Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

New Exact Solutions of a Variable-Coefficient (2+1)-Dimensional Kadomtsev- Petviashvili Equation Using a Combination of Lie Symmetry Analysis and the Generalized Expansion Method

A. S. Abo-Eleneen

a_sherif59@std.mans.edu.eg

Mansoura University

I. L. El-Kalla

Mansoura University

E. S. El-Shazly

Delta University for Science & Technology

Article

Keywords: Variable-coefficient Kadomtsev-Petviashvili, Lie symmetry analysis, Similarity reduction, Soliton solutions, Generalized Expansion Method, Introduction

Posted Date: November 2nd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11231752/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

New Exact Solutions of a Variable-Coefficient (2+1)-Dimensional Kadomtsev-Petviashvili Equation Using a Combination of Lie Symmetry Analysis and the Generalized Expansion Method

A.S. Abo-Eleneen[1,2*], I. L. El-Kalla[2] and E.S. El-Shazly[3]

[1]Misr Higher Institute for Engineering and Technology, Mansoura 35511, Egypt. [2]Mathematics & Engineering Physics Department, Faculty of Engineering, Mansoura University, Mansoura 35511, Egypt. [3]Basic Science Department, Faculty of Engineering, Delta University for Science & Technology, P.O. Box 11152, Mansoura, Egypt. Email: a_sherif59@std.mans.edu.eg.

Abstract

In this paper, we apply the variable-coefficient (2+1)-dimensional Kadomtsev-Petviashvili (vcKP) equation by using a combination of Lie symmetry analysis and the generalized expansion method. First, by using the method of Lie symmetry analysis, three similarity transformations associated with three Lie symmetry generators are obtained, which reduce the original equation into three different (1+1)-dimensional nonlinear evolution equations. These reduced equations are then solved analytically by the Generalized Expansion Method (GEM). During the solution process, different admissible classes of the variable coefficients are obtained in a natural way without any a priori assumptions about their forms. The resulting classification establishes the correspondence between the chosen Lie symmetry generators, their similarity transformations, and the corresponding classes of variable coefficients. Based on this classification, several families of exact solutions are derived, including bright solitons, dark solitons, kink-like solitary waves, double-dark solitons, M-shaped bright solitons, and singular periodic solitons. The influence of the variable coefficients on the wave structures is analyzed, and the dynamical behavior of the obtained solutions is illustrated through two-dimensional, three-dimensional, and contour plots.

Keywords: Variable-coefficient Kadomtsev-Petviashvili; Lie symmetry analysis; Similarity reduction; Soliton solutions; Generalized Expansion Method. Introduction

1. Introduction

In many scientific and engineering fields, complicated physical phenomena are described by Nonlinear Partial Differential Equations (NLPDEs) [1] . Recently, many researchers have focused on NLPDEs with variable coefficients due to their importance [2]. Although there are several different kinds of nonlinear evolution equations that can be studied, the Kadomtsev-Petviashvili (KP) equation is considered the most interesting since it describes the interaction and propagation of weakly nonlinear dispersive waves in two-dimensional media. The KP equation is a natural generalization of the classical Korteweg-de Vries (KdV) equation with nonlinearity, dispersion and weak transverse perturbations and it is a fundamental model in nonlinear wave theory [3]. The KP equation has been effectively used in several fields, including fluid dynamics, plasma physics, and nonlinear optics. Specifically, the variable coefficient Kadomtsev-Petviashvili (vcKP) equations describe the nonlinear waves in inhomogeneous media, where the system's physical parameters vary over time or space [4]. A crucial property of nonlinear evolution equations is the possibility to generate soliton solutions, Certain nonlinear evolution equations admit soliton solutions, namely localized waves that propagate while preserving their shape, amplitude, and

velocity. Due to their remarkable stability and robustness, solitons have widespread applications in optical communications, photonic devices, plasma physics, fluid dynamics and quantum systems [5]. The interaction between nonlinearity and dispersion leads to various classes of nonlinear wave solutions. Depending on the structure of the governing equation and its parameter values, these solutions may appear as bright and dark solitons, double-dark and M-shaped profiles, or as singular, periodic, and kink-type solitary waves with different propagation characteristics.
The analytical investigation of these exact solutions provides valuable insight into the underlying dynamics of nonlinear wave propagation, energy localization, wave interactions, and stability in complex multidimensional systems [6].
Due to the inherent complexity of the NLPDEs with variable coefficients, many analytical methods have been developed to construct their exact and closed-form solutions, such as the Generalized Expansion Method (GEM) [7], the F-expansion method [8], multiple exp-function methods [9], the simplest equation method [10], the generalized (G'/G)-expansion method [11], the auxiliary equation method [12], the Kudryashov's integration scheme [13,14], the Jacobi elliptic functions [15], and many other mathematical techniques. These methods have played an important role in the understanding of nonlinear wave phenomena and the construction of various classes of exact solutions.

Motivated by the above theoretical and practical considerations, the present study investigates the vcKP equation

$$a(t)u_x^2 + a(t)uu_{xx} + b(t)u_{xxxx} - c(t)u_{yy} + u_{xt} = 0, \quad (1)$$

where $u(x, y, t)$ denotes the wave amplitude with two scaled spatial variables and a temporal variable x, y and t, and $a(t)$, $b(t)$ and $c(t)$ are the functional parameters of the variable. This equation can be used to model long wavelength water waves with weak nonlinear restoring forces and dispersive effects and wave propagation in ferromagnetic and other inhomogeneous media [16]. Many analytical methods have been used to solve the KP equation and its extensions. The vcKP equations are of much current interest. Liu et al. [16] developed the interaction solutions for the vcKP equation and studied the effect of variable coefficients on nonlinear wave interactions. Their results showed that the changes of coefficients have a strong effect on the evolution and interaction behaviors of localized wave structures. Kumar et al. [17] studied two classes of vcKP equations by using the Lie symmetry analysis and obtained several exact solutions through similarity reductions. Their work highlighted the utility of symmetry-based methods in investigating nonlinear wave phenomena. Deka et al. [18] found a new exact solution of the vcKP equation using the generalized (G'/G)-expansion method. Their findings enhanced the family of analytical solutions and provided more additional insights into the complex dynamics of nonlinear wave propagation. Demirbilek et al. [19] investigated the generalized extended (2+1)-dimensional KP equation. They found some exact analytical solutions and performed sensitivity and stability analyses. They found different new wave structures. They also showed how the generalized KP models could be used in fluid dynamics and nonlinear wave theory.

Although the vcKP equations admit various exact solutions and symmetry reductions, the complete Lie group classification of their variable coefficients and the associated soliton dynamics remain insufficiently explored. This study aims to address this gap by employing Lie symmetry analysis combined with the GEM approach to establish a comprehensive group classification and derive new exact soliton solutions of Eq. (1). This paper is structured as follows: Section 2 presents the Lie symmetry analysis for Eq. (1) and obtains the corresponding symmetry generators. Section 3 presents the similarity reductions of Eq. (1) obtained through the derived Lie symmetry generators. Section 4 introduces the analysis of the GEM and investigating the exact solutions of

the reduced equations by using GEM. Section 5 discusses the dynamic characteristics of the solutions graphically. Finally, Section 6 summarizes the conclusions of the study.

2. Lie symmetry analysis

To apply the Lie symmetry analysis method [20-22] to Eq. (1), we utilize the Lie group of infinitesimal transformations with one parameter σ for the independent and dependent variables, which are given by

$$\bar{x} = x + \sigma\,\xi(x,y,t,u) + O(\sigma^2), \tag{2}$$
$$\bar{y} = y + \sigma\,\varphi(x,y,t,u) + O(\sigma^2), \tag{3}$$
$$\bar{t} = t + \sigma\,\tau(x,y,t,u) + O(\sigma^2), \tag{4}$$
$$\bar{u} = u + \sigma\,\eta(x,y,t,u) + O(\sigma^2). \tag{5}$$

Then it is required that the set of solutions of Eq. (1) be invariant under the transformation Eqs. (2-5). This requirement results in the formation of a system of linear equations for the infinitesimals ξ, φ, τ and η. The associated Lie algebra of infinitesimal symmetries is the set of infinitesimal generators of the form:

$$X = \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial y} + \tau\,\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}. \tag{6}$$

With the infinitesimal generator defined by Eq. (6), the invariance condition of Eq. (1) under the Lie group of point transformations

$$Pr^{(4)}X(\Delta)|_{\Delta=0} = 0. \tag{7}$$

Where $Pr^{(4)}X(\Delta)$ represents the fourth prolongation of the infinitesimal generator. This leads to a system of determining equations, whose solution yields

$$\xi = \frac{f'_1(t)}{2c(t)}y + f_2(t),$$
$$\varphi = f_1(t),$$
$$\tau = 0,$$
$$\eta = \frac{1}{a(t)}\left(\frac{f''_1(t)}{2c(t)}y - \frac{f'_1(t)c'(t)}{2c^2(t)}y + f'_2(t)\right). \tag{8}$$

After calculating these equations stated in the above Eq. (8), we derive the following infinitesimal generators listed as:

$$X_1 = \frac{\partial}{\partial x} + c_1\frac{\partial}{\partial y}, \tag{9}$$

where, c_1 are arbitrary constant.

$$X_2 = f_1(t)\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial y} + f_2(t)\frac{\partial}{\partial u}, \tag{10}$$

where, $a(t) = \frac{f'_1(t)}{f_2(t)}$.

$$X_3 = (c_1 y + c_2)\frac{\partial}{\partial x} + f_1(t)\frac{\partial}{\partial y}, \tag{11}$$

where, $c(t) = \frac{f'_1(t)}{2c_1}$. Using the symmetry generators provided in Eqs. (9-11), Eq. (1) is reduced to three distinct (1+1)-dimensional NLPDEs with variable coefficients.

