

Supplementary Note for "Non-Abelian Quantum Metric Governed Topological Boundary-Mode Localization"

King-Lei MA,¹ Jin-Xin Hu,^{1,*} and K. T. Law^{1,†}

¹*Department of Physics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China*

(Dated: October 1, 2026)

CONTENTS

I. Supplementary Note I: Abelian and non-Abelian quantum metric tensor	1
A. Proof for Eqs. (S4) and (S6).	2
II. Supplementary Note II: Spatial behaviors of topological boundary modes	4
A. Eigen-equations: From full Hamiltonian to effective model	4
B. TBMs wave functions	5
C. Two phases of behaviors of TBMs	7
D. Universal bound to the spatial extent of TBMs	8
III. Supplementary Note III: Constructed topological flat-band models	9
A. Lieb-SSH model	9
B. Lieb- $p + ip$ model	10
C. Ten-band model for magic-angle twisted bilayer graphene	11
IV. Supplementary Note IV: More discussions and applications of the QML	13
A. Alternative coupling schemes in constructing topological flat bands	13
B. Non-local resonance tunneling	13
References	15

I. SUPPLEMENTARY NOTE I: ABELIAN AND NON-ABELIAN QUANTUM METRIC TENSOR

We start by revisiting the quantum geometric tensor for a given Hamiltonian $H(\boldsymbol{\kappa})$ living in a d -dimensional parameter space $\mathcal{K}$, with coordinates $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \dots, \kappa_d)$. Let us take an eigenspace $\mathcal{H}_E$ from the h -dimensional Hilbert space of $H(\boldsymbol{\kappa})$ with eigenvalue $E(\boldsymbol{\kappa})$ and degeneracy g . Suppose $\mathcal{H}_E$ is well isolated; i.e., $E(\boldsymbol{\kappa})$ remains separated from other eigenvalues across the parameter space. Provided a complete orthonormal basis $\mathcal{B} = \{\psi_1(\boldsymbol{\kappa}), \dots, \psi_g(\boldsymbol{\kappa})\}$ in $\mathcal{H}_E$, any arbitrary state in the eigenspace can be expressed as

$$\Psi_{0,\boldsymbol{\kappa}} = \sum_{\alpha}^g C_{\alpha}(\boldsymbol{\kappa}) \psi_{\alpha}(\boldsymbol{\kappa}). \quad (\text{S1})$$

Under the basis set $\mathcal{B}$, the state can also be represented as a vector $\mathcal{C}_{\boldsymbol{\kappa}} = [C_1(\boldsymbol{\kappa}), \dots, C_g(\boldsymbol{\kappa})]^T$, and any two states can be related by a $U(g)$ local gauge transformation, rendering $\mathcal{H}_E$ a $U(g)$ vector bundle over $\mathcal{K}$. As a section in the vector bundle, $\Psi_{0,\boldsymbol{\kappa}}$ can be further equipped with intrinsic structures, one of which is the metric. To define it, consider the distance $D(\boldsymbol{\kappa})$ between two neighboring states on $\mathcal{K}$:

$$\begin{aligned} D_{\Psi_0}(\boldsymbol{\kappa}) &= 1 - \left| \Psi_{0,\boldsymbol{\kappa}}^{\dagger} \Psi_{0,\boldsymbol{\kappa}+\text{d}\boldsymbol{\kappa}} \right|^2 \\ &= \sum_{ab}^d \Re \left[\partial_a \Psi_{0,\boldsymbol{\kappa}}^{\dagger} \left(1 - \Psi_{0,\boldsymbol{\kappa}} \Psi_{0,\boldsymbol{\kappa}}^{\dagger} \right) \partial_b \Psi_{0,\boldsymbol{\kappa}} \right] \text{d}\kappa_a \text{d}\kappa_b \\ &= \sum_{ab}^d \mathcal{G}_{ab}^0(\boldsymbol{\kappa}) \text{d}\kappa_a \text{d}\kappa_b, \end{aligned} \quad (\text{S2})$$

* jhuphy@ust.hk

† phlaw@ust.hk

where $\partial_a := \partial/\partial\kappa_a$. Here, $\mathcal{G}_{ab}^0(\kappa)$ is the metric tensor for the state $\Psi_{0,\kappa}$ on $\mathcal{K}$. In particular, when $\mathcal{H}_E$ is non-degenerate, i.e., $g = 1$, $\mathcal{G}_{ab}^0(\kappa)$ recovers the Abelian quantum metric tensor in the original basis $\mathcal{B}$:

$$\mathcal{G}_{ab}(\kappa) = \Re \left[\partial_a \psi_1^\dagger(\kappa) \left(1 - \psi_1(\kappa) \psi_1^\dagger(\kappa) \right) \partial_b \psi_1(\kappa) \right]. \quad (\text{S3})$$

When $g > 1$, the metric tensor $\mathcal{G}_{ab}^0(\kappa)$ can then be related to the non-Abelian quantum metric $\mathcal{G}_{ab}^{\alpha\alpha'}(\kappa)$ in the degenerate manifold by:

$$\begin{aligned} \mathcal{G}_{ab}^0(\kappa) &= \sum_{\alpha\alpha'}^g \mathcal{C}_{\kappa,\alpha}^\dagger \mathcal{G}_{ab}^{\alpha\alpha'}(\kappa) \mathcal{C}_{\kappa,\alpha'} \\ &+ \sum_{0_\perp \in \mathcal{H}_E}^{g-1} \Re \left(\partial_a \Psi_{0_\perp,\kappa}^\dagger \Psi_{0,\kappa} \Psi_{0,\kappa}^\dagger \partial_b \Psi_{0_\perp,\kappa} \right), \end{aligned} \quad (\text{S4})$$

where $\Psi_{0_\perp,\kappa}$ denotes one of the $g - 1$ states in $\mathcal{H}_E$ that is perpendicular to $\Psi_{0,\kappa}$. Note that the second term is non-vanishing unless considering the quantum adiabatic limit [S1], and this has not been clarified in [S2, S3]. To see this, we decompose $\partial_a \Psi_{0,\kappa} = D_a \Psi_{0,\kappa} + \sum_\alpha^g C_\alpha(\kappa) [1 - P_0(\kappa)] \partial_a \psi_\alpha(\kappa)$, with $D_a \Psi_{0,\kappa} = \sum_\alpha^g [\partial_a C_\alpha(\kappa)] \psi_\alpha(\kappa) + \sum_\alpha^g C_\alpha(\kappa) P_0(\kappa) \partial_a \psi_\alpha(\kappa)$ where the second term is related to the Christoffel symbol in [S4], and the projector $P_0(\kappa) := \sum_\alpha^g \psi_\alpha(\kappa) \psi_\alpha^\dagger(\kappa)$. If $\Psi_{0,\kappa}$ evolves in the quantum adiabatic limit in the parameter space with [S5]

$$D_a \Psi_{0,\kappa} = 0, \quad (\text{S5})$$

the second term in Eq. (S4) will vanish and lead to $\mathcal{G}_{ab}^0(\kappa) = \sum_{\alpha\alpha'}^g \mathcal{C}_{\kappa,\alpha}^\dagger \mathcal{G}_{ab}^{\alpha\alpha'}(\kappa) \mathcal{C}_{\kappa,\alpha'}$. The non-Abelian quantum metric tensor $\mathcal{G}_{ab}^{\alpha\alpha'}(\kappa)$ is defined as follows:

$$\begin{aligned} \mathcal{G}_{ab}^{\alpha\alpha'}(\kappa) &\equiv \Re \left[\partial_a \psi_\alpha^\dagger \left(1 - \sum_{\beta \in \mathcal{H}_E}^g \psi_\beta \psi_\beta^\dagger \right) \partial_b \psi_{\alpha'} \right] \\ &= \sum_{\beta \notin \mathcal{H}_E}^{h-g} \Re \left[\frac{\psi_\alpha^\dagger \partial_a H(\kappa) \psi_\beta \psi_\beta^\dagger \partial_b H(\kappa) \psi_{\alpha'}}{(E(\kappa) - E_\beta(\kappa))^2} \right], \end{aligned} \quad (\text{S6})$$

which, together with the non-Abelian Berry curvature $\Omega_{ab}(\kappa)$, constitutes the non-Abelian quantum geometric tensor,

$$\mathcal{Q}_{ab}^{\alpha\alpha'}(\kappa) = \mathcal{G}_{ab}^{\alpha\alpha'}(\kappa) + \frac{1}{2i} \Omega_{ab}^{\alpha\alpha'}(\kappa). \quad (\text{S7})$$

In 2D, both the real part and imaginary part are nontrivial, and their interesting interplay is the focus of this work. Numerically we use the second line in Eq. (S6) for calculation to avoid the gauge problem. The proof for Eq. (S4) and Eq. (S6) is given in Appendix IA.

A. Proof for Eqs. (S4) and (S6).

Starting from the Hamiltonian $H(\kappa)$ with g -fold degenerate eigenspace $\mathcal{H}_E$, we decompose it with three sets of projectors: $P_{0,\kappa} \equiv \Psi_{0,\kappa} \Psi_{0,\kappa}^\dagger$ for the target state; $P_{0_\perp,\kappa} \equiv \sum_{0_\perp \in \mathcal{S}_\mathcal{F}}^{g-1} \Psi_{0_\perp,\kappa} \Psi_{0_\perp,\kappa}^\dagger$ for the states in $\mathcal{H}_E$ orthogonal to $\Psi_{0,\kappa}$; and $Q_{0,\kappa} \equiv 1 - (P_{0,\kappa} + P_{0_\perp,\kappa}) = \sum_{\beta \notin \mathcal{S}_\mathcal{F}}^{h-g} \Psi_{\beta,\kappa} \Psi_{\beta,\kappa}^\dagger$ for states outside $\mathcal{H}_E$, satisfying $H(\kappa) Q_{0,\kappa} = \sum_{\beta \notin \mathcal{S}_\mathcal{F}}^{h-g} E_\beta(\kappa) \Psi_{\beta,\kappa} \Psi_{\beta,\kappa}^\dagger$. The Hamiltonian can then be expressed as:

$$H(\kappa) = E(\kappa) P_{0,\kappa} + E(\kappa) P_{0_\perp,\kappa} + Q_{0,\kappa} H(\kappa) Q_{0,\kappa}. \quad (\text{S8})$$

