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Non-Abelian Quantum Metric Governed Topological Boundary-Mode Localization

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The presence of localized boundary modes is an unambiguous hallmark of topological quantum matter. While these modes are typically protected by topological invariants such as the Chern number, here we demonstrate that the *quantum metric length* (QML), a quantity inherent in multi-band topological systems, governs the spatial extent of flat-band topological boundary modes. We introduce a framework for constructing topological flat bands from a degenerate manifold with large non-Abelian quantum metric, and find that the topological boundary modes exhibit two sequential localization behaviors: a conventional oscillatory decay arising from bare band dispersion, followed by another exponential decay controlled by quantum geometry. Crucially, the QML, derived from the non-Abelian quantum metric of the subspace manifold, sets a lower bound on the spatial spread of boundary states in the flat-band limit. Applying our framework to concrete models, we validate the universal role of the QML in shaping the long-range localization of topological boundary modes. Furthermore, by tuning the QML, we unveil observable non-local transport signatures, including QML-shaped quantum Hall plateaus and anomalous Fraunhofer patterns. Our theoretical framework paves the way for engineering boundary-mode localization in topological flat-band systems.

I. INTRODUCTION

Hilbert space geometry endows electronic states with intrinsic characteristic scales [1–6], fundamentally reshaping conventional paradigms—particularly in the flat-band regime [7–9]. With their highly quenched kinetic energy, flat bands herald a paradigmatic shift, as traditional frameworks for understanding crystalline properties break down when the Fermi velocity vanishes. Consequently, the geometric properties of Bloch states have profound implications in various phenomena, such as unconventional superfluidity [10–17], optical responses [18–21], and other correlated phases of matter [22–30].

At the heart of these phenomena lies the quantum geometric tensor, $\mathcal{Q} \equiv \mathcal{G} + \frac{1}{2i}\Omega$, where the real part $\mathcal{G}$ (quantum metric) and the imaginary part Ω (Berry curvature) encodes distinct geometric properties of Bloch states [31–35]. While Berry curvature characterizes the phase difference that determines the global band topology [36], the quantum metric defines the infinitesimal distance between states in the parameter space [37, 38]. In the past several decades, the interplay between these geometric counterparts has raised wide interest and been explored in several dimensions: ideal quantum geometry in fractional Chern insulators [39, 40], inequality relations between them [41–43], and their manifestation in optical responses [44, 45], etc. In the context of the bulk-boundary correspondence [46], it has been firmly established that localized boundary modes are protected by bulk topological invariants since the discovery of the integer quantum Hall effect [47]. Therefore, a fundamental question arises: Does the bulk quantum metric, which generally hosts a non-Abelian structure, play any role in the localized boundary modes?

In this work, we develop a framework to construct topological flat bands with nontrivial quantum metric, and identify how the quantum metric influences the localization behaviors of topological boundary modes (TBMs). It is found that multi-band TBMs typically exhibit two sequential localization behaviors: an initial oscillatory decay driven by the band dispersion, followed by an exponential decay arising from the Wannier orbitals encoding the quantum geometry. They are thus dubbed the conventional and the geometric behavior, respectively. Importantly, we show that in the flat-band limit where the conventional behavior is suppressed, the TBMs spread is lower bounded by the *quantum metric length* (QML), which originates from a degenerate subspace manifold with non-Abelian quantum metric. This generalized QML is validated across various flat-band models and it underpins significant non-local transport phenomena.

Our framework begins with degenerate flat bands from a parent Hamiltonian H_f , and perturbative couplings H_c are introduced to make the flat bands topological. The total Hamiltonian H_t in general takes the form

$$H_t(\mathbf{k}) = H_f(\mathbf{k}) + H_c(\mathbf{k}), \quad (1)$$

where the coupling term H_c has subdominant energy scales compared to the original flat-band model. The topological bands inherit the quantum geometry from the original flat bands of H_f , profoundly shaping the characteristics of their TBMs.

To be specific, Fig. 1 illustrates the construction of topological bands within two distinct paradigms: (1) In the upper panel, we consider two atomic orbitals (A and B) with identical on-site energies, which form degenerate flat bands with trivial non-Abelian quantum metric. When these orbitals (and consequently the flat bands) are coupled in a 2D array, the degeneracy is lifted and dispersion emerges. During this process, the band gap may close and reopen, leading to an inversion of orbital com-

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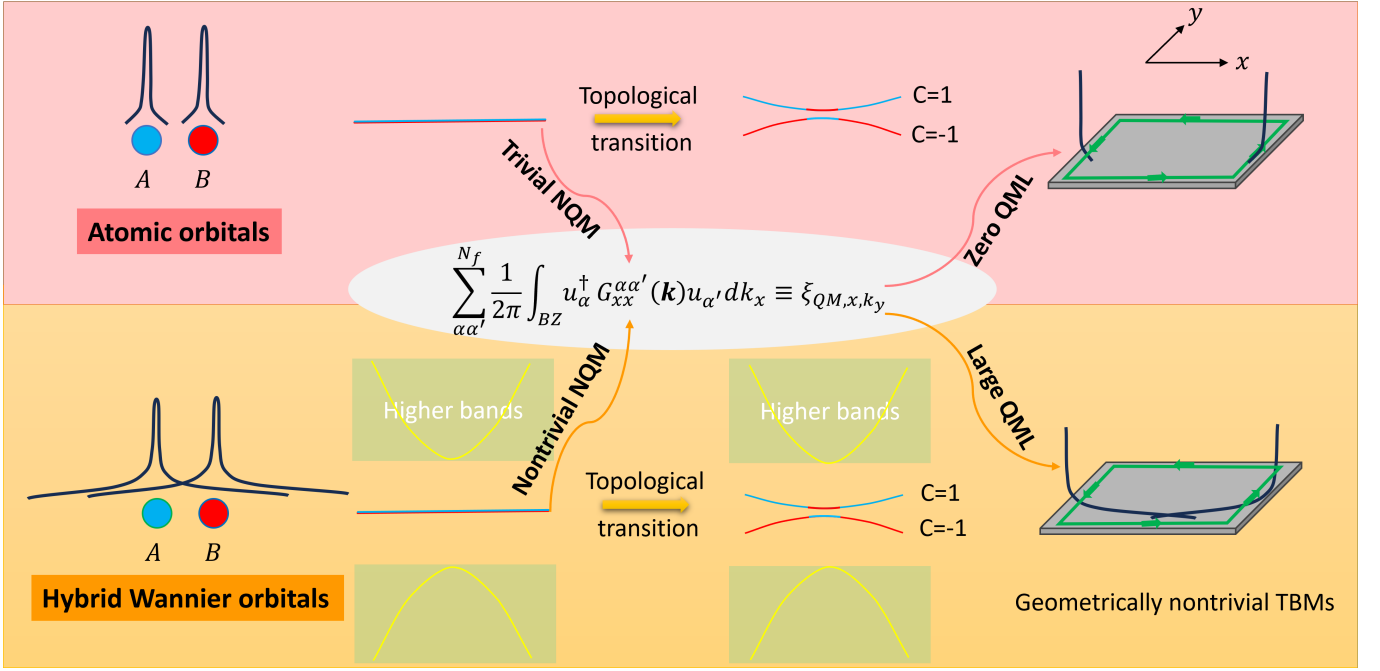


