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Singh Akansha Ravindrabahadur

akanshasingh950305@gmail.com

Banaras Hindu University <https://orcid.org/0009-0007-9276-4815>

Rajesh Singh

Banaras Hindu University

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Estimation of Neutrosophic Population Mean Incorporating Auxiliary Information under Simple Random Sampling

Singh Akansha Ravindrabahadur¹, Rajesh Singh², Madhulika Mishra^{3,*}

^{1,2}Department of Statistics, Institute of Science, Banaras Hindu University, Varanasi-221005, Uttar Pradesh, India

^{3,*}Symbiosis Statistical Institute, Symbiosis International (Deemed University), Pune, India

Emails: akanshasingh950305@gmail.com, rsinghstat@gmail.com

*Correspondence: madhulika1707@gmail.com, madhulika.mishra@ssi.edu.in

Abstract: In traditional statistics, the population mean is estimated using exact and well-defined data whenever auxiliary information is present. Still, the resulting estimates commonly suffer from bias. The goal of this research is to formulate a neutrosophic estimator that yields a lower Mean Squared Error (MSE) for the unknown population mean and addresses the shortcomings of classical statistical methods when confronted with ambiguous data. The concept of neutrosophic statistics was introduced by Smarandache (Smarandache 2014), as an extension of classical statistics to address uncertainty, vagueness, and indeterminacy of data. This research constructs an almost unbiased neutrosophic estimator that incorporates known neutrosophic auxiliary information for estimating the study variable's population mean. To assess the proposed estimator's efficiency, its bias, MSE, and minimum MSE are mathematically derived under first-order approximations. Through these derivations, we can assess how effective the estimator is and how well it performs in estimating the population mean. The research confirms that the suggested estimator is almost unbiased, because its first-order bias is extremely small. The suggested estimator provides superior performance over traditional estimators in terms of both MSE and Percentage Relative Efficiency (PRE). For practical applications, priority should be given to the estimator that yields the highest PRE or exhibits the least MSE. The conclusions drawn theoretically are confirmed by an empirical study based on real datasets. Additionally, a simulation study was carried out, demonstrating that the estimator we proposed yields superior results compared to those already existing.

Keyword: Neutrosophic statistics, Auxiliary information, Population mean, almost unbiased estimator, Mean squared error, Percentage relative efficiency.

1. Introduction

The purpose of sampling theory includes achieving greater precision in estimating the study variable by incorporating auxiliary information. The auxiliary information such as coefficient of variation (C_{xN}), kurtosis $\beta_{2(x)N}$, and correlation coefficient ρ_{yxN} can contribute positively to sample selection, stratification, choice of estimators, and estimation. In sampling theory, population parameters are estimated by employing auxiliary variable correlated with the study variable. Some well-known methods for estimating population parameters include the ratio, product, and regression methods. If the auxiliary variable and study variable are linearly related, passes through the origin, and the correlation between them is highly positive, then ratio method of estimation is generally used; otherwise, when the auxiliary variable and study variable are linearly related, but doesn't pass through the origin, then regression method of

estimation is used. When the auxiliary and study variables exhibit a high negative correlation, the product method of estimation is typically used.

In survey sampling, ratio and product estimators are frequently used to improve the accuracy of the population mean estimates when auxiliary information is available. The concept of using auxiliary information through the ratio estimator was initially proposed by (Cochran 1940). The ratio estimator uses the ratio of the sample means of the study and auxiliary variables to estimate the population mean by incorporating auxiliary information that is linearly correlated with the study variable.

On the other hand, (Murthy 1964) proposed the product estimator as another method for incorporating auxiliary information into estimation. In this method of estimation, the product of the sample means of the study and auxiliary variables serves as the basis for the estimate. Over the years, researchers have proposed various improvements and modifications to traditional ratio and product estimators. (Bahl and Tuteja 1991), introduced exponential ratio and product estimators for estimating the unknown population mean of study variable by making use of auxiliary information.

Significant work in this domain has been accomplished by several researchers. The concept of ratio estimators in simple random sampling was presented by (Kadilar and Cingi 2004), while (Singh et al. 2024) developed almost unbiased estimators for the population coefficient of variation based on auxiliary information. More recently, (Ravindrabahadur et al. 2025) proposed a logarithmic ratio-type estimator for estimating the finite population mean within a neutrosophic framework. (Yadav et al. 2025) presented an approach of employing two neutrosophic auxiliary variables for population mean estimation under uncertainty, while (Singh and Tiwari 2025) introduced an improved estimator based on the known medians of two neutrosophic auxiliary variables.

In traditional statistics, data are expressed as exact or crisp numbers, assuming no uncertainty in the measurement of observations. In many situations, though, the data may contain indeterminacy, vagueness, or ambiguity. Examples of indeterminate data in everyday life include temperature measurements, decision support and information systems, weather forecasts, and monitoring economic growth, etc. To handle this type of indeterminate data, new approaches must be implemented. It is important to distinguish interval-valued data from indeterminacy: an interval-valued observation represents a value within a specified numerical range, whereas indeterminacy refers to incomplete, ambiguous, or uncertain information whose precise status or value cannot be determined. Thus, although an interval may be used to represent uncertainty, the presence of an interval alone does not necessarily imply indeterminacy. Conventional fuzzy approaches can address ambiguous and imprecise observations; however, they do not explicitly represent indeterminacy. In such cases, the neutrosophic approach provides a more flexible framework for incorporating indeterminate information. This distinction motivates the use of approaches that can explicitly accommodate indeterminate information.

Whereas traditional statistics works with datasets containing well-defined or crisp values, neutrosophic statistics allows observations to contain both determinate and indeterminate components. In the absence of indeterminacy, neutrosophic statistics is equivalent to classical statistics, functioning as its extension (Smarandache 2014). As an extension of fuzzy logic, neutrosophic logic facilitates the evaluation of observations by incorporating both determinate and indeterminate components. According to (Smarandache 2014), it has particular importance in the analysis of ambiguous or imprecise data.

Neutrosophic statistics can be applied to various real-world problems, such as the analysis of road traffic accidents (Aslam 2020). In the domain of neutrosophic statistics, (Aslam 2021) presented a goodness-of-fit procedure, whereas (Aslam 2019) examined neutrosophic ANOVA applied to student-based datasets. The sample size within neutrosophic statistics is not always precisely defined and may be unknown as an exact number (Smarandache 2010). According to (Jan et al. 2019), fuzzy logic methods are advancing rapidly and are commonly implemented in decision-making

applications. The development of fuzzy sets led to complex fuzzy sets, which are further generalized by complex neutrosophic sets.

Classical statistical methods may face difficulties when the available information is vague, incomplete, or indeterminate and cannot be adequately represented by precise values. Thus, neutrosophic statistics provides a broader framework for statistical analysis when such information is present. Compared with the extensive literature on classical estimation methods, the use of neutrosophic techniques for estimating population parameters in sampling theory remains relatively less explored (Varshney et al. 2023). Despite its potential, neutrosophic sampling remains less studied than classical sampling methods for the estimation of population parameters. Motivated by this gap, the present study proposes an almost unbiased estimator that utilizes neutrosophic information.

The structure of our paper is organized as follows: The introduction is covered in Section 1. Section 2, presents the notation and terminology used in neutrosophic statistics. Section 3 provides an overview of existing neutrosophic estimators. Section 4 presents the formulation of the proposed neutrosophic almost unbiased estimator. Section 5 discusses different variants of the proposed estimator. Section 6 presents the empirical findings, while Section 7 describes the simulation study. Section 8 presents the discussion, and Section 9 concludes the paper.

2. Neutrosophic Notation and Terminology

A number of neutrosophic observations were provided, where the numerical data indicated that the value may lie within an uncertain interval (a, b). The interval value of a neutrosophic number may be expressed using multiple ways. According to (Singh et al. 2024) and (Masood et al. 2024), neutrosophic interval values are represented as $Z_N = Z_L + Z_U I_N$, where $I_N \in [I_L, I_U]$. The indeterminacy level, expressed by I_N , falls within the range of 0 to 1. Additionally, the neutrosophic data considered here are expressed using the interval notation from (Yadav and Smarandache 2023) as $Z_N \in [Z_L, Z_U]$, where Z_L and Z_U denotes lower and upper bound of the variable Z_N respectively.

