0; \beta > 0, \alpha > 0.$$

Then $g(x) = \beta x^{\alpha}$, $g^{-1}(x) = \left(\frac{x}{\beta}\right)^{\frac{1}{\alpha}}$ and so

$$Q_{W,\kappa}(u) = \left(-\frac{\log_{\kappa}(1-u)}{\beta} \right)^{\frac{1}{\alpha}}$$

as obtained in [Clementi et al. \(2008\)](#) by a different argument. Note that

$$\begin{aligned} Q_{W,\kappa}(u) &= \beta^{-\frac{1}{\alpha}} \left[\frac{(1-u)^{-\kappa} - (1-u)^{\kappa}}{2\kappa} \right]^{\frac{1}{\alpha}} \\ &= \frac{1}{(\kappa\beta)^{1/\alpha}} [\sinh(-\kappa \log(1-u))]^{1/\alpha} \end{aligned} \quad (3.9)$$

and when $\alpha = 1$, we have the κ -exponential law

$$Q_{E,\kappa}(u) = \frac{1}{\beta} \left[\frac{(1-u)^{-\kappa} - (1-u)^{\kappa}}{2\kappa} \right]$$

and when $\alpha = 2$, the κ -Rayleigh distribution is obtained. The descriptive properties of (3.9) using the distribution function, are available in [Clementi et al. \(2008\)](#) and the references therein. In Section 3, we have discussed the quantile-based characteristics.

2. Generalized Pareto model.

$$\bar{F}_{GP}(x) = \left(1 + \frac{ax}{b}\right)^{-\frac{a+1}{a}}, x > 0; a > -1, b > 0. \quad (3.10)$$

Notice that

$$g(x) = \frac{a+1}{a} \log \left(1 + \frac{ax}{b}\right) \text{ and } g^{-1}(x) = \frac{b}{a} \left(e^{\frac{ax}{a+1}} - 1\right).$$

Further (3.10) reduces to the Pareto II, with $\bar{F} = \left(1 + \frac{x}{\alpha}\right)^{-C}$, $x > 0$ and rescaled beta $\bar{F}(x) = \left(1 - \frac{x}{R}\right)^\alpha$, $0 \leq x \leq R$, $R > 0$, when $b = a\alpha$, $a = (C-1)^{-1}$ and $R = -ba$ and $d = -\frac{1}{C+1}$. The κ -quantile function is

$$\begin{aligned} Q_{GP,\kappa}(u) &= \log_\kappa \left[\exp \frac{b}{a} \left((1-u)^{-\frac{a}{a+1}} - 1 \right) \right] \\ &= \frac{\left\{ \exp \frac{b}{a} (1-u)^{-\frac{a}{a+1}} - 1 \right\}^\kappa - \left\{ \exp \frac{b}{a} (1-u)^{-\frac{a}{a+1}} - 1 \right\}^{-\kappa}}{2\kappa} \\ &= \frac{1}{\kappa} \sinh \left[k \frac{b}{a} (1-u)^{-\frac{a}{a+1}} - 1 \right]. \end{aligned} \quad (3.11)$$

For the Pareto II distribution mentioned above

$$\begin{aligned} Q_{PII,\kappa}(u) &= \frac{\{\exp[\alpha(1-u)^{-\frac{1}{C}} - 1]\}^\kappa - \{\exp[\alpha(1-u)^{-\frac{1}{C}} - 1]\}^{-\kappa}}{2\kappa} \\ &= \frac{1}{\kappa} \sinh[\kappa\{\alpha(1-u)^{-\frac{1}{C}} - 1\}]. \end{aligned}$$

The survival function is

$$\bar{F}_{PII,\kappa}(x) = \left[\sqrt{1 + C^2 \kappa^2 \left(\log \left(1 + \frac{x}{\alpha} \right) \right)^2} - \kappa C \log \left(1 + \frac{x}{\alpha} \right) \right]^{\frac{1}{\kappa}}. \quad (3.12)$$

Similarly for the beta distribution with survival function $\bar{F}(x) = \left(1 - \frac{x}{R}\right)^d$, we have the expressions,

$$\bar{F}_{B,\kappa}(x) = \left[\sqrt{1 + d^2 \kappa^2 \left(\log \left(1 - \frac{x}{R} \right) \right)^2} - \kappa d \log \left(1 - \frac{x}{R} \right) \right]^{\frac{1}{\kappa}} \quad (3.13)$$

and

$$Q_{B,\kappa} = \frac{1}{\kappa} \sinh \left[\kappa R (1 - (1-u)^{\frac{1}{d}}) \right]. \quad (3.14)$$

