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Asymptotic Mode-I crack-tip fields in an incompressible finite-strain viscoelastic solid

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Abstract

The Mode-I crack-tip fields for an incompressible finite-strain viscoelastic solid with neo-Hookean equilibrium and nonequilibrium branches are analyzed. The focus is on the branch where the viscous right Cauchy–Green tensor $\mathbf{C}_v$ shares the leading singularity of the total deformation, while the transformed internal variable $\tilde{\mathbf{D}} = \mathbf{F}^{-T} \mathbf{C}_v \mathbf{F}^{-1}$ remains bounded. In this branch, the vertical crack-opening field y_2 retains the square-root radial dependence characteristic of the incompressible neo-Hookean crack-tip field, with its amplitude determined by an asymptotic path integral governed by the equilibrium shear modulus. The horizontal field $y_1 \sim r^c g(\theta, t)$ presents a distinct order-selection problem, with mesh-refined calculations supporting the $c = 1$ branch. The angular function of y_1 remains coupled to the internal-variable evolution, rendering the finite-time angular field non-universal. In the fully relaxed limit, the coupled equations simplify to the incompressible neo-Hookean crack-tip field. Fixed-stretch calculations reveal that the opening amplitude p_2 and the angular root-mean-square (RMS) of the normalized horizontal field y_1/r relax at different rates.

Keywords: finite viscoelasticity, crack-tip fields, large-deformation fracture, asymptotic analysis, internal variables, neo-Hookean solids

1. Introduction

Soft elastomers, gels, and biological tissues often fracture under large deformations, where the deformation gradient near a crack tip may be far outside the range described by linear elastic fracture mechanics. A finite-strain crack-tip field is therefore needed to characterize the local deformation and stress states that drive rupture.

The finite-deformation crack-tip problem has a long history in hyperelasticity. Classical asymptotic analyses showed that near-tip fields in nonlinear elastic solids differ fundamentally from the Williams fields of linear elasticity (Knowles and Sternberg, 1973; Knowles, 1977). Subsequent studies extended these results to plane-stress sheets, generalized neo-Hookean solids, and strain-hardening soft materials (Geubelle and Knauss, 1994; Long et al., 2011). More recently, Li and Moran (2025) examined the nested linear and nonlinear crack-tip regions that arise in a neo-Hookean sheet at small far-field strain. That study concerns a purely elastic material, whereas the present problem couples the crack-tip field to a time-dependent viscous internal variable. For a plane-stress crack in an incompressible neo-Hookean solid, the leading-order y_2 field takes the form

$$y_2 \sim p_2 r^{1/2} \sin \frac{\theta}{2}, \quad (1.1)$$

and the associated Cauchy stress has an r^{-1} singularity, which is stronger than the $r^{-1/2}$ singularity of linear elasticity.

An asymptotic path-integral formulation for evaluating crack-tip amplitudes in an incompressible neo-Hookean material is described by Liu and Moran (2020a). The same formulation has also been applied to fiber-reinforced sheets (Liu and Moran, 2020b). We use it below to evaluate the amplitude p_2 .

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The corresponding finite-strain viscoelastic crack-tip problem is more subtle. From a constitutive perspective, internal-variable formulations based on multiplicative kinematics and generalized standard materials are widely used for finite viscoelasticity in elastomers and soft solids (Sidoroff, 1974; Le Tallec et al., 1993; Reese and Govindjee, 1998; Bergstrom and Boyce, 1998; Kumar and Lopez-Pamies, 2016). Recent studies have emphasized formulations in which the viscous right Cauchy–Green internal variable $\mathbf{C}_v$ is isochoric, $\det \mathbf{C}_v = 1$, with time-integration schemes that preserve this constraint exactly (Wijaya et al., 2023; Lefevre et al., 2024). Fracture studies, on the other hand, often focus on energy dissipation, cohesive-zone or phase-field descriptions, and rate-dependent fracture behavior (Guo et al., 2020; Shen and Vernerey, 2020; Shrimali and Lopez-Pamies, 2022; Kamasamudram et al., 2023; Kamarei et al., 2025). These works provide constitutive and energetic descriptions, but the local asymptotic structure of the internal variable near a finite-strain crack tip is still not well characterized.

In this work, we study Mode-I crack-tip fields in an incompressible finite-strain viscoelastic solid. The material is modeled as the finite-strain analogue of a standard viscoelastic solid, consisting of a neo-Hookean equilibrium branch in parallel with a neo-Hookean nonequilibrium branch with constant viscosity (Kumar and Lopez-Pamies, 2016; Wijaya et al., 2023; Lefevre et al., 2024). Writing $\mathbf{F}$ for the deformation gradient and $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ for the right Cauchy–Green tensor, the leading crack-opening field produces an $O(r^{-1})$ contribution to $\mathbf{C}$ near the crack tip. We characterize the viscous internal state relative to this singular deformation through the transformed internal variable

$$\tilde{\mathbf{D}} = \mathbf{F}^{-T} \mathbf{C}_v \mathbf{F}^{-1}. \quad (1.2)$$

The subsequent asymptotic analysis is developed for the bounded- $\tilde{\mathbf{D}}$ branch. The resulting asymptotic problem determines the leading y_2 field, its path-integral amplitude, and the admissible radial orders of y_1 . The field y_1 is represented locally as $y_1 \sim r^c g(\theta, t)$, with c denoting its radial order. At $c = 1$, the equilibrium and nonequilibrium stress contributions enter at the same leading order, while g remains coupled to the evolution of $\tilde{\mathbf{D}}$. The coupled equations recover the incompressible neo-Hookean crack-tip field in the fully relaxed state. The analytical results are examined using mesh-refined finite element calculations under continuous loading and after the applied stretch is held fixed.

The remainder of the paper is organized as follows. Section 2 introduces the constitutive model. Section 3 gives the behavior of y_2 and y_1 at the crack tip, the asymptotic J -integral, the coupled equations for g and $\tilde{\mathbf{D}}$, and the fully relaxed crack-tip field. Section 4 presents the numerical results and comparisons. Section 5 concludes the paper. The necessary higher-order condition associated with the interval $1 < c < 3/2$ and the asymptotic consistency of the bounded- $\tilde{\mathbf{D}}$ branch are discussed in the appendices.

2. Constitutive model

Let $\mathbf{X}$ denote the reference position and $\mathbf{y}(\mathbf{X}, t)$ the current position. The deformation gradient and right Cauchy–Green tensor are defined as

$$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{X}}, \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}. \quad (2.1)$$

The material is assumed to be incompressible, so that

$$\det \mathbf{F} = 1. \quad (2.2)$$

Under plane-stress conditions, the in-plane deformation gradient is

$$\mathbf{F}_{2D} = \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix}, \quad (2.3)$$

and incompressibility determines the out-of-plane stretch as

$$F_{33} = \lambda = \frac{1}{\det \mathbf{F}_{2D}}. \quad (2.4)$$

To model viscoelastic effects, we adopt the two-potential finite-viscoelastic framework (Kumar and Lopez-Pamies, 2016; Wijaya et al., 2023; Lefevre et al., 2024), which belongs to the broader class of multiplicative internal-variable

models (Sidoroff, 1974; Le Tallec et al., 1993; Reese and Govindjee, 1998; Bergstrom and Boyce, 1998). Its rheological representation is shown in Fig. 1(a). An equilibrium branch acts in parallel with a dissipative branch consisting of a nonequilibrium elastic element and a viscous element. This arrangement is the finite-deformation counterpart of the classical Zener, or standard solid model. The equilibrium and nonequilibrium elastic elements store the energies Ψ_{eq} and Ψ_{neq} , respectively, while viscous dissipation is characterized by the potential ϕ_v .

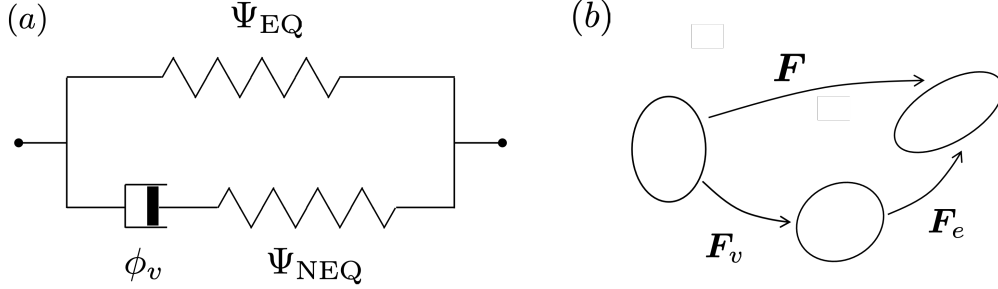


Figure 1: (a) Rheological representation of the two-potential finite-viscoelastic model. The equilibrium branch is connected in parallel with a nonequilibrium Maxwell branch composed of an elastic element and a viscous element in series. (b) Multiplicative decomposition of the nonequilibrium branch.