3. Similarity reductions of the variable-coefficient KP equation

In this section, we perform symmetry generators from the previous section, we derive three separate similarity reductions. Each reduction reduced the original (2+1)-dimensional vcKP equation into a (1+1)-dimensional nonlinear evolution equation while keeping its essential nonlinear characteristics.

3.1. Reduction of the vcKP equation using X_1

The characteristic equation is in the following form for X_1:

$$\frac{dx}{c_2} = \frac{dy}{c_1} = \frac{dt}{0} = \frac{du}{0}. \tag{12}$$

The corresponding results for the specified variable are as follows:

$$z = \frac{-c_1 x + c_2 y}{c_2}, \;\; u = v(z,t), \;\; c_2 = 1. \tag{13}$$

according to Eq. (13), Eq. (1) can be reduced to the following form

$$c_1{}^2 a(t)\, v_z{}^2 - c_1\, v_{\mathrm{zt}} - c(t)\, v_{\mathrm{zz}} + c_1{}^2\, a(t)\, v\, v_{\mathrm{zz}} + c_1{}^4\, b(t)\, v_{\mathrm{zzzz}} = 0. \tag{14}$$

3.2. Reduction of the vcKP equation using X_2

The characteristic equation is in the following form for X_2:

$$\frac{dx}{f_1(t)} = \frac{dy}{c_1} = \frac{dt}{0} = \frac{du}{f_2(t)}. \tag{15}$$

The corresponding results for the specified variable are as follows:

$$z = y - \frac{c_1 x}{f_1(t)}, \;\; \psi = \frac{x}{f_1(t)}, \;\; u = f_2(t)\psi + v(z,t), \;\; a(t) = \frac{f'_1(t)}{f_2(t)}, \tag{16}$$

according to Eq. (16), Eq. (1) can be reduced to the following form

$$\frac{f_2{}'(t)}{f_1(t)} - \frac{c_1\, f_1{}'(t)\, v_z}{f_1{}^2(t)} + \frac{c_1{}^2\, f_1{}'(t)\, v_z{}^2}{f_1{}^2(t)\, f_2(t)} - \frac{c_1\, v_{zt}}{f_1(t)} + \frac{c_1{}^2\, v\, f_1{}'(t)\, v_{zz}}{f_1{}^2(t)\, f_2(t)} + \frac{c_1{}^4\, b(t)\, v_{\mathrm{zzzz}}}{f_1{}^4(t)} = c(t)\, v_{\mathrm{zz}}. \tag{17}$$

Where $v = v(z,t)$, $f_1(t)$, and $f_2(t)$ are arbitrary function with respect to t.

3.3. Reduction of the vcKP equation using X_3

The characteristic equation is in the following form for X_3:

$$\frac{dx}{c_1 y + c_2} = \frac{dy}{f_1(t)} = \frac{dt}{0} = \frac{du}{0}. \tag{18}$$

The corresponding results for the specified variable are as follows:

$$z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2 f_1(t)}, \;\; u = v(z,t), \;\; c(t) = \frac{f'_1(t)}{2c_1}, \tag{19}$$

according to Eq. (19), Eq. (1) can be reduced to the following form

$$f_1'(t)\left(4\,v_z+\frac{(1-8\,c_1\,c_2)\,v_{zz}}{c_1\,f_1'(t)}\right)=\ 8f_1(t)(-v_{zt}+a(t)\,(v_z{}^2+vv_{zz})+b(t)\,v_{zzzz}). \quad (20)$$

Where $v=v(z,t)$, c_1 and c_2 are arbitrary constants and $f_1(t)$ is an arbitrary function with respect to t.

4. Investigating solutions using Generalized Expansion Method (GEM)

Consider a general (1+1) dimensional nonlinear evolution equation with variable coefficients in the form [7, 23].

$$H(v,v_t,v_z,v_{zt},v_{zz},\dots)=0, \quad (21)$$

Where $v=v(z,t)$ denotes the dependent variable, z and t are independent variables.
Assume that Eq. (21) has a solution of the form

$$v(z,t)=A_0(z,t)+\sum_{i=1}^{n}A_i(z,t)\,\omega^i\,,\quad i=1,2,\dots \quad (22)$$

Where $A_i(z,t), i=0.1,2,\dots$ are unknown functions and n is a positive integer to be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (21). The auxiliary function $\omega=\omega(\xi)$ is assumed to satisfy the following ODE:

$$\left(\frac{d\,\omega}{d\,\xi}\right)^2=r_0+r_1\omega+r_2\,\omega^2+r_3\,\omega^3+r_4\,\omega^4, \qquad r_4\neq 0 \quad (23)$$

Inserting Eqs. (22) and (23) into Eq. (21) and setting the coefficients of each power of ω^j to zero, an overdetermined system of partial differential equations is obtained for the unknown functions $A_i(z,t)$, $i=0,1,\dots,n$, and $\xi(z,t)$. The solution of this system determines the explicit forms of these functions and imposes the corresponding compatibility constraints on the governing parameters.

4.1 Exact Solutions of Equation (14)

In Eq. (14), utilizing the balancing principle between v_{zzzz} and $v_z{}^2$ we get

$$2n+2=n+4\ \Rightarrow\ n=2 \quad (24)$$

Then the solution of Eq. (14) is given by:

$$v(z,t)=A_0(z,t)+A_1(z,t)\omega+A_2(z,t)\,\omega^2,\ A_2\neq 0 \quad (25)$$

where $\omega\ =\omega(\xi)$, and $\xi=\alpha(t)z\ +\ \beta(t)+\ \theta$.

Inserting Eq. (25) with Eq. (23) into Eq. (14), and setting the coefficients of ω^i and $\omega^i\sqrt{\sum_{j=0}^{4}r_j\,\omega^j}$ to zero, a system of algebraic equations is derived. By solving this system, four results were obtained. Result (1 and 3) produced a dark, bright and singular periodic solutions, results (2 and 4) produced a singular periodic and kink-like solitary solutions. Each solution is presented as follows:

Result (1):

$$r_1 = r_3 = 0, \alpha(t) = \frac{d_2}{d_1 - \int a(t)\,dt}, b(t) = \rho a(t)\big(d_1 - \int a(t)\,dt\big),$$
$$A_2(z,t) = \frac{-1}{a(t)}\big(12\,{c_1}^2 r_4\, b(t)\alpha^2(t)\big), A_1(z,t) = 0, \text{ and}$$
$$A_0(z,t) = \frac{1}{4{c_1}^2 r_4 a(t)\alpha(t)}\Big(4r_4 c(t)\alpha(t) - 16{c_1}^4 r_2 r_4 b(t)\alpha(t)^3 + 4c_1 r_4\big(z\alpha'(t) + \beta'(t)\big)\Big). \quad (26)$$

Where, ρ is constant, d_1 and d_2 are an integration constants.

Case 1: If $r_0 = \frac{r_2^2}{4r_4}, r_2 < 0,$ and $r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{2r_4}}\, tanh\left(\sqrt{\frac{-r_2}{2}}\,\xi\right). \quad (27)$$

When $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have dark solution

$$u(x,y,t) = v(z,t) = \frac{d_2 c(t) + c_1\beta'(t)(d_1 - \int a(t)\,dt)}{{c_1}^2 d_2 a(t)} + \frac{c_1}{{c_1}^2(d_1 - \int a(t)\,dt)}\Bigg(-c_1 x + y - 4{c_1}^3 {d_2}^2 r_2\rho +$$
$$6{c_1}^3 {d_2}^2 r_2 \rho\, tanh^2\left(\sqrt{\frac{-r_2}{2}}\left(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\right)\right)\Bigg). \quad (28)$$

Case 2: If $r_0 = 0, r_4 < 0, r_2 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sech(\sqrt{r_2}\,\xi). \quad (29)$$

When $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have bright solution

$$u(x,y,t) = v(z,t) = \frac{d_2 c(t) + c_1\beta'(t)(d_1 - \int a(t)\,dt)}{{c_1}^2 d_2 a(t)} + \frac{c_1}{{c_1}^2(d_1 - \int a(t)\,dt)}\Bigg(-c_1 x + y - 4{c_1}^3 {d_2}^2 r_2\rho +$$
$$12{c_1}^3 {d_2}^2 r_2 \rho\, sech^2\left(\sqrt{r_2}\left(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\right)\right)\Bigg). \quad (30)$$

Case 3: If $r_0 = 0, r_2 < 0, r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sec(\sqrt{-r_2}\,\xi). \quad (31)$$

When $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have singular periodic solution

$$u(x,y,t) = v(z,t) = \frac{d_2 c(t) + c_1\beta'(t)(d_1 - \int a(t)\,dt)}{{c_1}^2 d_2 a(t)} + \frac{c_1}{{c_1}^2(d_1 - \int a(t)\,dt)}\Bigg(-c_1 x + y - 4{c_1}^3 {d_2}^2 r_2\rho +$$
$$12{c_1}^3 {d_2}^2 r_2 \rho\, sec^2\left(\sqrt{-r_2}\left(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\right)\right)\Bigg). \quad (32)$$