3. Power-Pareto distribution

This distribution is specified by the quantile function

$$Q_{PP}(u) = C \frac{u^{\lambda_1}}{(1-u)^{\lambda_2}}, \quad C > 0, \lambda_1 > 0, \lambda_2 > 0, \quad (3.15)$$

with one of λ_1 and λ_2 may be zero. It is derived as the product of the quantile functions of the Pareto I distribution $Q_{PI}(u) = (1-u)^{-\lambda_2}$ and the power law $Q_p(u) = u^{\lambda_1}$. The properties and applications are discussed in [Hankin and Lee \(2006\)](#) and [Nair et al. \(2013\)](#). There is no closed form distribution function for (3.15). We have the κ -quantile function as

$$\begin{aligned} Q_{PP,\kappa}(u) &= \log_{\kappa} \exp \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}} \\ &= \frac{\left(\exp \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}} \right)^{\kappa} - \left(\exp \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}} \right)^{-\kappa}}{2\kappa} \end{aligned} \quad (3.16)$$

$$= \frac{1}{\kappa} \sinh \left(\kappa \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}} \right). \quad (3.17)$$

In particular, the Pareto I deformed distribution is specified by

$$Q_{PI,\kappa} = -\frac{1}{2\kappa} \left[\exp(C(1-u)^{-\lambda_2}) \right]^{\kappa} - \left[\exp C(1-u)^{\lambda_2} \right]^{-\kappa} \quad (3.18)$$

$$= -\frac{1}{\kappa} \sinh(C\kappa(1-u)^{\lambda_2}) \quad (3.19)$$

or equivalently by

$$\begin{aligned} \bar{F}_{PI,\kappa} &= \exp_{\kappa} \left[-\frac{1}{\lambda_2} \log \frac{x}{C} \right] \\ &= \left[\sqrt{1 + \frac{1}{\lambda_2^2} \kappa^2 \left(\log \left(\frac{x}{C} \right) \right)^2} - \frac{1}{\lambda_2} \kappa \log \left(\frac{x}{C} \right) \right]^{\frac{1}{\kappa}}. \end{aligned} \quad (3.20)$$

Similarly for the power distribution we get

$$\begin{aligned} Q_{P,\kappa}(u) &= \frac{\left[\exp Cu^{\lambda_1} \right]^{\kappa} - \left[\exp Cu^{\lambda_1} \right]^{-\kappa}}{2\kappa} \\ &= \frac{1}{\kappa} \sinh [\kappa Cu^{\lambda_1}] \end{aligned} \quad (3.21)$$

and

$$\bar{F}_{P,\kappa}(x) = \left[\sqrt{1 + \kappa^2 \left(\log \left(1 - \left(\frac{x}{C} \right)^{\frac{1}{\lambda_1}} \right) \right)^2} + \kappa \log \left(1 - \left(\frac{x}{C} \right)^{\frac{1}{\lambda_1}} \right) \right]^{\frac{1}{\kappa}}. \quad (3.22)$$

4. Linear hazard quantile distribution

Arising in the reliability context, the linear hazard quantile distribution is characterized by the quantile function (Nair et al. (2013))

$$Q_{LH}(u) = (a(1+\theta))^{-1} \log \frac{1+\theta u}{1-u}, \quad \theta > -1, \quad a > 1. \quad (3.23)$$

The quantile function of κ -deformed model is

$$\begin{aligned} Q_{LH,\kappa}(u) &= \log_{\kappa} \left(\frac{1+\theta u}{1-u} \right)^{(a(1+\theta))^{-1}} \\ &= \frac{\left[\left(\frac{1+\theta u}{1-u} \right)^{\frac{\kappa}{a(1+\theta)}} - \left(\frac{1+\theta u}{1-u} \right)^{\frac{-\kappa}{a(1+\theta)}} \right]}{2\kappa} \\ &= \frac{1}{\kappa} \sinh \left[\kappa \log \left(\frac{1+\theta u}{1-u} \right)^{\frac{1}{a(1+\theta)}} \right]. \end{aligned}$$

The function (3.23) subsumes some important special cases in the form of the exponential when $\theta = 0$ as

$$Q_B(u) = -\frac{1}{a} \log(1-u),$$

the half-logistic

$$Q_{HL}(u) = \frac{1}{\sigma} \log \frac{1+u}{1-u}$$

when $\theta = 1$ and $a = \frac{1}{2\sigma}$. In the latter case,

$$\begin{aligned} Q_{HL,\kappa}(u) &= \log_{\kappa} \left(\frac{1+u}{1-u} \right)^{\frac{1}{\sigma}} \\ &= \frac{\left[\left(\frac{1+u}{1-u} \right)^{\frac{\kappa}{\sigma}} - \left(\frac{1+u}{1-u} \right)^{\frac{-\kappa}{\sigma}} \right]}{2\kappa} \\ &= \frac{1}{\kappa} \sinh \left(\kappa \log \left(\frac{1+u}{1-u} \right)^{\frac{1}{\sigma}} \right). \end{aligned}$$

The survival function of the κ -half logistic model is

$$\bar{F}_{LH,\kappa}(x) = \left[\sqrt{1 + \kappa^2 \left(\log \frac{1+e^{x/\sigma}}{2} \right)^2} - \kappa \log \frac{1+e^{x/\sigma}}{2} \right]^{\frac{1}{\kappa}}$$

and the density function is

$$f_{HL,\kappa}(x) = \frac{\bar{F}_{\kappa}(x)}{\sigma \sqrt{1 + \kappa^2 \left(\log \frac{1+e^{x/\sigma}}{2} \right)^2}} \frac{e^{\frac{x}{\sigma}}}{1 + e^{x/\sigma}}, \quad x > 0.$$