As shown in Fig. 1(b), the nonequilibrium branch is modeled through the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_v, \quad (2.5)$$

where $\mathbf{F}_e$ and $\mathbf{F}_v$ are the elastic and viscous parts of the deformation gradient. Both parts are taken to be isochoric, so that $\det \mathbf{F}_e = \det \mathbf{F}_v = 1$, consistent with $\det \mathbf{F} = 1$. We define

$$\begin{aligned} \mathbf{C}_e &= \mathbf{F}_e^T \mathbf{F}_e = \mathbf{F}_v^{-T} \mathbf{C} \mathbf{F}_v^{-1}, \\ \mathbf{C}_v &= \mathbf{F}_v^T \mathbf{F}_v, \quad \det \mathbf{C}_v = 1. \end{aligned} \quad (2.6)$$

The free energy is written as

$$\Psi(\mathbf{C}, \mathbf{C}_e) = \Psi_{\text{eq}}(\mathbf{C}) + \Psi_{\text{neq}}(\mathbf{C}_e). \quad (2.7)$$

In this work, both elastic branches are neo-Hookean, with μ_0 and μ_1 denoting the shear moduli of the equilibrium and nonequilibrium branches, respectively:

$$\begin{aligned} \Psi_{\text{eq}}(\mathbf{C}) &= \frac{\mu_0}{2} (\text{tr} \mathbf{C} - 3), \\ \Psi_{\text{neq}}(\mathbf{C}_e) &= \frac{\mu_1}{2} (\text{tr} \mathbf{C}_e - 3). \end{aligned} \quad (2.8)$$

The first invariant of $\mathbf{C}_e$ in the dissipative branch is

$$I_1^e = \text{tr} \mathbf{C}_e = \text{tr}(\mathbf{C}_v^{-1} \mathbf{C}). \quad (2.9)$$

Differentiating the free energy with respect to $\mathbf{C}$ gives the second Piola–Kirchhoff stress, and the first Piola–Kirchhoff stress follows from $\mathbf{P} = \mathbf{F}\mathbf{S}$:

$$\begin{aligned} \mathbf{S} &= \mu_0 \mathbf{I} + \mu_1 \mathbf{C}_v^{-1} - p \mathbf{C}^{-1}, \\ \mathbf{P} &= \mu_0 \mathbf{F} + \mu_1 \mathbf{F} \mathbf{C}_v^{-1} - p \mathbf{F}^{-T}. \end{aligned} \quad (2.10)$$

Here p enforces incompressibility.

For constant viscosity, the evolution equation for $\mathbf{C}_v$ is (Kumar and Lopez-Pamies, 2016; Wijaya et al., 2023; Lefevre et al., 2024)

$$\tau \dot{\mathbf{C}}_v = \mathbf{C} - \frac{I_1^e}{3} \mathbf{C}_v, \quad I_1^e = \text{tr}(\mathbf{C}_v^{-1} \mathbf{C}), \quad (2.11)$$

where $\tau = \eta/\mu_1$ is the relaxation time. The evolution preserves $\det \mathbf{C}_v$ because

$$\begin{aligned} \frac{d}{dt} \ln \det \mathbf{C}_v &= \text{tr}(\mathbf{C}_v^{-1} \dot{\mathbf{C}}_v) \\ &= \frac{1}{\tau} \left[\text{tr}(\mathbf{C}_v^{-1} \mathbf{C}) - I_1^e \right] = 0. \end{aligned} \quad (2.12)$$

In the fully relaxed state, Eq. (2.11), together with $\det \mathbf{C} = \det \mathbf{C}_v = 1$, gives

$$\mathbf{C}_v = \mathbf{C}, \quad \mathbf{C}_e = \mathbf{I}. \quad (2.13)$$

The stress then becomes

$$\mathbf{P} = \mu_0 \mathbf{F} - (p - \mu_1) \mathbf{F}^{-T}. \quad (2.14)$$

The relaxed response is therefore that of an incompressible neo-Hookean solid with shear modulus μ_0 .

The small-strain limit gives the standard linear solid. If

$$\mathbf{C} = \mathbf{I} + 2\mathbf{E}, \quad \mathbf{C}_v = \mathbf{I} + 2\mathbf{E}_v, \quad \text{tr } \mathbf{E} = \text{tr } \mathbf{E}_v = 0, \quad (2.15)$$

then, to first order,

$$\begin{aligned} \mathbf{s} &= 2\mu_0 \mathbf{E} + 2\mu_1 (\mathbf{E} - \mathbf{E}_v), \\ \dot{\mathbf{E}}_v &= \frac{1}{\tau} (\mathbf{E} - \mathbf{E}_v). \end{aligned} \quad (2.16)$$

For a step strain, the shear relaxation modulus is

$$G(t) = \mu_0 + \mu_1 \exp\left(-\frac{t}{\tau}\right). \quad (2.17)$$

Accordingly, the instantaneous and long-time shear moduli are $\mu_0 + \mu_1$ and μ_0 , respectively.

3. Crack tip fields

3.1. Asymptotic behavior of y_2 at the crack tip

For the local radial asymptotics at a fixed time, assume that the leading deformed coordinates have the form

$$\begin{aligned} y_2(r, \theta, t) &\sim r^m f(\theta, t), \\ y_1(r, \theta, t) &\sim r^c g(\theta, t). \end{aligned} \quad (3.1)$$

For a traction-free Mode-I crack, we assume that the gradient of y_2 is singular and asymptotically dominates that of y_1 :

$$0 < m < 1, \quad c > m, \quad |y_{2,\alpha}| \gg |y_{1,\alpha}| \quad \text{as } r \rightarrow 0. \quad (3.2)$$

We consider the plane-stress case in which the thickness stretch satisfies $\lambda \rightarrow 0$.

For the crack-tip branch considered here, $\mathbf{C}_v$ carries the same leading singular factor as $\mathbf{C}$. Equivalently, we write

$$\mathbf{C}_v = \mathbf{F}^T \tilde{\mathbf{D}} \mathbf{F}, \quad \tilde{\mathbf{D}} = \mathbf{F}^{-T} \mathbf{C}_v \mathbf{F}^{-1}, \quad (3.3)$$

where $\tilde{\mathbf{D}} = O(1)$ and $\tilde{\mathbf{D}}^{-1} = O(1)$. We refer to this as the bounded- $\tilde{\mathbf{D}}$ crack-tip branch. The factorization in Eq. (3.3) separates the singular deformation associated with $\mathbf{F}$ from the bounded asymptotic viscous state represented by $\tilde{\mathbf{D}}$.

Since $\det \mathbf{F} = \det \mathbf{C}_v = 1$, Eq. (3.3) gives

$$\det \tilde{\mathbf{D}} = 1, \quad \mathbf{C}_v^{-1} = \mathbf{F}^{-1} \tilde{\mathbf{D}}^{-1} \mathbf{F}^{-T}, \quad I_1^e = \text{tr } \tilde{\mathbf{D}}^{-1}. \quad (3.4)$$

The stress can be written as

$$\mathbf{P} = \mu_0 \mathbf{F} + (\mu_1 \tilde{\mathbf{D}}^{-1} - p \mathbf{I}) \mathbf{F}^{-T}. \quad (3.5)$$

For plane stress, write

$$\tilde{\mathbf{D}} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & \zeta \end{bmatrix}, \quad \zeta = (d_{11}d_{22} - d_{12}^2)^{-1}. \quad (3.6)$$

The condition $P_{33} = 0$ gives

$$p = \mu_0 \lambda^2 + \mu_1 \zeta^{-1}. \quad (3.7)$$

Thus $p = O(1)$ provided that $\tilde{\mathbf{D}}$ is bounded and $\lambda \rightarrow 0$.