Under the SRSWOR framework, a neutrosophic sample of size $n_N \in [n_L, n_U]$ is selected from a finite population consisting of N_N units, represented as $(P_{1N}, P_{2N}, P_{3N}, \dots, P_{N_N})$. Let $y_N \in [y_L, y_U]$ and $x_N \in [x_L, x_U]$, denote the i^{th} observed neutrosophic values of the study variable $y_N(i)$ and the auxiliary variable $x_N(i)$, respectively. The neutrosophic sample means for the study variable y_N and the auxiliary variable x_N are denoted by $\bar{y}_N(i) \in [\bar{y}_L, \bar{y}_U]$ and $\bar{x}_N(i) \in [\bar{x}_L, \bar{x}_U]$ respectively, where, neutrosophic auxiliary variable x_N is correlated to neutrosophic study variable y_N . The population means of the neutrosophic variables y_N and x_N , represents the average values of the data sets, defined as $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N \in [X_L, X_U]$. For the variables y_N and x_N , the corresponding neutrosophic coefficients of variation are $C_{yN} \in [C_{yL}, C_{yU}]$ and $C_{xN} \in [C_{xL}, C_{xU}]$ respectively. For the study variable y_N and auxiliary variable x_N , the neutrosophic correlation coefficient is given by $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$ respectively. For the auxiliary variable x_N , the neutrosophic coefficient of kurtosis is given by $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$. For computing the estimator's bias and MSE, let $\bar{e}_{0N} \in [\bar{e}_{0L}, \bar{e}_{0U}]$ and $\bar{e}_{1N} \in [\bar{e}_{1L}, \bar{e}_{1U}]$ represents the mean errors in the neutrosophic framework for the variables y_N and x_N along with $\bar{y}_N = \bar{Y}_N(1 + \bar{e}_{0N})$ and $\bar{x}_N = \bar{X}_N(1 + \bar{e}_{1N})$ respectively. The expected values of these error terms, corresponding to different orders, are as follows,

$$E(\bar{e}_{0N}) = E(\bar{e}_{1N}) = 0 ;$$

$$E(\bar{e}_{0N}^2) = f_N C_{yN}^2, E(\bar{e}_{1N}^2) = f_N C_{xN}^2 \text{ and } E(\bar{e}_{0N} \bar{e}_{1N}) = f_N \rho_{yxN} C_{xN} C_{yN}$$

where,

$$\bar{e}_{0N}^2 \in [\bar{e}_{0L}^2, \bar{e}_{0U}^2], \bar{e}_{1N}^2 \in [\bar{e}_{1L}^2, \bar{e}_{1U}^2] \text{ and } \bar{e}_{0N} \bar{e}_{1N} \in [\bar{e}_{0L} \bar{e}_{1L}, \bar{e}_{0U} \bar{e}_{1U}].$$

$f_N = \frac{N_N - n_N}{n_N N_N}$, where, $f_N \in [f_L, f_U]$.

$C_{yN}^2 = \frac{S_{yN}^2}{\bar{Y}_N^2}$, $C_{xN}^2 = \frac{S_{xN}^2}{\bar{X}_N^2}$, and $S_{yxN} = \rho_{yxN} C_{xN} C_{yN}$, where, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $S_{yN}^2 \in [S_{yL}^2, S_{yU}^2]$, $S_{xN}^2 \in [S_{xL}^2, S_{xU}^2]$, $S_{yxN} \in [S_{yxL}, S_{yxU}]$ and $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$.

Similarly, within a neutrosophic framework, the bias and MSE are expressed as $Bias(\bar{y}_N) \in [Bias(\bar{y}_L), Bias(\bar{y}_U)]$ and $MSE(\bar{y}_N) \in [MSE(\bar{y}_L), MSE(\bar{y}_U)]$ respectively.

3. Existing Neutrosophic Estimators

The neutrosophic sample mean is used to estimate the neutrosophic population mean $\bar{Y}_N$ as given below,

$$t_{1N} = \bar{y}_N \quad (3.1)$$

The variance of the estimator t_{1N} up to the first order approximation, is as follows,

$$V(t_{1N}) = f_N \bar{Y}_N^2 C_{yN}^2 \quad ; \text{ where, } t_{1N} \in [t_{1L}, t_{1U}]. \quad (3.2)$$

As per (Tahir et al. 2021), the conventional neutrosophic ratio estimator used for estimating $\bar{Y}_N$ relies on the known neutrosophic population mean of X_N , is given by,

$$t_{2N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N} \right) \quad (3.3)$$

The bias and MSE associated with estimator t_{2N} are respectively given by,

$$Bias(t_{2N}) = f_N \bar{Y}_N (C_{xN}^2 - \rho_{yxN} C_{xN} C_{yN}) \quad (3.4)$$

$$MSE(t_{2N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + C_{xN}^2 - 2\rho_{yxN} C_{xN} C_{yN}) \quad ; \text{ where, } t_{2N} \in [t_{2L}, t_{2U}]. \quad (3.5)$$

Based on the approach in (Murthy 1964), the neutrosophic product estimator is expressed as,

$$t_{3N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N} \right) \quad (3.6)$$

The bias and MSE for estimator t_{3N} are provided below,

$$Bias(t_{3N}) = f_N \bar{Y}_N \rho_{yxN} C_{yN} C_{xN} \quad (3.7)$$

$$MSE(t_{3N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + C_{xN}^2 + 2\rho_{yxN} C_{xN} C_{yN}) \quad ; \text{ where, } t_{3N} \in [t_{3L}, t_{3U}]. \quad (3.8)$$

Drawing inspiration from (Sisodia and Dwivedi 1981), (Tahir et al. 2021) developed a neutrosophic ratio estimator based on the coefficient of variation (CV) of X in the neutrosophic setting,

$$t_{4N} = \bar{y}_N \left(\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}} \right) \quad (3.9)$$

Up to the first order approximation, the bias and MSE of the estimator t_{4N} is as follows,

$$Bias(t_{4N}) = f_N \bar{Y}_N (\theta_{1N}^2 C_{xN}^2 - \theta_{1N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.10)$$

$$MSE(t_{4N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + \theta_{1N}^2 C_{xN}^2 - 2\theta_{1N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.11)$$

where, $\theta_{1N} = \frac{\bar{X}_N}{\bar{x}_N + C_{xN}}$ and $t_{4N} \in [t_{4L}, t_{4U}]$.

With reference to (Bahl and Tuteja 1991), (Tahir et al. 2021) devised neutrosophic exponential ratio-type estimators for estimating the neutrosophic population mean, which are provided in equations (3.12) and (3.15),

$$t_{5N} = \bar{y}_N \exp\left(\frac{\bar{x}_N - \bar{x}_N}{\bar{x}_N + \bar{x}_N}\right) \quad (3.12)$$

For estimator t_{5N} , the bias and MSE are stated as follows,

$$\text{Bias}(t_{5N}) = f_N \bar{Y}_N \left(\frac{3}{8} C_{xN}^2 - \frac{1}{2} \rho_{yxN} C_{xN} C_{yN} \right) \quad (3.13)$$

$$\text{MSE}(t_{5N}) = f_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{xN}^2}{4} - \rho_{yxN} C_{xN} C_{yN} \right) \quad ; \text{where, } t_{5N} \in [t_{5L}, t_{5U}]. \quad (3.14)$$

$$t_{6N} = \bar{y}_N \exp\left(\frac{(\bar{x}_{N+1}) - (\bar{x}_n + 1)}{(\bar{x}_{N+1}) + (\bar{x}_n + 1)}\right) \quad (3.15)$$

The bias and MSE corresponding to estimator t_{6N} are defined as follows,

$$\text{Bias}(t_{6N}) = f_N \bar{Y}_N \left(\frac{3}{8} \theta_{2N}^2 C_{xN}^2 - \frac{1}{2} \theta_{2N} \rho_{yxN} C_{xN} C_{yN} \right) \quad (3.16)$$

$$\text{MSE}(t_{6N}) = f_N \bar{Y}_N^2 \left(C_{yN}^2 + \theta_{2N}^2 C_{xN}^2 - 2\theta_{2N} \rho_{yxN} C_{xN} C_{yN} \right) \quad (3.17)$$

where, $\theta_{2N} = \frac{\bar{x}_N}{\bar{x}_N + 1}$ and $t_{6N} \in [t_{6L}, t_{6U}]$.

Inspired by (Bahl and Tuteja 1991), (Yadav and Prasad 2024) recommended the following neutrosophic product-type exponential estimator,

$$t_{7N} = \bar{y}_N \exp\left(\frac{\bar{x}_N - \bar{x}_N}{\bar{x}_N + \bar{x}_N}\right) \quad (3.18)$$

The corresponding expressions for its bias and MSE are given as follows,

$$\text{Bias}(t_{7N}) = f_N \bar{Y}_N \left(\frac{3}{8} C_{xN}^2 + \frac{1}{2} \rho_{yxN} C_{xN} C_{yN} \right) \quad (3.19)$$

$$\text{MSE}(t_{7N}) = f_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{xN}^2}{4} + \rho_{yxN} C_{xN} C_{yN} \right) \quad ; \text{where, } t_{7N} \in [t_{7L}, t_{7U}]. \quad (3.20)$$

Based on the methodology in (Singh and Tailor 2005), (Yadav and Smarandache 2023) developed the neutrosophic ratio estimator t_{8N} , which incorporates the known population correlation coefficient ρ_{yxN} , given as:

$$t_{8N} = \bar{y}_N \left(\frac{\bar{x}_N + \rho_{yxN}}{\bar{x}_N + \rho_{yxN}} \right) \quad (3.21)$$

The expressions for the bias and MSE of estimator t_{8N} are presented as follows,

$$\text{Bias}(t_{8N}) = f_N \bar{Y}_N (\theta_{3N}^2 C_{xN}^2 - \theta_{3N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.22)$$

$$\text{MSE}(t_{8N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + \theta_{3N}^2 C_{xN}^2 - 2\theta_{3N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.23)$$

where, $\theta_{3N} = \frac{\bar{x}_N}{\bar{x}_N + \rho_{yxN}}$ and $t_{8N} \in [t_{8L}, t_{8U}]$.

Following the approach in (Singh et al. 2007), (Tahir et al. 2021) recommended the following definition for the neutrosophic modified exponential ratio estimator,

$$t_{9N} = \bar{y}_N \exp\left(\frac{(a_N \bar{x}_N + b_N) - (a_N \bar{x}_n + b_N)}{(a_N \bar{x}_N + b_N) + (a_N \bar{x}_n + b_N)}\right) \quad (3.24)$$

where, a_N and b_N represent the auxiliary neutrosophic parameters.