Result (2):

$$r_2 = \frac{{r_3}^3 + 8r_1{r_4}^2}{4r_3r_4}, \alpha(t) = \frac{d_2}{d_1 - \int a(t)\,dt}, b(t) = \rho a(t)\big(d_1 - \int a(t)\,dt\big),$$
$$A_2(z,t) = \frac{-1}{a(t)}\big(12\,{c_1}^2 r_4\, b(t)\alpha^2(t)\big), A_1(z,t) = \frac{-1}{a(t)}\big(6\,{c_1}^2 r_3\, b(t)\alpha^2(t)\big), \text{ and}$$
$$A_0(z,t) = \frac{1}{4{c_1}^2 r_4 a(t)\alpha(t)}\Big(4r_4 c(t)\alpha(t) + {c_1}^4(3{r_3}^2 - 16r_2r_4)b(t)\alpha(t)^3 + 4c_1r_4\big(z\alpha'(t) + \beta'(t)\big)\Big). \quad (33)$$

Case 1: If $r_0 = 1, r_1 = -4, r_2 = 8 - m^2, r_3 = 2(m^2 - 4), r_4 = 2(2 - m^2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{sn(\xi, m) + cn(\xi, m)}. \quad (34)$$

When $m = 0$, $c_2 = 1$ and $z = \frac{-c_1x + c_2y}{c_2}$. We have singular periodic solution

$$u(x,y,t) = v(z,t) = \frac{d_2c(t) + c_1\beta'(t)\big(d_1 - \int a(t)\,dt\big)}{{c_1}^2 d_2 a(t)} + \frac{c_1\Big(-c_1x + y - 20{c_1}^3{d_2}^2\rho + (-c_1x + y + 4{c_1}^3{d_2}^2\rho)sin\Big(\frac{2d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + 2\beta(t) + 2\theta\Big)\Big)}{{c_1}^2\big(d_1 - \int a(t)\,dt\big)\Big(cos\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big) + sin\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big)\Big)^2}. \quad (35)$$

Case 2: If $r_0 = 1, r_1 = -4, r_2 = 5 - 6m - m^2, r_3 = 2(m^2 + 6m - 1), r_4 = m(4m^2 + 7m - 2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{1 + sn(\xi, m) + m\,sn^2(\xi, m)}. \quad (36)$$

When $m = 1$, $c_2 = 1$ and $z = \frac{-c_1x + c_2y}{c_2}$. We have kink-like solitary wave solution

$$u(x,y,t) = v(z,t) = \frac{c_1\big(-c_1x + y + 20{c_1}^3{d_2}^2\rho\big)}{{c_1}^2\big(d_1 - \int a(t)\,dt\big)} + \frac{d_2c(t) + c_1\beta'(t)\big(d_1 - \int a(t)\,dt\big)}{{c_1}^2 d_2 a(t)} - \frac{72{c_1}^2{d_2}^2\rho tanh\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big)}{\big(d_1 - \int a(t)\,dt\big)\Big(1 + tanh\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big) + tanh^2\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big)\Big)} - \frac{108{c_1}^2{d_2}^2\rho tanh^2\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big)}{\big(d_1 - \int a(t)\,dt\big)\Big(1 + tanh\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big) + tanh^2\Big(\frac{d_2(-c1x+y)}{d_1 - \int a(t)\,dt} + \beta(t) + \theta\Big)\Big)^2}. \quad (37)$$

Result (3):

$$r_1 = r_3 = 0\,, \alpha(t) = \mu, b(t) = \lambda a(t),$$
$$A_2(z,t) = -12{c_1}^2 r_4\lambda\mu^2,\ A_1(z,t) = 0, \text{ and}$$
$$A_0(z,t) = \frac{1}{{c_1}^2\mu a(t)}\ \big(-4{c_1}^4 r_2\lambda\mu^3 a(t) + \mu c(t) + c_1\beta'(t)\big). \quad (38)$$

Where, μ *and* λ are an integration constants.

Case 1: If $r_0 = \frac{r_2^2}{4r_4}, r_2 < 0, r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{2r_4}}\, tanh\left(\sqrt{\frac{-r_2}{2}}\, \xi\right). \tag{39}$$

When $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have dark solution

$$u(x,y,t) = v(z,t) = \frac{1}{{c_1}^2 \mu a(t)}\left(-4{c_1}^4 r_2 \lambda \mu^3 a(t) + \mu c(t) + c_1 \beta'(t)\right) +$$
$$6{c_1}^2 r_2 \lambda \mu^2 tanh^2\left(\sqrt{\frac{-r_2}{2}}\,(\mu(-\mathrm{c1}x + y) + \beta(t) + \theta)\right). \tag{40}$$

Case 2: If $r_0 = 0, r_4 < 0, r_2 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sech(\sqrt{r_2}\, \xi). \tag{41}$$

When $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have bright solution

$$u(x,y,t) = v(z,t) = \frac{1}{{c_1}^2 \mu a(t)}\left(-4{c_1}^4 r_2 \lambda \mu^3 a(t) + \mu c(t) + c_1 \beta'(t)\right) +$$
$$12{c_1}^2 r_2 \lambda \mu^2 sech^2\left(\sqrt{r_2}(\mu(-\mathrm{c1}x + y) + \beta(t) + \theta)\right). \tag{42}$$

Case 3: If $r_0 = 0, r_2 < 0, r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sec(\sqrt{-r_2}\, \xi). \tag{43}$$

When $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have singular periodic solution

$$u(x,y,t) = v(z,t) = \frac{1}{{c_1}^2 \mu a(t)}\left(-4{c_1}^4 r_2 \lambda \mu^3 a(t) + \mu c(t) + c_1 \beta'(t)\right) +$$
$$12{c_1}^2 r_2 \lambda \mu^2 sec^2\left(\sqrt{-r_2}(\mu(-c1x + y) + \beta(t) + \theta)\right). \tag{44}$$

Result (4):

$$r_2 = \frac{{r_3}^3 + 8 r_1 {r_4}^2}{4 r_3 r_4}, \alpha(t) = \mu, b(t) = \lambda a(t),$$
$$A_2(z,t) = -12{c_1}^2 r_4 \lambda \mu^2,\ A_1(z,t) = -6{c_1}^2 r_3 \lambda \mu^2, \text{ and}$$
$$A_0(z,t) = -\frac{1}{4 r_3 r_4}\left({c_1}^2({r_3}^3 + 32 r_1 {r_4}^2)\lambda \mu^2\right) + \frac{1}{{c_1}^2 \mu a(t)}\left(\mu c(t) + c_1 \beta'(t)\right). \tag{45}$$

Case 1: If $r_0 = 1, r_1 = -4, r_2 = 8 - m^2, r_3 = 2(m^2 - 4), r_4 = 2(2 - m^2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{sn(\xi, m) + cn(\xi, m)}. \tag{46}$$

When $m = 0$, $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have singular periodic solution

$$u(x,y,t) = v(z,t) = -20{c_1}^2 \lambda \mu^2 + \frac{1}{{c_1}^2 \mu a(t)}\left(\mu c(t) + c_1 \beta'(t)\right) +$$
$$\frac{48{c_1}^2 \lambda \mu^2 sin(\mu(-c1x+y)+\beta(t)+\theta)}{cos(\mu(-c1x+y)+\beta(t)+\theta)+sin(\mu(-c1x+y)+\beta(t)+\theta)} - \frac{48{c_1}^2 \lambda \mu^2 sin^2(\mu(-c1x+y)+\beta(t)+\theta)}{\left(cos(\mu(-c1x+y)+\beta(t)+\theta)+sin(\mu(-c1x+y)+\beta(t)+\theta)\right)^2}. \tag{47}$$

Case 2: If $r_0 = 1, r_1 = -4, r_2 = 5 - 6m - m^2, r_3 = 2(m^2 + 6m - 1), r_4 = m(4m^2 + 7m - 2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{1 + sn(\xi, m) + m\, sn^2(\xi, m)}. \tag{48}$$

When $m = 1$, $c_2 = 1$ and $z = \frac{-c_1 x + c_2 y}{c_2}$. We have kink-like solitary wave solution

$$u(x,y,t) = v(z,t) = 20{c_1}^2\lambda\mu^2 + \frac{1}{{c_1}^2\mu a(t)}\big(\mu c(t) + c_1\beta'(t)\big) - \frac{72{c_1}^2\lambda\mu^2 tanh(\mu(-c1x+y)+\beta(t)+\theta)}{1+tanh(\mu(-c1x+y)+\beta(t)+\theta)+tanh^2(\mu(-c1x+y)+\beta(t)+\theta)} - \frac{108{c_1}^2\lambda\mu^2 tanh^2(\mu(-c1x+y)+\beta(t)+\theta)}{\big(1+tanh(\mu(-c1x+y)+\beta(t)+\theta)+tanh^2(\mu(-c1x+y)+\beta(t)+\theta)\big)^2}. \quad (49)$$

4.2 Exact Solutions of Equation (17)

Following the same procedures described in the previous subsection, in this case four results were obtained. Result (1 and 3) produced a dark, bright and singular periodic solutions, and results (2 and 4) produced a singular periodic and kink-like solitary wave solutions. Each result is presented as follows:

Result (1):

$$r_1 = r_3 = 0\,, \alpha(t) = \rho\,,\ f_2(t) = \mu\,, \psi = \frac{x}{f_1(t)}, a(t) = \frac{1}{\mu}{f_1}'(t),$$

$$b(t) = \lambda\, f_1(t)\, {f_1}'(t), A_1(z,t) = 0, A_2(z,t) = -\frac{1}{f_1(t)}(12c_1^2 r_4\lambda\mu\rho^2), \text{ and}$$

$$A_0(z,t) = \frac{1}{\rho c_1^2 f_1(t){f_1}'(t)}\Big(\mu\Big(\rho c(t){f_1}^3(t) - 4{c_1}^4 r_2\lambda\rho^3\,{f_1}'(t) + c_1{f_1}^2(t)\big(z\alpha'(t) + \beta'(t)\big)\Big)\Big). \quad (50)$$

Where, $\rho, \mu\ and\ \lambda$ are an integration constants.