For the general orders in Eq. (3.1),

$$\det \mathbf{F}_{2D} \sim r^{c+m-2} (c g f_\theta - m f g_\theta), \quad (3.8)$$

and hence

$$\lambda \sim r^{2-c-m}. \quad (3.9)$$

The condition $\lambda \rightarrow 0$ requires $c + m < 2$. The rows of the in-plane inverse transpose then have the orders

$$\mathbf{F}_{1\alpha}^{-T} = O(r^{1-c}), \quad \mathbf{F}_{2\alpha}^{-T} = O(r^{1-m}). \quad (3.10)$$

Consequently, the second term in Eq. (3.5) is subdominant in the second row of the nominal stress, so that

$$P_{2\alpha} \sim \mu_0 y_{2,\alpha}. \quad (3.11)$$

The leading equilibrium equation and crack-face condition are

$$y_{2,\alpha\alpha} = 0, \quad y_{2,2}(r, \pm\pi, t) = 0. \quad (3.12)$$

Mode-I symmetry gives

$$f(\theta, t) = p_2(t) \sin(m\theta). \quad (3.13)$$

The traction-free condition requires $\cos(m\pi) = 0$. Together with $0 < m < 1$, this selects

$$m = \frac{1}{2}, \quad y_2 \sim p_2(t) r^{1/2} \sin \frac{\theta}{2}. \quad (3.14)$$

Combining $m = 1/2$ with $c > m$ in Eq. (3.2) and $c + m < 2$ following Eq. (3.9) restricts the radial order of y_1 to

$$\frac{1}{2} < c < \frac{3}{2}. \quad (3.15)$$

3.2. Asymptotic J -integral

The leading-order stress components are

$$\begin{aligned} P_{21} &= -\frac{\mu_0 p_2}{2} r^{-1/2} \sin \frac{\theta}{2}, \\ P_{22} &= \frac{\mu_0 p_2}{2} r^{-1/2} \cos \frac{\theta}{2}. \end{aligned} \quad (3.16)$$

The corresponding contribution to the strain-energy density and the associated nominal stress are

$$W^{(2)} = \frac{\mu_0}{2} (y_{2,1}^2 + y_{2,2}^2), \quad P_j^{(2)} = P_{2j}. \quad (3.17)$$

For a contour $\lim_{\epsilon \rightarrow 0} \Gamma_\epsilon$ surrounding the crack tip, the corresponding path integral is defined by

$$J_1^{(2)}(\epsilon) = \int_{\Gamma_\epsilon} (W^{(2)} \delta_{1j} - P_j^{(2)} y_{2,1}) n_j d\Gamma. \quad (3.18)$$

The divergence of the integrand is

$$\begin{aligned} \left(W^{(2)} \delta_{1j} - P_j^{(2)} y_{2,1} \right)_{,j} &= \mu_0 y_{2,k1} y_{2,k} - P_{2j,j} y_{2,1} - P_{2j} y_{2,1j} \\ &= \mu_1 \lambda \zeta d_{12} (y_{2,2} y_{2,11} - y_{2,1} y_{2,12}) + \lambda \left[\mu_1 (\zeta d_{11} - \zeta^{-1}) - \mu_0 \lambda^2 \right] (y_{1,2} y_{2,11} - y_{1,1} y_{2,12}). \end{aligned} \quad (3.19)$$

Here, the equilibrium equation gives $P_{2j,j} = 0$, while the leading-order contribution $\mu_0 y_{2,j}$ to P_{2j} cancels the first term. With $m = 1/2$, we have

$$\lambda = O(r^{3/2-c}), \quad y_{1,\alpha} = O(r^{c-1}), \quad y_{2,\alpha} = O(r^{-1/2}), \quad y_{2,\alpha\beta} = O(r^{-3/2}). \quad (3.20)$$

For the bounded- $\tilde{\mathbf{D}}$ branch, the two terms in Eq. (3.19) are at most of order $O(r^{-c-1/2})$ and $O(r^{-1})$, respectively. Since $c < 3/2$ by Eq. (3.15), their integral over a crack-tip neighborhood gives

$$\int_{A_\epsilon} \left(W^{(2)} \delta_{1j} - P_j^{(2)} y_{2,1} \right)_{,j} dA = O(\epsilon^{3/2-c}) + O(\epsilon) \longrightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.21)$$

It follows that the opening contribution is asymptotically path independent in the crack-tip limit. Evaluating the path integral with the leading-order opening field gives

$$J_1^{(2)} = \frac{\pi \mu_0 p_2^2}{4}. \quad (3.22)$$

This relation is akin to the neo-Hookean case (Liu and Moran, 2020a), with the equilibrium shear modulus μ_0 governing the opening contribution.

3.3. Asymptotic behavior of y_1 at the crack tip

We now determine the order of y_1 . Write

$$y_1 \sim r^c g(\theta, t). \quad (3.23)$$

Using the leading opening field,

$$\det \mathbf{F}_{2D} \sim \frac{p_2}{2} \left(c g \cos \frac{\theta}{2} - g_\theta \sin \frac{\theta}{2} \right) r^{c-3/2}, \quad (3.24)$$

so that

$$\lambda \sim \frac{2}{p_2} \left(c g \cos \frac{\theta}{2} - g_\theta \sin \frac{\theta}{2} \right)^{-1} r^{3/2-c}. \quad (3.25)$$

Writing $\mathbf{H} = \mu_1 \tilde{\mathbf{D}}^{-1} - p \mathbf{I}$ in Eq. (3.5), the three contributions to the first row of $\mathbf{P}$ scale as

$$\mu_0 F_{1\alpha} = O(r^{c-1}), \quad H_{11} F_{1\alpha}^{-T} = O(r^{1-c}), \quad H_{12} F_{2\alpha}^{-T} = O(r^{1/2}). \quad (3.26)$$

The two subintervals in Eq. (3.15) lead to different dominant balances.

- i) $1/2 < c < 1$. In this interval, the elastic contribution is the only singular leading term in the first row of the stress. The leading equilibrium problem is therefore

$$\nabla^2 y_1 = 0, \quad y_{1,2}(r, \pm\pi, t) = 0. \quad (3.27)$$

Substitution of $y_1 = r^c g(\theta, t)$ gives

$$g_{\theta\theta} + c^2 g = 0, \quad g_\theta(\pm\pi, t) = 0. \quad (3.28)$$

Mode-I symmetry requires $g(-\theta, t) = g(\theta, t)$, and hence

$$g(\theta, t) = q_c(t) \cos(c\theta), \quad \sin(c\pi) = 0. \quad (3.29)$$

Thus $c = 0, 1, 2, \dots$, and there is no nontrivial Mode-I eigenfunction with $1/2 < c < 1$.

- ii) $1 < c < 3/2$. In this interval, $H_{11}F_{1\alpha}^{-T}$ is more singular than the elastic contribution. At leading order, equilibrium and the traction-free condition can therefore be satisfied only if the leading coefficient of this term vanishes. Removing this more singular contribution does not determine the admissible radial order. A nontrivial field in this interval must therefore also satisfy an additional higher-order radial condition, whose necessary form is derived in Appendix A.

Mesh-refinement results in Section 4.4 reveal no stable exponent within $1 < c < 3/2$, supporting the $c = 1$ branch. Thus, we adopt

$$c = 1. \quad (3.30)$$

In the corresponding neo-Hookean elastic problem, the order $c = 1$ directly follows from the Mode-I eigenvalue problem.

3.4. Asymptotic equations for g and $\tilde{\mathbf{D}}$

With this result, the asymptotic y_1 field takes the form

$$y_1 \sim rg(\theta, t). \quad (3.31)$$

Define

$$A(\theta, t) = g \cos \frac{\theta}{2} - g_\theta \sin \frac{\theta}{2}, \quad K(\theta, t) = \mu_1 (\zeta d_{22} - \zeta^{-1}). \quad (3.32)$$

Then

$$\lambda \sim \frac{2}{p_2 A} r^{1/2}, \quad (3.33)$$

Moreover,

$$F_{1\alpha}^{-T} = \lambda(y_{2,2}, -y_{2,1}) \sim \frac{1}{A} \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right). \quad (3.34)$$

The asymptotic P_{1j} are therefore

$$\begin{aligned} P_{11} &\sim \mu_0 (g \cos \theta - g_\theta \sin \theta) + K \frac{\cos(\theta/2)}{A}, \\ P_{12} &\sim \mu_0 (g \sin \theta + g_\theta \cos \theta) + K \frac{\sin(\theta/2)}{A}. \end{aligned} \quad (3.35)$$

They satisfy

$$-\sin \theta (P_{11})_\theta + \cos \theta (P_{12})_\theta = 0, \quad P_{12}(\pm\pi, t) = 0. \quad (3.36)$$

Using Eqs. (3.3) and (2.11), we obtain

$$\tau (\dot{\tilde{\mathbf{D}}} + \mathbf{L}^T \tilde{\mathbf{D}} + \tilde{\mathbf{D}} \mathbf{L}) = \mathbf{I} - \frac{\text{tr } \tilde{\mathbf{D}}^{-1}}{3} \tilde{\mathbf{D}}. \quad (3.37)$$

Here $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient.

For the crack-tip fields, the velocity gradient satisfies

$$L_{12} = O(r^{1/2}), \quad L_{21} = 0, \quad L_{11} = \frac{\dot{A}}{A}, \quad L_{22} = \frac{\dot{p}_2}{p_2}, \quad L_{33} = -(L_{11} + L_{22}). \quad (3.38)$$

The velocity gradient in Eq. (3.38) contains no negative powers of r . This property holds throughout the admissible range $1/2 < c < 3/2$, and the resulting asymptotic consistency of the bounded- $\tilde{\mathbf{D}}$ branch is discussed in Appendix B.