Using a first-order approximation, the bias and MSE of the neutrosophic exponential ratio estimator t_{9N} are given as follows,

$$\text{Bias}(t_{9N}) = f_N \bar{Y}_N \left(\frac{3}{8} \theta_{4N}^2 C_{xN}^2 - \frac{1}{2} \theta_{4N} \rho_{yxN} C_{xN} C_{yN} \right) \quad (3.25)$$

$$\text{MSE}(t_{9N}) = f_N \bar{Y}_N^2 \left(C_{yN}^2 + \theta_{4N}^2 C_{xN}^2 - 2 \theta_{4N} \rho_{yxN} C_{xN} C_{yN} \right) \quad (3.26)$$

where, $\theta_{4N} = \frac{a_N \bar{X}_N}{2(a_N \bar{X}_N + b_N)}$ and $t_{9N} \in [t_{9L}, t_{9U}]$.

According to (Singh and Pal n.d.), the chain ratio–ratio exponential-type estimator for estimating the population mean is given as,

$$t_{10N} = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N} \right) \exp \left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N} \right) \quad (3.27)$$

Up to a first-order approximation, the bias and MSE associated with the estimator t_{10N} are as follows,

$$\text{Bias}(t_{10N}) = \frac{3}{8} f_N \bar{Y}_N C_{xN}^2 \left(5 - 4 \rho_{yxN} \frac{C_{yN}}{C_{xN}} \right) \quad (3.28)$$

$$\text{MSE}(t_{10N}) = f_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{3}{4} C_{xN}^2 \left(3 - 4 \rho_{yxN} \frac{C_{yN}}{C_{xN}} \right) \right) ; \text{ where, } t_{10N} \in [t_{10L}, t_{10U}]. \quad (3.29)$$

By using coefficient of kurtosis of X_N , (Tahir et al. 2021) developed a neutrosophic ratio estimator following the approach proposed by (Singh et al. 2004).

$$t_{11N} = \bar{y}_N \left(\frac{\bar{X}_N + \beta_{2(x)N}}{\bar{x}_N + \beta_{2(x)N}} \right) \quad (3.30)$$

The corresponding bias and MSE to the estimator t_{11N} are as follows,

$$\text{Bias}(t_{11N}) = f_N \bar{Y}_N (\theta_{5N}^2 C_{xN}^2 - \theta_{5N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.31)$$

$$\text{MSE}(t_{11N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + \theta_{5N}^2 C_{xN}^2 - 2 \theta_{5N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.32)$$

where, $\theta_{5N} = \frac{\bar{X}_N}{\bar{x}_N + \beta_{2(x)N}}$ and $t_{11N} \in [t_{11L}, t_{11U}]$.

Drawing inspiration from (Upadhyaya and Singh 1999), (Yadav and Smarandache 2023) introduced neutrosophic ratio estimators that incorporate the CV and kurtosis of the auxiliary variable X_N , defined as,

$$t_{12N} = \bar{y}_N \left(\frac{\beta_{2(x)N} \bar{X}_N + C_{xN}}{\beta_{2(x)N} \bar{x}_N + C_{xN}} \right) \quad (3.33)$$

$$t_{13N} = \bar{y}_N \left(\frac{C_{xN} \bar{X}_N + \beta_{2(x)N}}{C_{xN} \bar{x}_N + \beta_{2(x)N}} \right) \quad (3.34)$$

For the estimators t_{12N} and t_{13N} , the corresponding biases and MSEs are as follows,

$$\text{Bias}(t_{12N}) = f_N \bar{Y}_N (\theta_{6N}^2 C_{xN}^2 - \theta_{6N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.35)$$

$$\text{Bias}(t_{13N}) = f_N \bar{Y}_N (\theta_{7N}^2 C_{xN}^2 - \theta_{7N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.36)$$

$$\text{MSE}(t_{12N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + \theta_{6N}^2 C_{xN}^2 - 2 \theta_{6N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.37)$$

$$\text{MSE}(t_{13N}) = f_N \bar{Y}_N^2 (C_{yN}^2 + \theta_{7N}^2 C_{xN}^2 - 2\theta_{7N} \rho_{yxN} C_{xN} C_{yN}) \quad (3.38)$$

$$\text{where, } \theta_{6N} = \frac{\beta_{2(x)N} \bar{X}_N}{\beta_{2(x)N} \bar{X}_N + C_{xN}}, \quad \theta_{7N} = \frac{C_{xN} \bar{X}_N}{C_{xN} \bar{X}_N + \beta_{2(x)N}} \quad \text{and } t_{12N} \in [t_{12L}, t_{12U}], t_{13N} \in [t_{13L}, t_{13U}].$$

4. Proposed Neutrosophic Almost Unbiased Estimators

Motivated by the work of (Singh et al. 2024) and (Singh et al. n.d.), we propose enhanced neutrosophic estimators to estimate the neutrosophic finite population mean. These estimators are defined as:

$$t_{0N}^* = \bar{y}_N, \quad t_{1N}^* = \left[\bar{y}_N \left(\frac{a_N \bar{X}_N + b_N}{a_N \bar{X}_N + b_N} \right)^{K_{1N}} \exp \left(K_{2N} \left[\frac{(a_N \bar{X}_N + b_N) - (a_N \bar{X}_N + b_N)}{(a_N \bar{X}_N + b_N) + (a_N \bar{X}_N + b_N)} \right] \right) \right], \quad t_{2N}^* = \left[\bar{y}_N \left(1 + \log \left(\frac{\bar{X}_N}{\bar{X}_N} \right)^{K_{3N}} \right) \right]$$

such that all three estimators $t_{0N}^*, t_{1N}^*, t_{2N}^* \in A_N$, where, A_N represents the collection of all possible estimators used for estimating the neutrosophic population mean $\bar{Y}_N$. The use of neutrosophic constants a_N and b_N in developing the proposed estimators is justified in (Singh and Kumar 2011). According to the definition, the set A_N can be regarded as a linear variety if

$$t_N^* = \sum_{i=0}^2 l_{iN} t_{iN}^* \in A_N, \quad (4.1)$$

$$\text{for } \sum_{i=0}^2 l_{iN} = 1 \text{ and } l_{iN} \in R_N. \quad (4.2)$$

Here, $l_{iN} (i = 0, 1, 2)$ represents statistical constants, R_N refers to the real numbers and K_{1N}, K_{2N} and K_{3N} take on distinct values from the set $\{-1, 0, 1\}$.

The bias and MSE, up to first order of approximation for the estimators t_{1N}^* and t_{2N}^* are respectively expressed as,

$$\text{Bias}(t_{1N}^*) = \theta_N f_N K_N \bar{Y}_N C_{yN} (\theta_N (K_N + 1) C_{yN} - \rho_{yxN} C_{xN}). \quad (4.3)$$

$$\text{MSE}(t_{1N}^*) = f_N \bar{Y}_N^2 (C_{yN}^2 + \theta_N^2 K_N^2 C_{xN}^2 - 2\theta_N K_N \rho_{yxN} C_{yN} C_{xN}). \quad (4.4)$$

$$\text{Bias}(t_{2N}^*) = \frac{K_{3N} f_N \bar{Y}_N C_{xN}}{2} (2\rho_{yxN} C_{yN} - C_{xN}). \quad (4.5)$$

$$\text{MSE}(t_{2N}^*) = f_N \bar{Y}_N^2 (C_{yN}^2 + K_{3N}^2 C_{xN}^2 + 2K_{3N} \rho_{yxN} C_{yN} C_{xN}). \quad (4.6)$$

$$\text{Where, } K_N = K_{1N} + \frac{K_{2N}}{2} \text{ and } \theta_N = \frac{a_N \bar{X}_N}{a_N \bar{X}_N + b_N}.$$

Now, the proposed neutrosophic estimator can be written as,

$$t_N^* = l_{0N} \bar{y}_N + l_{1N} \bar{y}_N \left(\frac{a_N \bar{X}_N + b_N}{a_N \bar{X}_N + b_N} \right)^{K_{1N}} \exp \left(K_{2N} \left[\frac{(a_N \bar{X}_N + b_N) - (a_N \bar{X}_N + b_N)}{(a_N \bar{X}_N + b_N) + (a_N \bar{X}_N + b_N)} \right] \right) + l_{2N} \bar{y}_N \left(1 + \log \left(\frac{\bar{X}_N}{\bar{X}_N} \right)^{K_{3N}} \right) \quad (4.7)$$

The bias and MSE of the estimator t_N^* are derived on the basis of the following assumptions:

$$\bar{y}_N = \bar{Y}_N (1 + e_{0N}), \quad \bar{x}_N = \bar{X}_N (1 + e_{1N}), \quad \text{with the condition that } E(e_{0N}) = E(e_{1N}) = 0.$$