Consider

$$\begin{aligned} \partial_a H(\kappa) P_{0,\kappa} &= \partial_a E(\kappa) P_{0,\kappa} + E(\kappa) \partial_a P_{0,\kappa} P_{0,\kappa} + E(\kappa) \partial_a P_{0_\perp,\kappa} P_{0,\kappa} - Q_{0,\kappa} H(\kappa) (\partial_a P_{0,\kappa} + \partial_a P_{0_\perp,\kappa}) P_{0,\kappa} \\ &= \partial_a E(\kappa) P_{0,\kappa} + E(\kappa) \partial_a P_{0,\kappa} P_{0,\kappa} + E(\kappa) \partial_a P_{0_\perp,\kappa} P_{0,\kappa} - H(\kappa) (\partial_a P_{0,\kappa} + \partial_a P_{0_\perp,\kappa}) P_{0,\kappa} \\ &\quad + E(\kappa) (P_{0,\kappa} + P_{0_\perp,\kappa}) \partial_a (P_{0,\kappa} + P_{0_\perp,\kappa}) P_{0,\kappa}. \end{aligned} \quad (\text{S9})$$

For any two projectors $P_{1,\kappa}$ and $P_{2,\kappa}$, we observe that

$$\begin{aligned} P_{1,\kappa} \partial_a P_{1,\kappa} P_{2,\kappa} &= P_{1,\kappa} (\partial_a P_{1,\kappa} P_{1,\kappa} + P_{1,\kappa} \partial_a P_{1,\kappa}) P_{2,\kappa} \\ &= P_{1,\kappa} \partial_a P_{1,\kappa} P_{1,\kappa} P_{2,\kappa} + P_{1,\kappa} \partial_a P_{1,\kappa} P_{2,\kappa}, \end{aligned} \quad (S10)$$

implying that $P_{1,\kappa} \partial_a P_{1,\kappa} P_{1,\kappa} P_{2,\kappa} = 0$. In the specific case where $P_{1,\kappa} = (P_{0,\kappa} + P_{0\perp,\kappa})$ and $P_{2,\kappa} = P_{0,\kappa}$, one has $P_{1,\kappa} P_{2,\kappa} = P_{2,\kappa}$, which leads to $P_{1,\kappa} \partial_a P_{1,\kappa} P_{2,\kappa} = 0$. As a result, the fourth term in the last step of Eq. (S9) vanishes, and Eq. (S9) then simplifies to:

$$\partial_a P_{0,\kappa} P_{0,\kappa} = -Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} \partial_a H(\kappa) P_{0,\kappa} - \partial_a P_{0\perp,\kappa} P_{0,\kappa}. \quad (S11)$$

Similarly,

$$P_{0,\kappa} \partial_a P_{0,\kappa} = -P_{0,\kappa} \partial_a H(\kappa) Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} - P_{0,\kappa} \partial_a P_{0\perp,\kappa}. \quad (S12)$$

Combining Eq. (S11) and Eq. (S12), it follows that

$$\partial_a P_{0,\kappa} = - \left[Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} \partial_a H(\kappa) P_{0,\kappa} + P_{0,\kappa} \partial_a H(\kappa) Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} + (\partial_a P_{0\perp,\kappa} P_{0,\kappa} + P_{0,\kappa} \partial_a P_{0\perp,\kappa}) \right]. \quad (S13)$$

Then, consider the operator trace for $\partial_a P_{0,\kappa} \partial_b P_{0,\kappa}$:

$$\begin{aligned} &\text{Tr} (\partial_a P_{0,\kappa} \partial_b P_{0,\kappa}) \\ &= \text{Tr} \left\{ \left[Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} \partial_a H(\kappa) P_{0,\kappa} + P_{0,\kappa} \partial_a H(\kappa) Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} + (\partial_a P_{0\perp,\kappa} P_{0,\kappa} + P_{0,\kappa} \partial_a P_{0\perp,\kappa}) \right] \right. \\ &\quad \cdot \left. \left[Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} \partial_b H(\kappa) P_{0,\kappa} + P_{0,\kappa} \partial_b H(\kappa) Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} + (\partial_b P_{0\perp,\kappa} P_{0,\kappa} + P_{0,\kappa} \partial_b P_{0\perp,\kappa}) \right] \right\} \\ &= \sum_{\beta \notin \mathcal{H}_E}^{h-g} \left[\frac{\Psi_0^\dagger \partial_a H(\kappa) \Psi_\beta \Psi_\beta^\dagger \partial_b H(\kappa) P_{0,\kappa}}{(E_\beta(\kappa) - E(\kappa))^2} + (a \leftrightarrow b) \right] + \text{Tr} (\partial_a P_{0\perp,\kappa} P_{0,\kappa} \partial_b P_{0\perp,\kappa} P_{0,\kappa} \\ &\quad + \partial_a P_{0\perp,\kappa} P_{0,\kappa} \partial_b P_{0\perp,\kappa} + P_{0,\kappa} \partial_a P_{0\perp,\kappa} \partial_b P_{0\perp,\kappa} P_{0,\kappa} + P_{0,\kappa} \partial_a P_{0\perp,\kappa} P_{0,\kappa} \partial_b P_{0\perp,\kappa}) \\ &\quad + \sum_{\beta \notin \mathcal{H}_E}^{h-g} \text{Tr} \left[\frac{P_{\beta,\kappa}}{E_\beta(\kappa) - E(\kappa)} \partial_a H(\kappa) P_{0,\kappa} \partial_b P_{0\perp,\kappa} + \partial_a P_{0\perp,\kappa} P_{0,\kappa} \partial_b H(\kappa) \frac{P_{\beta,\kappa}}{E_\beta(\kappa) - E(\kappa)} + (a \leftrightarrow b) \right], \end{aligned} \quad (S14)$$

where, in the second step, we have used:

$$Q_{0,\kappa} \frac{1}{H(\kappa) - E(\kappa)} Q_{0,\kappa} = \sum_{\beta \notin S_{\mathcal{F}}}^{h-g} \frac{1}{E_\beta(\kappa) - E(\kappa)} \Psi_\beta(\kappa) \Psi_\beta^\dagger(\kappa). \quad (S15)$$

Note

$$\begin{aligned} P_{0,\kappa} \partial_a P_{0\perp,\kappa} &= P_{0,\kappa} \partial_a P_{0\perp,\kappa} P_{0\perp,\kappa}, \\ \partial_a P_{0\perp,\kappa} P_{0,\kappa} &= P_{0\perp,\kappa} \partial_a P_{0\perp,\kappa} P_{0,\kappa}, \end{aligned} \quad (S16)$$

and

$$P_{0,\kappa} \partial_a P_{0\perp,\kappa} = \partial_a (P_{0,\kappa} P_{0\perp,\kappa}) - \partial_a P_{0,\kappa} P_{0\perp,\kappa} = -\partial_a P_{0,\kappa} P_{0\perp,\kappa}. \quad (S17)$$

Then Eq. (S14) can be further simplified:

$$\begin{aligned} \text{Tr} (P_{0,\kappa} \partial_b P_{0,\kappa}) &= \sum_{\beta \notin \mathcal{H}_E}^{h-g} \left[\frac{\Psi_0^\dagger \partial_a H(\kappa) \Psi_\beta \Psi_\beta^\dagger \partial_b H(\kappa) P_{0,\kappa}}{(E_\beta(\kappa) - E(\kappa))^2} + (a \leftrightarrow b) \right] + \text{Tr} (\partial_a P_{0\perp,\kappa} P_{0,\kappa} \partial_b P_{0\perp,\kappa} + \partial_a P_{0,\kappa} P_{0\perp,\kappa} \partial_b P_{0,\kappa}) \\ &= \sum_{\beta \notin \mathcal{H}_E}^{h-g} \left[\frac{\Psi_0^\dagger \partial_a H(\kappa) \Psi_\beta \Psi_\beta^\dagger \partial_b H(\kappa) P_{0,\kappa}}{(E_\beta(\kappa) - E(\kappa))^2} + (a \leftrightarrow b) \right] + \sum_{0\perp \in \mathcal{H}_E}^{g-1} 2\Re \left(\partial_a \Psi_{0\perp,\kappa}^\dagger \Psi_{0,\kappa} \Psi_{0,\kappa}^\dagger \partial_b \Psi_{0\perp,\kappa} \right). \end{aligned} \quad (S18)$$

In the above procedures we have repeatedly applied the following useful relation:

$$\partial_a P_{0,\kappa} = \partial_a (P_{0,\kappa} P_{0,\kappa}) = \partial_a P_{0,\kappa} P_{0,\kappa} + P_{0,\kappa} \partial_a P_{0,\kappa}. \quad (\text{S19})$$

With Eq. (S18), let us re-examine the quantum metric tensor of the states $\Psi_{0,\kappa}$ on the degenerate eigenspace $\mathcal{H}_E$, defined in Eq. (S2):

$$\begin{aligned} \mathcal{G}_{ab}^0(\kappa) &= \Re \left[\partial_a \Psi_{0,\kappa}^\dagger \left(1 - \Psi_{0,\kappa} \Psi_{0,\kappa}^\dagger \right) \partial_b \Psi_{0,\kappa} \right] \\ &= \frac{1}{2} \text{Tr}(\partial_a P_{0,\kappa} \partial_b P_{0,\kappa}) \\ &= \frac{1}{2} \left\{ \sum_{\beta \notin \mathcal{H}_E}^{h-g} \left[\frac{\Psi_{0,\kappa}^\dagger \partial_a H(\kappa) \Psi_{\beta,\kappa} \Psi_{\beta,\kappa}^\dagger \partial_b H(\kappa) \Psi_{0,\kappa}}{(E_\beta(\kappa) - E(\kappa))^2} + (a \leftrightarrow b) \right] + \sum_{0_\perp \in \mathcal{H}_E}^{g-1} 2\Re \left(\partial_a \Psi_{0_\perp,\kappa}^\dagger \Psi_{0,\kappa} \Psi_{0,\kappa}^\dagger \partial_b \Psi_{0_\perp,\kappa} \right) \right\} \\ &= \left[\sum_{\alpha\alpha'}^g C_{\kappa,\alpha}^\dagger \mathcal{G}_{ab}^{\alpha\alpha'}(\kappa) C_{\kappa,\alpha'} + \sum_{0_\perp \in \mathcal{H}_E}^{g-1} \Re \left(\partial_a \Psi_{0_\perp,\kappa}^\dagger \Psi_{0,\kappa} \Psi_{0,\kappa}^\dagger \partial_b \Psi_{0_\perp,\kappa} \right) \right], \end{aligned} \quad (\text{S20})$$

where the non-Abelian quantum metric $\mathcal{G}_{ab}^{\alpha\alpha'}(\kappa)$ is defined as:

$$\begin{aligned} \mathcal{G}_{ab}^{\alpha\alpha'}(\kappa) &= \Re \left[\partial_a \psi_\alpha^\dagger \left(1 - \sum_\beta^g \psi_\beta \psi_\beta^\dagger \right) \partial_b \psi_{\alpha'} \right] \\ &= \sum_{\beta \notin \mathcal{H}_E} \Re \left[\frac{\psi_\alpha^\dagger \partial_a H(\kappa) \psi_\beta \psi_\beta^\dagger \partial_b H(\kappa) \psi_{\alpha'}}{(E_f - E_\beta)^2} \right]. \end{aligned} \quad (\text{S21})$$

This finishes the proof for Eqs. (S4) and (S6).