Retaining all three independent in-plane components of $\tilde{\mathbf{D}}$ in Eq. (3.6), substitution into Eq. (3.37) gives

$$\begin{aligned} \tau (\dot{d}_{11} + 2L_{11}d_{11}) &= 1 - \frac{T}{3}d_{11}, \\ \tau (\dot{d}_{22} + 2L_{22}d_{22}) &= 1 - \frac{T}{3}d_{22}, \\ \tau [\dot{d}_{12} + (L_{11} + L_{22})d_{12}] &= -\frac{T}{3}d_{12}, \end{aligned} \quad (3.39)$$

where

$$T = \text{tr } \tilde{\mathbf{D}}^{-1} = \zeta (d_{11} + d_{22}) + \zeta^{-1}, \quad \zeta = (d_{11}d_{22} - d_{12}^2)^{-1}. \quad (3.40)$$

Equations (3.36) and (3.39) form a coupled system for $g(\theta, t)$ and the asymptotic internal state. In general, no closed-form solution for g is available.

3.5. Crack-tip field in the fully relaxed state

In the fully relaxed state, $\mathbf{C}_v = \mathbf{C}$, so that

$$\tilde{\mathbf{D}}_\infty = \mathbf{I}, \quad K_\infty = 0. \quad (3.41)$$

With $K_\infty = 0$, the leading first-row stress in Eq. (3.35) reduces to its equilibrium contribution, and the angular equation for y_1 becomes

$$g_{\infty, \theta\theta} + g_\infty = 0, \quad g_{\infty, \theta}(0) = g_{\infty, \theta}(\pi) = 0. \quad (3.42)$$

The corresponding Mode-I crack-tip fields are

$$\begin{aligned} y_2(r, \theta, t) &\longrightarrow p_{2, \infty} r^{1/2} \sin \frac{\theta}{2}, \\ y_1(r, \theta, t) &\longrightarrow q_{1, \infty} r \cos \theta, \end{aligned} \quad t \rightarrow \infty, \quad (3.43)$$

with

$$J_{1, \infty}^{(2)} = \frac{\pi \mu_0 (p_{2, \infty})^2}{4}. \quad (3.44)$$

The amplitudes $p_{2, \infty}$ and $q_{1, \infty}$ are determined by the outer boundary-value problem and are the same as those for an incompressible neo-Hookean solid with shear modulus μ_0 under the same final boundary displacement.

4. Numerical results and discussion

4.1. Set-up and finite element model

As shown in Fig. 2(a), we consider a long plane-stress strip of length L_0 and height H_0 , with $L_0 = 4H_0$. An edge crack of length $a = H_0$ lies along the midline. Length, stress, and time are normalized by L_0 , μ_0 , and τ , respectively, so that $H_0/L_0 = a/L_0 = 0.25$ and $\mu_1/\mu_0 = 1$ unless otherwise stated. Overbars are omitted from normalized spatial quantities below. Let t_r denote the duration of a linear ramp from $\lambda_s = 1$ to $\lambda_s = 2$. We introduce

$$\bar{t} = \frac{t}{\tau}, \quad \text{De} = \frac{\tau}{t_r}, \quad \frac{\partial \mathbf{C}_v}{\partial \bar{t}} = \mathbf{C} - \frac{I_1^e}{3} \mathbf{C}_v. \quad (4.1)$$

During the ramp, the applied stretch is prescribed as

$$\lambda_s = 1 + \text{De } \bar{t}, \quad 0 \leq \bar{t} \leq \frac{1}{\text{De}}. \quad (4.2)$$

The stretch is applied in the x_2 -direction through the top and bottom boundaries, while the x_1 displacement is constrained on these boundaries. The crack faces are traction free, and the crack remains stationary throughout the simulation.

The quasi-static equilibrium equations and constitutive evolution are solved using a total Lagrangian finite element formulation. The nonlinear algebraic equations are solved by Newton–Raphson iterations. Eight-node quadrilateral elements with four Gauss integration points are used. As shown in Fig. 2(b), the mesh is concentrically refined toward the crack tip. The primary mesh is level 1, with a minimum crack-tip mesh size of approximately $10^{-6}a$. Levels 2 and 3 are generated recursively by adding a self-similar annulus and reducing the inner mesh scale by a factor of 0.1 at each level; their minimum mesh sizes are approximately $10^{-7}a$ and $10^{-8}a$, respectively. Unless otherwise stated, field comparisons use level 1, while levels 2 and 3 are used to assess the crack-tip order under mesh refinement. General details of nonlinear finite element formulations are given in Belytschko et al. (2014).

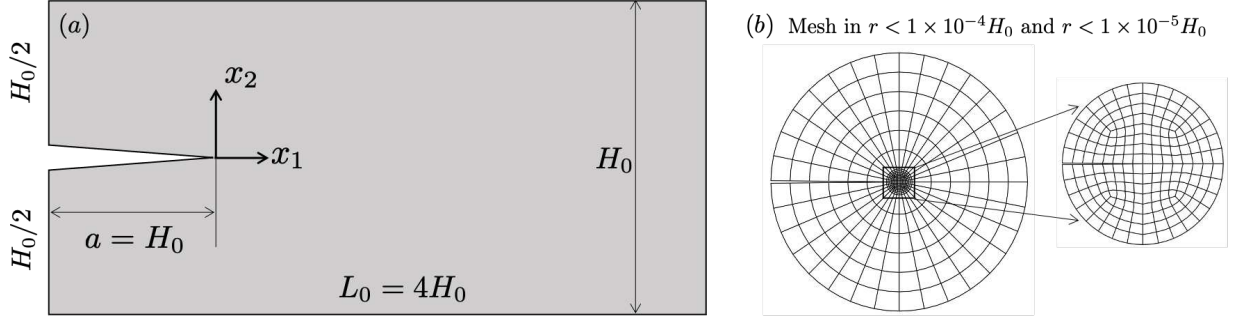


Figure 2: Geometry and finite element model: (a) the long cracked strip; (b) the concentrically refined meshes near the crack tip.

At the beginning of each simulation, the body is undeformed and stress free, with $C_v(X, 0) = I$. The constitutive evolution is then integrated implicitly at the Gauss points, following the constrained formulation of Le Tallec et al. (1993), with the isochoric constraint $\det C_v = 1$ enforced throughout the calculation.

The numerical cases used for the comparisons are summarized in Table 1. Cases V1–V3 use the same ramp with different De, spanning slow, intermediate, and fast loading relative to the constitutive relaxation time. Cases H_f and H_m reach the same applied stretch through rapid and moderate ramps and are then held at fixed displacement. Case R is calculated separately using the static neo-Hookean formulation on the same mesh and provides the equilibrium reference at the same stretch.

Table 1: Numerical cases for the crack tip field comparisons.

Case	μ_1/μ_0	De	Loading history
V1	1	0.02	$\lambda_s = 1 + \text{De } \bar{t}, 0 \leq \bar{t} \leq 1/\text{De}$
V2	1	0.2	$\lambda_s = 1 + \text{De } \bar{t}, 0 \leq \bar{t} \leq 1/\text{De}$
V3	1	2	$\lambda_s = 1 + \text{De } \bar{t}, 0 \leq \bar{t} \leq 1/\text{De}$
H_f	1	20	Linear ramp to $\lambda_s = 2$, followed by hold
H_m	1	2	Linear ramp to $\lambda_s = 2$, followed by hold
R	0	–	Static neo-Hookean solution at $\lambda_s = 2$

4.2. Numerical validation of the asymptotic y_2 field

The asymptotic analysis in Section 3 gives

$$P_{2\alpha} \sim \mu_0 y_{2,\alpha}, \quad y_2 \sim p_2(t) r^{1/2} \sin \frac{\theta}{2}, \quad J_1^{(2)} = \frac{\pi \mu_0 p_2^2}{4}. \quad (4.3)$$

For the comparisons below, the amplitude $p_2(R, t)$ is evaluated using a circular domain of radius R ,

$$p_2(R, t) = 2 \left[\frac{J_1^{(2)}(R, t)}{\pi \mu_0} \right]^{1/2}. \quad (4.4)$$

We use the reference radius $r_0 = 2.5 \times 10^{-5} = 10^{-4}a$, which lies within the concentrically refined crack-tip mesh, and write $p_2(t)$ for the corresponding domain-integral value $p_2(r_0, t)$.

Figure 3 compares the numerical y_2 field along $\theta = \pi/2$ with the prediction

$$y_2 = p_2(t) r^{1/2} \sin \frac{\theta}{2}. \quad (4.5)$$

At each De, the states $\lambda_s = 1.2, 1.4, 1.6$, and 1.8 are sampled at $\bar{t} = (\lambda_s - 1)/\text{De}$. The numerical profiles approach the predicted $r^{1/2}$ field as r decreases for all three values of De. The amplitude p_2 increases with the applied stretch. At fixed stretch, larger De gives a larger p_2 , reflecting the greater retention of the nonequilibrium response.