FIG. 1. **An illustration of two paradigms in constructing 2D topological flat bands.** Upper panel: Decoupled atomic orbitals form degenerate flat bands with trivial non-Abelian quantum metric (NQM), resulting in well-localized TBMs with zero QML after topological transition. Lower panel: Degenerate flat bands with nontrivial NQM, the same process leads to geometrically nontrivial TBMs with long-range localization controlled by the large QML.

positions that yields nontrivial topology, such as topological insulators (TIs) [48]. In this scenario, while the bulk states possess nontrivial topology, we nevertheless regard their gapless boundary modes as geometrically trivial, since their localization is entirely determined by band dispersion. In the flat-band limit, the localization length approaches zero. (2) In the lower panel, we show that, in contrast, when starting from degenerate flat bands with nontrivial non-Abelian quantum metric, the same coupling process produces topological bands whose boundary modes inherit an additional length scale from the spatially extended hybrid Wannier orbitals. Notably, despite of Wannier obstruction on the topological bands [49–52], the TBMs are safe from it since they are built from the parent manifold that is topologically trivial. Specifically, we demonstrate that this length scale is, in general, controlled by the QML:

$$\xi_{QM,i,\tilde{k}} \equiv \sum_{\alpha\alpha'} \frac{1}{2\pi} \int_{BZ} u_{\alpha}^* \mathcal{G}_{ii}^{\alpha\alpha'}(\mathbf{k}) u_{\alpha'} d\mathbf{k}_i, \quad (2)$$

which sets a lower bound for the spatial spread of the TBMs at channel $\tilde{k}$ in the i -direction. Here, $\mathcal{G}_{ii}^{\alpha\alpha'}(\mathbf{k})$ is the non-Abelian quantum metric of the N_f degenerate flat bands [53], and $|u\rangle = (u_1, \dots, u_{N_f})$ is a vector parameterizing the low-energy effective Hamiltonian. The QML governs the long-range behavior of flat-band TBMs and can readily exceed lattice scales. This principle can be extended to other topological classes, motivating our general framework for engineering TBMs in topological flat

bands and for exploring the rich physical consequences enabled by the QML.

II. RESULTS

A. Construction of topological flat bands with non-trivial quantum metric

Various methods have been developed to construct exactly flat bands on a lattice [54–59], one of which is to use bipartite crystalline lattices (BCL), which naturally host degenerate flat bands due to their inherent chiral symmetry [59]. When partitioned into sublattices L_1 (N_{L_1} orbitals) and L_2 (N_{L_2} orbitals) with $N_{L_1} < N_{L_2}$, the BCL Hamiltonian exhibits a block off-diagonal structure $H_f = \sum_{\mathbf{k}} \Psi_{L_1,\mathbf{k}}^\dagger S_{\mathbf{k}} \Psi_{L_2,\mathbf{k}} + \text{h.c.}$, where $S_{\mathbf{k}}$ is a $N_{L_1} \times N_{L_2}$ matrix, and $\Psi_{L_1,\mathbf{k}} (\Psi_{L_2,\mathbf{k}})$ is a $N_{L_1} (N_{L_2})$ -dimensional spinor. This structure guarantees $N_f = N_{L_2} - N_{L_1}$ degenerate flat bands at zero energy. To readily engineer flat-band quantum geometry, we further partition L_2 into $L_{2,A}$ (N_{L_1} orbitals) and $L_{2,B}$ (N_f orbitals), and confine the minimal coupling H_c to $L_{2,B}$. Accordingly, $S_{\mathbf{k}}$ is partitioned as $S_{\mathbf{k}} = (S_{1,\mathbf{k}}, S_{2,\mathbf{k}})$, with $S_{1,\mathbf{k}}$ ($S_{2,\mathbf{k}}$) being the size of $N_{L_1} \times N_{L_1}$ ($N_{L_1} \times N_{L_f}$). The flat-band eigenvectors $\mathcal{F}_{\mathbf{k},\alpha}$ ($N_{L_1} < \alpha \leq N_{L_2}$) reside exclusively on L_2 and

take the form

$$\mathcal{F}_{\mathbf{k},\alpha} = \begin{pmatrix} 0 \\ \phi_{1,\mathbf{k},\alpha} \\ \phi_{2,\mathbf{k},\alpha} \end{pmatrix}, \quad (3)$$

with components satisfying the eigenvalue equation $S_{1,\mathbf{k}}\phi_{1,\mathbf{k},\alpha} + S_{2,\mathbf{k}}\phi_{2,\mathbf{k},\alpha} = 0$. The quantum geometry of the flat bands can be engineered by imposing specific constraints on a quantum geometry indicator $\lambda_{\mathbf{k}}$ (see Methods). By imposing the *local atomic constraint* $\lambda_{\mathbf{k}} \ll 1$, we find that each flat-band eigenvector can be locally (e.g., the Γ -point) identified with a single atomic orbital,

with $\mathcal{F}_{\mathbf{k} \rightarrow 0, \alpha} \approx (0, \dots, \overset{(2N_{L_1} + \alpha) \text{th}}{1}, \dots, 0)^T$. Therefore, near the Γ -point, we introduce the minimal topological coupling among the flat bands [suppose $N_f = 2^g$ ($g \in \mathbb{Z}^+$)], and the total Hamiltonian becomes

$$H_t(\mathbf{k}) = \begin{pmatrix} 0 & S_{1,\mathbf{k}} & S_{2,\mathbf{k}} \\ S_{1,\mathbf{k}}^\dagger & 0 & 0 \\ S_{2,\mathbf{k}}^\dagger & 0 & H_d^{(g)}(\mathbf{k}) \end{pmatrix} \quad (4)$$