Another special case is the exponential geometric distribution obtained when $a = \lambda(1-p)^{-1}$, $\theta = -p$, $0 < p < 1$, $\lambda > 0$; as

$$Q_{EG}(u) = \frac{1}{\lambda} \log \frac{1-pu}{1-u}$$

as a model for lifetime with special properties discussed in [Adamidis and Loukas \(1998\)](#) with survival function

$$\bar{F}(x) = \frac{(1-p)e^{-\lambda x}}{1-pe^{-\lambda x}} \quad x > 0.$$

The κ -deformed version is specified by

$$\begin{aligned} Q_{EG,\kappa}(u) &= \log_{\kappa} \left(\frac{1-pu}{1-u} \right)^{\frac{1}{\lambda}} \\ &= \left[\left(\frac{1-pu}{1-u} \right)^{\frac{\kappa}{\lambda}} - \left(\frac{1-pu}{1-u} \right)^{\frac{-\kappa}{\lambda}} \right] / 2\kappa \\ &= \frac{1}{\kappa} \sinh \left[\kappa \log \left(\frac{1-pu}{1-u} \right)^{\frac{1}{\lambda}} \right] \end{aligned}$$

or by the survival function

$$\bar{F}_{EG,\kappa}(x) = \left[\sqrt{1 + \kappa^2 \left(\log \frac{(1-p)e^{-\lambda x}}{1-pe^{-\lambda x}} \right)^2} - \kappa \log \left(\frac{(1-p)e^{-\lambda x}}{1-pe^{-\lambda x}} \right) \right]^{\frac{1}{\kappa}}.$$

3.1. Tail comparison

An important distinction of the κ -deformed families from the baseline distributions, as pointed by various authors, is the difference in the tail behaviour consequent on the introduction of the parameter κ . We now examine this aspect in the case of some selected distributions. The distributions selected are Pareto I with a decreasing density, the power law which has increasing density and the Weibull with a density that is increasing first and then decreasing.

The relevant theory prescribed by [Parzen \(1979\)](#) which appears to be basic in this context, is used to classify distributions according to tail behaviour. Accordingly it is said that a distribution is short tailed if $\alpha < 1$, of medium type if $\alpha = 1$ and is long tailed if $\alpha > 1$, where

$$\alpha = \lim_{u \rightarrow 1} \frac{(1-u)J(u)}{f(Q(u))} \quad (3.24)$$

and $J(u) = \frac{f'(Q(u))}{f(Q(u))}$, f being the density function. When the distribution function F is continuous and strictly increasing $F(Q(u)) = u$ and hence differentiating we have

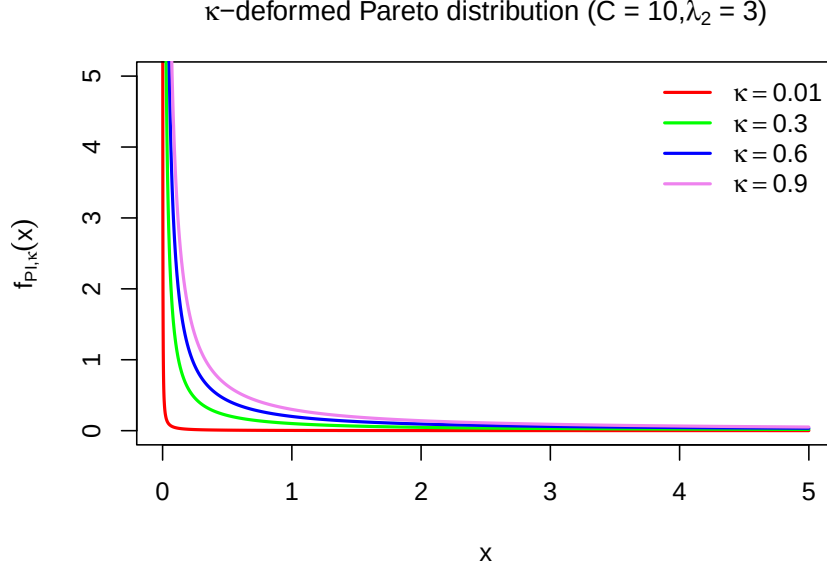


Figure 1: Density plot of κ -deformed Pareto I model.

$$f(Q(u))q(u) = 1$$

where $q(u) = Q'(u)$, with prime denoting differentiation. This relationship render (3.24) in the form

$$\alpha = \lim_{u \rightarrow 1} (1-u)q'(u) = \lim_{u \rightarrow 1} (1-u)Q''(u). \quad (3.25)$$

When X is Pareto, $Q_{PI}(u) = (1-u)^{-\lambda_2}$ from (3.15) and so,

$$\alpha = \lim_{u \rightarrow 1} \frac{\lambda_2(\lambda_2 + 1)}{(1-u)^{\lambda_2+1}} = \infty.$$