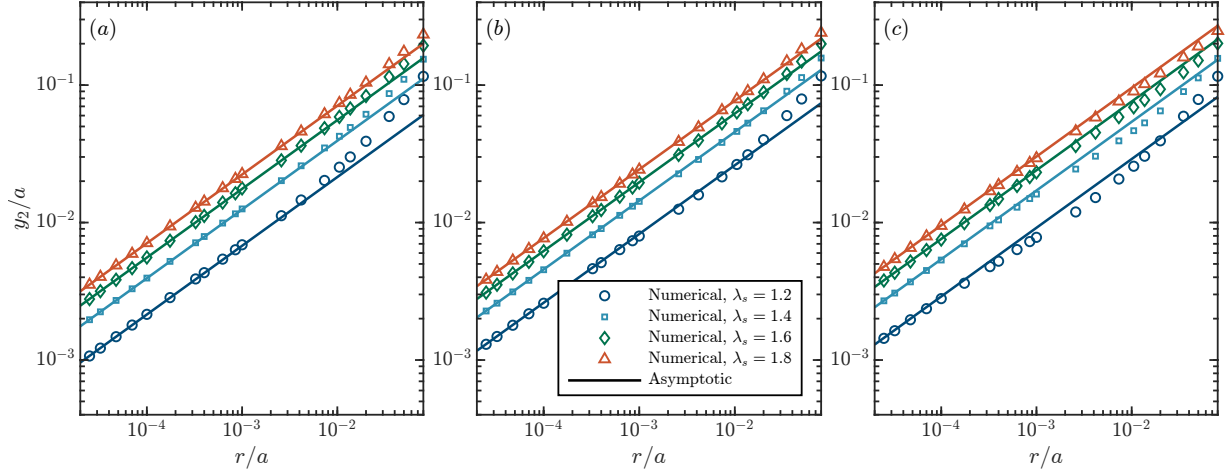


Figure 3: Numerical y_2 field (symbols) and the asymptotic field (solid lines) along $\theta = \pi/2$ at $\lambda_s = 1.2, 1.4, 1.6$, and 1.8 : (a) $De = 0.02$; (b) $De = 0.2$; (c) $De = 2$. The amplitudes of the solid lines are evaluated from the domain integral.

The angular dependence at $\lambda_s = 1.8$ is examined using

$$\widehat{y}_2(r, \theta, t) = \frac{y_2(r, \theta, t)}{p_2(t) \sqrt{r}}. \quad (4.6)$$

Figure 4 shows $\widehat{y}_2$ on three circles with different radii. The numerical curves agree closely with $\sin(\theta/2)$ over the full angular range at all three radii, supporting the predicted angular dependence.

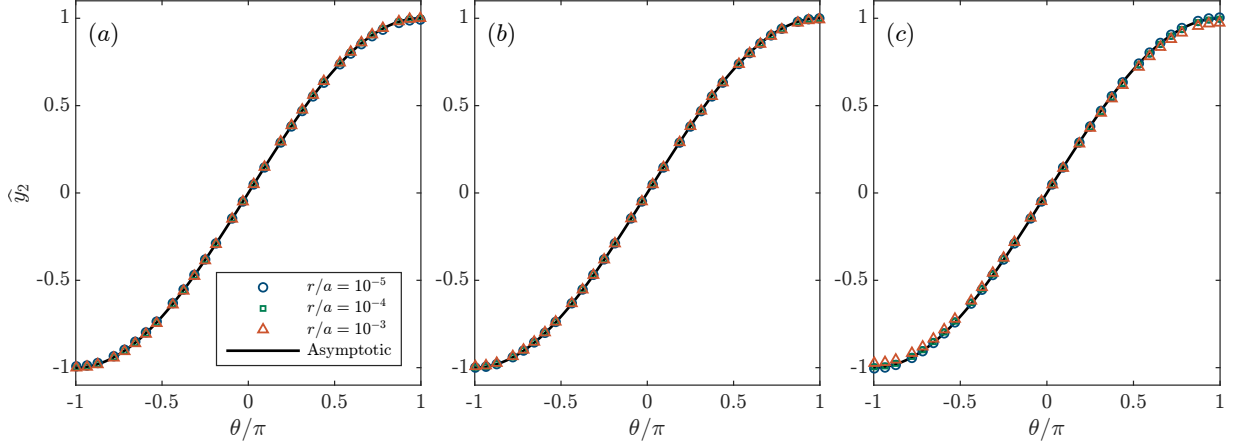


Figure 4: Angular variation of the normalized y_2 field at $\lambda_s = 1.8$ on three crack-tip circles: (a) $De = 0.02$; (b) $De = 0.2$; (c) $De = 2$. The solid line is $\sin(\theta/2)$.

To assess the dependence on the integration-domain radius, we use concentric circular domains with outer radius R . Taking r_0 as the reference, we define

$$\delta p_2(R) = 100 \left[\frac{p_2(R)}{p_2(r_0)} - 1 \right]. \quad (4.7)$$

Figure 5 illustrates $\delta p_2(R)$ at $\lambda_s = 1.8$. Within $10^{-5} \leq R/a \leq 10^{-3}$, the variation in p_2 remains below 1.4% for all De values. This result, alongside prior field comparisons, confirms the dominance of the asymptotic y_2 field over the considered radial range and validates the asymptotic crack-tip integral as a reliable method for evaluating p_2 .

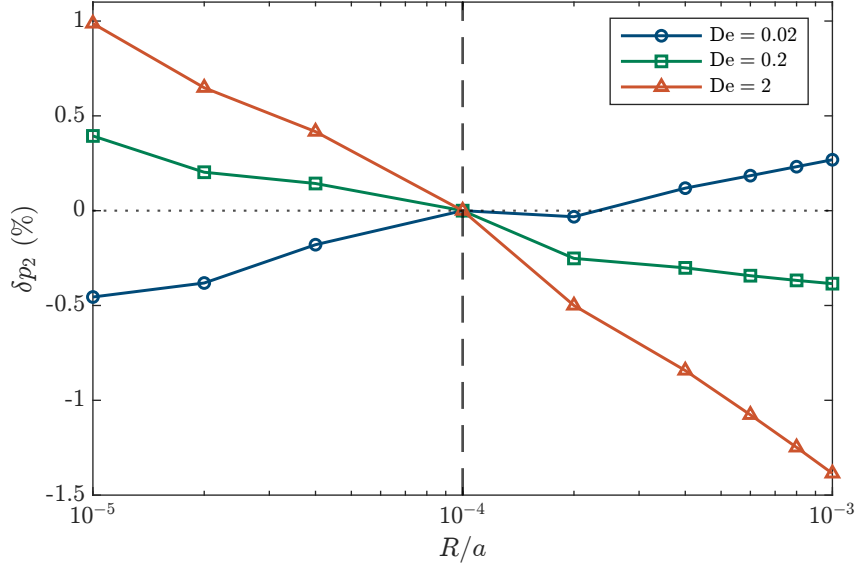


Figure 5: Percentage difference in p_2 relative to its value at $r_0/a = 10^{-4}$, plotted against the outer radius R of the integration domain at $\lambda_s = 1.8$. The vertical dashed line marks the reference radius.

4.3. Numerical verification of the leading stress balance for y_2

The asymptotic field y_2 is based on the leading stress balance $P_{2\alpha} \sim \mu_0 y_{2,\alpha}$. We quantify its residual using the annular RMS residual

$$\mathcal{R}_2(r, t) = \frac{\|P_{2\alpha} - \mu_0 y_{2,\alpha}\|_{\text{rms}}}{\|\mu_0 y_{2,\alpha}\|_{\text{rms}}}. \quad (4.8)$$

Figure 6 shows the radial variation of $\mathcal{R}_2$ at $\lambda_s = 1.8$. At the innermost resolved radius, $\mathcal{R}_2$ is below 2% for all three values of De . This provides numerical support for the leading stress balance assumed in Eq. (4.3).