with $H_d^{(g)}(\mathbf{k}) = \sum_{i=1}^d v_i k_i \gamma_i^{(g)} + m \gamma_{d+1}^{(g)}$,

where d is the spatial dimension, and $2^g \times 2^g$ Gamma matrices $\{\gamma_i^{(g)}\}$ generate the Clifford algebra $\text{Cl}(2g + 1, 0)$. The Dirac form $H_d^{(g)}(\mathbf{k})$ provides maximal flexibility for realizing different Altland-Zirnbauer symmetry classes [60] with dimension $d < 2g + 1$. Projecting $H_t(\mathbf{k})$ onto the flat-band subspace spanned by $\mathcal{F}_{\mathbf{k}} = (\mathcal{F}_{\mathbf{k}, N_{L_1}+1}, \mathcal{F}_{\mathbf{k}, N_{L_1}+2}, \dots, \mathcal{F}_{\mathbf{k}, N_{L_2}})$, we can obtain the effective model

$$H^{\text{eff}}(\mathbf{k}) = \mathcal{F}_{\mathbf{k}}^\dagger H_t \mathcal{F}_{\mathbf{k}} \xrightarrow{\mathbf{k} \rightarrow 0} H_d^{(g)}(\mathbf{k}), \quad (5)$$

demonstrating that the low-energy physics is directly inherited from $H_d^{(g)}(\mathbf{k} \rightarrow 0)$. For instance, the mass term m controls the topological transitions, while the velocities v_i set the bandwidth scale.

Furthermore, by imposing additional constraints for the quantum geometry indicator at a certain point where the gap separating flat bands and dispersive bands is located, e.g., the X -point, we find that the xx -component of the non-Abelian quantum metric for the flat bands at X -point can significantly exceed the lattice scale (see Methods), $\mathcal{G}_{xx}^{\alpha\alpha}(\mathbf{k}_X) \gg a^2$, and it can be further enhanced by reducing the band gap. In contrast, the Γ point does not exhibit large non-Abelian quantum metric due to the local atomic constraint. Therefore, a quantum metric hot spot near X in the Brillouin Zone (BZ) is expected to emerge. As a remark, the BZ distribution of the non-Abelian quantum metric for the flat bands later determines the QML at different momentum channels, profoundly shaping the spatial behaviors of the TBMs.

B. Manifested QML in the TBMs

Multi-band TBMs in general can be expressed as linear combinations of hybrid Wannier orbitals $\mathcal{W}_\alpha(x - R_x, \tilde{\mathbf{k}})$ located at different lattice sites, modulated by a vector $\mathcal{U}_\alpha(R_x, \tilde{\mathbf{k}})$:

$$\Psi^B(x, \tilde{\mathbf{k}}) = \sum_{\alpha} \sum_{R_x} \mathcal{U}_\alpha(R_x, \tilde{\mathbf{k}}) \mathcal{W}_\alpha(x - R_x, \tilde{\mathbf{k}}), \quad (6)$$

where we have imposed open boundary conditions (OBC) along the x -direction so that $\tilde{\mathbf{k}} = (k_y, \dots, k_d)$ remains a good quantum number. The envelop vector $\mathcal{U}_\alpha(R_x, \tilde{\mathbf{k}})$ is an eigenvector of the flat-band effective Hamiltonian in Eq. (5) with the transformation $k_x \rightarrow -i\partial_{R_x}$. It typically takes an oscillatory decay with length scale ξ_c , which is entirely driven by bare band dispersion when the Wannier basis is optimally localized (see Supplementary Note II A). On the other hand, the Wannier basis in Eq. (6)—Fourier transformed from the flat-band eigenvectors in Eq. (3)—contains the information of the quantum geometry. In the limit $\xi_c \rightarrow 0$, the Wannier orbitals at the boundary site form observable eigenstates for the system, which naturally adopts an exponential decay due to the translation symmetry [61, 62]. Therefore, taking a specific form for the exponentially localized Wannier orbitals with length scale ξ_g , we find that the TBMs have a simple form

$$\Psi^B(x, \tilde{\mathbf{k}}) = A_c e^{-x/\xi_c} + A_g e^{-x/\xi_g}, \quad (7)$$

where the coefficients A_c and A_g , implicitly dependent on $\tilde{\mathbf{k}}$, are determined by the details of the Wannier orbitals (see Supplementary Note II C). Notably, the general form of Eq. (7) is not tied to any specific model. This suggests that multi-band TBMs universally exhibit two phases of behaviors, namely, a conventional behavior and a geometric behavior, corresponding to the two terms in Eq. (7). Near the boundary, the states initially undergo a conventional oscillatory decay with a diminishing oscillation amplitude. Subsequently, a pure exponential decay emerges. The competition between them depends on the relative scale of ξ_c and ξ_g .

In particular, in the flat-band limit when $\xi_c \rightarrow 0$, we find that the spread for the TBMs, $\Omega_{\Psi_\alpha^B}^x \equiv \int_x ((\Delta \hat{x})_{|\Psi_\alpha^B(x)|^2}^2 dx$, satisfies the following inequality (see Supplementary Note II D):

$$\Omega_{\Psi_\alpha^B}^x \geq \frac{a}{2\pi} \sum_{\alpha\alpha'}^{N_f} \int_{\text{BZ}} u_\alpha^* \mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k}) u_{\alpha'} d\mathbf{k}_x, \quad (8)$$

where $\mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k})$ is the non-Abelian quantum metric of the original N_f degenerate flat bands, and $|u\rangle = (u_1, \dots, u_{N_f})$ is an eigenvector of $\gamma_i^g \gamma_{d+1}^g$ (e.g., in the case of 2 topological flat bands with x -OBC in 2D, one has $g = 1$, $i = 1$, $d = 2$, the eigenvector of $\gamma_1^g \gamma_{d+1}^g = \sigma_1 \sigma_3$ is $|u\rangle = (1, \pm i)^T / \sqrt{2}$). The BZ-integrated non-Abelian

quantum metric on the right-hand side of Eq. (8) sets a lower bound for the TBMs spread, which naturally defines a length scale, namely, the QML [38, 63, 64]. A general QML is defined with respect to some localizing direction (e.g., the OBC-direction) $\hat{i}$ with a certain momentum channel $\tilde{\mathbf{k}}$, as given in Eq. (2). This also aligns with the QML definition in Ref. [38], as discussed in Supplementary Note II D.