For the κ -Pareto,

$$\begin{aligned} Q_{PI, \kappa}(u) &= [\exp C(1-u)^{-\lambda_2 \kappa} - \exp C(1-u)^{\lambda_2 \kappa}] / 2\kappa \\ &= \frac{1}{\kappa} \sinh C(1-u)^{\lambda_2 \kappa} \end{aligned}$$

giving

$$\begin{aligned} \alpha &= \lim_{u \rightarrow 1} \frac{1}{\kappa} [\cosh C(1-u)^{\lambda_2 \kappa}] \lambda_2 \kappa C(1-u)^{\lambda_2 \kappa - 1} \\ &= \begin{cases} 0 & \kappa \lambda_2 > 1 \\ C & \kappa \lambda_2 = 1 \\ \infty & \kappa \lambda_2 < 1. \end{cases} \end{aligned}$$

Thus the κ -Pareto I model differs from the standard Pareto in tail behaviour as it can be short tailed, medium type or long tailed. A comparison between the

models can also be read from Figure 1, where the Pareto models are shown for different κ values. Note that the density function of κ -Pareto I model is

$$f_{PI,\kappa}(x) = \left[\sqrt{1 + \frac{\kappa^2}{\lambda_2^2} \left(\log \frac{x}{C} \right)^2} - \frac{\kappa}{\lambda_2} \log \frac{x}{C} \right]^{\frac{1}{\kappa}} \frac{\kappa C}{\lambda_2 x \sqrt{1 + \frac{\kappa^2}{\lambda_2^2} \left(\log \frac{x}{C} \right)^2}}.$$

Considering the power distribution (3.21)

$$(1-u)Q''(u) = \lambda_1 \kappa C u^{\lambda_1-2} [\lambda_1 \kappa C u^{\lambda_1} \sinh(\kappa C u^{\lambda_1}) + (\lambda_1 - 1) \cosh(\kappa C u^{\lambda_1})] (1-u)$$

giving

$$\alpha_1 = 0 \text{ as } u \rightarrow 1 \text{ and } \lambda_1 > 1,$$

so that the power distribution is short tailed. The plots of the density function

$$f_{P,\kappa}(x) = \left[\sqrt{1 + \kappa^2 \left[\log \left(1 - \left(\frac{x}{C} \right)^{\frac{1}{\lambda_1}} \right) \right]^2} - \kappa \log \left(1 - \left(\frac{x}{C} \right)^{\frac{1}{\lambda_1}} \right) \right]^{\frac{1}{\kappa}} \frac{\left(\frac{x}{C} \right)^{\frac{1}{\lambda_1}-1}}{\lambda_1 C \sqrt{1 + \kappa^2 \left(\log \left(1 - \left(\frac{x}{C} \right)^{\frac{1}{\lambda_1}} \right) \right)^2}}. \quad (3.26)$$

Finally, we have the Weibull model

$$Q_w(u) = \left(\frac{-\log(1-u)}{\beta} \right)^{\frac{1}{\alpha}}$$

for which

$$(1-u)Q''_w(u) = \frac{(1-\alpha) \left[\frac{-\log(1-u)}{\beta} \right]^{\frac{1}{\alpha}-2} + \left[\frac{-\log(1-u)}{\beta} \right]^{\frac{1}{\alpha}-1}}{\alpha \beta^2 (1-u)^2}$$

giving $\alpha = \infty$. Further

$$(1-u)Q''_{w,\kappa}(u) = \frac{\kappa \sinh(-\kappa \log(1-u) + \cosh(-\kappa \log(1-u)))}{\beta^{\frac{1}{\alpha}} (1-u)^2}.$$

Hence for the κ -Weibull $\alpha = \infty$. The density function is

$$f_{w,\kappa}(x) = \frac{\bar{F}_\kappa(x)}{2\sqrt{1 + \kappa^2} \cdot e^{-2\beta x^\alpha}} \beta \alpha x^{\alpha-1}. \quad (3.27)$$

The plots of the density for $\kappa = 0$ (the baseline distribution), $\kappa = 0.3$, $\kappa = 0.6$ and $\kappa = 0.9$ of (3.26) and (3.27) can be seen in Figures 2 and 3, for the same parameter values in each case. For the power distribution, increasing κ progressively reduces the sharp concentration near the upper boundary $x = 1$, while for the Weibull distribution it produces a slower decay in the right tail without materially changing the modal location. Thus, compared with the classical models ($\kappa \rightarrow 0$), larger κ values provide greater flexibility and heavier right-tail behaviour. Overall, the figures demonstrate the ability of the κ -deformation parameter to control departures from the classical density shapes, particularly in the tail region.