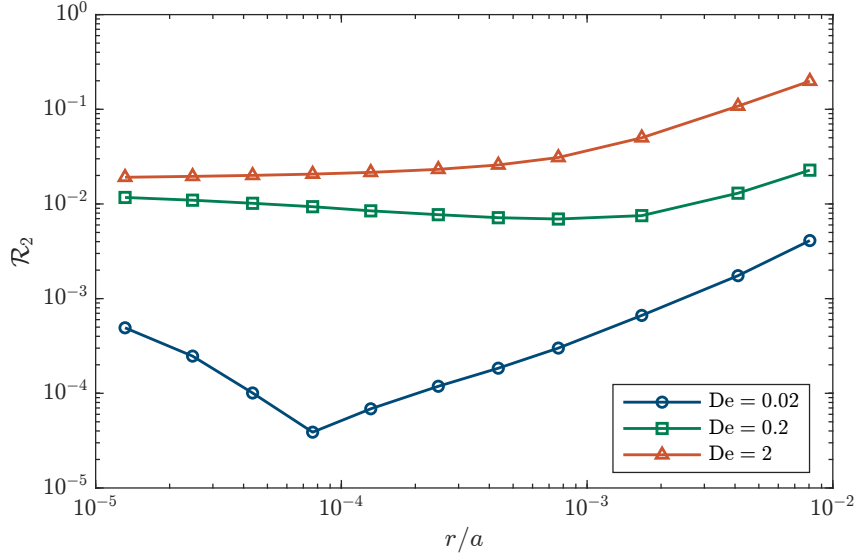


Figure 6: Relative residual of the leading second-row stress balance at $\lambda_s = 1.8$.

4.4. Numerical assessment of the radial order of y_1

To assess the radial order adopted in Eq. (3.31), we define

$$M_0(r, t) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} y_1^2(r, \theta, t) d\theta \right]^{1/2}, \quad (4.9)$$

$$c_{\text{eff}}(r, t) = \frac{d \ln M_0}{d \ln r}.$$

If

$$y_1(r, \theta, t) = r^c g(\theta, t) + o(r^c), \quad (4.10)$$

then

$$M_0(r, t) = r^c \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} g^2(\theta, t) d\theta \right]^{1/2} + o(r^c). \quad (4.11)$$

The logarithmic slope of M_0 therefore provides an angularly averaged estimate of c without relying on a particular ray.

Figure 7 shows M_0 for the four loading levels used in Fig. 3. The numerical curves are approximately parallel to the r^1 reference over the resolved near-tip range. Fits over $3.96 \times 10^{-5} < r/a < 3.68 \times 10^{-4}$ give $c_{\text{eff}} = 1.01$ – 1.05 for $\text{De} = 0.02$, 0.93 – 0.96 for $\text{De} = 0.2$, and 0.94 – 0.96 for $\text{De} = 2$ at $\lambda_s \geq 1.4$. At the lower stretch $\text{De} = 2$, $\lambda_s = 1.2$, the fitted value is $c_{\text{eff}} \approx 0.87$, indicating the resolved radial range does not sufficiently extend into the crack-tip asymptotic regime at this loading.

The radial order is directly verified by recalculating on three successively refined self-similar meshes at $\text{De} = 2$ and $\bar{t} = 0.172$, corresponding to $\lambda_s = 1.344$. Figure 8 compares M_0 and its local logarithmic slope at this loading state. The fitted exponents at the innermost resolved region are 0.894 , 0.968 , and 0.976 on levels 1–3, respectively, with the relative fitting residual decreasing from 1.69×10^{-2} to 1.67×10^{-4} . The mesh-refinement trend supports the $c = 1$ branch and does not exhibit a stable exponent within the formally admissible interval $1 < c < 3/2$ discussed in Appendix A. We consequently use $c = 1$ in Eqs. (3.31)–(3.39). This result concerns the radial order only and does not require the finite-time angular function $g(\theta, t)$ to be known.

4.5. Crack-tip relaxation at fixed applied stretch

The preceding ramp calculations contain simultaneous changes in the applied stretch and the internal variable. To separate these effects, cases H_f and H_m are ramped to the same stretch $\lambda_s = 2$ and then held at fixed boundary

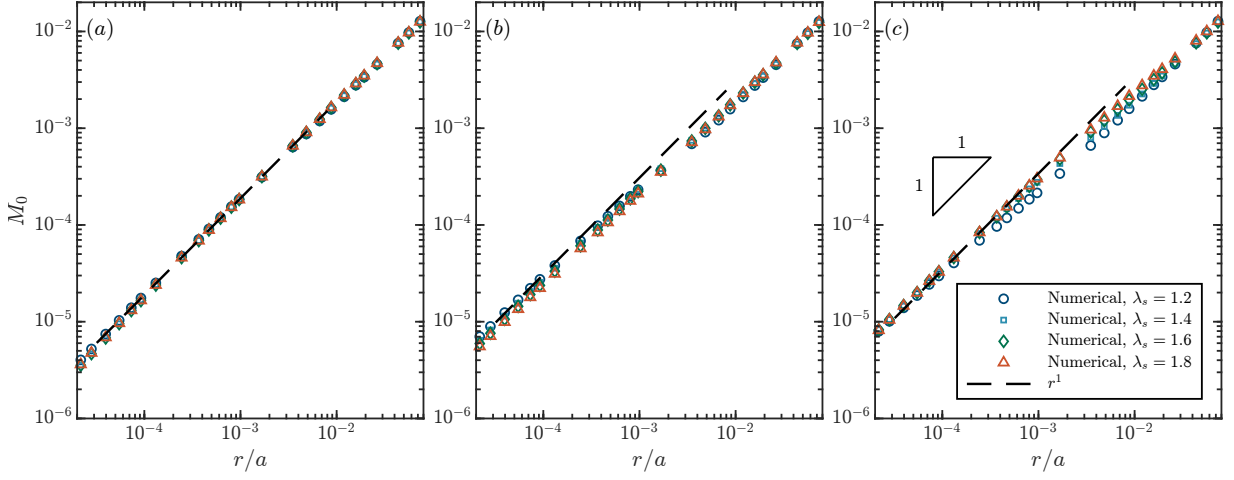


Figure 7: Angular RMS quantity M_0 for y_1 at $\lambda_s = 1.2, 1.4, 1.6$, and 1.8 : (a) $De = 0.02$; (b) $De = 0.2$; (c) $De = 2$. The dashed lines indicate the radial order r^1 .

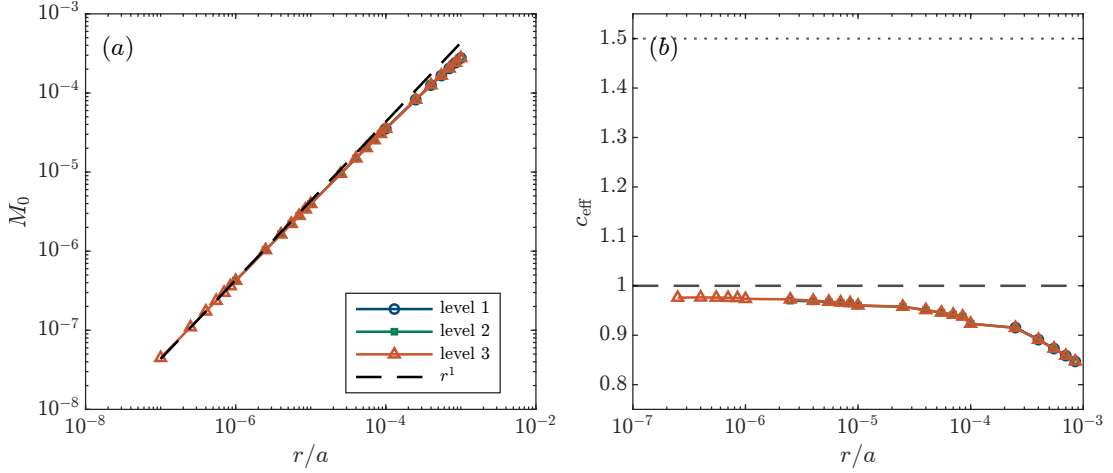


Figure 8: Mesh-refinement assessment of the radial order of y_1 : (a) the angular RMS quantity M_0 ; (b) the local logarithmic slope c_{eff} .

displacement. In terms of $\bar{t} = t/\tau$, the two histories are

$$\lambda_s^{(i)}(\bar{t}) = \begin{cases} 1 + De_i \bar{t}, & 0 \leq \bar{t} \leq 1/De_i, \\ 2, & \bar{t} > 1/De_i, \end{cases} \quad De_f = 20, \quad De_m = 2. \quad (4.12)$$

Thus, case H_f loads more rapidly relative to the constitutive relaxation time than case H_m . Once the stretch is fixed, changes in the crack-tip amplitudes result from the evolution of C_v and the associated re-equilibration of the deformation field.

We define the hold time by

$$s^{(i)} = \frac{t - t_r^{(i)}}{\tau} = \bar{t} - \frac{1}{De_i}, \quad i = f, m, \quad 0 \leq s^{(i)} \leq 100. \quad (4.13)$$

For notational simplicity, the superscript i is suppressed below; thus, s denotes the hold time for each loading history, and $y_1(r_0, \theta, s)$ and $p_2(s)$ denote the corresponding crack-tip quantities. We follow the response to $s = 100$ to resolve both the initial relaxation and the approach to the long-time limit.