C. Topological flat band models

We exemplify our framework by constructing topological flat bands on a variant Lieb lattice. Specifically, we extend sublattice C of the original Lieb lattice [65] into a set of 2^g ($g \in \mathbb{Z}^+$) equivalent sublattice replicas, giving rise to 2^g degenerate flat bands. By incorporating a coupling matrix $H_d^{(g)}(\mathbf{k})$ among the extended sublattices, we can realize various topological classes depending on the choice of d and g . The Hamiltonian, as defined in Eq. (4), is given by

$$\begin{aligned} S_{1,\mathbf{k}} &= J[(1 + \delta) + (1 - \delta)e^{ik_1a}] \\ S_{2,\mathbf{k}} &= \alpha J[(1 + \delta) - (1 - \delta)e^{ik_2a}] \mathcal{I}_{1 \times 2^g}. \end{aligned} \quad (9)$$

Here, the parameter δ ($0 < \delta \ll 1$) controls the staggered hoppings between unit cells, determining the band gap and the non-Abelian quantum metric of the flat bands. The parameter α ($0 < \alpha \lesssim 1$) shapes the energy spectrum of the flat bands. As expected from our framework, the variant Lieb lattice exhibits nontrivial quantum metric. Specifically, in terms of the local quantum geometry indicator (see Methods), at the Γ -point, $\lambda_\Gamma = \alpha\delta \ll 1$, which satisfies the local atomic constraint. In contrast, at the X -point where the direct band gap is located, one has $\lambda_X \sim 1$. This results in a peak for the quantum metric distribution in the BZ, controllable by tuning the band gap via δ , as demonstrated in the subsequent models.

As a concrete example, taking $g = 1$, $d = 2$, the 2×2 block is well known as the Qi-Wu-Zhang (QWZ) model which is a prototypical lattice model for Chern insulators [66]:

$$H_2^{(1)}(\mathbf{k}) = 2t \sin(k_1a)\sigma_x + 2\tilde{t} \sin(k_2a)\sigma_y + M(\mathbf{k})\sigma_z, \quad (10)$$

with $M(\mathbf{k}) = m - 2\tilde{t}(\cos(k_1a) + \cos(k_2a))$. The flatness condition requires $\max(t, \tilde{t}, m) \ll \alpha\delta J$. By substituting Eqs. (9), (10) into Eq. (4), we embed the QWZ model within a variant-Lieb lattice, yielding the four-band Lieb-QWZ model. Figure 2(a) displays the full band structure, where two dispersive bands sandwich a pair of flat bands in the middle, with a zoomed-in view in Fig. 2(b). The Lieb-QWZ model belongs to the class A in the AZ classification [60], characterized by the Chern number. In Fig. 2(c), we compute the Chern number of one flat band as a function of m/t , revealing a topological nontrivial regime for $|m| < 4t$. The flat bands inherit the quantum metric from the original degenerate manifold, which can be recovered by turning off $H_d^{(g)}$ (i.e.,

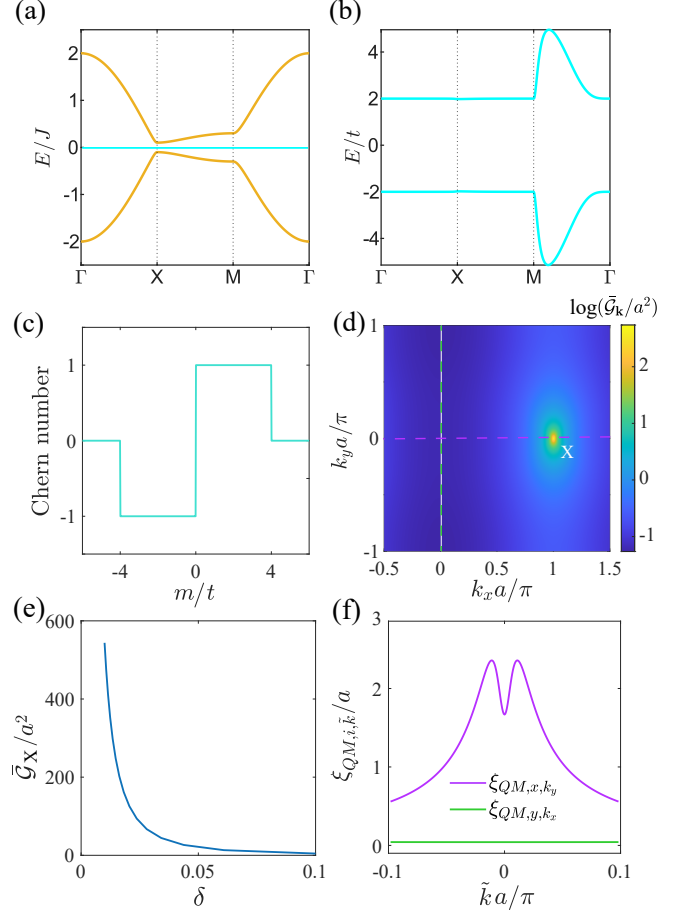


FIG. 2. Band structure, Topology and QML of the Lieb-QWZ model. (a) Band structure of the Lieb-QWZ model along high-symmetry points. (b) Zoomed-in view of the two topological flat bands. (c) The Chern number evolution of a flat band as a function of the mass term m/t . (d) The BZ distribution of the traced non-Abelian quantum metric tensor $\bar{\mathcal{G}}_{\mathbf{k}} \equiv \frac{1}{N_f} \sum_{\alpha\alpha'} \text{Tr}[\mathcal{G}_{ab}^{\alpha\alpha'}(\mathbf{k})]$ of the exactly degenerate flat bands, featuring a hot spot at the X -point. The green (purple) dashed line represents $k_x = 0$ ($k_y = 0$) cut lines respectively. (e) The $\bar{\mathcal{G}}_X$ as a function of δ ($\propto \Delta_X$). (f) The (x,y) -directional QML ξ_{QM,x,k_y} (ξ_{QM,y,k_x}) around the Γ -channel. Parameters: $(m, \tilde{t}, t, \alpha, \delta) = (4 \times 10^{-4}, 2 \times 10^{-4}, 2 \times 10^{-4}, 0.1, 0.05)$ for (a) to (c); $(m, \tilde{t}, t, \alpha, \delta) = (0, 0, 0, 0.5, 0.01)$ for (d) to (f). $J = 1$ for all panels.

$t = \tilde{t} = m = 0$). We then compute the traced non-Abelian quantum metric for the degenerate flat bands, defined as $\bar{\mathcal{G}}_{\mathbf{k}} \equiv \frac{1}{N_f} \text{Tr} \left[\sum_{\alpha\alpha'} \mathcal{G}_{ab}^{\alpha\alpha'}(\mathbf{k}) \right]$, and plot its distribution across the BZ in Fig. 2(d). The quantum metric peaks prominently near the X -point, as expected. Figure 2(e) shows the traced non-Abelian quantum metric $\bar{\mathcal{G}}_X$ at the X -point, demonstrating enhancement by reducing the band gap Δ_X .