Case 1: If $r_0 = \frac{r_2^2}{4r_4}, r_2 < 0, r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{2r_4}}\, tanh\left(\sqrt{\frac{-r_2}{2}}\,\xi\right). \quad (51)$$

When $z = y - \frac{c_1 x}{f_1(t)}$. We have dark solution

$$u(x,y,t) = \psi f_2(t) + v(z,t) = \frac{\mu x}{f_1(t)} + \frac{1}{{c_1}^2\rho f_1(t){f_1}'(t)}\Big(\mu\Big(\rho c(t){f_1}^3(t) - 4{c_1}^4 r_2\lambda\rho^3 {f_1}'(t) + c_1{f_1}^2(t)\beta'(t)\Big)\Big) + \frac{6{c_1}^2 r_2\lambda\mu\rho^2}{f_1(t)}\, tanh^2\left(\sqrt{\frac{-r_2}{2}}\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right)\right). \quad (52)$$

Case 2: If $r_0 = 0, r_4 < 0, r_2 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sech(\sqrt{r_2}\,\xi). \quad (53)$$

When $z = y - \frac{c_1 x}{f_1(t)}$. We have bright solution

$$u(x,y,t) = \psi f_2(t) + v(z,t)$$

$$= \frac{\mu x}{f_1(t)} + \frac{1}{{c_1}^2 \rho f_1(t) {f_1}'(t)} \left(\mu \left(\rho c(t) {f_1}^3(t) - 4{c_1}^4 r_2 \lambda \rho^3 {f_1}'(t) + c_1 {f_1}^2(t) \beta'(t) \right) \right)$$

$$+ \frac{12{c_1}^2 r_2 \lambda \mu \rho^2}{f_1(t)} sech^2 \left(\sqrt{r_2} \left(\rho \left(y - \frac{c_1 x}{f_1(t)} \right) + \beta(t) + \theta \right) \right). \quad (54)$$

Case 3: If $r_0 = 0, r_2 < 0, r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sec(\sqrt{-r_2}\, \xi). \quad (55)$$

When $z = y - \frac{c_1 x}{f_1(t)}$. We have singular periodic solution

$$u(x,y,t) = \psi f_2(t) + v(z,t) = \frac{\mu x}{f_1(t)} + \frac{1}{{c_1}^2 \rho f_1(t) {f_1}'(t)} \left(\mu \left(\rho c(t) {f_1}^3(t) - 4{c_1}^4 r_2 \lambda \rho^3 {f_1}'(t) + \right. \right.$$

$$\left. \left. c_1 {f_1}^2(t) \beta'(t) \right) \right) + \frac{12{c_1}^2 r_2 \lambda \mu \rho^2}{f_1(t)} sec^2 \left(\sqrt{-r_2} \left(\rho \left(y - \frac{c_1 x}{f_1(t)} \right) + \beta(t) + \theta \right) \right). \quad (55)$$

Result (2):

$$r_2 = \frac{{r_3}^3 + 8 r_1 {r_4}^2}{4 r_3 r_4}, \alpha(t) = \rho\,,\ f_2(t) = \mu\,, \psi = \frac{x}{f_1(t)}, a(t) = \frac{1}{\mu} {f_1}'(t),$$

$$b(t) = \lambda\, f_1(t)\, {f_1}'(t), A_1(z,t) = -\frac{1}{f_1(t)} (6c_1^2 r_3 \lambda \mu \rho^2), A_2(z,t) = -\frac{1}{f_1(t)} (12 c_1^2 r_4 \lambda \mu \rho^2),$$

and

$$A_0(z,t) = -\frac{1}{4 r_3 r_4 f_1(t)} ({c_1}^2 ({r_3}^3 + 32 r_1 {r_4}^2) \lambda \mu \rho^2) + \frac{1}{{c_1}^2\, {f_1}'(t)} \left(\mu c(t) {f_1}^2(t) \right) +$$

$$\frac{1}{c_1 \rho\, {f_1}'(t)} \left(\mu f_1(t) \beta'(t) \right). \quad (56)$$

Case 1: If $r_0 = 1, r_1 = -4, r_2 = 8 - m^2, r_3 = 2(m^2 - 4), r_4 = 2(2 - m^2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{sn(\xi, m) + cn(\xi, m)}. \quad (57)$$

When $m = 0, z = y - \frac{c_1 x}{f_1(t)}$. We have singular periodic solution

$$u(x,y,t) = \psi f_2(t) + v(z,t) = \frac{\mu x}{f_1(t)} - \frac{20{c_1}^2 \lambda \mu \rho^2}{f_1(t)} + \frac{\mu c(t) {f_1}^2(t)}{{c_1}^2 {f_1}'(t)} + \frac{\mu f_1(t) \beta'(t)}{c_1 \rho {f_1}'(t)} +$$

$$\frac{48{c_1}^2 \lambda \mu \rho^2 sin\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right)}{f_1(t)\left(cos\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right) + sin\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right)\right)} -$$

$$\frac{48 c1^2 \lambda \mu \rho^2 sin^2\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right)}{f_1(t)\left(cos\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right) + sin\left(\rho\left(y - \frac{c_1 x}{f_1(t)}\right) + \beta(t) + \theta\right)\right)^2}. \quad (58)$$

Case 2: If $r_0 = 1, r_1 = -4, r_2 = 5 - 6m - m^2, r_3 = 2(m^2 + 6m - 1), r_4 = m(4m^2 + 7m - 2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{1 + sn(\xi, m) + m\, sn^2(\xi, m)}. \quad (59)$$

When $m = 1, z = y - \frac{c_1 x}{f_1(t)}$. We have kink-like solitary wave solution

$$u(x,y,t)=\psi f_2(t)+v(z,t)=\frac{\mu x}{f_1(t)}+\frac{20{c_1}^2\lambda\mu\rho^2}{f_1(t)}+\frac{\mu c(t){f_1}^2(t)}{{c_1}^2{f_1}'(t)}+\frac{\mu f_1(t)\beta'(t)}{c_1\rho{f_1}'(t)}-\frac{72{c_1}^2\lambda\mu\rho^2\tanh\left(\rho\left(y-\frac{c_1x}{f_1(t)}\right)+\beta(t)+\theta\right)}{f_1(t)\left(1+\tanh\left(\rho\left(y-\frac{c_1x}{f_1(t)}\right)+\beta(t)+\theta\right)+\tanh^2\left(\rho\left(y-\frac{c_1x}{f_1(t)}\right)+\beta(t)+\theta\right)\right)}-\frac{108{c_1}^2\lambda\mu\rho^2\tanh^2\left(\rho\left(y-\frac{c_1x}{f_1(t)}\right)+\beta(t)+\theta\right)}{f_1(t)\left(1+\tanh\left(\rho\left(y-\frac{c_1x}{f_1(t)}\right)+\beta(t)+\theta\right)+\tanh^2\left(\rho\left(y-\frac{c_1x}{f_1(t)}\right)+\beta(t)+\theta\right)\right)^2}. \quad (60)$$

Result (3):

$$r_1=r_3=0\ ,a(t)=\frac{{f_1}'(t)}{\kappa\left(d_2+d_1f_1(t)\right)^2},\ f_2(t)=\kappa\left(d_2+d_1f_1(t)\right)^2,$$

$$\psi=\frac{x}{f_1(t)},\alpha(t)=d_2+d_1f_1(t),b(t)=\frac{\gamma f_1(t){f_1}'(t)}{\left(d_2+d_1f_1(t)\right)^3},$$

$$A_1(z,t)=0,A_2(z,t)=-\frac{1}{f_1(t)}\left(12\gamma\kappa c_1^2r_4\left(d_2+d_1f_1(t)\right)\right),\text{ and}$$

$$A_0(z,t)=\frac{1}{{c_1}^2f_1(t){f_1}'(t)}\left(\kappa\left(d_2+d_1f_1(t)\right)\left(-4{c_1}^4r_2\gamma{f_1}'(t)+{f_1}^2(t)\left(c(t)f1(t)\left(d_2+d_1f_1(t)\right)+c_1\left(d_1z{f_1}'(t)+\beta'(t)\right)\right)\right)\right). \quad (61)$$

Where, $\kappa, d_1\ and\ d_2$ are an integration constants.