II. SUPPLEMENTARY NOTE II: SPATIAL BEHAVIORS OF TOPOLOGICAL BOUNDARY MODES

A. Eigen-equations: From full Hamiltonian to effective model

With open boundary conditions (OBC) or a domain wall, e.g., along the x -direction, $\tilde{\mathbf{k}} = (k_y, \dots, k_d)$ remains a good quantum number. The Hamiltonian can be separated into a $\mathbf{k}$ -dependent part $H_t(\mathbf{k})$ and a spatially varying part $U(x)$, resulting in the following eigen-equation of the topological boundary modes (TBMs):

$$[H_t(\mathbf{k}) + U(x)] \Psi^B(x, \tilde{\mathbf{k}}) = E^B \Psi^B(x, \tilde{\mathbf{k}}). \quad (\text{S22})$$

To project the full Hamiltonian onto the flat band manifold, we express the TBMs wave functions in terms of the hybrid Wannier orbitals of the flat bands:

$$\Psi^B(x, \tilde{\mathbf{k}}) = \sum_\alpha \sum_{R'_x}^{N_f} \mathcal{U}_\alpha(R'_x) \mathcal{W}_\alpha(x - R'_x, \tilde{\mathbf{k}}), \quad (\text{S23})$$

with

$$\mathcal{W}_\alpha(x - R_x, \tilde{\mathbf{k}}) \equiv \frac{1}{\sqrt{N}} \sum_{k_x} e^{ik_x(x - R_x)} \mathcal{F}_{\mathbf{k},\alpha}, \quad (\text{S24})$$

where R_x denotes the lattice translation vector in the x -direction, and $\mathcal{F}_{\mathbf{k},\alpha}$ are Bloch vectors of the original exact flat bands. The hybrid Wannier orbitals form orthonormal bases in the flat-band manifold:

$$\int \mathcal{W}_\alpha^\dagger(x - R_x, \tilde{\mathbf{k}}) \mathcal{W}_\beta(x - R'_x, \tilde{\mathbf{k}}) dx = \delta_{\alpha\beta} \delta_{R_x R'_x}. \quad (\text{S25})$$

We then plug Eq.(S23) into Eq.(S22), multiply it by $\mathcal{W}^\alpha(x - R_x, \tilde{\mathbf{k}})$ from the left and then integrate over x , which results in the following eigen-equation:

$$\sum_\alpha \sum_{R_x} \left[\mathcal{H}_{tf}^{\alpha\beta}(R'_x - R_x, \tilde{\mathbf{k}}) - E^B \delta_{R_x R'_x} \delta^{\alpha\beta} + U^{\alpha\beta}(R'_x, R_x) \right] \mathcal{U}_\alpha(R_x) = 0, \quad (\text{S26})$$

where we have used:

$$\begin{aligned}
& \int dx \mathcal{W}_\alpha^\dagger(x - R_x, \tilde{\mathbf{k}}) H_t \mathcal{W}_\beta(x - R'_x, \tilde{\mathbf{k}}) \\
&= \frac{1}{N_x} \sum_{k_x} e^{ik_x(R_x - R'_x)} \mathcal{F}_{\mathbf{k}}^{\alpha\dagger} H_t(\mathbf{k}) \mathcal{F}_{\mathbf{k}}^\beta \\
&= \frac{1}{N_x} \sum_{k_x} e^{ik_x(R_x - R'_x)} H_{\alpha\beta}^{\text{eff}}(\mathbf{k}) \\
&\equiv \mathcal{H}_{\alpha\beta}(R'_x - R_x, \tilde{\mathbf{k}}),
\end{aligned} \tag{S27}$$

with $\mathcal{H}_{\alpha\beta}(R_x, \tilde{\mathbf{k}}) \equiv \frac{1}{N_x} \sum_{k_x} e^{-ik_x R_x} H_{\alpha\beta}^{\text{eff}}(\mathbf{k})$. In the second step of Eq. (S27), we have projected the full Hamiltonian onto the original flat-band manifold with $H_{\alpha\beta}^{\text{eff}}(\mathbf{k}) \equiv \mathcal{F}_{\mathbf{k}}^{\alpha\dagger} H_t(\mathbf{k}) \mathcal{F}_{\mathbf{k}}^\beta$. We have also defined $U^{\alpha\beta}(R_x, R'_x) \equiv \int \mathcal{W}_\alpha^\dagger(x - R_x, \tilde{\mathbf{k}}) U(x) \mathcal{W}_\beta(x - R'_x, \tilde{\mathbf{k}}) dx$ in Eq. (S26). In general, the projected potential $U^{\alpha\beta}(R_x, R'_x)$ depends on the specific Wannier basis one choose. However, notice that for topologically trivial bands, one can choose a gauge for $\mathcal{F}_{\mathbf{k},\alpha}$ in Eq. (S24) that makes the Bloch states maximally smooth in k-space, producing Wannier orbitals that sharply peak at the orbital site. So effectively the Wannier orbital can be viewed as a delta function with most of its weight concentrated at that peak (despite possessing a long-range tail). Therefore, consider maximally localized wannier orbitals as the basis in Eq. (S23), the transformed potential approximates to the on-site form of $U(x)$ as:

$$\begin{aligned}
U_{\alpha\beta}(R_x, R'_x) &\equiv \int \mathcal{W}_\alpha^\dagger(x - R_x, \tilde{\mathbf{k}}) U(x) \mathcal{W}_\beta(x - R'_x, \tilde{\mathbf{k}}) dx \\
&\approx m(x) \Big|_{x=R_x} \delta_{\alpha\beta} \delta_{R_x R'_x},
\end{aligned} \tag{S28}$$

where the external potential has been replaced by the spatially varying mass term $m(x)$ controlling the topological regime.

Note

$$\begin{aligned}
\sum_{R'_x} \mathcal{H}^{\alpha\beta}(R'_x - R_x, \tilde{\mathbf{k}}) \mathcal{U}_\alpha(R'_x) &= \sum_{R'_x} \mathcal{H}^{\alpha\beta}(R'_x, \tilde{\mathbf{k}}) \mathcal{U}_\alpha(R'_x + R_x) \\
&= \sum_{R'_x} \mathcal{H}^{\alpha\beta}(R'_x, \tilde{\mathbf{k}}) \mathcal{U}_\alpha(x + R'_x) \Big|_{x=R_x} \\
&= \sum_{R'_x} \mathcal{H}^{\alpha\beta}(R'_x, \tilde{\mathbf{k}}) \left[\mathcal{U}_\alpha(x) + R'_x \partial_x \mathcal{U}_\alpha(x) + \frac{1}{2} R'^2_x \partial_x^2 \mathcal{U}_\alpha(x) + \dots \right] \Big|_{x=R_x} \\
&= \sum_{R'_x} \mathcal{H}^{\alpha\beta}(R'_x, \tilde{\mathbf{k}}) e^{iR'_x(-i\partial_x)} \mathcal{U}_\alpha(x) \Big|_{x=R_x} \\
&= H_{\alpha\beta}^{\text{eff}}(-i\partial_x, \tilde{\mathbf{k}}) \mathcal{U}_\alpha(x) \Big|_{x=R_x},
\end{aligned} \tag{S29}$$

Eq. (S26) is transformed into the following effective eigen-equation:

$$\left[H_{\alpha\beta}^{\text{eff}}(-i\partial_{R_x}, \tilde{\mathbf{k}}) + m(R_x) \delta_{\alpha\beta} \right] \mathcal{U}^\beta(R_x) = E^B \mathcal{U}_\alpha(R_x), \tag{S30}$$

the solution to which gives the modulation vector in the TBMs wave functions in Eq. (S23).

B. TBMs wave functions

Consider TBMs arising at a domain wall that separates regions $x < 0$ and $x > 0$, each characterized by a distinct topological regime. The mass terms on the two sides are:

$$m(R_x) = \begin{cases} m_l & \text{if } R_x < 0 \\ m_r & \text{if } R_x > 0, \end{cases} \tag{S31}$$

with $|m_r| < |\sum_i \tilde{t}_i|$ and $|m_l| > |\sum_i \tilde{t}_i|$. To get an analytical form of the TBMs wave functions, we separate the effective Hamiltonian in Eq. (S30) into the quasi-1d part along the OBC direction $H_1^{(g)}(\mathbf{k})$ and the parallel-direction part $H_{d-1}^{(g)}(\mathbf{k})$:

$$H_{\text{eff}}^{\alpha\beta}(-i\partial_{R_x}, \tilde{\mathbf{k}}) + m(R_x) = H_1^{(g)}(-i\partial_{R_x}, \tilde{\mathbf{k}}) + H_{d-1}^{(g)}(\tilde{\mathbf{k}}), \quad (\text{S32})$$

with

$$\begin{aligned} H_1^{(g)}(-i\partial_{R_x}, \tilde{\mathbf{k}}) &= 2t_1 \sin(-i\partial_{R_x}) \gamma_1^{(g)} + \left[m(R_x) - 2\tilde{t}_1 \cos(-i\partial_{R_x}) - \sum_{i=2}^d 2\tilde{t}_i \cos(\tilde{\mathbf{k}}) \right] \gamma_{d+1}^{(g)} \\ H_{d-1}^{(g)}(\tilde{\mathbf{k}}) &= \sum_{i=2}^d 2t_i \sin(\tilde{\mathbf{k}}) \gamma_i^{(g)}. \end{aligned} \quad (\text{S33})$$