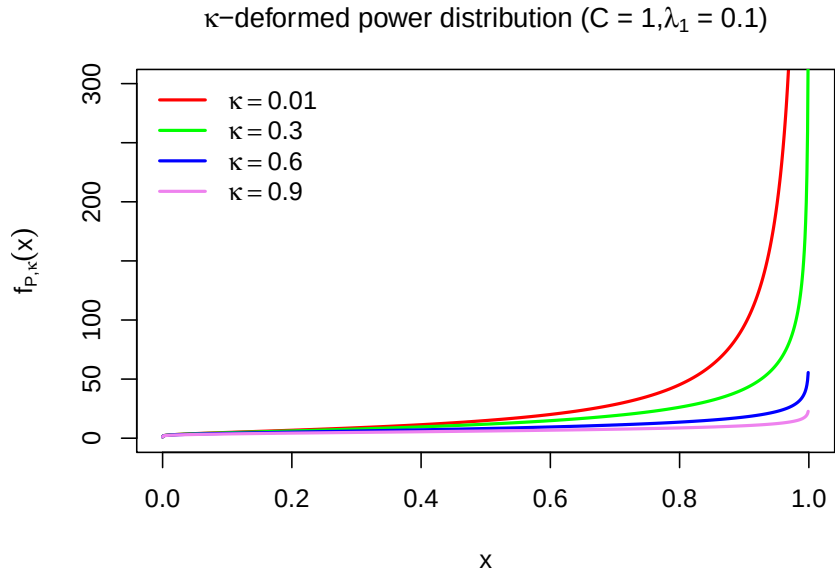


Figure 2: Density plot of κ -deformed power model.

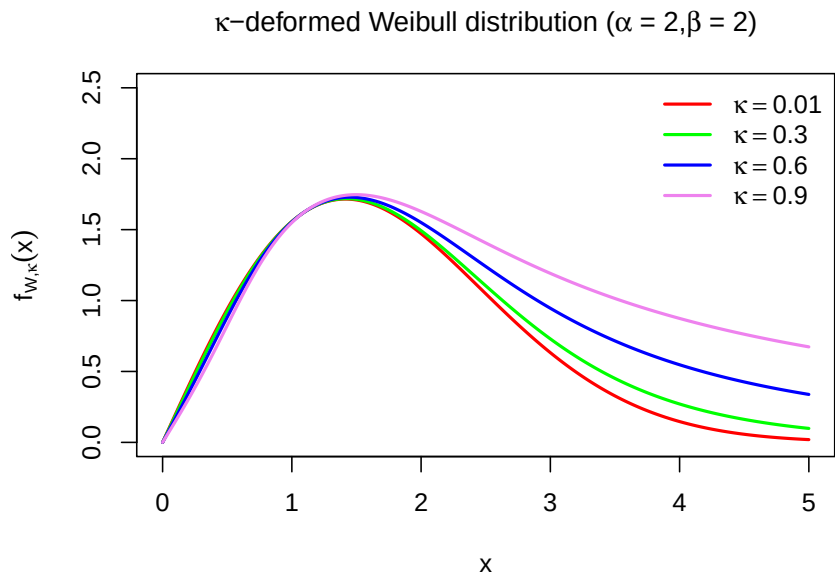


Figure 3: Density plot of κ -deformed Weibull model.

4. Properties of κ -quantile functions

In this section we present some special properties of the $Q_\kappa(u)$ that are not easy to derive from $\bar{F}_\kappa(x)$. This points out the advantages of using the quantile approach and some of the motivations for the present work.

1. We notice that $-\log_\kappa(1-u)$ is the quantile function of the κ -exponential so that we can write

$$Q_\kappa(u) = g_\kappa^{-1}(Q_{E,\kappa}(u)). \quad (4.1)$$

It is known from [Parzen \(1979\)](#) that for a continuous random variable X and a continuous non-decreasing function $a(X)$, the quantile function $Q_X(u)$ of X satisfies

$$Q_{a(X)}(u) = a(Q_X(u)). \quad (4.2)$$

From (4.1) and (4.2)

$$g_\kappa^{-1}(Q_{E,\kappa}(u)) = Q_{E,g_\kappa}(u) \quad (4.3)$$

meaning that by replacing u by $g_\kappa^{-1}(u)$ in $Q_{E,\kappa}(u)$, a new quantile function can be generated. An obvious example is the κ -Weibull from the κ -exponential by the transformation $a(x) = x^\alpha$. One can also utilize (4.2) in the case of κ -quantile functions.

Example 4.1. Consider the κ -Pareto II survival function

$$\bar{F}_{PII}(x) = (1+x)^C, \quad x > 0; C > 0$$

with κ -quantile function

$$Q_{PII,\kappa}(u) = \frac{1}{\kappa} \sinh \left[\kappa(1-u)^{-\frac{1}{C}} - 1 \right]$$

obtained from (3.12). Then the distribution of $a(x) = x^m$ is

$$Q_{a(x),\kappa}(u) = \frac{1}{\kappa} \sinh \left[\frac{k}{m} (1-u)^{-\frac{1}{C}} - 1 \right]$$

that corresponds to the Burr *XII* distribution $\bar{F}_B(x) = (1+x^m)^{-C}$, $x > 0; m, C > 0$.