Case 1: If $r_0=\frac{r_2^2}{4r_4}, r_2<0, r_4>0$, Eq. (23) has the solution

$$\omega(\xi)=\sqrt{\frac{-r_2}{2r_4}}\,tanh\left(\sqrt{\frac{-r_2}{2}}\,\xi\right). \quad (62)$$

When $z=y-\frac{c_1x}{f_1(t)}$. We have dark solution

$$u(x,y,t)=\psi f_2(t)+v(z,t)=\frac{\kappa\left(d_2+d_1f_1(t)\right)}{{c_1}^2f_1(t){f_1}'(t)}\left(c(t){f_1}^3(t)\left(d_2+d_1f_1(t)\right)+c_1{f_1}'(t)\left(c_1d_2x-4{c_1}^3r_2\gamma+d_1y{f_1}^2(t)\right)+c_1{f_1}^2(t)\beta'(t)\right)+\frac{6{c_1}^2r_2\gamma\kappa\left(d_2+d_1f_1(t)\right)}{f_1(t)}\tanh^2\left(\sqrt{\frac{-r_2}{2}}\left(\left(y-\frac{c_1x}{f_1(t)}\right)\left(d_2+d_1f_1(t)\right)+\beta(t)+\theta\right)\right). \quad (63)$$

Case 2: If $r_0=0, r_4<0, r_2>0$, Eq. (23) has the solution

$$\omega(\xi)=\sqrt{\frac{-r_2}{r_4}}\,sech(\sqrt{r_2}\,\xi). \quad (64)$$

When $z=y-\frac{c_1x}{f_1(t)}$. We have bright solution

$$u(x,y,t)=\psi f_2(t)+v(z,t)=\frac{\kappa(d_2+d_1f_1(t))}{{c_1}^2f_1(t){f_1}'(t)}\Big(c(t){f_1}^3(t)\big(d_2+d_1f_1(t)\big)+c_1{f_1}'(t)\Big(c_1d_2x-$$
$$4{c_1}^3r_2\gamma+d_1y{f_1}^2(t)\Big)+c_1{f_1}^2(t)\beta'(t)\Big)+\frac{12{c_1}^2r_2\gamma\kappa(d_2+d_1f_1(t))}{f_1(t)}sech^2\Bigg(\sqrt{r_2}\Bigg(\Big(y-\frac{c_1x}{f_1(t)}\Big)\big(d_2+$$
$$d_1f_1(t)\big)+\beta(t)+\theta\Bigg)\Bigg). \tag{65}$$

Case 3: If $r_0=0, r_2<0, r_4>0$, Eq. (23) has the solution

$$\omega(\xi)=\sqrt{\frac{-r_2}{r_4}}\,sec(\sqrt{-r_2}\,\xi). \tag{66}$$

When $z=y-\frac{c_1x}{f_1(t)}$. We have singular periodic solution

$$u(x,y,t)=\psi f_2(t)+v(z,t)=\frac{\kappa(d_2+d_1f_1(t))}{{c_1}^2f_1(t){f_1}'(t)}\Big(c(t){f_1}^3(t)\big(d_2+d_1f_1(t)\big)+c_1{f_1}'(t)\Big(c_1d_2x-$$
$$4{c_1}^3r_2\gamma+d_1y{f_1}^2(t)\Big)+c_1{f_1}^2(t)\beta'(t)\Big)+\frac{12{c_1}^2r_2\gamma\kappa(d_2+d_1f_1(t))}{f_1(t)}sec^2\Bigg(\sqrt{-r_2}\Bigg(\Big(y-$$
$$\frac{c_1x}{f_1(t)}\Big)\big(d_2+d_1f_1(t)\big)+\beta(t)+\theta\Bigg)\Bigg). \tag{67}$$

Result (4):

$$r_2=\frac{{r_3}^3+8r_1{r_4}^2}{4r_3r_4}\,,a(t)=\frac{{f_1}'(t)}{\kappa\big(d_2+d_1f_1(t)\big)^2}\,,\ f_2(t)=\kappa\big(d_2+d_1f_1(t)\big)^2,$$
$$\psi=\frac{x}{f_1(t)},\alpha(t)=d_2+d_1f_1(t),b(t)=\frac{\gamma f_1(t){f_1}'(t)}{\big(d_2+d_1f_1(t)\big)^3},$$
$$A_1(z,t)=-\frac{1}{f_1(t)}\Big(6\gamma\kappa c_1^2r_3\big(d_2+d_1f_1(t)\big)\Big),A_2(z,t)=-\frac{1}{f_1(t)}\Big(12\gamma\kappa c_1^2r_4\big(d_2+$$
$$d_1f_1(t)\big)\Big),\text{ and}$$
$$A_0(z,t)=\frac{1}{4r_3r_4{c_1}^2f_1(t){f_1}'(t)}\Bigg(\kappa\big(d_2+d_1f_1(t)\big)\Big(4r_3r_4c(t){f_1}^3(t)\big(d_2+d_1f_1(t)\big)-$$
$$\Big({c_1}^4({r_3}^3+32r_1{r_4}^2)\gamma-4c_1d_1r_3r_4z{f_1}^2(t)\Big){f_1}'(t)+4c_1r_3r_4{f_1}^2(t)\beta'(t)\Big)\Bigg). \tag{68}$$

Case 1: If $r_0=1, r_1=-4, r_2=8-m^2, r_3=2(m^2-4), r_4=2(2-m^2)$, Eq. (23) has the solution

$$\omega(\xi)=\frac{sn(\xi,m)}{sn(\xi,m)+cn(\xi,m)}. \tag{69}$$

When $m=0, z=y-\frac{c_1x}{f_1(t)}$. We have singular periodic solution

$$u(x,y,t)=\psi f_2(t)+v(z,t)=\frac{\kappa(d_2+d_1f_1(t))}{{c_1}^2f_1(t){f_1}'(t)}\Big(c(t){f_1}^3(t)\big(d_2+d_1f_1(t)\big)+c_1{f_1}'(t)\Big(c_1d_2x-20{c_1}^3\gamma+d_1y{f_1}^2(t)\Big)+c_1{f_1}^2(t)\beta'(t)\Big)+\frac{48{c_1}^2\gamma\kappa(d_2+d_1f_1(t))sin\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)}{f_1(t)\Big(cos\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)+sin\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)\Big)}-\frac{48{c_1}^2\gamma\kappa(d_2+d_1f_1(t))sin^2\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)}{f_1(t)\Big(cos\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)+sin\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)\Big)^2}. \tag{70}$$

Case 2: If $r_0=1, r_1=-4, r_2=5-6m-m^2, r_3=2(m^2+6m-1), r_4=m(4m^2+7m-2)$, Eq. (23) has the solution

$$\omega(\xi)=\frac{sn(\xi,m)}{1+sn(\xi,m)+m\,sn^2(\xi,m)}. \tag{71}$$

When $m=1, z=y-\frac{c_1x}{f_1(t)}$. We have kink-like solitary wave solution

$$u(x,y,t)=\psi f_2(t)+v(z,t)=\frac{\kappa(d_2+d_1f_1(t))}{{c_1}^2f_1(t){f_1}'(t)}\Big(c(t){f_1}^3(t)\big(d_2+d_1f_1(t)\big)+c_1{f_1}'(t)\Big(c_1d_2x+20{c_1}^3\gamma+d_1y{f_1}^2(t)\Big)+c_1{f_1}^2(t)\beta'(t)\Big)-\frac{72{c_1}^2\gamma\kappa(d_2+d_1f_1(t))tanh\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)}{f_1(t)\Big(1+tanh\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)+tanh^2\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)\Big)}-\frac{108{c_1}^2\gamma\kappa(d_2+d_1f_1(t))tanh^2\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)}{f_1(t)\Big(1+tanh\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)+tanh^2\Big(\big(y-\frac{c_1x}{f_1(t)}\big)(d_2+d_1f_1(t))+\beta(t)+\theta\Big)\Big)^2}. \tag{72}$$

4.3 Exact Solutions of Equation (20)

In this case, two results were obtained. Result (1) produced a double- dark soliton, M-shaped bright soliton and singular periodic solutions, and result (2) produced a singular periodic and double-dark soliton solutions. Each result is presented as follows:

Result (1):

$$r_1=r_3=0\ \ ,\alpha(t)=\eta, b(t)=\frac{\nu a(t)}{\sqrt{f_1(t)}}, c(t)=\frac{{f_1}'(t)}{2c_1},$$

$$A_2(z,t)=-\frac{12r_4\nu\eta^2}{\sqrt{f_1(t)}}, A_1(z,t)=0, \text{ and}$$

$$A_0(z,t)=-\frac{1}{\sqrt{f_1(t)}}(4r_2\eta^2\nu)+\frac{1}{8c_1a(t){f_1}^2(t)}\Big((1-8c_1c_2){f_1}'(t)\Big)+\frac{1}{\eta a(t)}\big(\beta'(t)\big). \tag{73}$$

Where, η *and* ν are an integration constants.