By solving the real-space quasi-1d eigen equation:

$$H_1^{(g)}(-i\partial_{R_x}) \mathcal{U}(R_x, \tilde{\mathbf{k}}) = 0, \quad (\text{S34})$$

we are able to obtain the TBMs in the momentum channel $\tilde{\mathbf{k}}$, with energy $E_{\tilde{\mathbf{k}}} = \sum_{i=2}^d 2t_i \sin(\tilde{\mathbf{k}})$ and wave functions $\Psi^B(x)$. To solve the infinite-order differential equation Eq. (S34), we plug in a trial function $\mathcal{U}_r(R_x, \tilde{\mathbf{k}}) = A_r e^{-R_x/\xi_{c,r}} u$ for region $R_x \geq 0$, where u is a $N_f \times 1$ vector and $\xi_{c,r}$ denotes the localization length on the right region. It yields:

$$2\eta t_1 \sinh(\xi_{c,r}^{-1}) + \left[m_r - 2\tilde{t}_1 \cosh(\xi_{c,r}^{-1}) - \sum_{i=2}^d 2\tilde{t}_i \cos(\tilde{\mathbf{k}}) \right] = 0, \quad (\text{S35a})$$

$$i\gamma_1^{(g)} \gamma_{d+1}^{(g)} u = \eta u. \quad (\text{S35b})$$

The solution to Eq. (S35a) yields the conventional length $\xi_{c,r}$:

$$\xi_{c,r}^{-1} = \begin{cases} \ln \left(\frac{-M \pm \sqrt{M^2 + 4(t_1 - \tilde{t}_1)(t_1 + \tilde{t}_1)}}{2(t_1 - \tilde{t}_1)} \right) & \text{for } \eta = 1, \\ \ln \left(\frac{M \pm \sqrt{M^2 + 4(t_1 - \tilde{t}_1)(t_1 + \tilde{t}_1)}}{2(t_1 + \tilde{t}_1)} \right) & \text{for } \eta = -1, \end{cases} \quad (\text{S36})$$

where $M = m_r - \sum_{i=2}^d 2\tilde{t}_i \cos(\tilde{\mathbf{k}})$. The conventional length $\xi_{c,r}$ is generally complex with the real part $\Re(\xi_{c,r}) > 0$, representing an oscillatory decaying profile. For example, when $M = 0$, the conventional length would be $\xi_{c,r} = 2/\ln \frac{t_1 - \tilde{t}_1}{t_1 + \tilde{t}_1}$ or $\tilde{\xi}_{c,r} = 2/(\ln \frac{t_1 - \tilde{t}_1}{t_1 + \tilde{t}_1} + i\pi)$. The general solution in the right region is a linear combination of the two solutions:

$$\mathcal{U}_r(R_x, \tilde{\mathbf{k}}) = \left(A_r e^{-R_x/\xi_{c,r}} + \tilde{A}_r e^{-R_x/\tilde{\xi}_{c,r}} \right) u, \quad (\text{S37})$$

which means that the conventional behavior of the TBMs is in general not an exponentially oscillatory decay. It will decay exponentially when only one solution to $\xi_{c,r}$ survives. For example, when $t_1 = \tilde{t}_1$, the conventional length can only be $\xi_{c,r} = 1/\ln \frac{M}{2t_1}$, resulting in a single exponentially decaying mode.

Similarly, for the left region $R_x < 0$, the solution to Eq. (S34) would be $\mathcal{U}_l(R_x, \tilde{\mathbf{k}}) = A_l e^{R_x/\xi_{c,l}} u$ where $\xi_{c,l}$ is also determined by Eq. (S36) with m_r replaced by m_l . Then combining the solutions in the two regions, we write down the complete wave functions for the TBMs

$$\Psi^B(x) = \sum_{\alpha} \sum_{R_x} u_{\alpha} \left[\left(A_r e^{-R_x/\xi_{c,r}} + \tilde{A}_r e^{-R_x/\tilde{\xi}_{c,r}} \right) \Theta(R_x) + \left(A_l e^{-R_x/\xi_{c,l}} + \tilde{A}_l e^{-R_x/\tilde{\xi}_{c,l}} \right) \Theta(-R_x - 1) \right] \mathcal{W}_{\alpha}(x - R_x, \tilde{\mathbf{k}}), \quad (\text{S38})$$

with u_{α} being the eigenvector in Eq. (S35b). When it is applied with open boundary conditions in a certain direction, the TBMs would take a similar form to Eq. (S38), except that the two terms in the square bracket will be arranged at opposite boundaries of the system and form two boundary modes individually. In this case, the

object we care about would no longer be the single OBC wave functions, but rather a rearranged form of the wave functions located at opposite boundaries, because they essentially share the entire Wannier orbitals. Specifically, let us denote the left and right boundary modes as $\psi_L(x)$ and $\psi_R(x - x_R)$ with $\psi_L(0)/\psi_R(0) \equiv q$. Then we rearrange the wave functions as $\tilde{\Psi}^B(x) = \sqrt{\frac{1}{1+\mathcal{N}_q}}(\psi_L(x) + q\psi_R(x)\Theta(-x-1))$, where $\mathcal{N}_q \equiv \sum_{x=-\infty}^{-1} |q\psi_R(x)|^2$ to normalize $\tilde{\Psi}^B(x)$. For example, in the case with an asymmetric Wannier orbital, one has $\psi_R(0) = 1$, $\tilde{\Psi}^B(x) = \psi_L(x)$. As a result, the remainder of the discussions follows with $\Psi^B(x)$ in Eq. (S38) or the rearranged wave functions $\tilde{\Psi}^B(x)$.

C. Two phases of behaviors of TBMs

Wannier orbitals are not unique, depending on the specific gauge imposed for the Bloch vectors. However, the Wannier orbitals serving as the bases for a physical observable state cannot be arbitrary. Consider the limit $\xi_c \rightarrow 0$, the TBMs wave functions in Eq. (S38) becomes a simple summation of the Wannier orbitals at the boundary site:

$$\Psi^B(x) \stackrel{\xi_c \rightarrow 0}{=} \sum_{\alpha} \sum_{R_x} u_{\alpha} \delta_{R_x,0} \mathcal{W}_{\alpha}(x - R_x, \tilde{\mathbf{k}}) = \sum_{\alpha} \sum_{R_x} u_{\alpha} \mathcal{W}_{\alpha}(x - 0, \tilde{\mathbf{k}}). \quad (\text{S39})$$

As eigen-states, the Wannier orbitals without topological obstruction naturally adopt an exponentially decaying form [S6, S7]. Therefore, we consider the exponentially localized Wannier orbitals, which are typically of the following form:

$$\begin{aligned} \mathcal{W}_{\alpha}(x - R_x, \tilde{\mathbf{k}}) &= A_{\alpha} \delta(x - R_x) \\ &+ B_{\alpha} e^{-(x-R_x)/\xi_g} \Theta(x - (R_x + 1)), \end{aligned} \quad (\text{S40})$$

where $\Theta(x)$ is the Heaviside step function and ξ_g is the decay length of the exponentially localized Wannier orbitals. Without loss of generality, we have assumed an asymmetric form of Wannier orbitals that decay rightward. ξ_g , A_{α} and B_{α} are also implicitly dependent on $\tilde{\mathbf{k}}$. In general, the coefficients satisfy $|A_{\alpha}| > |B_{\alpha}|$. When we plug Eqs.(S37) (we consider a simplified case with $\xi_c = \tilde{\xi}_c$) and (S40) into Eq. (S23), the boundary-mode wave functions become ($\xi_c \neq \xi_g$):

$$\Psi^B(x, \tilde{\mathbf{k}}) = A_c e^{-x/\xi_c} + A_g e^{-x/\xi_g}, \quad (\text{S41})$$

with

$$A_c = \sum_{\alpha} u_{\alpha} \left[A_{\alpha} - B_{\alpha} (1 - e^{-(\frac{1}{\xi_c} - \frac{1}{\xi_g})})^{-1} \right] \quad (\text{S42a})$$

$$A_g = \sum_{\alpha} u_{\alpha} B_{\alpha} \left(1 - e^{-(\frac{1}{\xi_c} - \frac{1}{\xi_g})} \right)^{-1}. \quad (\text{S42b})$$

Near the boundary, the states initially undergo a conventional oscillatory decay with diminishing oscillation amplitude. Beyond this region, once the oscillation fades, a pure exponential decay emerges. However, these two behaviors may not always be observed simultaneously, as their prominence depends on the relative scales of the conventional length ξ_c and the geometric length ξ_g . When the conventional length dominates, i.e., $\xi_c \gg \xi_g > 0$, Eq. (S42) simplifies to:

$$A_c = \sum_{\alpha} u_{\alpha} (A_{\alpha} + B_{\alpha} \xi_g), A_g = - \sum_{\alpha} u_{\alpha} B_{\alpha} \xi_g. \quad (\text{S43})$$

In this case, the first term in Eq. (S41) overwrites the second, effectively masking the geometric behavior. Conversely, the geometric length will be manifested when $\xi_g > \xi_c$ and it becomes the dominant length scale when $\xi_g \gg \xi_c > 0$. In this regime, Eq. (S42) becomes:

$$A_c = \sum_{\alpha} u_{\alpha} (A_{\alpha} - B_{\alpha} \xi_c), A_g = \sum_{\alpha} u_{\alpha} B_{\alpha} \xi_c. \quad (\text{S44})$$

Examining the two terms in Eq. (S41), it is evident that, at long range, the boundary modes are ultimately governed by the geometric behavior. However, over the first few sites near the boundary, either term may prevail. If the conventional decay initially dominates, it will soon transition to a slower exponential decay dictated by ξ_g . When $\xi_c \rightarrow 0$, the topological boundary modes only follow the geometric behavior.