By defining the estimator t_N^* in the form of error terms e_{0N} and e_{1N} , we get,

$$t_N^* = l_{0N} \bar{Y}_N (1 + e_{0N}) + l_{1N} \left[\bar{Y}_N (1 + e_{0N}) \left(\frac{a_N \bar{X}_N + b_N}{a_N \bar{X}_N (1 + e_{1N}) + b_N} \right)^{K_{1N}} \exp \left(K_{2N} \left[\frac{(a_N \bar{X}_N + b_N) - (a_N \bar{X}_N (1 + e_{1N}) + b_N)}{(a_N \bar{X}_N + b_N) + (a_N \bar{X}_N (1 + e_{1N}) + b_N)} \right] \right) \right] + l_{2N} \left[\bar{Y}_N (1 + e_{0N}) \left(1 + \log \left(\frac{\bar{X}_N (1 + e_{1N})}{\bar{X}_N} \right)^{K_{3N}} \right) \right] \quad (4.8)$$

$$= \bar{Y}_N(1 + e_{0N}) \left[\frac{l_{0N} + l_{1N}(1 + \theta_N e_{1N})^{-K_{1N}} \exp(K_{2N}[-\theta_N e_{1N}(2 + \theta_N e_{1N})^{-1}])}{l_{2N}(1 + \log(1 + e_{1N})^{K_{3N}})} \right] + \quad (4.9)$$

Expanding the expression (4.9) on the right-hand side, and taking into consideration the error terms up to second order, we obtain,

$$t_N^* = \bar{Y}_N \left[(l_{0N} + l_{1N} + l_{2N}) + (l_{0N} + l_{1N} + l_{2N})e_{0N} + (l_{2N}K_{3N} - l_{1N}\theta_N K_N)e_{1N} + (l_{2N}K_{3N} - l_{1N}\theta_N K_N)e_{1N}e_{0N} - \frac{1}{2}(l_{2N}K_{3N} - l_{1N}\theta_N^2 K_N(K_N + 1))e_{1N}^2 \right] \quad (4.10)$$

$$t_N^* = \bar{Y}_N \left[1 + e_{0N} + l_N e_{1N} + l_N e_{1N}e_{0N} - \frac{1}{2}(l_{2N}K_{3N} - l_{1N}\theta_N^2 K_N(K_N + 1))e_{1N}^2 \right] \quad (4.11)$$

$$\text{Where, } l_N = (l_{2N}K_{3N} - l_{1N}\theta_N K_N). \quad (4.12)$$

By subtracting $\bar{Y}_N$ from equation (4.11) and taking the expectation on both sides, the bias of estimator t_N^* is derived to the first order of approximation as,

$$\text{Bias}(t_N^*) = E(t_N^* - \bar{Y}_N) = f_N \bar{Y}_N \left[l_N \rho_{yxN} \bar{X}_N \bar{Y}_N C_{yN} C_{xN} - \frac{1}{2}(l_{2N}K_{3N} - l_{1N}\theta_N^2 K_N(K_N + 1))C_{xN}^2 \right] \quad (4.13)$$

From equation (4.11), it follows that,

$$t_N^* - \bar{Y}_N = \bar{Y}_N(e_{0N} + l_N e_{1N}) \quad (4.14)$$

Squaring equation (4.14) and then taking expectation on both sides, we get the first-order approximation of the MSE for the estimator t_N^* as,

$$MSE(t_N^*) = f_N \bar{Y}_N^2 (C_{yN}^2 + l_N^2 C_{xN}^2 + 2l_N \rho_{yxN} C_{yN} C_{xN}). \quad (4.15)$$

By differentiating equation (4.15) with respect to l_N and equating it to zero, we get, the optimum value of l_N .

$$l_N = -\rho_{yxN} \frac{C_{yN}}{C_{xN}} \quad (4.16)$$

The minimum MSE attained by the estimator t_N^* , under the optimum value of l_N is,

$$\text{Min. } MSE(t_N^*) = f_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{yxN}^2) \quad (4.17)$$

The resulting MSE is equivalent to that of the traditional linear regression estimator.

From equation (4.12) and (4.16), we get,

$$(l_{2N}K_{3N} - l_{1N}\theta_N K_N) = l_N = -\rho_{yxN} \frac{C_{yN}}{C_{xN}} \quad (4.18)$$

Based on equations (4.2) and (4.18), there are only two equations with three unknowns. The constants l_{iN}' s ($i=0,1,2$) cannot be uniquely determined. To obtain unique values of l_{iN}' s, we shall apply the linear constraint.

$$\sum_{i=0}^2 l_{iN} \text{Bias}(t_{iN}^*) = 0. \quad (4.19)$$

Where, $\text{Bias}(t_{iN}^*)$ denotes the bias of i^{th} estimator.

The matrix representation of equations (4.2), (4.18), and (4.19) can be rewritten as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -\theta_N K_N & K_{3N} \\ 0 & \text{Bias}(t_{1N}^*) & \text{Bias}(t_{2N}^*) \end{bmatrix} \begin{bmatrix} l_{0N} \\ l_{1N} \\ l_{2N} \end{bmatrix} = \begin{bmatrix} 1 \\ l_N \\ 0 \end{bmatrix} \quad (4.20)$$

By solving equation (4.20), we obtain the unique values of l_{iN} 's ($i=0,1,2$) as,

$$l_{0N} = \frac{\theta_N K_N \text{Bias}(t_{2N}^*) + K_{3N} \text{Bias}(t_{1N}^*) + l_N \text{Bias}(t_{2N}^*) - l_N \text{Bias}(t_{1N}^*)}{\theta_N K_N \text{Bias}(t_{2N}^*) + K_{3N} \text{Bias}(t_{1N}^*)} \quad (4.21)$$

$$l_{1N} = \frac{-l_N \text{Bias}(t_{2N}^*)}{\theta_N K_N \text{Bias}(t_{2N}^*) + K_{3N} \text{Bias}(t_{1N}^*)} \quad (4.22)$$

$$l_{3N} = \frac{l_N \text{Bias}(t_{1N}^*)}{\theta_N K_N \text{Bias}(t_{2N}^*) + K_{3N} \text{Bias}(t_{1N}^*)} \quad (4.23)$$

such that,

$$\sum_{i=0}^2 l_{iN} = 1 \text{ and } l_{iN} \in R_N$$

Using these l_{iN} ($i=0,1,2$) constants in equation (4.7) ensures that bias is eliminated up to the first-order term $O(n^{-1})$.

5. Several variants of the suggested estimator t_N^*

A number of estimators may be formulated using the proposed estimator t_N^* from equation (4.7) through appropriate choices of the constants a_N , b_N , K_{1N} , K_{2N} , K_{3N} and l_{iN} 's ($i=0,1,2$) as shown in the scheme below.

Members of suggested Estimators	Values of							
	l_{0N}	l_{1N}	l_{2N}	a_N	b_N	K_{1N}	K_{2N}	K_{3N}
$t_{1N} = \bar{y}_N$	1	0	0	1	1	1	1	1
$t_{2N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{x}_N} \right)$	0	1	0	1	0	1	0	1
$t_{3N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{x}_N} \right)$	0	1	0	1	0	-1	0	1
$t_{4N} = \bar{y}_N \left(\frac{\bar{x}_N + C_{xN}}{\bar{x}_N + C_{xN}} \right)$	0	1	0	1	C_{xN}	1	0	1
$t_{5N} = \bar{y}_N \exp \left(\frac{\bar{x}_N - \bar{x}_N}{\bar{x}_N + \bar{x}_N} \right)$	0	1	0	1	0	0	1	1
$t_{6N} = \bar{y}_N \exp \left(\frac{(\bar{x}_N + 1) - (\bar{x}_N + 1)}{(\bar{x}_N + 1) + (\bar{x}_N + 1)} \right)$	0	1	0	1	1	0	1	1
$t_{7N} = \bar{y}_N \exp \left(\frac{\bar{x}_N - \bar{x}_N}{\bar{x}_N + \bar{x}_N} \right)$	0	1	0	1	0	0	-1	1
$t_{8N} = \bar{y}_N \left(\frac{\bar{x}_N + \rho_{yxN}}{\bar{x}_N + \rho_{yxN}} \right)$	0	1	0	1	ρ_{yxN}	1	0	1
$t_{9N} = \bar{y}_N \exp \left(\frac{(a_N \bar{x}_N + b_N) - (a_N \bar{x}_N + b_N)}{(a_N \bar{x}_N + b_N) + (a_N \bar{x}_N + b_N)} \right)$	0	1	0	a_N	b_N	0	1	1
$t_{10N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{x}_N} \right) \exp \left(\frac{\bar{x}_N - \bar{x}_N}{\bar{x}_N + \bar{x}_N} \right)$	0	1	0	1	0	1	1	1
$t_{11N} = \bar{y}_N \left(\frac{\bar{x}_N + \beta_{2(x)N}}{\bar{x}_N + \beta_{2(x)N}} \right)$	0	1	0	1	$\beta_{2(x)N}$	1	0	1
$t_{12N} = \bar{y}_N \left(\frac{\beta_{2(x)N} \bar{x}_N + C_{xN}}{\beta_{2(x)N} \bar{x}_N + C_{xN}} \right)$	0	1	0	$\beta_{2(x)N}$	C_{xN}	1	0	1
$t_{13N} = \bar{y}_N \left(\frac{C_{xN} \bar{x}_N + \beta_{2(x)N}}{C_{xN} \bar{x}_N + \beta_{2(x)N}} \right)$	0	1	0	C_{xN}	$\beta_{2(x)N}$	1	0	1

6. Empirical Study

For the purpose of empirical analysis, three real-world data sets with indeterminacy were selected. A description of each dataset is provided below:

6.1 Dataset 1: The dataset comprises daily stock price information for NVIDIA Corporation (NVDA), obtained from the publicly accessible Yahoo Finance historical data at <https://finance.yahoo.com/quote/NVDA/history/>. For this study, the daily stock prices from May 19, 2023 to May 19, 2024 are considered as the neutrosophic study variable (Y_N), while the corresponding daily stock prices from May 19, 2022 to May 19, 2023 are considered as the neutrosophic auxiliary variable (X_N). The neutrosophic population size is $N_N \in (251, 251)$, and a neutrosophic sample of size $n_N = (50, 50)$ was selected through SRSWOR.