Case 1: If $r_0=\frac{r_2^2}{4r_4}, r_2<0, r_4>0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{2r_4}}\, tanh\left(\sqrt{\frac{-r_2}{2}}\, \xi\right). \tag{74}$$

When $z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2f_1(t)}$. We have double-dark soliton solution

$$u(x,y,t) = v(z,t) = \frac{\beta'(t)}{\eta a(t)} + \frac{f_1{}'(t)}{8c_1 a(t) f_1^2(t)} - \frac{c_2 f_1{}'(t)}{a(t) f_1^2(t)} - \frac{4\eta^2 \nu r_2}{\sqrt{f_1(t)}} +$$

$$\frac{6\eta^2 \nu r_2}{\sqrt{f_1(t)}} tanh^2\left(\sqrt{\frac{-r_2}{2}}\left(-\eta x + \frac{\eta(2c_2 + y + c_1 y^2)}{2f_1(t)} + \beta(t) + \theta\right)\right). \tag{75}$$

Case 2: If $r_0 = 0, r_4 < 0, r_2 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sech(\sqrt{r_2}\, \xi). \tag{76}$$

When $z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2f_1(t)}$. We have M-shaped bright soliton solution

$$u(x,y,t) = v(z,t) = -\frac{4\eta^2 \nu r_2}{\sqrt{f_1(t)}} + \frac{\beta'(t)}{\eta a(t)} + \frac{f_1{}'(t)}{8c_1 a(t) f_1^2(t)} - \frac{c_2 f_1{}'(t)}{a(t) f_1^2(t)} +$$

$$\frac{12\eta^2 \nu r_2}{\sqrt{f_1(t)}} sech^2\left(\sqrt{r_2}\left(-\eta x + \frac{\eta(2c_2 + y + c_1 y^2)}{2f_1(t)} + \beta(t) + \theta\right)\right). \tag{77}$$

Case 3: If $r_0 = 0, r_2 < 0, r_4 > 0$, Eq. (23) has the solution

$$\omega(\xi) = \sqrt{\frac{-r_2}{r_4}}\, sec(\sqrt{-r_2}\, \xi). \tag{78}$$

When $z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2f_1(t)}$. We have singular periodic solution

$$u(x,y,t) = v(z,t) = -\frac{4\eta^2 \nu r_2}{\sqrt{f_1(t)}} + \frac{\beta'(t)}{a\eta} + \frac{f_1{}'(t)}{8c_1 a(t) f_1^2(t)} - \frac{c_2 f_1{}'(t)}{a(t) f_1^2(t)} +$$

$$\frac{12\eta^2 \nu r_2}{\sqrt{f_1(t)}} sec^2\left(\sqrt{-r_2}\left(-\eta x + \frac{\eta(2c_2 + y + c_1 y^2)}{2f_1(t)} + \beta(t) + \theta\right)\right). \tag{79}$$

Result (2):

$$r_2 = \frac{{r_3}^3 + 8 r_1 {r_4}^2}{4 r_3 r_4}\ , \alpha(t) = \eta, b(t) = \frac{\nu a(t)}{\sqrt{f_1(t)}}, c(t) = \frac{f_1{}'(t)}{2c_1},$$

$$A_2(z,t) = -\frac{12 r_4 \nu \eta^2}{\sqrt{f_1(t)}}, A_1(z,t) = -\frac{6 r_3 \nu \eta^2}{\sqrt{f_1(t)}}, \text{ and}$$

$$A_0(z,t) = -\frac{1}{4r_4\sqrt{f_1(t)}}({r_3}^2 \eta^2 \nu) - \frac{1}{r_3\sqrt{f_1(t)}}(8 r_1 r_4 \eta^2 \nu) + \frac{1}{8c_1 a(t) {f_1}^2(t)}\left(f_1{}'(t)\right) -$$

$$\frac{1}{a(t){f_1}^2(t)}\left(c_2 f_1{}'(t)\right) + \frac{1}{\eta a(t)}\left(\beta'(t)\right). \tag{80}$$

.

Case 1: If $r_0 = 1, r_1 = -4, r_2 = 8 - m^2, r_3 = 2(m^2 - 4), r_4 = 2(2 - m^2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{sn(\xi, m) + cn(\xi, m)}. \tag{81}$$

When $m = 0, z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2 f_1(t)}$. We have singular periodic solution

$$u(x,y,t) = v(z,t) = \frac{\beta'(t)}{\eta a(t)} + \frac{f_1{}'(t)}{8c_1 a(t) f_1^2(t)} - \frac{c_2 f_1{}'(t)}{a(t) f_1^2(t)} - \frac{20\eta^2 \nu}{\sqrt{f_1(t)}\left(cos\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)+sin\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)\right)^2} + \frac{4\eta^2 \nu sin\left(-2\eta x+\frac{\eta(2c_2+y+c_1y^2)}{f_1(t)}+2\beta(t)+2\theta\right)}{\sqrt{f_1(t)}\left(cos\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)+sin\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)\right)^2}. \quad (82)$$

Case 2: If $r_0 = 1, r_1 = -4, r_2 = 5 - 6m - m^2, r_3 = 2(m^2 + 6m - 1), r_4 = m(4m^2 + 7m - 2)$, Eq. (23) has the solution

$$\omega(\xi) = \frac{sn(\xi, m)}{1 + sn(\xi, m) + m\, sn^2(\xi, m)}. \quad (83)$$

When $m = 1, z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2 f_1(t)}$. We have double-dark soliton solution

$$u(x,y,t) = v(z,t) = \frac{20\eta^2 \nu}{\sqrt{f_1(t)}} + \frac{\beta'(t)}{\eta a(t)} + \frac{f_1{}'(t)}{8c_1 a(t) f_1^2(t)} - \frac{c_2 f_1{}'(t)}{a(t) f_1^2(t)} - \frac{72\eta^2 \nu tanh\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)}{\sqrt{f_1(t)}\left(1+tanh\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)+tanh^2\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)\right)} - \frac{108\eta^2 \nu tanh^2\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)}{\sqrt{f_1(t)}\left(1+tanh\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)+tanh^2\left(\frac{\eta\left(2c_2+y+c_1y^2-2xf_1(t)\right)}{2f_1(t)}+\beta(t)+\theta\right)\right)^2}. \quad (84)$$

The analytical procedure developed in this study leads naturally to a number of admissible cases for the variable coefficients $a(t), b(t)$, and $c(t)$, as summarized in Table 1. These cases are not obtained through a separate classification procedure; rather, they emerge progressively during the analytical treatment. The Lie symmetry determining equations impose certain constraints on the variable coefficients, while additional restrictions may arise from the compatibility conditions obtained through the GEM. Their combination leads to the admissible coefficient regimes and corresponding group-invariant reductions listed in Table (1).

Table (1). Admissible variable-coefficient regimes and corresponding group-invariant reductions.

Case	$\boldsymbol{a(t)}$	$\boldsymbol{c(t)}$	$\boldsymbol{b(t)}$	**Generator**	**Group invariant solution**
1	∀	∀	$\rho a(t)\left(d_1 - \int a(t)\, dt\right)$	X_1	$u(x,y,t) = v(z,t),$ $z = -c_1 x + y$
2	∀	∀	$\lambda a(t)$	X_1	$u(x,y,t) = v(z,t),$ $z = -c_1 x + y$

3	$\frac{1}{\mu}f_1{}'(t)$	$\forall$	$\forall$	X_2	$z = y - \frac{c_1 x}{f_1(t)}$, $\psi = \frac{x}{f_1(t)}$, $u(x,y,t) = f_2(t)\psi + v(z,t)$
4	$\frac{f_1{}'(t)}{\kappa\left(d_2 + d_1 f_1(t)\right)}$	$\forall$	$\frac{\gamma f_1(t) f_1{}'(t)}{\left(d_2 + d_1 f_1(t)\right)^3}$	X_2	$z = y - \frac{c_1 x}{f_1(t)}$, $\psi = \frac{x}{f_1(t)}$, $u(x,y,t) = f_2(t)\psi + v(z,t)$
5	$\forall$	$\frac{f_1{}'(t)}{2c_1}$	$\frac{va(t)}{\sqrt{f_1(t)}}$	X_3	$z = \frac{c_1 y^2 - 2x f_1(t) + 2c_2 + y}{2f_1(t)}$, $u(x,y,t) = v(z,t)$

Here, d_1, d_2, ρ, c_1, c_2, μ, λ, κ, γ, and ν are arbitrary constants.

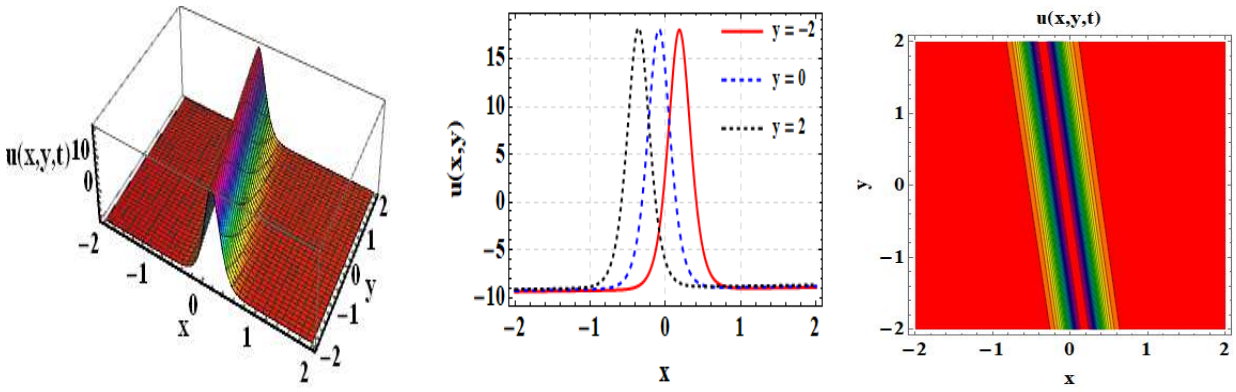
5. Results and discussion

We present a comprehensive graphical representation of the newly obtained soliton solutions for Eq. (1). To provide deeper physical insight into the results, it is essential to interpret the graphical plots, as the geometric composition of each wave determines its propagation characteristics and nonlinear behavior. In this section, several distinct types of soliton solutions are illustrated, including bright soliton, dark soliton, singular periodic, kink-like solitary wave, double-dark, and M-shaped bright soliton structures.

The bright soliton solutions, derived from Eqs. (30), (42), (54), and (65), and shown in Fig. (1), exhibit smooth, localized, bell-shaped profiles with a well-defined amplitude and spatial localization during propagation, reflecting the characteristic structure of bright solitary waves. The dark soliton solutions, derived from Eqs. (28), (40), (52), and (63), and shown in Fig. (2), exhibit smooth, localized intensity dips embedded within a continuous nonzero background, forming characteristic dark solitary-wave profiles. The singular periodic solutions, derived from Eqs. (32), (35), (44), (47), (55), (58), (67), (70), (79), and (82), and shown in Fig. (3), exhibit periodic wave profiles with singular behavior at specific points, resulting in repeated wave patterns separated by singularities. The kink-like solitary wave solutions, derived from Eqs. (37), (49), (60), and (72), and shown in Fig. (4), exhibit monotonic transition profiles connecting two different asymptotic states, producing characteristic kink-shaped structures.