D. Universal bound to the spatial extent of TBMs

In the flat band limit with $\xi_c \rightarrow 0$, we look at the spread function of the TBMs:

$$\begin{aligned}
\Omega_{\Psi^B}^x &\equiv \int_x ((\Delta \hat{x})_{|\Psi^B(x)|^2}^2 dx \\
&= \int_x \Psi^{B\dagger}(x) \hat{x}^2 \Psi^B(x) dx - \left| \int_x \Psi^{B\dagger}(x) \hat{x} \Psi^B(x) dx \right|^2 \\
&\geq \sum_{\alpha, \alpha'} u_{\alpha}^* u_{\alpha'} \int_x \mathcal{W}_{\alpha}^{\dagger}(x-0, \tilde{\mathbf{k}}) \hat{x}^2 \mathcal{W}_{\alpha'}(x-0, \tilde{\mathbf{k}}) dx - \sum_{R_x} \left| \sum_{\alpha, \alpha'} u_{\alpha}^* u_{\alpha'} \int_x \mathcal{W}_{\alpha}^{\dagger}(x-R_x, \tilde{\mathbf{k}}) \hat{x} \mathcal{W}_{\alpha'}(x-0, \tilde{\mathbf{k}}) dx \right|^2 \\
&= \sum_{\alpha, \alpha'} u_{\alpha}^* u_{\alpha'} \frac{1}{N_x} \sum_{k_x} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha} - \sum_{\alpha, \alpha'} \sum_{\beta, \beta'} u_{\alpha}^* u_{\alpha'} u_{\beta} u_{\beta'}^* \frac{1}{N_x} \sum_{k_x} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \beta'}^{\dagger} \mathcal{F}_{\mathbf{k}, \beta} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha'} \\
&= \frac{1}{N_x} \sum_{\alpha, \alpha'} \sum_{k_x} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \left(u_{\alpha}^* u_{\alpha'} - \sum_{\beta, \beta'} u_{\beta'}^* u_{\alpha'} u_{\beta} u_{\alpha}^* \mathcal{F}_{\mathbf{k}, \beta} \mathcal{F}_{\mathbf{k}, \beta'}^{\dagger} \right) \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha'} \\
&= \frac{1}{N_x} \sum_{\alpha, \alpha'} u_{\alpha}^* u_{\alpha'} \sum_{k_x} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \left(1 - \sum_{\beta, \beta'} u_{\beta'}^* u_{\beta} \mathcal{F}_{\mathbf{k}, \beta} \mathcal{F}_{\mathbf{k}, \beta'}^{\dagger} \right) \partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha'} \\
&\equiv \frac{1}{N_x} \sum_{k_x} \mathcal{G}_{xx}^0(\mathbf{k}),
\end{aligned} \tag{S45}$$

where in the fourth step we have used:

$$\begin{aligned}
&\int_x \mathcal{W}_{\alpha}^{\dagger}(x-R'_x, \tilde{\mathbf{k}}) \hat{x} \mathcal{W}_{\beta}(x-R_x, \tilde{\mathbf{k}}) dx \\
&= \frac{1}{N_x^2} \int_x \sum_{k_x, k'_x} e^{-ik_x(x-R'_x)} e^{ik'_x(x-R_x)} x \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \mathcal{F}_{\mathbf{k}', \beta} dx \\
&= \frac{1}{N_x^2} \int_x \sum_{k_x, k'_x} (i \partial_{k_x} e^{-i(k_x-k'_x)x}) e^{ik_x R'_x} e^{-ik'_x R_x} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \mathcal{F}_{\mathbf{k}', \beta} dx \\
&= -\frac{i}{N_x^2} \sum_{k_x, k'_x} \int_x e^{-i(k_x-k'_x)x} dx e^{ik_x R'_x} e^{-ik'_x R_x} (\partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger}) \mathcal{F}_{\mathbf{k}', \beta} \\
&= -\frac{i}{N_x} \sum_{k_x} e^{ik_x(R'_x-R_x)} (\partial_{k_x} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger}) \mathcal{F}_{\mathbf{k}, \beta}, \\
&= \frac{i}{N_x} \sum_{k_x} e^{ik_x(R'_x-R_x)} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \partial_{k_x} \mathcal{F}_{\mathbf{k}, \beta}
\end{aligned} \tag{S46}$$

and

$$\int_x \mathcal{W}_{\alpha}^{\dagger}(x-R'_x, \tilde{\mathbf{k}}) \hat{x}^2 \mathcal{W}_{\beta}(x-R_x, \tilde{\mathbf{k}}) dx = -\frac{1}{N_x} \sum_{k_x} e^{ik_x(R'_x-R_x)} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \partial_{k_x}^2 \mathcal{F}_{\mathbf{k}, \beta}. \tag{S47}$$

After the inequality scaling in Eq. (S45), we obtain a simple form of the lower bound for the spread of the boundary modes. It is the xx-component of the BZ-integrated quantum metric tensor $\mathcal{G}_{ab}^0(\mathbf{k})$, which is defined as:

$$\begin{aligned}
\mathcal{G}_{ab}^0(\mathbf{k}) &\equiv \frac{1}{2} \left[\sum_{\alpha, \alpha'}^{N_f} u_{\alpha}^* u_{\alpha'} \partial_{k_a} \mathcal{F}_{\mathbf{k}, \alpha}^{\dagger} \left(1 - \sum_{\beta, \beta'}^{N_f} u_{\beta'}^* u_{\beta} \mathcal{F}_{\mathbf{k}, \beta} \mathcal{F}_{\mathbf{k}, \beta'}^{\dagger} \right) \partial_{k_b} \mathcal{F}_{\mathbf{k}, \alpha'} + (a \leftrightarrow b) \right] \\
&= \Re \left[\partial_{k_a} \Psi_0(\mathbf{k}) \left(1 - \Psi_0(\mathbf{k}) \Psi_0^{\dagger}(\mathbf{k}) \right) \partial_{k_b} \Psi_0(\mathbf{k}) \right],
\end{aligned} \tag{S48}$$

where we have defined flat-band eigen-states $\Psi_0(\mathbf{k}) \equiv \sum_{\alpha}^{N_f} u_{\alpha} \mathcal{F}_{\mathbf{k}, \alpha}$. From Eq. (S2) one can readily tell that $\mathcal{G}_{ab}^0(\mathbf{k})$ is the metric that measures the quantum distance between two infinitesimally adjacent states $\Psi_0(\mathbf{k})$ and $\Psi_0(\mathbf{k}+d\mathbf{k})$.

As $\Psi_0(\mathbf{k})$ is a vector in the degenerate flat-band manifold $S_{\mathcal{F}}$, its quantum metric can be related to the non-Abelian quantum metric of the degenerate manifold, as given in Eq. (S4). With that, we find a further inequality between the xx-component of the two metric:

$$\begin{aligned} \mathcal{G}_{xx}^0(\mathbf{k}) &\equiv \sum_{\alpha\alpha'}^{N_f} u_{\alpha}^{\dagger} \mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k}) u_{\alpha'} + \left| \partial_x \Psi_{0\perp, \mathbf{k}}^{\dagger} \Psi_{0, \mathbf{k}} \right|^2 \\ &\geq \sum_{\alpha\alpha'}^{N_f} u_{\alpha}^{\dagger} \mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k}) u_{\alpha'}, \end{aligned} \quad (\text{S49})$$

which, together with the inequality in Eq. (S45), amounts to the core result of this work:

$$\Omega_{\Psi_{\alpha}^B}^x \geq \frac{a}{2\pi} \sum_{\alpha\alpha'}^{N_f} \int_{\text{BZ}} u_{\alpha}^{\dagger} \mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k}) u_{\alpha'} dk_x, \quad (\text{S50})$$

where u_{α} is an eigenvector of $\gamma_1^g \gamma_{d+1}^g$, and $\mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k})$ is the non-Abelian quantum metric of the original N_f degenerate flat bands.

More generally, by defining the i -directional quantum metric length (QML) as:

$$\xi_{QM, i, \bar{\mathbf{k}}} \equiv \sum_{\alpha\alpha'}^{N_f} \frac{1}{2\pi} \int_{\text{BZ}} u_{\alpha}^{\dagger} \mathcal{G}_{ii}^{\alpha\alpha'}(\mathbf{k}) u_{\alpha'} dk_i, \quad (\text{S51})$$

the i -directional spread function $\Omega_{\Psi_{\alpha}^B}^i$ will be lower bounded by the corresponding QML (up to a lattice constant) as: $\Omega_{\Psi_{\alpha}^B}^i \geq a \xi_{QM, i, \bar{\mathbf{k}}}$, with u_{α} being an eigenvector of $\gamma_i^g \gamma_{d+1}^g$.

To better understand above definition for QML, we now add a few remarks. First, as a concrete example, let us consider 2D TIs with coupling matrix $H_2^{(1)}$ under the y -OBC. In this case, u_{α} , the eigenvector of $i\sigma_3^{(2)}\sigma_2^{(2)} \propto \sigma_1$, would have uniform components $u_{\alpha} \propto (1, 1)/\sqrt{2}$, resulting in a rather simplified expression for the QML: $\xi_{QM, x, k_y} = \frac{1}{2\pi N_f} \sum_{\alpha\alpha'}^{N_f} \int_{\text{BZ}} \mathcal{G}_{xx}^{\alpha\alpha'}(\mathbf{k}) dk_x$. Additionally, for topological superconductors with a single flat band in the normal state, such as the Lieb- $p + ip$ model provided in Supplementary Note III, the doubly degenerate flat bands arise from enforced particle-hole symmetry. As a result, the integrand in Eq. (S51) corresponds directly to the Abelian quantum metric of the single normal flat band, $\mathcal{G}_{ab}^f$, as $\frac{1}{N_f} \sum_{\alpha\alpha'} u_{\alpha}^{\dagger} \mathcal{G}_{ab}^{\alpha\alpha'} u_{\alpha'} = \mathcal{G}_{ab}^f$. This ensures that Eq. (S51) aligns with the QML definition in [S8].