Example 4.2. The exponential geometric law

$$\bar{F}_{EG}(x) = \frac{(1-p)e^{-\lambda x}}{1-pe^{-\lambda x}}, \quad x > 0; \lambda > 0, 0 < p < 1,$$

considered in the previous section has

$$Q_{EG,\kappa}(u) = \frac{1}{\lambda} \log_\kappa \left(\frac{1-pu}{1-u} \right).$$

The transformation $a(x) = e^{bx}$ provides

$$Q_{a(x),\kappa}(u) = b \left[\log_k \left(\frac{1-pu}{1-u} \right)^\alpha \right], \alpha = b\lambda^{-1}$$

generating a new distribution

$$\bar{F}_\kappa(x) = \frac{(1-p) \exp_\kappa \left(\frac{\lambda\kappa}{b} \log x \right)}{1 - p \exp_\kappa \left(\frac{\lambda\kappa}{b} \log x \right)}, x > 0.$$

2. Since the sum of two quantile functions is again a quantile function, we have that for two κ -quantile functions $Q_{\kappa,1}(u)$ and $Q_{\kappa,2}(u)$, the sum $Q_{\kappa,1}(u) + Q_{\kappa,2}(u)$ is a quantile function. However, it is not known whether the κ -sum, $Q_{\kappa,1} \oplus Q_{\kappa,2}(u)$, can be a quantile function. To ascertain that, we have from Section 2,

$$Q_{\kappa,1}(u) \oplus Q_{\kappa,2}(u) = Q_{\kappa,1}(u) \sqrt{1 + Q_{\kappa,2}(u)} + Q_{\kappa,2}(u) \sqrt{1 + Q_{\kappa,1}(u)}.$$

Using the relationship

$$\sinh^{-1}(x) = \log \left(x + \sqrt{1 + x^2} \right),$$

we have

$$\begin{aligned} & \sinh^{-1} \kappa Q_{\kappa,1}(u) + \sinh^{-1} \kappa Q_{\kappa,2}(u) \\ &= \log \left\{ \left(\kappa Q_{\kappa,1} + \sqrt{1 + \kappa^2 Q_{\kappa,1}^2} \right) \right\} + \log \left\{ \left(\kappa Q_{\kappa,2} + \sqrt{1 + \kappa^2 Q_{\kappa,2}^2} \right) \right\} \\ &= \log \left(\kappa Q_{\kappa,1} + \sqrt{1 + \kappa^2 Q_{\kappa,1}^2} \right) \left(\kappa Q_{\kappa,2} + \sqrt{1 + \kappa^2 Q_{\kappa,2}^2} \right). \end{aligned}$$

Hence

$$\begin{aligned} & \sinh \left(\sinh^{-1} \kappa Q_{\kappa,1} + \sinh^{-1} \kappa Q_{\kappa,2} \right) \\ &= \frac{1}{2} \left[\left(\kappa Q_{\kappa,1} + \sqrt{1 + \kappa^2 Q_{\kappa,1}^2} \right) \left(\kappa Q_{\kappa,2} + \sqrt{1 + \kappa^2 Q_{\kappa,2}^2} \right) \right. \\ & \quad \left. - \left(\kappa Q_{\kappa,1} - \sqrt{1 + \kappa^2 Q_{\kappa,1}^2} \right) \left(\kappa Q_{\kappa,2} - \sqrt{1 + \kappa^2 Q_{\kappa,2}^2} \right) \right] \\ &= \kappa Q_{\kappa,1}(u) \sqrt{1 + \kappa^2 Q_{\kappa,2}(u)} + \kappa Q_{\kappa,2}(u) \sqrt{1 + \kappa^2 Q_{\kappa,1}(u)} \\ &= \kappa [Q_{\kappa,1}(u) \oplus Q_{\kappa,2}(u)]. \end{aligned} \tag{4.4}$$

Any function $Q : [0, 1] \rightarrow \mathbb{R}$ which is left continuous and non decreasing qualifies to be quantile function. In (4.4) $\sinh(\cdot)$ and $\sinh^{-1}(\cdot)$ are continuous and increasing functions and the same properties hold for $\sinh Q_{\kappa,i}$ and $\sinh^{-1} Q_{\kappa,i}$ and their inverses for $i = 1, 2$. Hence the left side is a quantile function, showing that the κ -sum of two κ -quantile function is also a quantile function. This result enables to create new κ -quantile functions from known ones.

3. Since the random variables under considerations are non-negative, the usual product of their quantile functions is also a quantile function. By virtue of the relationship

$$x \otimes y = \frac{1}{\kappa} (\sinh^{-1} \kappa x + \sinh^{-1} \kappa y)$$

and the arguments given for (4.4) it follows that the κ -product $Q_{\kappa,1}(u) \otimes Q_{\kappa,2}(u)$ is also a quantile function.

4. The quantile density function of $Q_{\kappa}(u)$ is defined as its derivative, $q_{\kappa}(u) = \frac{d}{du_{\kappa}} Q_{\kappa}(u)$. It then holds that if $q_{\kappa,1}(u)$ and $q_{\kappa,2}(u)$ are two quantile density functions, their sum is also a quantile density function.
5. It is comparatively easier to find the descriptive properties of a distribution from the quantile functions than the distribution functions since in the latter we often need integration to derive them. For quantile functions, they are explicitly calculated from them or the g -function. We illustrate some typical cases.