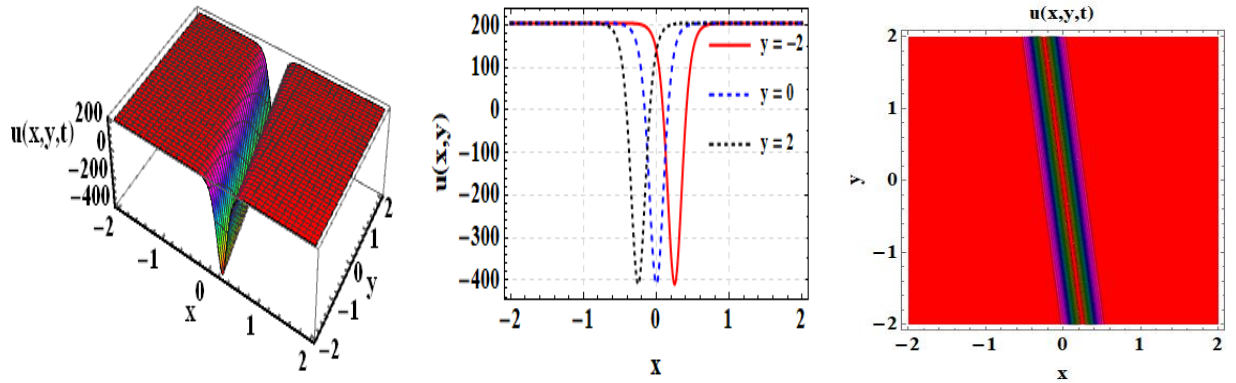
In addition to these wave structures, the analytical procedure also yields double-dark and M-shaped bright soliton solutions, which represent further forms of localized nonlinear wave profiles. The double-dark solutions, derived from Eqs. (75), and (84), and shown in Fig. (5), exhibit two distinct localized intensity depressions within a nonzero background, whereas the M-shaped bright soliton solution, derived from Eq. (77), and shown in Fig. (6), exhibits a localized profile consisting of two prominent peaks separated by a central depression, resulting in a characteristic M-shaped structure. Overall, the graphical analysis demonstrates that Eq. (1) admits a diverse

range of nonlinear wave structures, including localized, periodic, singular, and kink-like profiles. The obtained graphical results are consistent with the analytical forms of the solutions and illustrate the influence of the variable coefficients and solution parameters on the resulting wave profiles.



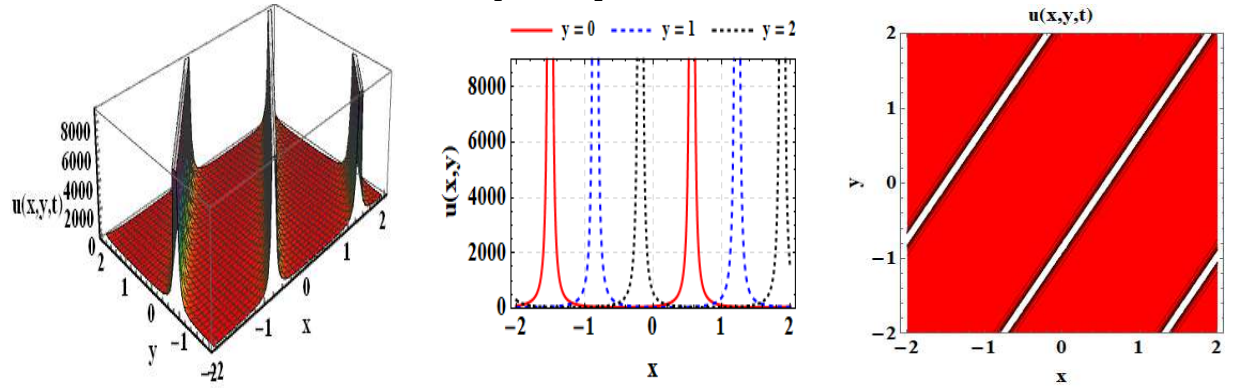
(a) (b) (c)

Fig.1: (a) 3D,(b) 2D and (c) contour visualizations of Eq. (65) $f_1(t) = e^t, \beta(t) = t, c(t) = t^2, t = 1,$ $d_1 = 1.4, d_2 = -3, c_1 = -20, \kappa = 1, \gamma = 1, r_2 = 0.1$ *and* $\theta = 1.$



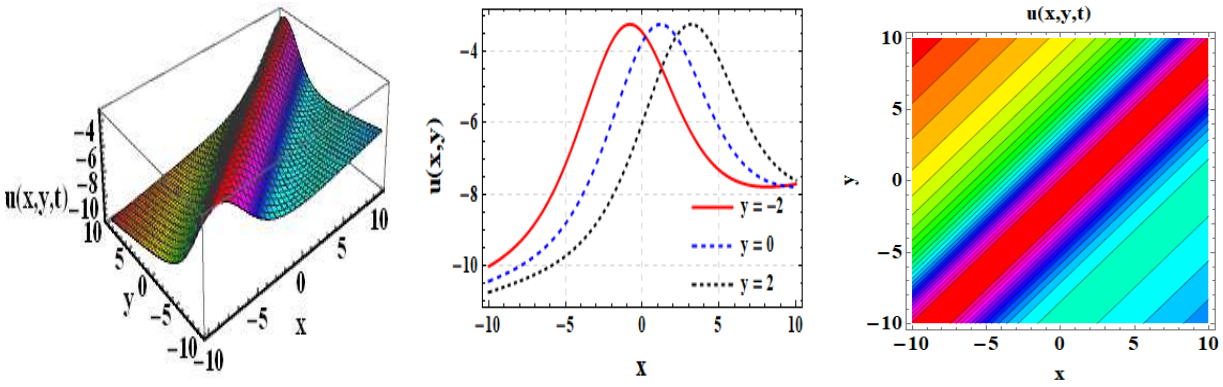
(a) (b) (c)

Fig.2: (a) 3D,(b) 2D and (c) contour visualizations of Eq. (40) $a(t) = e^t, \beta(t) = t, c(t) = t^2, t = 1, \mu = 4,$ $\lambda = -1, c_1 = -8, r_2 = -0.1$ *and* $\theta = -1.$



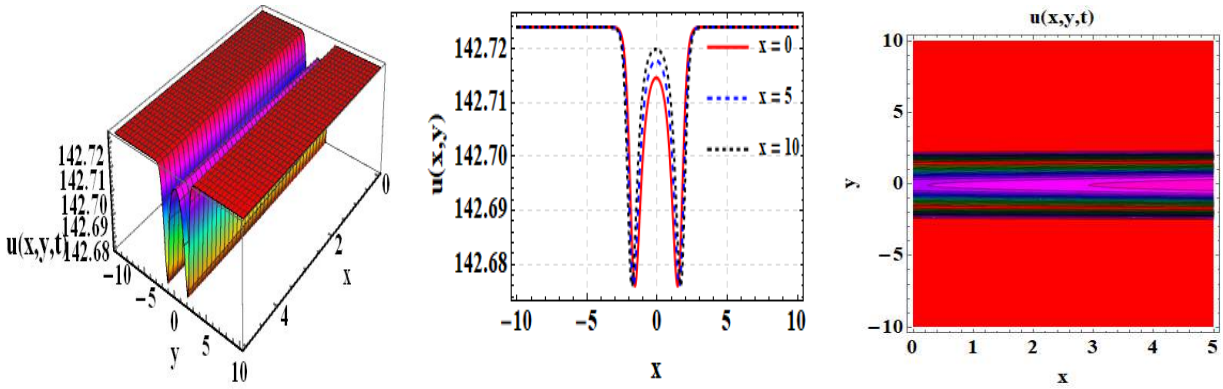
(a) (b) (c)

Fig.3: (a) 3D,(b) 2D and (c) contour visualizations of Eq. (58) $f_1(t) = e^t, \beta(t) = t, c(t) = t^2, t = 0.5, \lambda = 1,$ $\mu = -1, c_1 = 2.5, \rho = -1 and\ \theta = 1.$



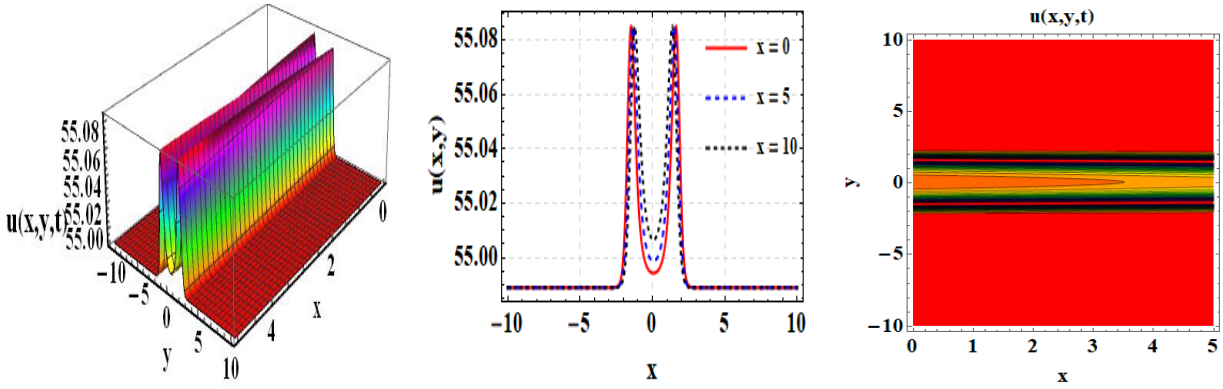
(a) (b) (c)