Integrating the non-Abelian quantum metric over the cut line along different k_y (k_x)-channels according to Eq. (2), we obtain the orientation-dependent QML

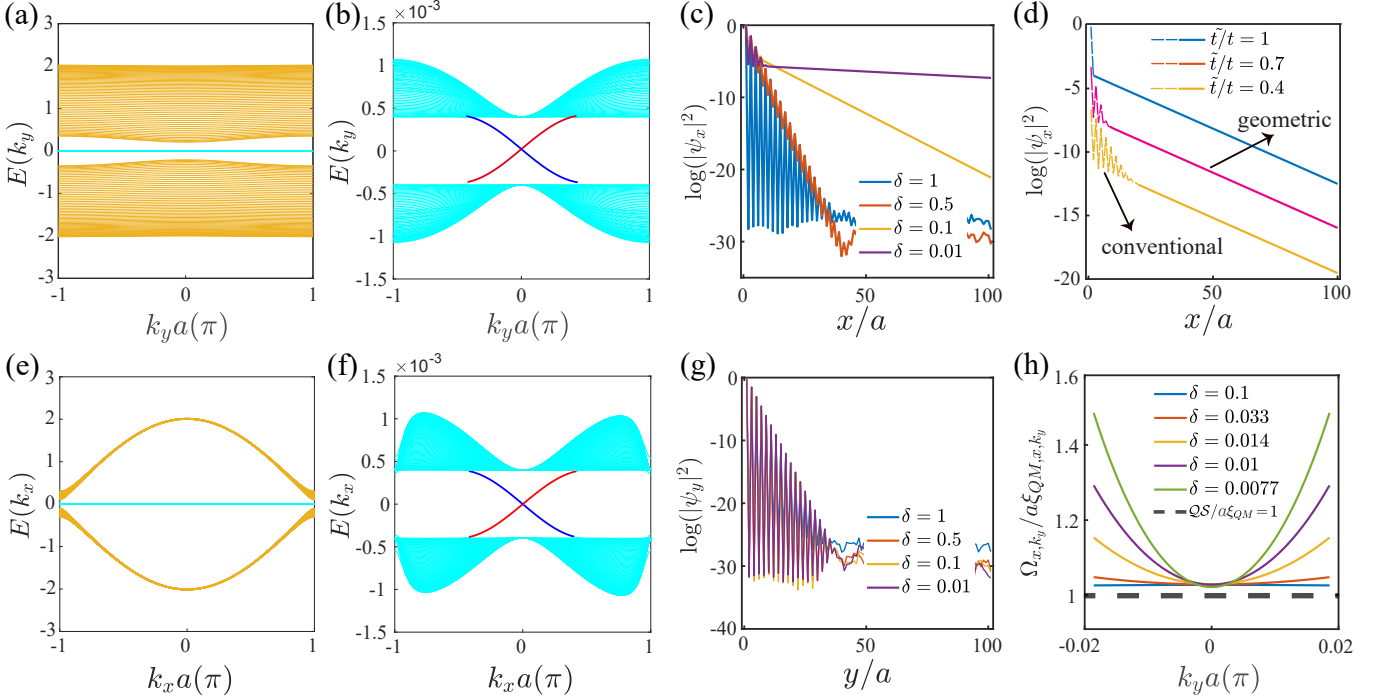


FIG. 3. **Spatial behaviors of TBMs in the Lieb-QWZ model.** (a,b,e,f) The spectrum for the Lieb-QWZ model with OBC (a) in the x -direction and (e) in the y -direction. (b) and (f) shows the corresponding flat bands zoomed in, with in-gap modes highlighted in red and blue for the left and right branches, respectively. (c,g) The logarithmic wave functions for different δ with OBC in (d) the x -direction and (h) the y -direction. To manifest both behaviors, we keep the parameters the same as before except $t = 0.5\tilde{t}$. (d) The logarithmic wave functions for different $\tilde{t}/t$ with OBC in the x -direction (vertically shifted for visual clarity). (h) In the case $t = \tilde{t}$, the ratio $\Omega_{x,k_y}/a\xi_{QM}$ (where a is the lattice constant) is consistently bounded below by 1 in the Lieb-QWZ model, in excellent agreement with Eq. (8). Parameters: $t = \tilde{t}$ for (a,b,e,f), and $J = 1$, $\tilde{t} = 2 \times 10^{-4}J$, $m = 2\tilde{t}$, $\delta = 0.05$ and $\alpha = 0.1$ for all panels.

ξ_{QM,x,k_y} (ξ_{QM,y,k_x}) around the Γ -channel in Fig. 2(f), exhibiting an anisotropic feature. Specifically, the x -directional QML $\xi_{QM,x,k_y=0}$, integrated over k_x along $k_y = 0$, intersects the X -point, acquiring a finite and tunable value, as indicated by the purple dashed line in Fig. 2(d). In contrast, the y -directional QML, $\xi_{QM,y,k_x=0}$ —integrated over k_y along $k_x = 0$ while bypassing $\mathbf{k}_X$ —remains zero and independent of δ , as shown by the green dashed line in the same figure. The Lieb-QWZ model provides a minimal topological model with tunable non-Abelian quantum metric. Additional topological flat band models, belonging to distinct topological classes, are constructed in Supplementary Note III to further illustrate our framework.

D. Boundary-modes localization in the Lieb-QWZ model

In the topological regime of the Lieb-QWZ model, in-gap TBMs emerge within the flat bands under x (y)-OBC, as shown in Fig. 3(b) [(f)], which are zoomed in from the full spectrum in Fig. 3(a) [(e)]. As clarified above, multi-band TBMs typically exhibit two sequen-

tial phases of spatial behaviors: A conventional oscillatory decay driven by bare band dispersion, and a geometrical exponential decay from the localized Wannier orbitals controlled by the QML. This is exactly what we observe in Fig. 3(d), where the conventional and geometrical behaviors are denoted in dashed and solid lines, respectively. However, depending on the relative scales of ξ_c and ξ_g in Eq. (7), these two behaviors may not always be observed simultaneously. When the conventional length dominates, i.e., $\xi_c \gg \xi_g > 0$, it is found that the first term in Eq. (7) dominates the second one, effectively overshadowing the geometrical behavior. This is well reflected in Fig. 3(c). By increasing δ , the geometrical length ξ_g for the x -directional TBMs diminishes, and eventually becomes invisible. In contrast, the TBMs under the y -OBC are insensitive to δ and only exhibit the conventional behavior, as shown in Fig. 3(g). This indicates that the y -directional geometrical length is much smaller than the conventional length: $\xi_g \ll \xi_c$. Their distinct behaviors are consistent with the anisotropic QML we found in Fig. 2(f).