During the hold, $p_2(s)$ is evaluated from Eq. (4.4) using an integration domain of radius r_0 , while the y_1 field is characterized on the circle $r = r_0$ by

$$G_0(s; r_0) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_1(r_0, \theta, s)}{r_0} \right)^2 d\theta \right]^{1/2}. \quad (4.14)$$

For the relaxed reference case R, the in-plane neo-Hookean field has the form $g^R(\theta) = q_1^R \cos \theta$. The coefficient q_1^R is obtained from the static numerical field using the procedure of Liu and Moran (2020a). The corresponding reference for Eq. (4.14) is

$$G_0^R = \frac{|q_1^R|}{\sqrt{2}}. \quad (4.15)$$

Figure 9 compares $p_2(s)$ with p_2^R and $G_0(s; r_0)$ with $|q_1^R|/\sqrt{2}$. During the hold, p_2 decreases monotonically toward its neo-Hookean limit. In contrast, G_0 exhibits a non-monotonic response: it decreases below the relaxed reference, reaches a minimum near $s = 23$, and subsequently approaches the reference from below. The differences between the two loading histories diminish during the hold, with both converging to the same long-time limits for the two observables. These results show that p_2 and G_0 relax on different time scales and that G_0 does not follow a single monotonic relaxation law.

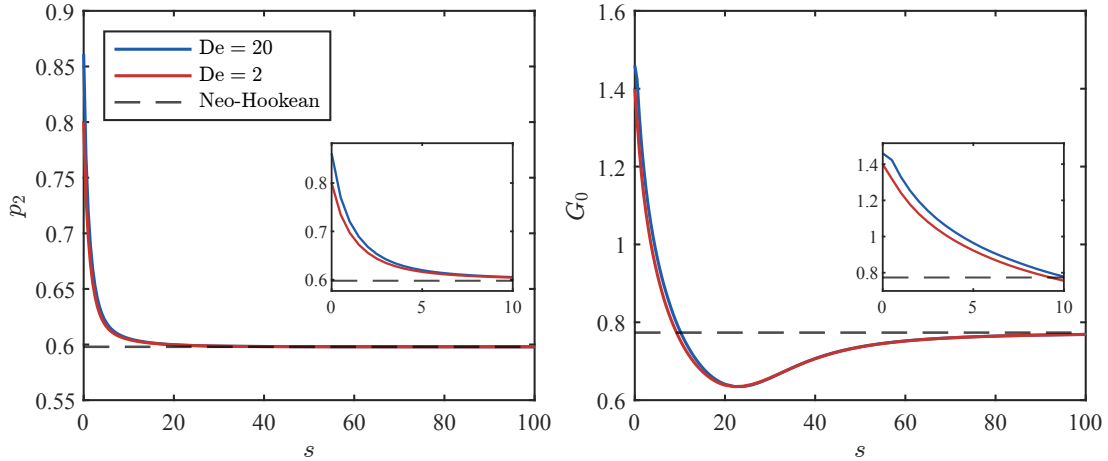


Figure 9: Relaxation after the stretch is held at $\lambda_s = 2$: (a) p_2 ; (b) the fixed-radius angular RMS quantity G_0 at $r_0 = 2.5 \times 10^{-5}$. The dashed lines denote the neo-Hookean references p_2^R and $|q_1^R|/\sqrt{2}$, respectively. The insets show the interval $0 \leq s \leq 10$.

Figure 10 illustrates the relaxation process for case H_f as summarized in Fig. 9. The rapid ramp results in a strongly loaded nonequilibrium branch at $s = 0$. During the hold, the applied stretch remains constant while the internal variable evolves, reducing the nonequilibrium stress and redistributing the near-tip field. The decrease in p_2 appears as a reduction in the vertical crack opening, while the horizontal mesh reflects both the magnitude and angular redistribution captured by G_0 . The deformed coordinates are normalized by the neo-Hookean references p_2^R and q_1^R , ensuring the relaxed outer boundary aligns directly with the neo-Hookean angular fields.

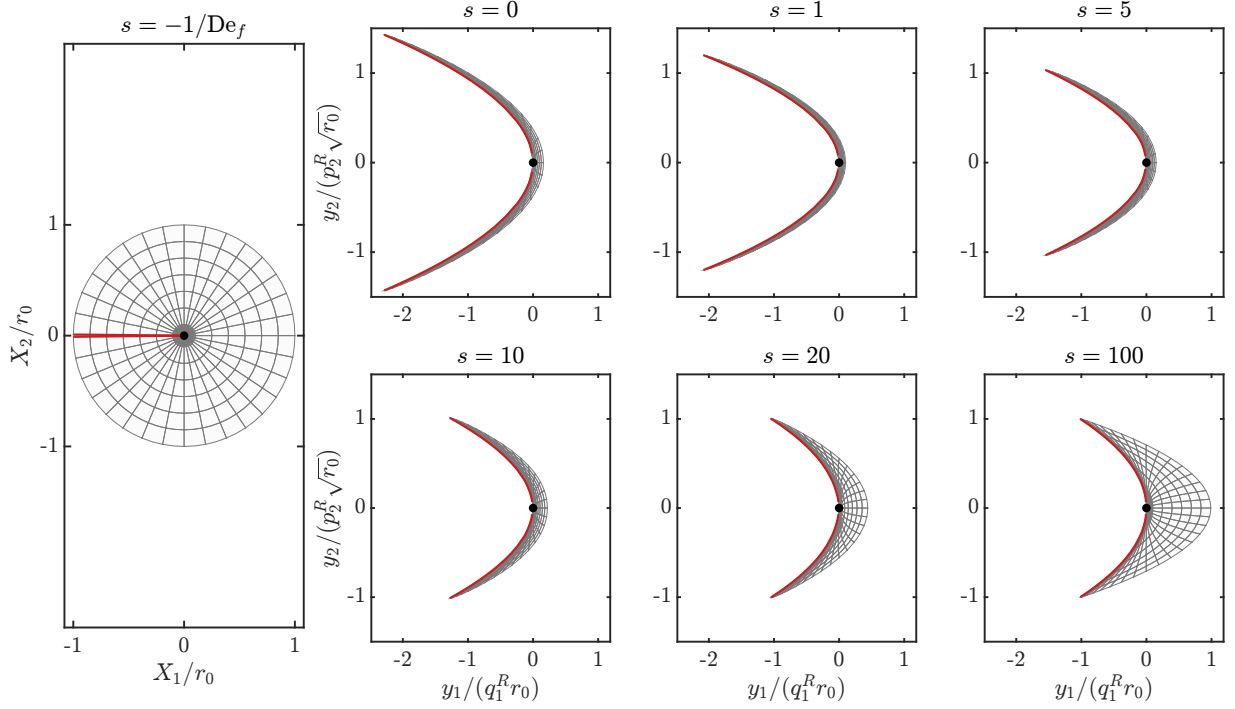


Figure 10: Evolution of the rescaled near-tip configuration for case H_f . The undeformed initial configuration corresponds to $s = -1/\text{De}_f$, and the subsequent panels show $s = 0, 1, 5, 10, 20$, and 100 . For $s > 0$, the applied stretch remains fixed and the configuration evolves through relaxation of the nonequilibrium branch. Each panel shows the same material region, $r \leq r_0$, where $r_0 = 2.5 \times 10^{-5} = 10^{-4}a$. The initial coordinates are normalized by r_0 ; in the deformed configurations, the horizontal and vertical coordinates are normalized by $q_1^R r_0$ and $p_2^R \sqrt{r_0}$, respectively. Red curves represent the crack faces.

5. Discussion and conclusions

We studied Mode-I fields near a nonpropagating crack in an incompressible finite-strain standard viscoelastic solid. The leading y_2 field and its associated asymptotic path-integral relation are

$$y_2 \sim p_2(t)r^{1/2} \sin \frac{\theta}{2}, \quad J_1^{(2)} = \frac{\pi\mu_0 p_2^2}{4}. \quad (5.1)$$

The finite element results recover both the radial and angular dependence of y_2 .

For y_1 , the Mode-I eigenvalue problem excludes $1/2 < c < 1$. Although the leading crack-face condition alone does not eliminate $1 < c < 3/2$, a nontrivial field in this interval must also satisfy an additional higher-order radial condition. The mesh-refinement calculations support the $c = 1$ branch and show no stable exponent in $1 < c < 3/2$ over the resolved range. For $c = 1$, the equilibrium and nonequilibrium terms enter the leading-order balance, while the angular function $g(\theta, t)$ remains coupled to the internal-variable evolution. A universal finite-time angular solution is therefore not obtained.