Fig.4: (a) 3D,(b) 2D and (c) contour visualizations of Eq. (37) $a(t) = e^t, \beta(t) = t, c(t) = t^2, t = 0.1,$ $d_1 = -7, d_2 = 1, c_1 = 1, \rho = -1 and\ \theta = -0.5.$



(a) (b) (c)

Fig.5: (a) 3D,(b) 2D and (c) contour visualizations of Eq. (75) $a(t) = e^t, \beta(t) = t, f_1(t) = t^2, t = 0.5,$ $\eta = -0.2, \nu = -1, c_1 = 7, r_2 = -0.1, c_2 = -15, and\ \theta = -1.$



(a) (b) (c)

Fig.6: (a) 3D,(b) 2D and (c) contour visualizations of Eq. (77) $a(t) = e^t, \beta(t) = t, f_1(t) = t^2, t = 0.5, \eta = -0.2, \nu = 1, c_1 = -7, r_2 = 0.1, c_2 = -6, and\ \theta = -12.$

6. Conclusion

In this paper, the vcKP equation was investigated by combining Lie point symmetry analysis with the GEM approach. First, by using the method of Lie symmetry analysis, three similarity transformations associated with three Lie symmetry generators were obtained, which reduced the original equation into three different (1+1)-dimensional nonlinear evolution equations. These reduced equations were then solved analytically by the GEM, and the resulting solutions were transformed back to the original variables to obtain exact analytical solutions of the original vcKP equation. The obtained solutions include different nonlinear wave structures, demonstrating the capability of the considered equation to support localized, periodic, singular, and kink-like profiles. The corresponding graphical representations further illustrate the characteristic features of these solutions and the effects of the variable coefficients and solution parameters on their wave profiles. It is worth noting that the admissible forms of the variable coefficients arise naturally during the analytical procedure. The constraints obtained from the Lie symmetry determining equations, together with the compatibility conditions encountered in the GEM, lead to the admissible cases considered in this article. Thus, the proposed Lie symmetry-GEM framework provides a systematic approach for deriving exact solutions of the considered variable-coefficient nonlinear evolution equation and can be extended to other higher-dimensional nonlinear models for which suitable symmetry reductions and auxiliary equations can be constructed.

Data availability

All data generated or analysed during this study are included in this manuscript.

Reference

[1] El-Shazly, E., Eissa, M. E., Elsonbaty, A., & Nisar, K. S. (2025). Optical Soliton Solutions for a Generalized Nonlocal Nonlinear Schrödinger-Type Equation with Kudryashov-Like Refractive Index Law. Franklin Open, 100475.

[2] Chen, S. J., Lü, X., & Tang, X. F. (2021). Novel evolutionary behaviors of the mixed solutions to a generalized Burgers equation with variable coefficients. *Communications in Nonlinear Science and Numerical Simulation*, *95*, 105628.

[3] Erbay, H. A., Erbay, S., & Erkip, A. (2022). On the full dispersion Kadomtsev–Petviashvili equations for dispersive elastic waves. Wave Motion, 114, 103015.

[4] Sakkaravarthi, K., Singh, S., & Karjanto, N. (2023). Exploring the dynamics of nonlocal nonlinear waves: analytical insights into the extended Kadomtsev–Petviashvili model. Frontiers in Physics, 11, 1168830.

[5] Wei, P., Zhao, Z., Ling, S., Gui, R., Chen, Y., Wang, W., & Ren, B. (2024). Multi-soliton, breather, lump and their interactions of the (2+ 1)-dimensional generalized potential Kadomtsev-Petviashvili equation. Journal of Nonlinear Mathematical Physics, 31(1), 82.

[6] Lv, N., Sun, J., Zhang, R., Yuan, X., & Yue, Y. (2025). Nonlinear characteristics of various local waves on nonzero backgrounds of a (2+ 1)-dimensional generalized Kadomtsev–Petviashvili equation with variable coefficients. Physica D: Nonlinear Phenomena, 472, 134501.

[7] Sabry, R., & El-Taibany, W. F. (2008). New exact solutions for a generalized variable-coefficient KdV equation. *Nonlinear Analysis: Theory, Methods & Applications*, *69*(8), 2763-2770.

[8] Hao, L., & Zhao, X. (2024). Extended F-expansion method and its application to the variable-coefficient fractional nonlinear Schrödinger equation. In *Proc 2024 Int Conf Mach Learn Intell Comput, PMLR* (Vol. 245, pp. 297-307).

[9] Fekadu, A. (2022). *Solution of (2+ 1) − Dimensional Nonlinear Schr¨ odinger Equation with Nonlinearity derivatives by Exp-function method* (Doctoral dissertation, Ambo University).

[10] Farahat, S. E., El Shazly, E. S., El-Kalla, I. L., & Abdel Kader, A. H. (2023). New exact solutions for perturbed nonlinear Schrödinger's equation with self-phase modulation of Kudryashov's sextic power law refractive index. *Optical and Quantum Electronics*, *55*(14), 1280.

[11] Hossain, A. K. S., Islam, M. K., Akter, H., & Akbar, M. A. (2024). Exact and soliton solutions of nonlinear evolution equations in mathematical physics using the generalized (G'/G)-expansion approach. *Physica Scripta*, *100*(1), 015269.

[12] Sirisubtawee, S., Thamareerat, N., & Iatkliang, T. (2021). Variable coefficient exact solutions for some nonlinear conformable partial differential equations using an auxiliary equation method. *Computation*, *9*(3), 31.

[13] Kudryashov, N. A. (2019). First integrals and general solution of the Fokas–Lenells equation. Optik, 195, 163135.

[14] Zayed, E. M., Shohib, R. M., & Alngar, M. E. (2020). New extended generalized Kudryashov method for solving three nonlinear partial differential equations. *Nonlinear Analysis: Modelling and Control*, *25*(4), 598-617.

[15] Farooq, A., Khan, M. I., & Ma, W. X. (2024). Exact solutions for the improved mKdv equation with conformable derivative by using the Jacobi elliptic function expansion method: A. Farooq et al. Optical and Quantum Electronics, 56(4), 542.

[16] Liu, J. G., Zhu, W. H., & Zhou, L. (2019). Interaction solutions for Kadomtsev-Petviashvili equation with variable coefficients. Communications in Theoretical Physics, 71(7), 793.

[17] Kumar, S., Dhiman, S. K., Baleanu, D., Osman, M. S., & Wazwaz, A. M. (2022). Lie symmetries, closed-form solutions, and various dynamical profiles of solitons for the variable coefficient (2+ 1)-dimensional KP equations. Symmetry, 14(3), 597.

[18] Mohanty, S. K., Kravchenko, O. V., Deka, M. K., Dev, A. N., & Churikov, D. V. (2023). The exact solutions of the 2+ 1–dimensional Kadomtsev-Petviashvili equation with variable coefficients by extended generalized G' G-expansion method. Journal of King Saud University-Science, 35(1), 102358.

[19] Demirbilek, U., Nadeem, M., Çelik, F. M., Bulut, H., & Şenol, M. (2024). Generalized extended (2 1)-dimensional Kadomtsev-Petviashvili equation in fluid dynamics: analytical solutions, sensitivity and stability analysis. Nonlinear Dynamics, 112(15), 13393.

[20] Latif, M. S. A., Shazly, E. E., Elsaid, A., & Nour, H. M. (2018). Group classification and some new periodic-like and soliton-like solutions of the generalised Fisher equation with time-variable coefficients. *International Journal of Dynamical Systems and Differential Equations*, *8*(4), 313-325.

[21] Abdel Latif, M. S., El-Shazly, E., Baleanu, D., Elsaid, A., & Nour, H. M. (2018). Some new soliton-like and doubly periodic-like solutions of Fisher equation with time-dependent coefficients. *Modern Physics Letters B*, *32*(33), 1850413.

[22] Hamed, A. A., El-Kalla, I. L., Latif, M. A., & Kader, A. A. (2024). Optical solutions for a quintic derivative nonlinear Schrödinger equation using symmetry analysis. *Optik*, *309*, 171834.

[23] Sabry, R., Zahran, M. A., & Fan, E. (2004). A new generalized expansion method and its application in finding explicit exact solutions for a generalized variable coefficients KdV equation. *Physics Letters A*, *326*(1-2), 93-101.

Acknowledgments

This work was supported by the Science, Technology & Innovation Funding Authority (STDF) in cooperation with the Egyptian Knowledge Bank (EKB).

Author Contributions

A.S. Abo-Eleneen conceived the study, developed the methodology, performed the analysis, and wrote the main manuscript text. I.L. El-Kalla contributed to the methodology, analysis, and interpretation of the results. E.S. El-Shazly contributed to the analysis and scientific review of the manuscript. All authors reviewed and approved the final version of the manuscript.

Funding

Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB). Open access funding provided by the authors gratefully acknowledge that this research was funded by the Science, Technology & Innovation Funding Authority (STDF) in cooperation with the Egyptian Knowledge Bank (EKB).

Declarations

Competing interests

The authors declare no competing interests.