Conversely, the geometrical length will manifest itself when $\xi_g > \xi_c$ and become the dominant length scale when $\xi_g \gg \xi_c > 0$. Examining the two terms in Eq. (7) in this

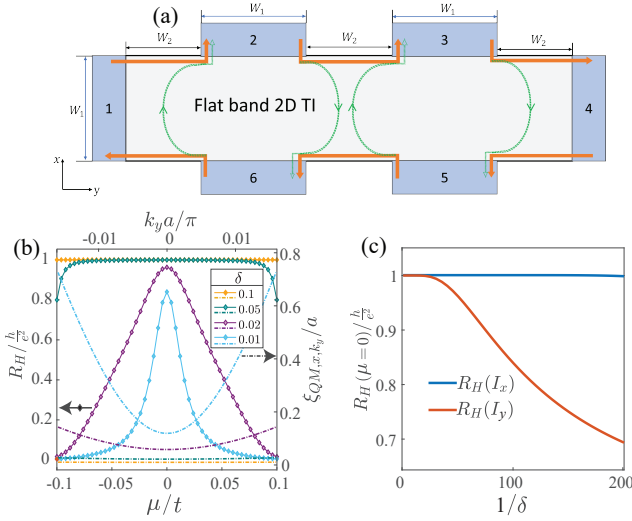


FIG. 4. **Quantum Hall resistivity and the QML.** (a) A 6-terminal Hall-bar device. The green dashed lines represent inter-edge scattering of the chiral edge states due to a large QML. (b) The Hall resistivity R_H (solid lines) and the QML ξ_{QM,x,k_y} (dash-dot lines) versus the chemical potential μ/t for different δ . The chemical potential in the lower axis is locked with the momentum in the upper axis due to the linear dispersion of TBMs. The width for the device and the leads is fixed at $W_1 = 30a$, and the spatial separation of the terminals is set as $W_2 = 400a$. Other parameters are the same as Fig. 3. (c) The zero-bias Hall resistivity versus $1/\delta$ for current driving along the x -direction and y -direction respectively. The y -directional zero-bias Hall resistivity decreases with an increasing QML while the x -direction response remains quantized.

regime, it is evident that in the long range, the boundary modes are ultimately governed by ξ_g . However, over the first few sites near the boundary, the conventional decay may dominate, and soon transitions to a slower exponential decay dictated by ξ_g . The conventional length ξ_c determines how quickly this transition occurs: a large ξ_c delays the transition, as shown in Fig. 3(d). When $\xi_c \rightarrow 0$, the TBMs only follow the geometrical behavior.

To verify the universal bound from the QML in Eq. (8), we compute the x -OBC spread function Ω_{x,k_y} of the in-gap modes near the Γ -channel ($\vec{k} = 0$), where the conventional length is negligible. The ratio $\Omega_{x,k_y}/a\xi_{QM,x,k_y}$, presented in Fig. 3(h), clearly demonstrates that the QML sets a lower bound on the spread of flat-band TBMs, with tighter saturation for larger δ . The underlying reason can be traced back to Eq. (8), where the dropped terms during inequality scaling diminish in the trivial quantum geometry limit (small quantum metric). Meanwhile, it is evident from Fig. 2(e) that the QML yields a smaller value for larger δ , leading to a tighter bound. In particular, in the trivial atomic limit (vanishing QML), the left and right sides of Eq. (8) both approach zero, concealing the geometric behavior.

E. Quantum Hall plateau shaped by QML

The quantum anomalous Hall effect is experimentally characterized by a quantized Hall plateau and vanishing longitudinal conductivity. The quantized plateau is robust against disorder but will deviate from perfect quantization due to finite-size effects [67–69]. As we point out in this work, the localization behavior of the chiral edge states can be controlled via the QML. A large QML promotes coupling between the edge states on opposite edges [denoted by the green lines in Fig. 4(a)]. This coupling leads to a deviation of the Hall resistivity from its quantized value h/e^2 .

Since gapless chiral edge states with linear dispersion exhibit energy-momentum locking, the in-gap Hall resistivity around zero energy serves as a marker for the QML. We now examine how the Hall plateau is affected by the QML using a 6-terminal Hall-bar device. As illustrated in Fig. 4(a), a current I_y is driven along the y -direction from terminal 1 to terminal 4. Four additional terminals, each with a width equal to that of the device (W_1), are arranged on opposite edges and separated by a length W_2 from each other. By measuring the voltage difference between terminal 2 and 3, $V_{23} = V_2 - V_3$, and the voltage difference between terminal 2 and 6, $V_{26} = V_2 - V_6$, we obtain the longitudinal and Hall resistivity as

$$R_{xx} = \frac{V_{23}}{I_x}, \quad R_H = \frac{V_{26}}{I_x}. \quad (11)$$

Taking the Lieb-QWZ model as an example, we perform the multi-terminal recursive Green's function method [70] to simulate the Hall response. As shown in Fig. 4(b), when the QML is small (large δ), the chiral edge states at opposite edges remain well-separated, yielding a perfectly quantized Hall plateau (orange solid line). When we increase the QML by reducing δ , the Hall resistivity drops below the quantized value due to a stronger overlap between TBMs on opposite edges. And the resistivity also varies with chemical potential μ . This deviation is shaped by the BZ-distribution of the QML, particularly near the Γ -channel where the conventional behavior vanishes. Specifically, since the in-gap TBMs exhibit linear dispersions with Fermi velocity $v_\Gamma = 2ta/\hbar$, the momentum range, e.g., $k_y a \in (-0.015\pi, 0.015\pi)$, is locked to an energy range of $\mu \in (-0.1t, 0.1t)$. Consequently, the QML distribution in the momentum space is projected onto the energy space, which is reflected in the Hall resistivity measurements. The shape of the quantum Hall plateau around zero-energy (solid lines) in Fig. 4(b) mirrors the inverse QML distribution around Γ -channel (dash-dot lines), in which a dip at $k_y = 0$ indicates a smaller QML and thus a poorer coupling between chiral edge states, leading to a larger Hall resistivity. Focusing on the zero-energy response (the Γ channel in momentum space), we plot the Hall resistivity as a function of $1/\delta$ in Fig. 4(c). As indicated by the red line, $R_H(I_y)$ decreases with increased $\xi_{QM,x,k_y=0}$ (via $1/\delta$), reflecting stronger couplings between the chiral edge states.