III. SUPPLEMENTARY NOTE III: CONSTRUCTED TOPOLOGICAL FLAT-BAND MODELS

A. Lieb-SSH model

With the Lieb-QWZ model introduced in the main text, we consider the channel $k_2 = 0$, and obtain an effective 1d model. Take $d = 1$ and $g = 1$, we have the coupling matrix as:

$$H_1^{(1)}(k) = [t + \tilde{t} \cos(ka)]\sigma_x + \tilde{t} \sin(ka)\sigma_y, \quad (\text{S52})$$

which itself is the SSH model. Embedding Eq. (S52) into the variant Lieb model yields a 1D Lieb-SSH model. When $S_{2, \mathbf{k}} = [(1 + \delta) - (1 - \delta)e^{ik_2 a}, 0]$, the Lieb-SSH model belongs to the AIII class in the AZ classification. It has chiral symmetry $\hat{C} H_{AIII} \hat{C}^{-1} = -H_{AIII}$ with the chiral operator represented as $D(\hat{C}) = \text{diag}(1, -1, -1, 1)$. A lattice structure for the Lieb-SSH model is shown in Fig. S1(a). Similar to the Lieb-QWZ model, the Lieb-SSH model also contains four bands with two flat bands in the middle, as shown in Fig. S1(b), where the flat bands are zoomed in the inset. In Fig. S1(c), we calculate and plot the Abelian quantum metric of one flat band, which peaks at the point $k = \pi$. By tuning δ , the quantum metric at $k = \pi$ varies accordingly, as presented in Fig. S1(d), as expected.

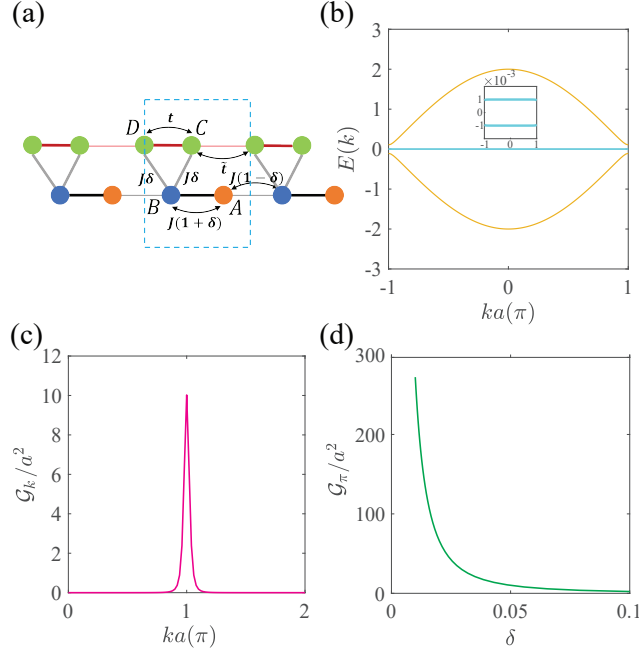


FIG. S1. (a) The real-space lattice structure of the Lieb-SSH model. The green sites represent the C,D orbital forming a SSH chain. (b) The band structure of the Lieb-SSH model featuring two dispersive bands (yellow) and two central flat bands (cyan), with the inset providing a zoomed-in view of the flat bands. The parameters are $J = 1$, $\tilde{t} = 2 \times 10^{-4} J$, $t = 0$, and $\delta = 0.05$. (c) The BZ distribution of the quantum metric tensor $\mathcal{G}_k$ for a single flat band. (d) The quantum metric at $ka = \pi$ is tunable by the parameter δ .

B. Lieb- $p + ip$ model

When the replicated sublattices are introduced by particle-hole symmetry. We can further realize a 2D topological superconductor (TSC). Specifically, a 2D Lieb- $p + ip$ model is realized when imposing:

$$\begin{aligned} S_{1,\mathbf{k}} &= [(1 + \delta) + (1 - \delta)e^{ik_1a}] \sigma_z \\ S_{2,\mathbf{k}} &= [(1 + \delta) - (1 - \delta)e^{ik_2a}] \mathcal{I}_{1 \times 2g} \sigma_z \\ H_2^{(1)}(\mathbf{k}) &= 2\Delta \sin(k_1a) \sigma_x + 2\Delta \sin(k_2a) \sigma_y + M(\mathbf{k}) \sigma_z, \end{aligned} \quad (\text{S53})$$

where $\sigma_{x,y,z}$ are Pauli matrices in the particle-hole basis, and $M(\mathbf{k}) = m - 2t(\cos(k_1a) + \cos(k_2a))$. Here, we also require $\max(t, \Delta, m) \ll \delta J$ to meet the flatness condition. At the M -point, where the gap between the flat bands and higher dispersive bands is located, the local quantum geometry indicator $\lambda_M = 1$ with the band gap $\Delta_g \sim \delta J$. Therefore, the M -point non-Abelian quantum metric can be enhanced by reducing Δ_g via parameter δ according to our framework. A general BdG Hamiltonian should contain pairings among all orbitals, so a realistic coupling matrix for the topological superconductor takes $H_c(\mathbf{k}) = \mathcal{I}_3 \otimes H_2^{(1)}(\mathbf{k})$ with the identity matrix $\mathcal{I}_3$ in the orbital basis. With this enlarged degrees of freedom, we no longer invoke the parameter α in the Lieb-QWZ model, since the band structure of the flat bands in this model is well described by $H_2^{(1)}(\mathbf{k})$ throughout the BZ. The 2D Lieb- $p + ip$ features six Bogoliubov bands with two flat bands in the middle.

To calculate the QML of the flat bands in the Lieb- $p + ip$ model, we need to turn off the coupling term by setting $\Delta = t = m = 0$. In this case, the flat bands become degenerate, enforced by particle-hole symmetry. For a better visual comparison, in Fig. S2(a) and (b), we present the QML for both the Lieb-QWZ model and the Lieb- $p + ip$ model. The QML for both the Lieb-QWZ model and the Lieb- $p + ip$ model is tunable via the parameter δ . To verify the relation in Eq. (S50), we compute the x -OBC spread function Ω_{x,k_y} of the in-gap modes for both models near the Γ -channel ($\tilde{\mathbf{k}} = 0$), where the conventional length is negligible. The results are presented in Fig. S2(c) and (d). Strikingly, both the Lieb-QWZ and the Lieb- $p + ip$ models exhibit a similar trend between the spread Ω_{x,k_y} and QML ξ_{x,k_y} over the momentum channel k_y , revealing an unambiguous origin of the long-range behavior of the flat-band TBMs – the exponentially localized Wannier orbitals. The ratio $\Omega_{x,k_y}/a\xi_{x,k_y}$, shown in Fig. S2(e) and (f), clearly demonstrates that the QML sets a lower bound on the spread of flat-band TBMs.

C. Ten-band model for magic-angle twisted bilayer graphene

Magic angle twisted bilayer graphene (MATBG) is a pioneering 2D platform renowned for its fascinating correlated phenomena, which is driven by the suppressed kinetic energy in the flat bands. This establishes MATBG a versatile arena for harnessing the rich flat-band physics [S10]. One of the questions that has interested the community for a long time is the role of quantum geometry in shaping the MATBG physics. In this subsection, we dig further into the intricate interplay of topology and the quantum metric of the flat bands in MATBG by exploring the quantum metric length underlying the TBMs of the system, which could be responsible for the transport phenomena discovered in twisted bilayer graphene, e.g., the Josephson current found in [S11]. Concretely, we apply our theory on the 10-band lattice model for the twisted bilayer graphene introduced in [S9]. This example not only demonstrates the universal role of the QML, but also suggests a possible platform for detecting the quantum metric length.

In the chiral limit of the MATBG, there are two exact flat bands in the middle of the 10 bands, as plotted in Fig. S3(a), and these flat bands have the same origin as the BCL introduced in the main text. In Fig. S3(b), a logarithmic plot of the BZ distribution of the traced non-Abelian quantum metric summing over the band index pronouncedly exhibits a hot spot region around the Γ point, where the band gap of the flat bands is located. Examining the Hamiltonian structure, we can see that it has a typical structure of the bipartite crystalline lattices (BCL). Interestingly, the off-diagonal block $S_{\mathbf{k}}$ can be partitioned into two parts, $S_{1,\mathbf{k}}$ and $S_{2,\mathbf{k}}$, with contrasting energy scales:

$$S_{1,\mathbf{k}} = J \begin{pmatrix} -(\omega + \phi_{11}\omega^* + \phi_{01})\zeta^*a & -i\omega d & -i\phi_{01}\omega^*d & -i\phi_{10}d \\ (1 + \phi_{11}\omega^* + \phi_{01}\omega)\zeta^*a & -i\omega^*d & -i\phi_{10}d & -i\phi_{01}\omega d \\ (\omega + \phi_{10}\omega^* + \phi_{11})\zeta a & i\omega d & i\omega^*d & -id \\ -(\omega + \phi_{10} + \phi_{11}\omega^*)\zeta a & i\omega^*d & i\omega d & id \end{pmatrix} \quad (\text{S54})$$

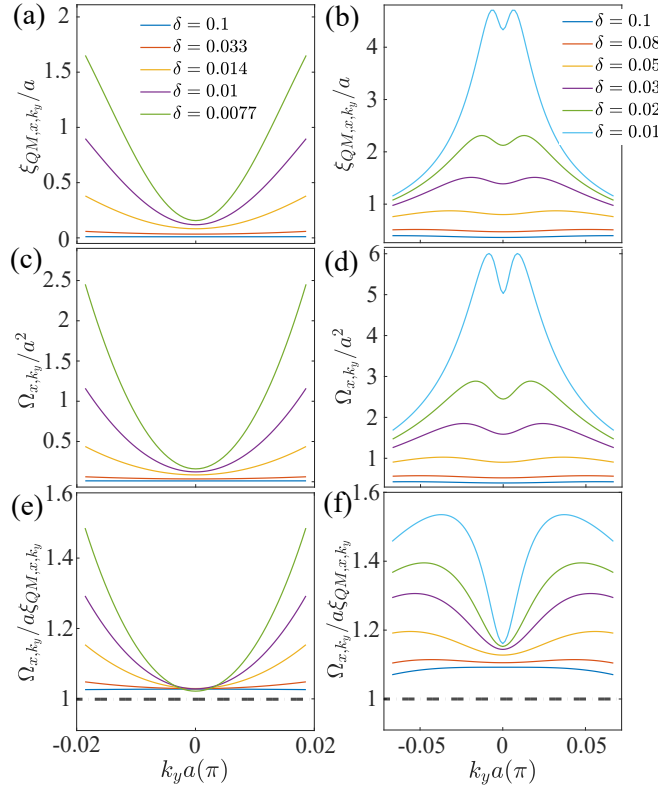


FIG. S2. (a, b) The x -direction QML ξ_{QM,x,k_y} versus different k_y channels for (a) the Lieb-QWZ model and (b) the Lieb- $p + ip$ model, tunable via the parameter δ , with shared parameters $J = 1$, $\tilde{t} = 2 \times 10^{-4}J$, $m = 2\tilde{t}$, $t = \Delta = \tilde{t}$, $\delta = 0.05$, and $\alpha = 0.1$ for the Lieb-QWZ model. (c, d) The x -directional spread function Ω_{x,k_y} for (c) the Lieb-QWZ model and (d) the Lieb- $p + ip$ model; The ratio $\Omega_{x,k_y}/a\xi_{QM}$ where a is the lattice constant, is consistently bounded below by 1 (black dashed line) in both (c) the Lieb-QWZ model and (d) the Lieb- $p + ip$ model, in excellent agreement with Eq. (S50).