As a measure of location, we have the median

$$M = Q_{\kappa} \left(\frac{1}{2} \right) = g_{\kappa}^{-1} \left(-\log \left(1 - \frac{1}{2} \right) \right) = g_{\kappa}^{-1}(\log 2).$$

To measure the dispersion, we have the semi-inter quartile range or quartile deviation

$$\begin{aligned} QD &= \frac{1}{2} \left(Q_{\kappa} \left(\frac{3}{4} \right) - Q_{\kappa} \left(\frac{1}{4} \right) \right) \\ &= \frac{1}{2} \left[g_{\kappa}^{-1}(\log 4) - g_{\kappa}^{-1} \left(\log \frac{4}{3} \right) \right]. \end{aligned}$$

Similarly, for skewness, Galton's measure

$$\begin{aligned} S_{\kappa} &= \frac{Q_{\kappa} \left(\frac{3}{4} \right) + Q_{\kappa} \left(\frac{1}{4} \right) - 2M}{Q_{\kappa}^{-1} \left(\frac{3}{4} \right) - Q_{\kappa} \left(\frac{1}{4} \right)} \\ &= \frac{g_{\kappa}^{-1}(\log 4) - g_{\kappa}^{-1} \left(\log \frac{4}{3} \right) - 2g_{\kappa}^{-1}(\log 2)}{g_{\kappa}^{-1}(\log 4) - g_{\kappa}^{-1} \left(\log \frac{4}{3} \right)} \end{aligned}$$

and to evaluate Kurtosis, the Moores' coefficient

$$\begin{aligned} T &= Q_{\kappa} \left(\frac{7}{8} \right) - Q_{\kappa} \left(\frac{5}{8} \right) + Q_{\kappa} \left(\frac{3}{8} \right) - Q_{\kappa} \left(\frac{1}{8} \right) / 2QD \\ &= g_{\kappa}^{-1}(\log 8) - g_{\kappa}^{-1} \left(\log \frac{8}{3} \right) + g_{\kappa}^{-1} \left(\log \frac{8}{5} \right) - g_{\kappa}^{-1} \left(\log \frac{8}{7} \right) / 2QD \end{aligned}$$

are used.

An advantage of these measures over their usual moment based counterpart is that they are robust, not affected by extreme values.

5. Applications

To examine the applicability of the proposed κ -deformed models, two real-life data sets were considered. The first data set consists of failure times of electronic components obtained from [Lawless \(2003\)](#). The second application concerns household income data from Thailand, originally reported from the Socio Economic Survey conducted by the National Statistical Office of Thailand during 1975–76. The two data sets provide different empirical settings for assessing the performance of the proposed models. Parameters of the proposed κ -deformed distributions were obtained, and the limiting cases were examined by allowing κ to approach zero. The adequacy of the fitted models was subsequently assessed using the appropriate goodness-of-fit tests and graphical diagnostics.

(a) Analysis of Electronic Component Failure-Time Data

The first data set comprises failure times, measured in minutes, for 15 electronic components subjected to an accelerated life test. This data set was originally reported by [Lawless \(2003\)](#). The present study uses the data to investigate the performance of the proposed κ -deformed model. For the proposed κ -deformed linear hazard quantile model, the parameters were estimated using the method of percentiles. The estimates obtained by this method possess desirable properties like consistency, asymptotic normality etc. which are discussed in detail in [Nair et al. \(2013\)](#). The estimated parameter values obtained for the proposed κ -deformed model were $\hat{\theta} = 8.621$, $\hat{a} = 0.013$, and $\hat{\kappa} = 0.058$, so $\hat{b} = 0.112$. When κ tends to zero, the proposed κ -deformed quantile function reduces to its corresponding undeformed limiting model. The estimates obtained under this limiting case were $\hat{\theta}_0 = 1.029$ and $\hat{a}_0 = 0.024$ so $\hat{b}_0 = 0.025$.

The fitted distributions were further examined using the Kolmogorov–Smirnov (KS) goodness-of-fit test. The null hypothesis states that the observed failure-time data are adequately described by the fitted distribution. The resulting KS statistic for the proposed κ -deformed model was KS Statistic=0.100, with corresponding $p = 0.994$. At the 5% significance level, a p -value greater than 0.05 indicates that there is insufficient evidence to reject the fitted distribution. The empirical adequacy of the model was also examined graphically through Q–Q plots in [Figure 4](#). Thus the test results provide a formal assessment that the proposed κ models are compatible with the data.

(b) Analysis of Household Income Data

The second application considers household income data from Thailand given in [Table 1](#). The data were obtained from the Socio Economic Survey conducted by the National Statistical Office of Thailand during 1975–76. The data considered 3,941 observations corresponding to the lower tail of a total sample of 12,189 household incomes.