At full relaxation, the coupled equations reduce exactly to the incompressible neo-Hookean crack-tip field with shear modulus μ_0 . The fixed-stretch relaxation calculations are consistent with this limit. The amplitude p_2 approaches its neo-Hookean reference monotonically, whereas the angular RMS quantity G_0 based on y_1/r first decreases below its relaxed value and then approaches the same reference from below. Thus y_2 and y_1 have different transient responses even though they recover the same relaxed constitutive limit.

The analysis concerns the finite-strain field near a stationary crack and does not address crack propagation.

Appendix A. Necessary higher-order condition for $1 < c < 3/2$

For $1 < c < 3/2$, the first row of the nominal stress can be written as

$$\begin{aligned} P_{1\alpha} &= \mu_0 y_{1,\alpha} + \mathcal{K}_c F_{1\alpha}^{-T} - \mu_1 \zeta d_{12} F_{2\alpha}^{-T}, \\ A_c &= c g \cos \frac{\theta}{2} - g_\theta \sin \frac{\theta}{2}, \quad \mathcal{K}_c = \mu_1 (\zeta d_{22} - \zeta^{-1}) - \mu_0 \lambda^2. \end{aligned} \quad (\text{A.1})$$

Here, the pressure contribution has been absorbed into the effective coefficient $\mathcal{K}_c$. The leading components of F^{-T} are

$$F_{1\alpha}^{-T} = \frac{r^{1-c}}{A_c} t_\alpha, \quad F_{2\alpha}^{-T} = O(r^{1/2}), \quad \mathbf{t} = \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right). \quad (\text{A.2})$$

For the bounded- $\tilde{\mathbf{D}}$ branch, the last term in Eq. (A.1) is at most $O(r^{1/2})$. Since $c - 1 < 1/2$, this term is subdominant to the elastic stress $O(r^{c-1})$ and does not enter the order selection below. If the $O(1)$ coefficient of $\mathcal{K}_c$ is nonzero, the potentially dominant stress contribution is $O(r^{1-c})$. With $\mathcal{S}_c^{(0)} = \mathcal{K}_c^{(0)}/A_c$, the leading equilibrium equation and its Mode-I solution are

$$-\sin \frac{\theta}{2} (\mathcal{S}_c^{(0)})_\theta + \left(\frac{3}{2} - c \right) \cos \frac{\theta}{2} \mathcal{S}_c^{(0)} = 0, \quad \mathcal{S}_c^{(0)} = C_c(t) \left| \sin \frac{\theta}{2} \right|^{3-2c}. \quad (\text{A.3})$$

The crack-face traction condition then gives

$$C_c(t) = 0, \quad \mathcal{K}_c^{(0)} = 0. \quad (\text{A.4})$$

Thus, the $O(1)$ coefficient of $\mathcal{K}_c$ must vanish. This cancellation does not by itself exclude $1 < c < 3/2$. Let the first nonzero radial correction, if it exists, be

$$\mathcal{K}_c(r, \theta, t) = r^\beta \widehat{\mathcal{K}}_c(\theta, t) + o(r^\beta), \quad \beta > 0. \quad (\text{A.5})$$

This correction enters the stress through F^{-T} at order $O(r^{1-c+\beta})$, whereas the elastic stress is $O(r^{c-1})$. If $\beta < 2c - 2$, this correction remains the only leading term; its homogeneous equilibrium equation, together with the crack-face condition, forces $\widehat{\mathcal{K}}_c = 0$, contradicting Eq. (A.5) as the first nonzero correction. If $\beta > 2c - 2$, the elastic term is the only leading term, and the Mode-I eigenvalue problem in Eq. (3.29) excludes a noninteger c in this interval. Therefore, the two contributions can balance only when

$$\beta = 2c - 2, \quad \mathcal{K}_c(r, \theta, t) = r^{2c-2} \widehat{\mathcal{K}}_c(\theta, t) + o(r^{2c-2}). \quad (\text{A.6})$$

Equation (A.6) is a necessary order condition rather than a radial correction derived from the internal-variable evolution. Determining $\widehat{\mathcal{K}}_c$ would require the corresponding higher-order radial problem for $\tilde{\mathbf{D}}$. The mesh-refinement results in Section 4.4 show no stable exponent in this interval and instead support the $c = 1$ branch.

Appendix B. Consistency of the bounded- $\tilde{\mathbf{D}}$ branch

All quantities in this appendix refer to the asymptotic crack-tip fields. With $m = 1/2$, the in-plane deformation gradient for a general admissible c can be written as

$$\mathbf{F}_{2D} = \begin{bmatrix} r^{c-1} & 0 \\ 0 & r^{-1/2} \end{bmatrix} \mathbf{B}(\theta, t), \quad A_c = c g \cos \frac{\theta}{2} - g_\theta \sin \frac{\theta}{2}. \quad (\text{B.1})$$

The second row of $\mathbf{B}$ has the fixed angular form in Eq. (3.14) and is proportional to $p_2(t)$. Direct evaluation of $\mathbf{L} = \dot{\mathbf{F}} \mathbf{F}^{-1}$ therefore gives

$$L_{12} = O(r^{c-1/2}), \quad L_{21} = 0, \quad L_{11} = \frac{\dot{A}_c}{A_c}, \quad L_{22} = \frac{\dot{p}_2}{p_2}, \quad L_{33} = -(L_{11} + L_{22}). \quad (\text{B.2})$$

Since $c > 1/2$, $\mathbf{L}$ contains no negative powers of r throughout the admissible range in Eq. (3.15). From the multiplicative decomposition and $\mathbf{C}_v = \mathbf{F}_v^T \mathbf{F}_v$,

$$\tilde{\mathbf{D}} = \mathbf{F}^{-T} \mathbf{C}_v \mathbf{F}^{-1} = \mathbf{F}_e^{-T} \mathbf{F}_e^{-1} = \mathbf{b}_e^{-1}, \quad \mathbf{b}_e = \mathbf{F}_e \mathbf{F}_e^T. \quad (\text{B.3})$$

Thus, $\tilde{\mathbf{D}}$ is symmetric positive definite for each $r > 0$. The isochoric conditions also give

$$\det \tilde{\mathbf{D}} = 1. \quad (\text{B.4})$$

Let the eigenvalues of $\tilde{\mathbf{D}}$ be $0 < d_1 \leq d_2 \leq d_3 = M$. Since $d_1 d_2 M = 1$,

$$T = \text{tr } \tilde{\mathbf{D}}^{-1} = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{M} \geq \frac{2}{\sqrt{d_1 d_2}} + \frac{1}{M} = 2\sqrt{M} + \frac{1}{M}. \quad (\text{B.5})$$

For a unit eigenvector $\mathbf{n}$ associated with M , projection of Eq. (3.37) along $\mathbf{n}$ gives

$$\tau \dot{M} = 1 - \frac{T}{3} M - 2\tau M \mathbf{n} \cdot \text{sym}(\mathbf{L}) \mathbf{n} \leq 1 + 2\tau \|\text{sym } \mathbf{L}\| M - \frac{T}{3} M \leq \frac{2}{3} + 2\tau \|\text{sym } \mathbf{L}\| M - \frac{2}{3} M^{3/2}. \quad (\text{B.6})$$

Since $c > 1/2$ and the nondegenerate asymptotic coefficients A_c and p_2 are regular and nonzero, Eq. (B.2) gives $L_* = \sup_{t \in [t_0, t_1]} \|\text{sym } \mathbf{L}(t)\| < \infty$ on each finite interval. Equation (B.6) then gives $\dot{M} < 0$ above a finite threshold M_* , and hence

$$M(t) \leq \max\{M(t_0), M_*\}, \quad t \in [t_0, t_1]. \quad (\text{B.7})$$

Moreover, Eq. (B.4) implies

$$\lambda_{\min}(\tilde{\mathbf{D}}) \geq M^{-2}, \quad \|\tilde{\mathbf{D}}^{-1}\| \leq M^2. \quad (\text{B.8})$$

Equations (B.6)–(B.8) show that the asymptotic evolution does not generate an additional radial singularity in $\tilde{\mathbf{D}}$ or $\tilde{\mathbf{D}}^{-1}$ when the kinematic coefficients in Eq. (3.38) remain finite. At a stationary relaxed state, $\mathbf{L} = \mathbf{0}$, and Eq. (3.37) reduces to

$$\mathbf{I} - \frac{\text{tr } \tilde{\mathbf{D}}^{-1}}{3} \tilde{\mathbf{D}} = \mathbf{0}. \quad (\text{B.9})$$

Together with $\det \tilde{\mathbf{D}} = 1$, the unique positive-definite solution is $\tilde{\mathbf{D}} = \mathbf{I}$. These results establish the consistency of the bounded- $\tilde{\mathbf{D}}$ scaling throughout $1/2 < c < 3/2$.

Statements and Declarations

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Competing interests

The authors declare that they have no competing interests.

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