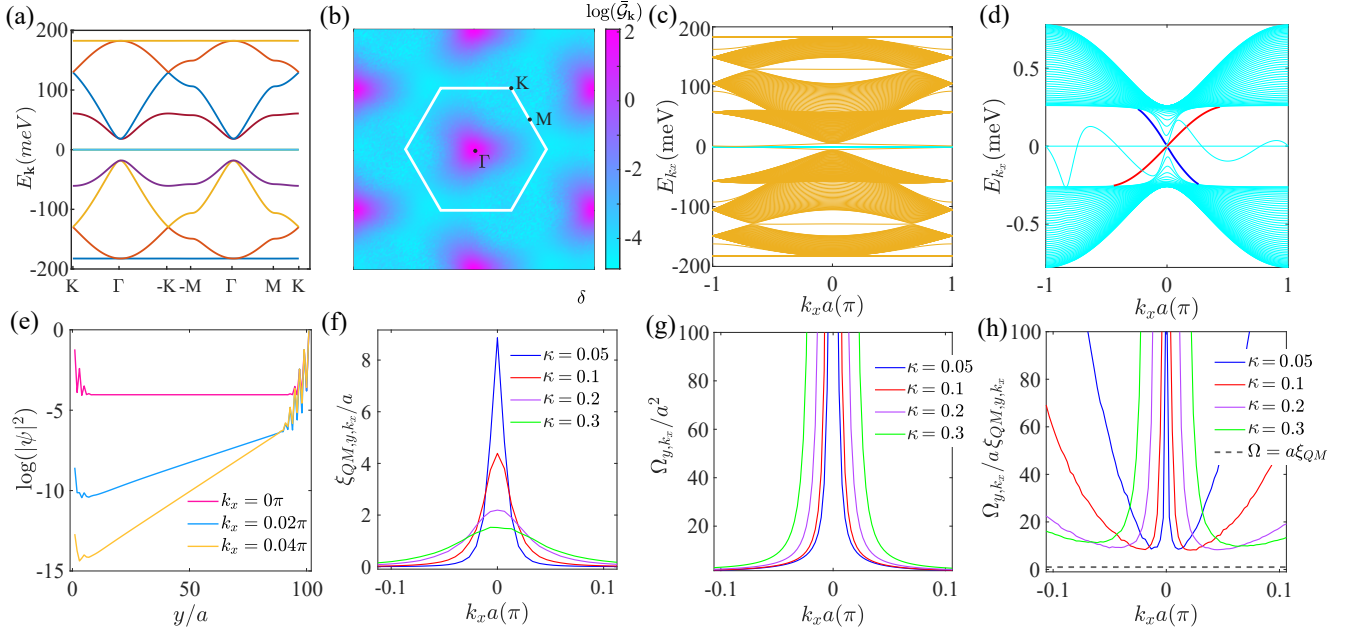


FIG. S3. (a) The band structure for the 10-band model in the chiral limit, featuring two exact flat bands in the middle, with the parameters provided in [S9]. (b) Logarithmic plot of the BZ distribution for the traced non-Abelian quantum metric tensor $\bar{\mathcal{G}}_{\mathbf{k}} \equiv \frac{1}{N_f} \sum_{\alpha\alpha'} \text{Tr}[\mathcal{G}_{ab}^{\alpha\alpha'}(\mathbf{k})]$, revealing a hotspot region with threefold rotation symmetry. (c) Under y -OBC, the full OBC spectrum (orange) sandwiches two topological flat bands (cyan), zoomed in (d), with in-gap TBMs highlighted in red and blue. The flat band coupling parameters are $\tilde{t} = 1 \times 10^{-3}J = 0.13$ meV, $m = 2\tilde{t}$, $t = \tilde{t}$ and $\kappa = 0.3$, and applied to subsequent panels unless otherwise specified. (e) The TBMs near the Γ -channel exhibit both the conventional oscillatory decay and an geometric behavior controlled by the QML. Parameter $t = 0.6\tilde{t}$. (f) The y -directional QML distribution around the Γ -channel, controlled by the gap-closing parameter κ . (g) The spread of the TBMs, also tunable by κ , aligns with the QML behavior away from the Γ -channel, where the conventional length is suppressed. (h) The ratio $\Omega_{y,k_x}/a\xi_{QM,y,k_x}$ of the spread to the QML is well bounded below by 1.

$$S_{2,\mathbf{k}} = \kappa J \begin{pmatrix} -(\omega^* + \phi_{11}\omega + \phi_{01})\zeta b & (1 + \phi_{11} + \phi_{01})c \\ (1 + \phi_{11} + \phi_{01})c & (1 + \phi_{11}\omega + \phi_{01}\omega^*)\zeta b \\ (\omega^* + \phi_{10}\omega + \phi_{11})\zeta^* b & (1 + \phi_{10} + \phi_{11})c \\ (1 + \phi_{10} + \phi_{11})c & (\omega^* + \phi_{10}\omega + \phi_{11}\omega)\zeta^* b \end{pmatrix}, \quad (\text{S55})$$

where $\omega = e^{i2\pi/3}\zeta$, $\zeta^2 = \omega$, and $\phi_{lm} \equiv e^{-i\mathbf{k} \cdot (l\mathbf{a}_1 + m\mathbf{a}_2)}$, with $\mathbf{a}_1$ and $\mathbf{a}_2$ being the crystal lattice vectors and $\bar{l} \equiv -l$. To reproduce the MATBG band structure, the parameters are taken as $J = 130\text{meV}$, $a = 0.11$, $b = c = 0.033$, $d = 0.573$, $\kappa = 1$. After partitioning, the block $S_{2,\mathbf{k}}$ has subdominant energy scales compared to $S_{1,\mathbf{k}}$, allowing us to implement our framework. Since the energy scale $\max(|S_{2,\mathbf{k}}|)$ controls the gap between the flat bands and the higher bands, the non-Abelian quantum metric of the flat bands at the Γ point could be enhanced by decreasing the energy scale of $S_{2,\mathbf{k}}$ using the parameter κ ($0 < \kappa \leq 1$). In Fig. S3(f), we calculated the y -directional quantum metric length ξ_{QM,y,k_x} near the Γ channel ($\tilde{\mathbf{k}} = k_x = 0$) with varying κ using Eq. (S51). As expected, the Γ channel QML is significantly enhanced when decreasing κ , while the QML for the channels away from Γ has an opposite effect from κ .

To make the flat bands topological, we introduce the coupling matrix $H_d^{(g)}$ with $d = 2, g = 1$ instead of the original coupling scheme in [S9] for two reasons. First, it allows flexible control of the conventional behavior, and the geometric behavior can be exhibited solely when exploring the role of the QML. Secondly, the definition of the QML in Eq. (S51) only involves the degenerate exact flat bands, making our primary conclusion independent of how specifically topology is introduced. Therefore, the topological flat bands employed here provide a streamlined counterpart to those in the realistic MATBG, while retaining the essential physics when inspecting the geometric behavior and the role of the QML in TBMs.

Let us examine the role of the QML for the TBMs in this system. Imposing an OBC in the y -direction, we obtain its spectrum in Fig. S3(c), where two flat bands are highlighted in cyan. A zoomed-in view of the topological flat bands, along with the in-gap edge spectrum, is presented in Fig. S3(d). Here, the bulk states around the Γ channel merge into the gap due to the comparable energy scales of flat bands' bandwidth and $S_{2,\mathbf{k}}$. It is also noticeable that, besides the highlighted ones, other in-gap boundary modes also emerge under OBC but do not possess a

topological origin. Those TBMs exhibit two phases of spatial behavior consistent with our discussions in the main text: A conventional oscillatory decay originating from the flat bands dispersion, followed by an exponential decay controlled by the QML. To elucidate the role of the QML, we focus on boundary modes that exhibit purely geometric behavior with $\xi_c = 0$ and examine their spatial spread Ω_{y,k_x} . Nevertheless, not all channels can be well controlled to solely exhibit the geometric behavior; e.g., the Γ channel has a very small gap, as explained earlier, making the flat-bands-determined conventional length a significant contribution to the spatial spread of the TBMs. Consequently, the Γ channel TBMs have a considerably large spatial spread, far exceeding the QML, as shown in Fig. S3(g). Conversely, away from the Γ point where the conventional length is suppressed, the spatial spread has the same dependence on κ as the QML, suggesting a quantum-metric origin of the TBMs behavior in this regime. The ratio between the spatial spread and the QML is presented in Fig. S3(h), clearly indicating that the QML serves as a loose, yet stringent, lower bound for the spread of the TBMs in MATBG. Through this ten-band model, we highlight the ubiquitous presence the QML and its universal role in boundary-mode localization in topological flat-band systems.