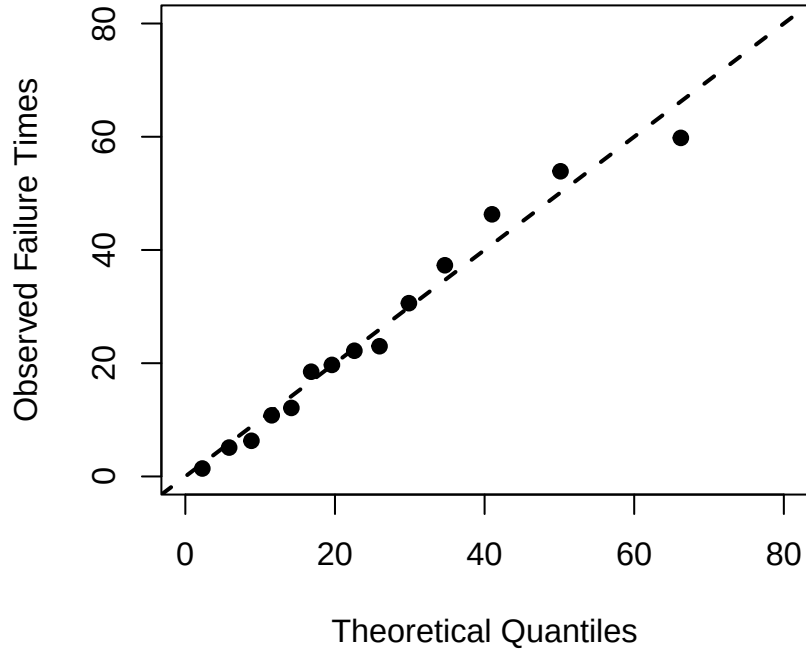


Figure 4: QQ plot for κ -deformed linear hazards model

Table 1: Household income of Thailand

Household:							
Class	0-249	250-374	375-499	500-624	625-749	750-874	875-999
Income (y) :							
in Baht	211	309	439	561	692	808	921

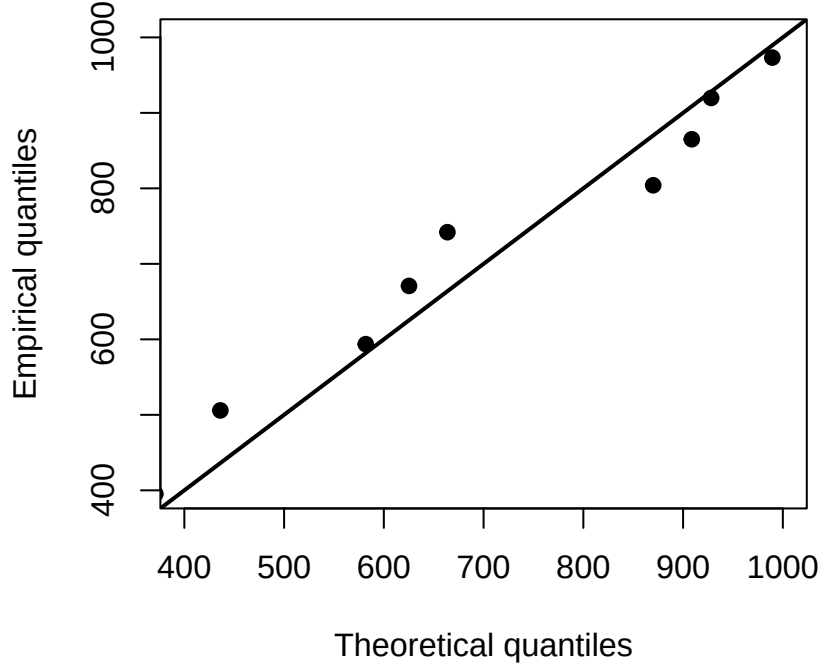


Figure 5: QQ Plot for κ -deformed power model

For the purpose of illustrating the analysis of income distributions, representative income values corresponding to the income classes were considered. The lower-tail nature of the data makes this application suitable for examining models designed to describe the behaviour of income distributions over a restricted range.

We considered the power distribution. As an initial model for the lower tail and estimated its parameters by the method of percentiles. In particular, the κ -power-deformed model were fitted to the observed income data.

For the κ -power-deformed model, the estimated parameters were $\hat{C} = 0.964$, $\hat{\lambda}_1 = 2.112$, and $\hat{\kappa} = 0.479$. When κ approaches zero the model is the classical power law with estimates calculated as $\hat{C}_0 = 0.994$ and $\hat{\lambda}_{10} = 2.047$. The goodness of fit of the fitted income distributions was assessed using the Chi-Square test. For the κ -power-deformed model, the Chi-square statistic was $\chi^2 = 1.343$, with corresponding $p = 0.719$. The QQ plot is given in Figure 5.

We have found in our data analysis that the baseline models and their κ versions provide satisfactory fits by standard criteria. Therefore to further investigate their relative supremacy we have used the Akaike information

criterion corrected(AICc) defined as $AICc = n \log(SSE/n) + 2K + \frac{2k(k+1)}{n-k-1}$ where k is the number of parameters and SSE is the sum of squares of error. It is found that for the lifetime data the AICc were -34.968 for the κ distribution and -8.696 when $\kappa = 0$. Similarly for the Thai data the κ distribution gave AICc = -140.525 while $\kappa = 0$ gave -120.45 . Thus in both cases the data gave more information when κ distributions were employed. The application of various quantile function models are under further investigation and will be reported separately.

Data Availability Statements

Authors can confirm that all relevant data are included in the article and/or its supplementary information files.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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