To further verify how the directional properties of the QML affect the Hall responses, we drive a current along the x -direction and evaluate the Hall resistivity $R_H(I_x)$ upon varying the QML, finding that the quantization of $R_H(I_x)$ is robust, as shown by the blue line in Fig. 4(c). This is consistent with the anisotropic QML of the Lieb-QWZ model presented in Fig. 2(f), in which $\xi_{QM,y,k_x=0}$ is negligibly small and nearly independent of δ . Thus, when the current is driven along the x -direction, the y -direction profiles of the chiral edge states remain localized and separated, sustaining quantized Hall resistivity. The suppressed Hall plateau clearly reflects the QML of the in-gap TBMs, allowing indirect detection for the flat band quantum geometry.

F. QML-enabled $2\Phi_0$ - Φ_0 crossover in Fraunhofer pattern

The critical supercurrent in a Josephson junction oscillates with applied out-of-plane magnetic flux, widely known as the Fraunhofer pattern [71]. Of particular interest is that when the weak link is a QAH insulator, the chiral Andreev edge states mediate the supercurrent. In this case, it has also been pointed out that a crossover from $2\Phi_0$ ($\Phi_0 = h/2e$) to Φ_0 in the periodicity of the Fraunhofer pattern is caused by edge-state coupling [72]. Practically, this crossover occurs when the junction length d is comparable to the conventional length scale ξ_c . In the two opposite limits, the Fraunhofer pattern exhibits $2\Phi_0$ (long junction with $d \gg \xi_c$) and Φ_0 (short junction with $d \ll \xi_c$) oscillations, corresponding to the supercurrent driven by the local Andreev reflection and crossed Andreev reflection processes respectively [72].

In contrast, in flat-band QAH Josephson junctions with large QML, this crossover is expected to happen even in the conventionally long junction limit ($d \gg \xi_c$). As illustrated in Fig. 5(a), the chiral Andreev states at the left and right edges extend deeply into the bulk of the weak link with long-range characteristic length ξ_g controllable by ξ_{QM} . When ξ_g surpasses the junction length d with $\xi_g \gg d$, even though the conventional length ξ_c might be significantly shorter than d , the local Andreev reflection process is still expected to dominate, leading to a Φ_0 Fraunhofer oscillation.

We numerically calculate the Josephson supercurrent using the recursive Green's function method. As shown in Fig. 5(b), the critical supercurrent oscillates with the magnetic flux Φ forming the Fraunhofer pattern. When the QML is short ($\delta = 0.05$), the chiral states on the left and right edges are well separated so that the crossed Andreev reflection process dominates, resulting in a conventional $2\Phi_0$ oscillation. As the QML increases with smaller δ , the QML-enabled long-range behavior of the edge states becomes prominent, and a Φ_0 oscillation emerges. The Fraunhofer pattern suggests that the supercurrent is mediated by Josephson vortices due to the presence of

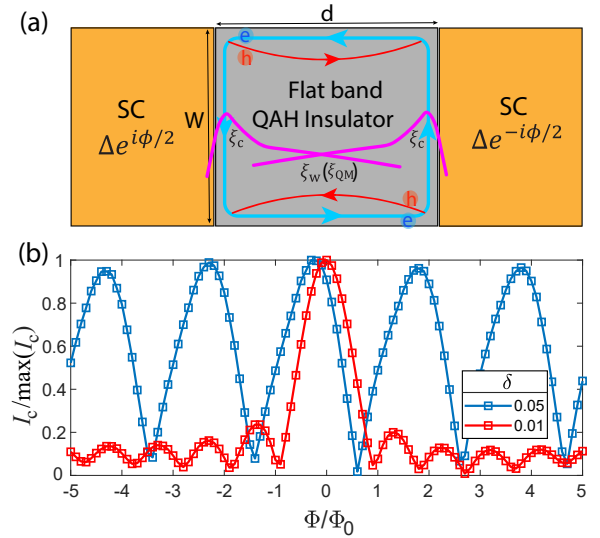


FIG. 5. **Fraunhofer pattern of flat-band QAH Josephson junctions.** (a) An illustration of a flat-band QAH Josephson junction with chiral Andreev edge states coupled to each other controlled by the QML, allowing local Andreev reflections in the junction. (b) A $2\Phi_0$ - Φ_0 crossover in the Fraunhofer pattern occurs when increasing the QML by reducing δ . The weak link has length $d = 100a$ and width $W = 30a$ and is composed of the Lieb-QWZ lattice with $\alpha = 0.5$, $\tilde{t} = 0.8t$.

strong inter-edge scattering. The $2\Phi_0$ - Φ_0 crossover unambiguously pins down the crucial role of the QML in transport measurements.

III. DISCUSSIONS

The pivotal role of the QML revealed in this work suggests that with only low-energy description of topological bands one misses essential quantum geometry. While full-model treatments are computationally intensive, we show that the QML is determined by the BZ-distribution of the non-Abelian quantum metric tensor—particularly dominated by hot-spot regions near direct gaps between flat and dispersive bands—allowing a reasonably accurate QML estimation through the inclusion of the topological flat bands and their dispersive neighbors. Consequently, for moiré materials, continuum models capturing these low-lying bands can also reliably predict boundary-mode localization.

In addition to the topological flat band models studied here, future works can be explored on the QML effects in 3D TIs, topological semimetals, and high-order topological insulators. In this work, we have applied our theory to the lattice model for magic-angle twisted bilayer graphene (more details, see Supplementary Note III), suggesting that 2D moiré materials with topological flat bands are plausible experimental systems to explore the physics of the QML. Furthermore, if we expand the scope to new platforms such as artificial systems, the sim-

ple flat-band topological models developed here could be readily implemented in well-controlled photonic and circuit lattices [73]. The detection of the QML in these systems would be rather promising due to their precise tunability.