6.2 Dataset 2: The dataset comprises daily stock price information for ServiceNow, Inc. (NOW), obtained from the publicly accessible Yahoo Finance historical data at <https://finance.yahoo.com/quote/NOW/history/>. For this study, the daily stock prices from July 25, 2023 to July 25, 2024 are considered as the neutrosophic study variable (Y_N), while the corresponding daily stock prices from July 25, 2022 to July 25, 2023 are considered as the neutrosophic auxiliary variable (X_N). The neutrosophic population size is $N_N \in (251, 251)$, and a neutrosophic sample of size $n_N = (30, 30)$ was selected through SRSWOR.

6.3 Dataset 3: The dataset contains Seasonal and Annual Minimum-Maximum Temperature records (1901-2017), obtained from <https://data.gov.in/resource/seasonal-and-annual-minimum-maximum-temperature-series-1901-2017>. For this study, March to May temperatures (Y_N) were taken as the study variable and January to February minimum and maximum temperatures (X_N) as the auxiliary variable, with a neutrosophic population size of $N_N \in (117, 117)$. Using SRSWOR, a neutrosophic sample of size $n_N = (15, 15)$ was drawn from the specified neutrosophic population.

6.4 Dataset 2: The natural growth rate dataset, comprising information from 21 bigger states, 9 smaller states, and 6 Union Territories (total population size: 36) was obtained from the SRS Bulletin 2020 at <https://censusindia.gov.in/nada/index.php/catalog/42687>. In this study, the natural growth rate is taken as the neutrosophic study variable (Y_N), while the birth rate as the auxiliary variable (X_N), with a neutrosophic population size of $N_N \in (36, 36)$. A sample of size $n_N = (9, 9)$ was selected from the neutrosophic population through SRSWOR.

Table 1: Descriptive statistic for datasets 1, 2, 3 and 4.

Parameters	Data Set 1 (NVIDIA Corporation (NVDA))	Data Set 2 (ServiceNow, Inc. (NOW))	Data Set 3 (Temperature)	Data Set 4 (Natural Growth Rate)
N_N	[251, 251]	[251, 251]	[117, 117]	[36,36]
n_N	[50, 50]	[30, 30]	[15, 15]	[9, 9]
$\bar{Y}_N$	[561.524, 581.600]	[675.847, 691.384]	[20.6685, 31.5176]	[9.7583, 11.7667]
$\bar{X}_N$	[182.736, 190.320]	[443.486, 457.759]	[13.8946, 24.6295]	[14.7083, 17.8167]
C_{yN}	[0.323, 0.327]	[0.123, 0.122]	[0.3447, 0.5261]	[0.3360, 0.4886]
C_{xN}	[0.273, 0.265]	[0.133, 0.128]	[0.4364, 0.7753]	[0.2531, 0.3607]
ρ_{yxN}	[0.906, 0.904]	[0.390, 0.370]	[0.6126, 0.6759]	[0.9652, 0.9585]
$\beta_{2(x)N}$	[2.358, 2.331]	[2.730, 2.746]	[5.2403, 6.0507]	[2.5910, 2.2163]

The PRE of the estimators is calculated by the formula,

$$PRE = \frac{MSE(\bar{y}_N)}{MSE(estimator)} \times 100.$$

Table 2: MSEs and PREs of the proposed estimator and various existing estimators for dataset 1.

Estimator	MSE	I _N	PRE
t_{1N}	[526.8585, 579.2908]	[0, 0.0905]	[100, 100]
t_{2N}	[96.3416, 110.9605]	[0, 0.1317]	[546.8650, 522.0694]
t_{3N}	[1710.1142, 1808.5128]	[0, 0.0544]	[30.8084, 32.0313]
t_{4N}	[96.4232, 111.0834]	[0, 0.1320]	[546.4021, 521.4916]
t_{5N}	[217.5077, 250.0142]	[0, 0.1300]	[242.2252, 231.7032]
t_{6N}	[96.6475, 111.4303]	[0, 0.1327]	[545.1344, 519.8685]
t_{7N}	[1024.3940, 1098.7904]	[0, 0.0677]	[51.4312, 52.7208]
t_{8N}	[96.6179, 111.3845]	[0, 0.1326]	[545.3011, 520.0822]
t_{9N}	[218.6821, 251.2407]	[0, 0.1296]	[240.9244, 230.5720]
t_{10N}	[163.3602, 162.1298]	[0, -0.0076]	[322.5135, 357.3007]
t_{11N}	[97.0925, 112.0796]	[0, 0.1337]	[542.6356, 516.8568]
t_{12N}	[96.3760, 111.0131]	[0, 0.1318]	[546.6696, 521.8220]
t_{13N}	[1013.3945, 1079.6139]	[0, 0.0613]	[51.9895, 53.6572]
t_N^*	[94.3941, 105.8851]	[0, 0.1085]	[558.1478, 547.0938]

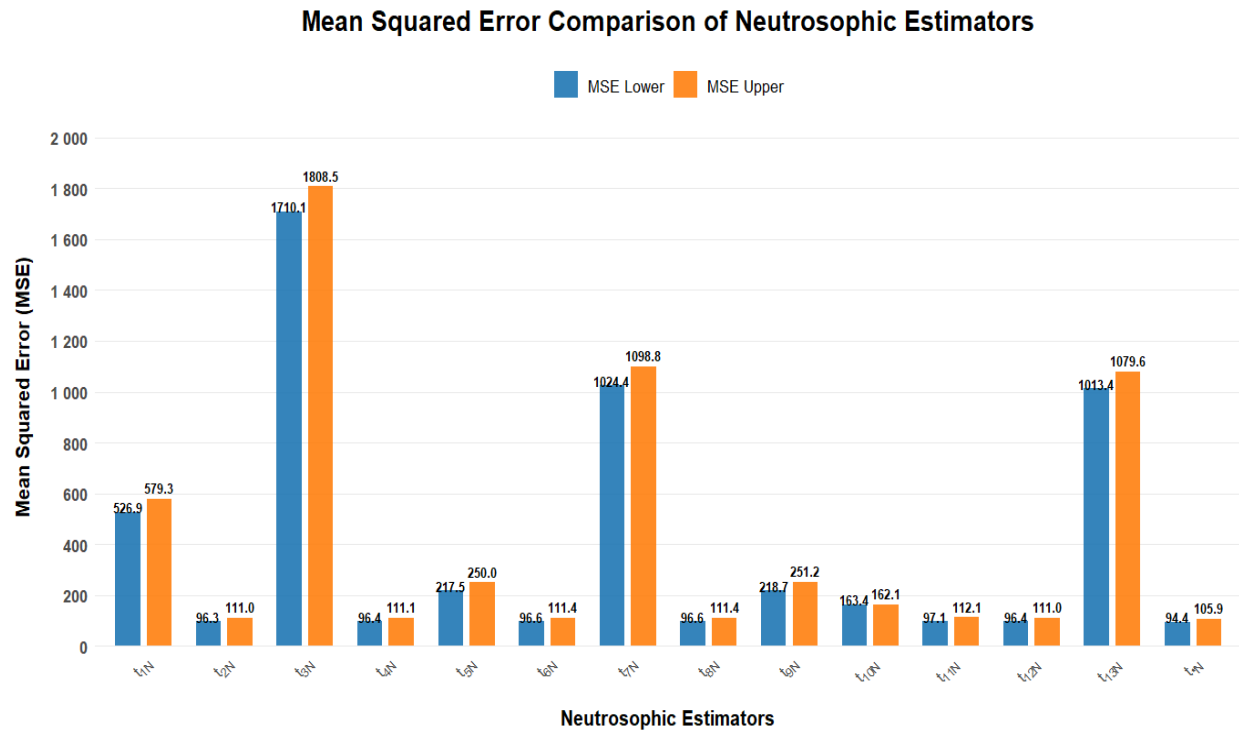


Fig. 1: MSE values for proposed estimator and various existing estimators for dataset 1.