IV. SUPPLEMENTARY NOTE IV: MORE DISCUSSIONS AND APPLICATIONS OF THE QML

A. Alternative coupling schemes in constructing topological flat bands

We have introduced the minimal topological coupling in the main text to generate topological flat bands by restricting the coupling term to act exclusively among the orbitals of $L_{2,B}$ in the partitioned BCL model as:

$$H_c(\mathbf{k}) = \begin{pmatrix} O_{N_{L_1} \times N_{L_1}} & O \\ O & H_d^{(g)}(\mathbf{k}) \end{pmatrix}, \quad (\text{S56})$$

with $H_d^{(g)}(\mathbf{k}) = \sum_{i=1}^d 2t_i \sin k_i \gamma_i^{(g)} + \left(m - \sum_{i=1}^d 2\tilde{t}_i \cos k_i\right) \gamma_{d+1}^{(g)}$. For instance, when $d = 1$, $g = 1$ and $N_{L_1} = 2$, the coupling matrix becomes:

$$H_{c,1}(\mathbf{k}) = \begin{pmatrix} O_{2 \times 2} & O_{2 \times 2} \\ O_{2 \times 2} & \begin{pmatrix} m - 2\tilde{t}_1 \cos k_1 & 2t_1 \sin k_1 \\ 2t_1 \sin k_1 & 2\tilde{t}_1 \cos k_1 \end{pmatrix} \end{pmatrix}. \quad (\text{S57})$$

Alternatively, we can introduce other coupling schemes on the degenerate manifolds to introduce topology by including more orbitals on the sublattice L_2 . The following is a concrete example:

$$H_{c,2}(\mathbf{k}) = \begin{pmatrix} O_{1 \times 1} & & O_{1 \times 3} \\ & 0 & 2t_1 \sin k_1 & 2t_1 \sin k_1 \\ O_{3 \times 1} & 2t_1 \sin k_1 & m - 2\tilde{t}_1 \cos k_1 & 2t_1 \sin k_1 \\ & 2t_1 \sin k_1 & 2t_1 \sin k_1 & -(m - 2\tilde{t}_1 \cos k_1) \end{pmatrix}. \quad (\text{S58})$$

Both coupling schemes in Eqs. (S57) and (S58) give rise to the same low-energy physics of the flat bands, thus preserving their topological properties. However, by including more orbitals in the coupling matrix such as Eq. (S58), the orbital components for later TBMs will be rearranged so that the boundary modes at opposite ends can have better overlaps, as presented in Fig. S4. The second coupling scheme is thus adopted in numerical calculations when exploring the resonance tunneling effect and the crossover in the flat-band QAH Fraunhofer patterns.

B. Non-local resonance tunneling

As pointed out in the main text, the TBMs generally possess two distinct and independent phases of behaviors: a conventional behavior resulting from bare band dispersion, and a geometric behavior controlled by the QML. The conventional length ξ_c will be suppressed in the flat band limit, leaving only the geometric behavior dominating the real space physics. In this section, we will introduce some consequences resulting from this geometric behavior of TBMs tunable via the QML. Transport measurements are one of the most effective experimental tools for investigating solid state physics. Here, we introduce a robust non-local resonance tunneling effect as the transport benchmark for detecting the QML in (quasi-)1D topological systems.

TBMs across the bulk can couple with each other when they are overlapped in real space, which enables non-local resonance tunneling inside the topological gap. With the QML this effect can be rather prominent. Specifically, consider a (quasi-) 1D TI hosting topological flat bands that is contacted by two metallic leads at its opposite

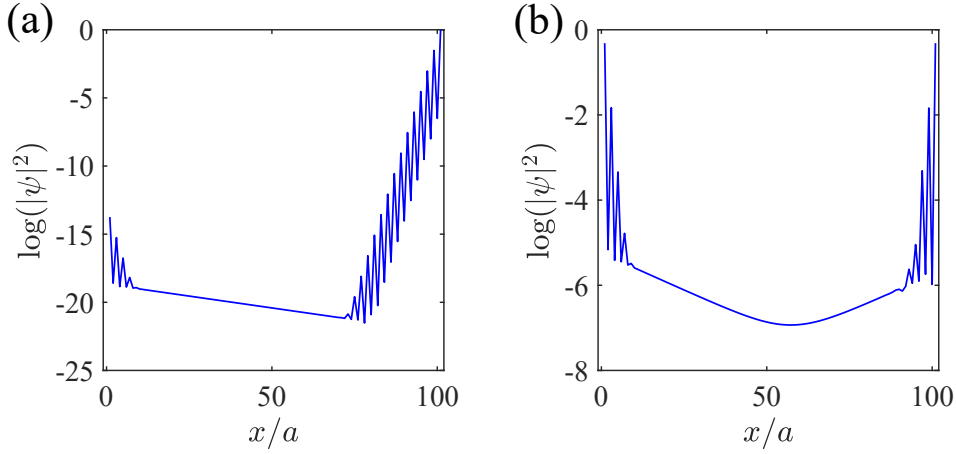


FIG. S4. Logarithmic plots for the TBMs wave functions under the coupling scheme (a) $H_{c,1}$ and (b) $H_{c,2}$.

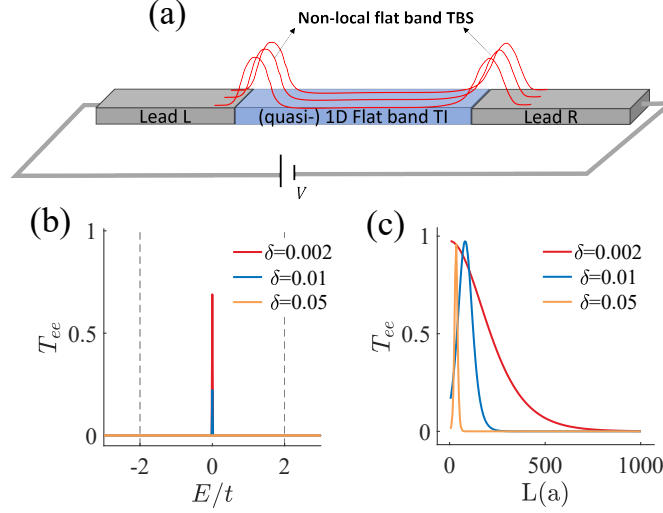


FIG. S5. (a) An illustration of the resonance tunneling in a flat-band TI. The red lines represent the non-local flat-band topological boundary modes that are coupled due to a large QML. (b) The tunneling conductance carried by the non-local boundary modes under a bias voltage $E = eV$. The two dashed lines represent the topological gap, inside which is a tunneling peak. The length for the device is fixed at $L = 150a$, the hopping strength inside the lead is $t_l = \tilde{t}$, and the coupling between the lead and device are $t_{ld} = 0.02\tilde{t}$, respectively. Other parameters are $t = \tilde{t}$, $J = 1$, $\tilde{t} = 2 \times 10^{-4}J$, $m = 2\tilde{t}$, $\delta = 0.05$ and $\alpha = 0.1$. (c) The tunneling conductance versus the device width L .

ends, as illustrated in Fig. S5(a). When an electric volt is applied between the leads, one would usually expect a tunneling current to only appear outside the insulating gap, even with the presence of in-gap boundary channels. These channels do not conduct because they typically extend a few nanometers into the bulk, while the device is usually on the scale of hundreds of nanometers to micrometers. Therefore, they will not carry electric current. However, in topological flat-band systems with large QML, the TBMs can spread so widely enabling tunneling current in rather long devices, easily reaching one thousand unit cells, which is about several hundred nanometers. Consequently, we expect that the electrons carried by these modes can tunnel through the TI, resulting in an in-gap conductance peak.

We simulate such an experimental setup numerically, with the 1D flat-band TI being the $k_y = 0$ channel of the proposed Lieb-QWZ model, and calculate the tunneling conductance using the recursive Green function method. The scattering matrix element from channel n in lead j to channel m in lead i at zero temperature is given by [S12],

$$t_{ij}^{mn} = -\delta_{ij}\delta_{mn} + i(\sqrt{\Gamma})_i^m (G^r)_{mn}^{ij} (\sqrt{\Gamma})_j^n. \quad (\text{S59})$$

Here, G^r is the retarded Green's function. The broadening function Γ relates to the imaginary part of the retarded self-energy Σ^r by $\Gamma = -2\Im(\Sigma^r)$. In the setup of Fig. S5(a), $i/j = L/R$ denotes the left/right lead and m, n are

both the e-channels. Therefore, the electron tunneling probability will be $T_{ee} = |t_{LR}^{ee}|^2$. As shown in Fig. S5(b), inside the topological gap denoted as the region between the dashed lines, a prominent zero-bias peak for T_{ee} arises. When decreasing the parameter δ with increased quantum metric length, the tunneling conductance is enhanced due to a stronger coupling between the boundary modes. On the other hand, a larger QML enables the tunneling process to persist in wider devices, as depicted in Fig. S5(c). The phenomenal ultra-long range resonance tunneling effect can be viewed as a benchmark for the large QML of the 1D topological flat-band system.

-
- [S1] G. Rigolin and G. Ortiz, Degenerate adiabatic perturbation theory: Foundations and applications, Phys. Rev. A **90**, 022104 (2014).
 - [S2] H.-T. Ding, C.-X. Zhang, J.-X. Liu, J.-T. Wang, D.-W. Zhang, and S.-L. Zhu, Non-abelian quantum geometric tensor in degenerate topological semimetals, Phys. Rev. A **109**, 043305 (2024).
 - [S3] H.-T. Ding, Y.-Q. Zhu, P. He, Y.-G. Liu, J.-T. Wang, D.-W. Zhang, and S.-L. Zhu, Extracting non-abelian quantum metric tensor and its related chern numbers, Phys. Rev. A **105**, 012210 (2022).
 - [S4] B. Hetényi and P. Lévy, Fluctuations, uncertainty relations, and the geometry of quantum state manifolds, Physical Review A **108**, 032218 (2023).
 - [S5] Y.-Q. Ma, S. Chen, H. Fan, and W.-M. Liu, Abelian and non-abelian quantum geometric tensor, Phys. Rev. B **81**, 245129 (2010).
 - [S6] W. Kohn, Analytic properties of bloch waves and wannier functions, Phys. Rev. **115**, 809 (1959).
 - [S7] L. He and D. Vanderbilt, Exponential decay properties of wannier functions and related quantities, Phys. Rev. Lett. **86**, 5341 (2001).
 - [S8] X. Guo, X. Ma, X. Ying, and K. T. Law, Majorana zero modes in the lieb-kitaev model with tunable quantum metric, Phys. Rev. Lett. **135**, 076601 (2025).
 - [S9] H. C. Po, L. Zou, T. Senthil, and A. Vishwanath, Faithful tight-binding models and fragile topology of magic-angle bilayer graphene, Phys. Rev. B **99**, 195455 (2019).
 - [S10] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, Nature **556**, 43 (2018).
 - [S11] A. Diez-Carlon, J. Diez-Merida, P. Rout, D. Sedov, P. Virtanen, S. Banerjee, R. Penttilä, P. Altpeter, K. Watanabe, T. Taniguchi, *et al.*, Probing the flat-band limit of the superconducting proximity effect in twisted bilayer graphene josephson junctions, Physical Review X **15**, 041033 (2025).
 - [S12] S. Datta, *Electronic transport in mesoscopic systems* (Cambridge university press, 1997).