IV. METHODS

A. Quantum geometry engineering on flat bands

The quantum geometry of the flat bands can be worked out through energy scale analysis. We denote $\mathcal{E}_{1,\mathbf{k}}$ ($\mathcal{E}_{2,\mathbf{k}}$) as the set of characteristic energy scales of $S_{1,\mathbf{k}}$ ($S_{2,\mathbf{k}}$) in Eq. (4), encompassing hopping amplitudes, on-site energies, Zeeman terms, etc. Then a *local quantum geometry indicator* can be defined as:

$$\lambda_{\mathbf{k}} \equiv \frac{\max(\mathcal{E}_{2,\mathbf{k}})}{\min(\mathcal{E}_{1,\mathbf{k}})} \sim \frac{\max(|\phi_{1,\mathbf{k},\alpha}|)}{\min(|\phi_{2,\mathbf{k},\alpha}|)}, \quad (12)$$

in which the second relation arises from the flat-band eigen-equation, and the last ratio represents orbital compositions at $\mathbf{k}$, reflecting the local quantum geometry of the flat bands. The ratio $\lambda_{\mathbf{k}}$ serves as a crucial tuning parameter throughout our construction. For instance, we can achieve trivial local quantum geometry near the Γ -point ($\mathbf{k} = 0$) by imposing the *local atomic constraint*:

$$\lambda_{\mathbf{k} \rightarrow 0} \ll 1, \quad (13)$$

which means that $S_{2,\mathbf{k} \rightarrow 0}$ has much smaller energy scales compared to $S_{1,\mathbf{k} \rightarrow 0}$. From the bare BCL part of Eq. (4), it is evident that Eq. (13) restricts the flat band eigenvectors to resemble atomic orbitals around the Γ point:

$\mathcal{F}_{\mathbf{k} \rightarrow 0, \alpha} \approx (0, \dots, \overset{(2N_{L_1} + \alpha) \text{th}}{\uparrow} 1, \dots, 0)^T$. A minimal topological coupling among the flat bands can be introduced with [suppose $N_f = 2^g$ ($g \in \mathbb{Z}^+$)]

$$H_c(\mathbf{k}) = \begin{pmatrix} O_{N_{L_1} \times N_{L_1}} & O \\ O & H_d^{(g)}(\mathbf{k}) \end{pmatrix} \quad (14)$$

$$\text{with } H_d^{(g)}(\mathbf{k}) = \sum_{i=1}^d v_i k_i \gamma_i^{(g)} + m \gamma_{d+1}^{(g)}.$$

With Eqs. (1), and (14), we obtain the total Hamiltonian $H_t(\mathbf{k})$ in Eq. (4), and the projected effective model is given in Eq. (5). Besides the minimal coupling given in Eq. (14), alternative coupling schemes and their comparisons can be found in Supplementary Note IV.

Eq. (13) trivializes the flat band states so as to be locally akin to atomic orbitals. To maintain global non-trivial quantum geometry across the BZ, the flat band eigenvectors must deviate atomic orbitals at somewhere away from the Γ point ($\mathbf{k} \neq 0$), e.g., the point where the direct gap separating flat bands and dispersive bands is located, supposed to be at the X -point:

$$\lambda_{\mathbf{k}_X} \sim 1. \quad (15)$$

Given the flat-band eigenvectors in Eq. (3), the conditions in Eqs. (13) and (15) ensure nontrivial quantum distance between Bloch vectors near the Γ - and X -point. Additionally, the X -point in general possesses a large non-Abelian quantum metric. This can be seen by analyzing the inter-band velocity. Consider a dispersive band eigenvector $\mathcal{D}_{\mathbf{k}, \alpha'} = (\varphi_{1,\mathbf{k},\alpha'}, \varphi_{2,\mathbf{k},\alpha'}, \varphi_{3,\mathbf{k},\alpha'})^T$, the inter-band velocity at the X -point is $V_{x,\mathbf{k}}^{\alpha'\alpha} \big|_{\mathbf{k}_X} = \mathcal{D}_{\mathbf{k}, \alpha'}^\dagger \partial_{k_x} H_f \mathcal{F}_{\mathbf{k}, \alpha} \big|_{\mathbf{k}_X} = \left(\varphi_{1,\mathbf{k}}^\dagger \partial_{k_x} S_{1,\mathbf{k}} \phi_{1,\mathbf{k}} + \varphi_{1,\mathbf{k}}^\dagger \partial_{k_x} S_{2,\mathbf{k}} \phi_{2,\mathbf{k}} \right) \big|_{\mathbf{k}_X} \sim a \max(\mathcal{E}_{1,\Gamma})$, which reflects the largest hopping amplitude in the system (assumed to be along x). This implies $V_{x,\mathbf{k}}^{\alpha'\alpha} \big|_{\mathbf{k}_X} \gg \Delta_X$, resulting in a large non-Abelian quantum metric at the X -point:

$$\mathcal{G}_{xx}^{\alpha\alpha}(\mathbf{k}_X) \sim \sum_{\alpha'} \frac{(V_{x,\mathbf{k}}^{\alpha'\alpha})^\dagger (V_{x,\mathbf{k}}^{\alpha'\alpha})}{(\Delta_{\mathbf{k}})^2} \bigg|_{\mathbf{k}_X} \gg a^2. \quad (16)$$

And it can be further enhanced by reducing the X -point band gap. Therefore, a hot spot region near the X -point is expected to emerge in the BZ distribution of the quantum metric.

In the variant Lieb lattice, for example, the energy scales for $S_{1,\mathbf{k}}$ and $S_{2,\mathbf{k}}$ are $\mathcal{E}_{1,\Gamma} = \{J\}$, $\mathcal{E}_{2,\Gamma} = \{\alpha\delta J\}$, yielding $\lambda_\Gamma = \alpha\delta \ll 1$, which satisfies the local atomic constraint in Eq. (13). Meanwhile, it hosts a direct band gap at the X -point, where the energy scales are $\mathcal{E}_{1,X} = \{\delta J\}$, $\mathcal{E}_{2,X} = \{\alpha\delta J\}$. For $\alpha \sim 1$, these scales are comparable, satisfying Eq. (15) and resulting in a band gap $\Delta_X \sim \delta J$. Consequently, the variant Lieb lattices feature a peak of the quantum metric distribution near the X -point which can be further enhanced by reducing the band gap Δ_X .

V. DATA AVAILABILITY

The datasets generated during this study are available from the corresponding authors upon request.

VI. CODE AVAILABILITY

The custom codes generated during this study are available from the corresponding authors upon request.

VII. ACKNOWLEDGEMENTS

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VIII. AUTHOR CONTRIBUTIONS

X.L.M., J.-X.H. and K.T.L. conceived the project idea. X.L.M. performed the calculations and wrote

the manuscript with input from J.-X.H. and K.T.L. All authors contributed to the scientific discussions and manuscript revisions.

IX. COMPETING INTERESTS

The authors declare no competing interests.

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