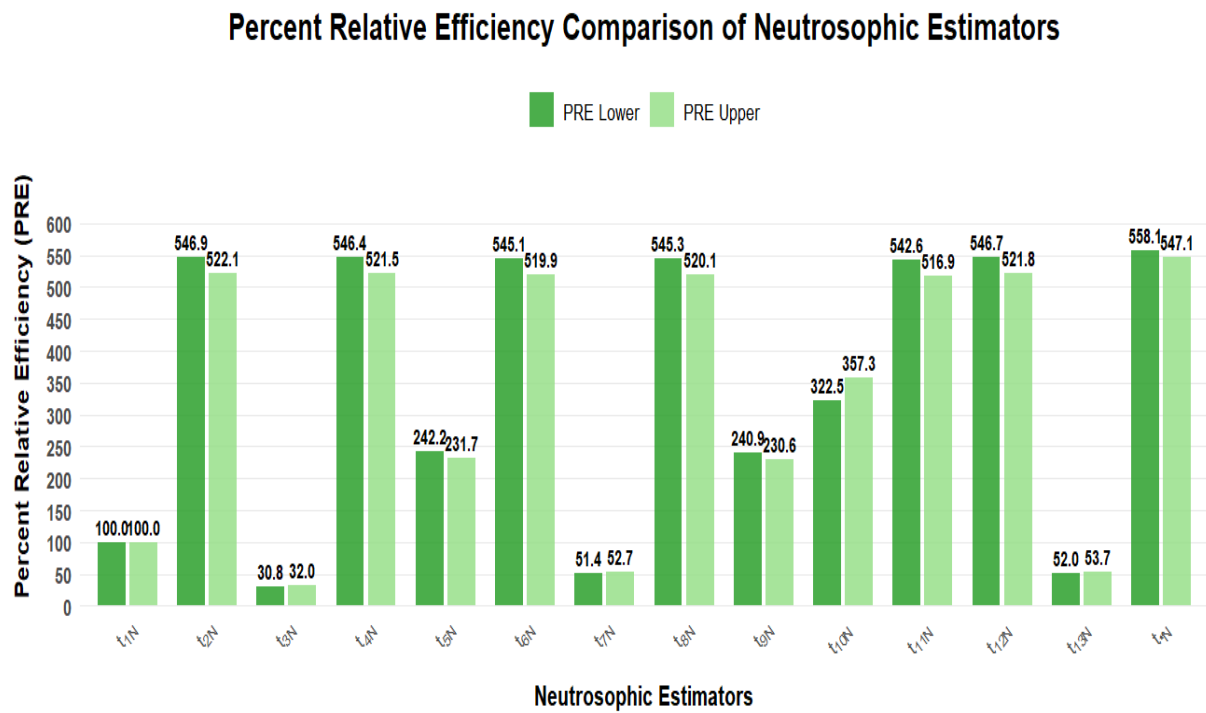


Fig. 2: PRE values for proposed estimator and various existing estimators for dataset 1.

Table 3: MSEs and PREs of the proposed estimator and various existing estimators for dataset 2.

Estimator	MSE	I _N	PRE
t_{1N}	[202.8170, 208.8121]	[0, 0.0287]	[100, 100]
t_{2N}	[268.8941, 276.5478]	[0, 0.0277]	[75.4263, 75.5067]
t_{3N}	[611.0117, 600.7884]	[0, -0.0170]	[33.1936, 34.7563]
t_{4N}	[268.8032, 276.4646]	[0, 0.0277]	[75.4518, 75.5294]
t_{5N}	[176.5716, 185.2159]	[0, 0.0467]	[114.8639, 112.7398]
t_{6N}	[268.2131, 275.9002]	[0, 0.0279]	[75.6178, 75.6839]
t_{7N}	[347.6304, 347.3362]	[0, -0.0008]	[58.3427, 60.1181]
t_{8N}	[268.6279, 276.3076]	[0, 0.0278]	[75.5011, 75.5723]
t_{9N}	[176.4975, 185.1424]	[0, 0.0467]	[114.9121, 112.7846]
t_{10N}	[479.7846, 482.8076]	[0, 0.0063]	[42.2725, 43.2495]
t_{11N}	[267.0479, 274.7814]	[0, 0.0281]	[75.9478, 75.9921]
t_{12N}	[268.8608, 276.5175]	[0, 0.0277]	[75.4357, 75.5150]
t_{13N}	[1668.0456, 1692.5161]	[0, 0.0145]	[12.1590, 12.3374]
t_N^*	[171.9685, 180.2257]	[0, 0.0458]	[117.9384, 115.8614]

Mean Squared Error Comparison of Neutrosophic Estimators

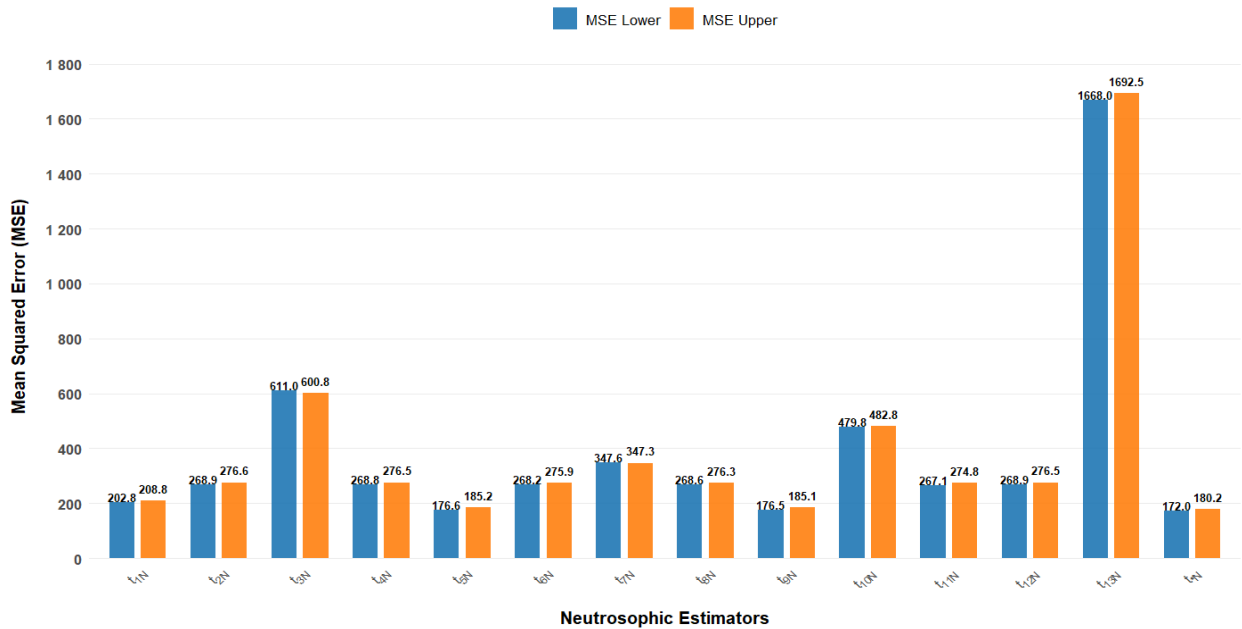


Fig. 3: MSE values for proposed estimator and various existing estimators for dataset 2.

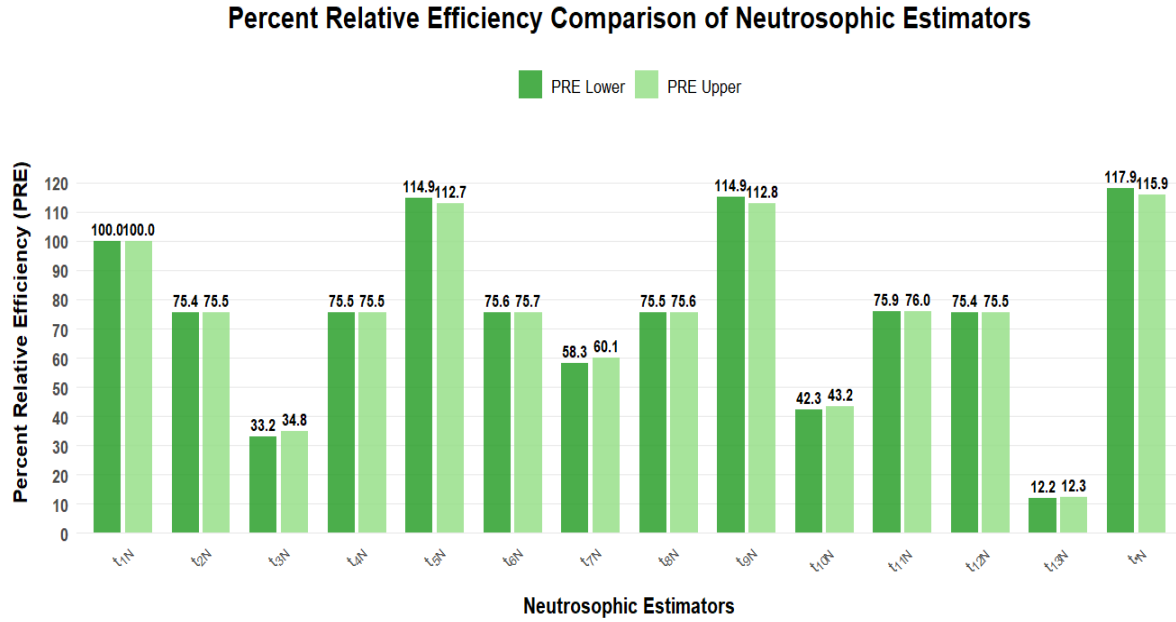


Fig. 4: PRE values for proposed estimator and various existing estimators for dataset 2.

Table 4: MSEs and PREs of the proposed estimator and various existing estimators for dataset 3.

Estimator	MSE	I_N	PRE
t_{1N}	[2.9500, 15.9796]	[0, 0.8154]	[100, 100]
t_{2N}	[3.1025, 18.8496]	[0, 0.8354]	[95.0851, 84.7743]
t_{3N}	[12.2542, 82.5159]	[0, 0.8515]	[24.0734, 19.3655]
t_{4N}	[2.9583, 17.7353]	[0, 0.8332]	[99.7214, 90.1008]
t_{5N}	[1.8442, 8.7388]	[0, 0.7890]	[159.9647, 182.8580]
t_{6N}	[2.7961, 17.4364]	[0, 0.8396]	[105.5038, 91.6451]
t_{7N}	[6.4200, 40.5720]	[0, 0.8418]	[45.9501, 39.3858]
t_{8N}	[2.9048, 17.8708]	[0, 0.8375]	[101.5556, 89.4175]
t_{9N}	[1.8444, 8.6960]	[0, 0.7879]	[159.9465, 183.7576]
t_{10N}	[6.7250, 46.3119]	[0, 0.8548]	[43.9465, 34.5043]
t_{11N}	[2.1204, 12.7893]	[0, 0.8342]	[139.1219, 124.9455]
t_{12N}	[3.0736, 18.6561]	[0, 0.8352]	[95.9795, 85.6537]

t_{13N}	[3.6131, 17.6178]	[0, 0.7949]	[81.6475, 90.7017]
t_N^*	[1.8429, 8.6795]	[0, 0.7877]	[160.0714, 184.1081]

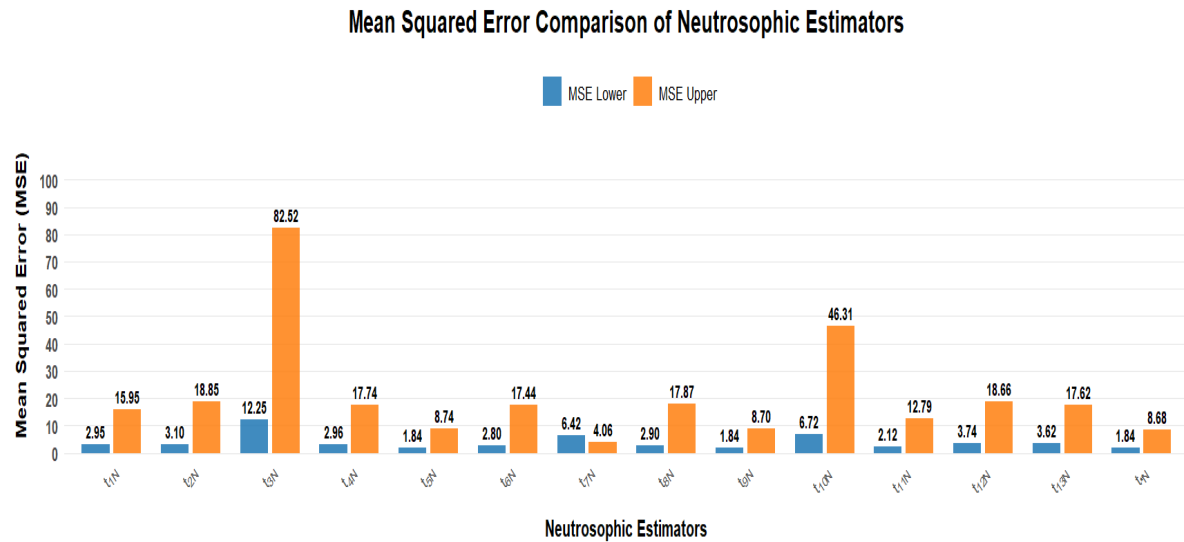


Fig. 5: MSE values for proposed estimator and various existing estimators for dataset 3.

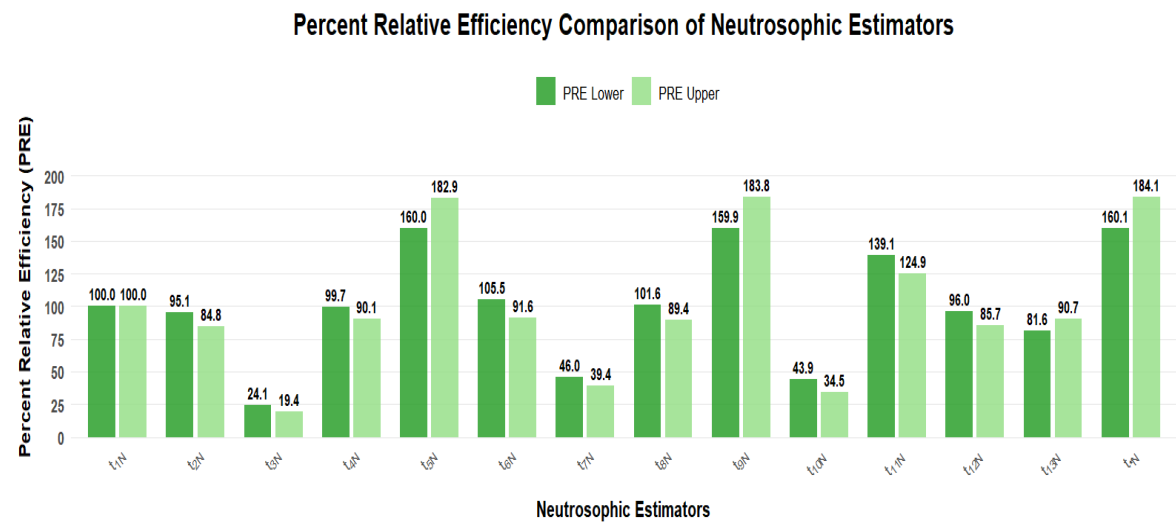


Fig. 6: PRE values for proposed estimator and various existing estimators for dataset 3.

Table 5: MSEs and PREs of the proposed estimator and various existing estimators for dataset 4.

Estimator	MSE	I_N	PRE
t_{1N}	[0.8959, 2.7545]	[0, 0.6748]	[100, 100]

t_{2N}	[0.1015, 0.3575]	[0, 0.7161]	[882.5995, 770.4405]
t_{3N}	[2.7069, 8.1537]	[0, 0.6680]	[33.0957, 33.7818]
t_{4N}	[0.1065, 0.3759]	[0, 0.7167]	[841.2890, 732.7946]
t_{5N}	[0.3716, 1.1807]	[0, 0.6853]	[241.0827, 233.2898]
t_{6N}	[0.1218, 0.4094]	[0, 0.7025]	[735.6911, 672.8634]
t_{7N}	[1.6743, 5.0788]	[0, 0.6703]	[53.5070, 54.2346]
t_{8N}	[0.1210, 0.4072]	[0, 0.7027]	[740.1097, 676.5026]
t_{9N}	[0.3974, 1.2455]	[0, 0.6809]	[225.4312, 221.1609]
t_{10N}	[0.0856, 0.2849]	[0, 0.6996]	[1046.9167, 966.8063]
t_{11N}	[0.1557, 0.4750]	[0, 0.6721]	[575.2085, 579.8919]
t_{12N}	[0.1034, 0.3657]	[0, 0.7173]	[866.3046, 753.1006]
t_{13N}	[0.8260, 2.1570]	[0, 0.6170]	[108.4577, 127.7006]
t_N^*	[0.0613, 0.2239]	[0, 0.7263]	[1462.2243, 1230.3490]

Mean Squared Error Comparison of Neutrosophic Estimators

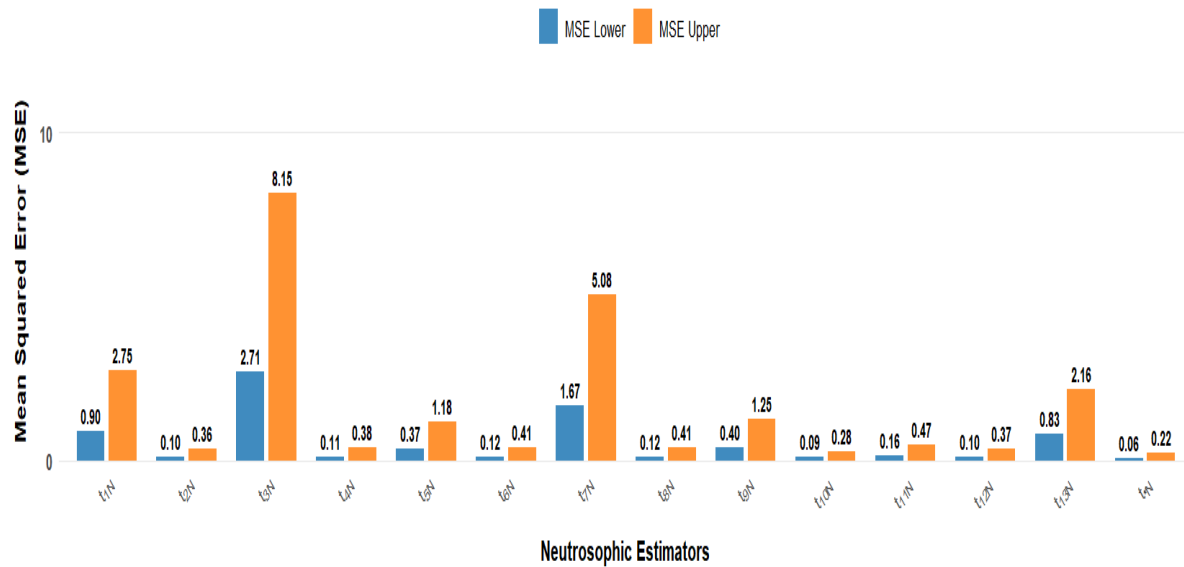


Fig. 7: MSE values for proposed estimator and various existing estimators for dataset 4.

Percent Relative Efficiency Comparison of Neutrosophic Estimators

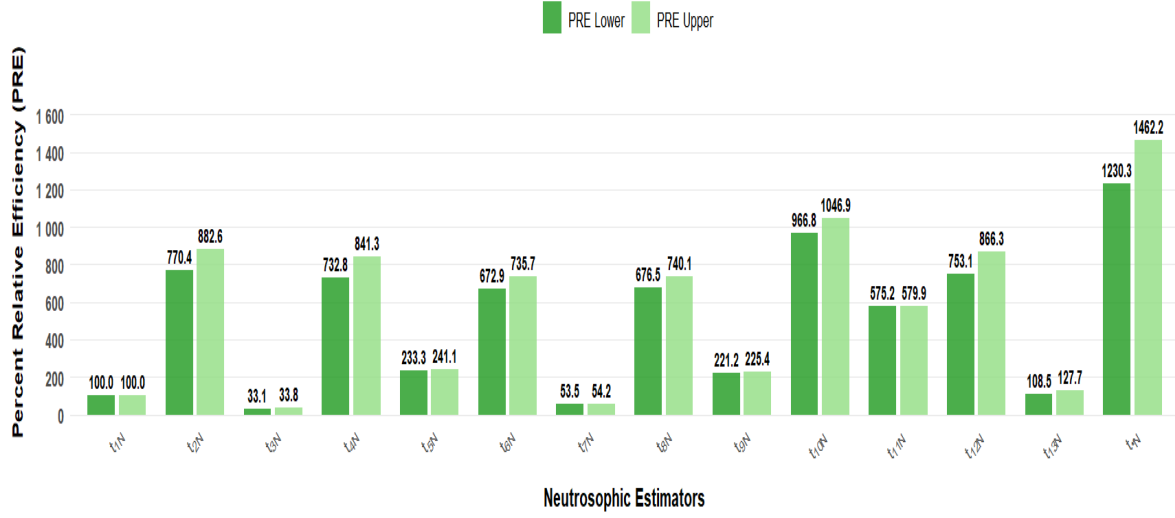


Fig. 8: PRE values for proposed estimator and various existing estimators for dataset 4.

7. Simulation Study

A simulation study based on neutrosophic data was conducted to evaluate the empirical performance and robustness of the proposed estimator under different distributional settings. The framework for generating neutrosophic data was based on the simulation procedure of (Raghav 2023) for the normal population and was subsequently extended to Gamma and Lognormal population settings. This extension was considered to examine the robustness of the proposed estimator under both symmetric and positively skewed population structures. Three populations with distinct distributional characteristics were considered in the simulation study:

For Population I, the neutrosophic auxiliary variable X_N was generated from a Neutrosophic Normal distribution, representing a case of indeterminacy in the population mean μ_{xN} such that $X_N \in (X_L, X_U)$, with $\mu_{xN} \in (0.649, 1.584)$ and $\sigma_{xN}^2 \in (1, 4.500)$. For Population II, the auxiliary variable X_N was generated from a Gamma distribution, with its shape and scale parameters given by: $\alpha_N = \frac{\mu_{xN}^2}{\sigma_{xN}^2}$, $\beta_N = \frac{\sigma_{xN}^2}{\mu_{xN}}$. For Population III, X_N was generated from a Lognormal distribution, with $\sigma_{\log,N}^2 = \ln\left(1 + \frac{\sigma_{xN}^2}{\mu_{xN}^2}\right)$ and $\mu_{\log,N} = \ln(\mu_{xN}) - \frac{1}{2}\sigma_{\log,N}^2$.

The three populations were selected to represent different distributional characteristics. Population I represents a symmetric population, whereas Populations II and III represent positively skewed populations with different distributional characteristics. Consequently, these settings enable an assessment of the robustness and relative performance of the proposed estimator under both symmetric and asymmetric population structures.

For each population, the neutrosophic study variable Y_N was generated according to the model $Y_N = X_N - 7.2e_N$, where $e_N \sim N_N(0, 1)$ represents a neutrosophic random error term. A neutrosophic population of size $N_N = 1000$ was considered under each distributional setting, from which simple random samples without replacement of sizes $n_{1N} = 100$, $n_{2N} = 200$, and $n_{3N} = 300$ were selected.

The simulation procedure was carried out as follows:

- **Step 1:** For each of the three considered population settings, values of μ_{xN} and σ_{xN}^2 were generated within their respective indeterminacy intervals. Using these values, the auxiliary variable X_N was generated according to the Normal, Gamma, and Lognormal distributions for Populations I, II, and III, respectively.
- **Step 2:** For each generated auxiliary population, the corresponding study variable Y_N was generated using the specified model. A population of size $N_N = 1000$ was considered for each distributional setting.
- **Step 3:** From each generated population, simple random samples without replacement of sizes $n_{1N} = 100$, $n_{2N} = 200$, and $n_{3N} = 300$ were selected independently. For each sample size and population distribution, the sampling and estimation procedure was repeated independently for 10,000 replications.
- **Step 4:** For every generated sample, the proposed estimator and the competing estimators were computed, and their corresponding MSE values were obtained from the 10,000 replications.
- **Step 5:** For each estimator, the minimum and maximum MSE values obtained across the replications were considered as the lower and upper bounds, respectively, thereby yielding the corresponding neutrosophic MSE interval. The I_N and PRE were also obtained from the simulation results. The procedure was repeated for all population distributions and sample sizes, and the resulting MSE intervals, I_N , and PRE values were used to compare the empirical performance of the proposed and existing estimators.

The results for MSE intervals, I_N , and PRE of the proposed and existing estimators are reported in Tables 5 to 7 for the Normal, Gamma, and Lognormal populations, respectively.

Table 6: MSE and PRE values of various existing and proposed estimators for different sample sizes under the normal population.

Estimator	$n_{1N} = 100$			$n_{2N} = 200$			$n_{3N} = 300$		
	MSE	I_N	PRE	MSE	I_N	PRE	MSE	I_N	PRE
t_{1N}	[0.3836, 0.5543]	[0, 0.3079]	[100, 100]	[0.1705, 0.2464]	[0, 0.3079]	[100, 100]	[0.0995, 0.1437]	[0, 0.3079]	[100, 100]
t_{2N}	[0.3726, 0.5175]	[0, 0.2801]	[102.9657, 107.1181]	[0.1656, 0.2299]	[0, 0.2801]	[102.9657, 107.1181]	[0.0966, 0.1342]	[0, 0.2801]	[102.9657, 107.1181]
t_{3N}	[0.4068, 0.8176]	[0, 0.5024]	[94.2967, 67.7967]	[0.1808, 0.3634]	[0, 0.5024]	[94.2967, 67.7967]	[0.1055, 0.2119]	[0, 0.5024]	[94.2967, 67.7968]
t_{4N}	[0.3771, 0.5367]	[0, 0.2974]	[101.7343, 103.2914]	[0.1676, 0.2385]	[0, 0.2974]	[101.7343, 103.2914]	[0.0978, 0.1391]	[0, 0.2974]	[101.7343, 103.2914]
t_{5N}	[0.3766, 0.5263]	[0, 0.2844]	[101.8584, 105.3327]	[0.1674, 0.2339]	[0, 0.2844]	[101.8584, 105.3328]	[0.0976, 0.1364]	[0, 0.2844]	[101.8584, 105.3328]
t_{6N}	[0.3765, 0.5271]	[0, 0.2858]	[101.8897, 105.1558]	[0.1673, 0.2343]	[0, 0.2858]	[101.8897, 105.1558]	[0.0976, 0.1367]	[0, 0.2858]	[101.8897, 105.1559]
t_{7N}	[0.3937, 0.6147]	[0, 0.3596]	[97.4429, 90.1738]	[0.1749, 0.2732]	[0, 0.3596]	[97.4428, 90.1738]	[0.1021, 0.1594]	[0, 0.3596]	[97.4429, 90.1738]
t_{8N}	[0.3735, 0.5225]	[0, 0.2851]	[102.7071, 106.1003]	[0.1659, 0.2322]	[0, 0.2851]	[102.7071, 106.1003]	[0.0968, 0.1355]	[0, 0.2851]	[102.7071, 106.1004]
t_{9N}	[0.3799, 0.5379]	[0, 0.2936]	[100.9575, 103.0521]	[0.1689, 0.2391]	[0, 0.2936]	[100.9575, 103.0521]	[0.0985, 0.1395]	[0, 0.2936]	[100.9575, 103.0521]
t_{10N}	[0.3699, 0.6512]	[0, 0.4320]	[103.7104, 85.1202]	[0.1644, 0.2894]	[0, 0.4320]	[103.7104, 85.1202]	[0.0959, 0.1688]	[0, 0.4320]	[103.7104, 85.1202]

t_{11N}	[0.3797, 0.5378]	[0, 0.2939]	[101.0237, 103.0751]	[0.1688, 0.2390]	[0, 0.2939]	[101.0237, 103.0751]	[0.0984, 0.1394]	[0, 0.2939]	[101.0237, 103.0751]
t_{12N}	[0.3745, 0.5261]	[0, 0.2882]	[102.4453, 105.3689]	[0.1664, 0.2338]	[0, 0.2882]	[102.4453, 105.3689]	[0.0971, 0.1364]	[0, 0.2882]	[102.4453, 105.3689]
t_{13N}	[0.3792, 0.5283]	[0, 0.2823]	[101.1734, 104.9278]	[0.1685, 0.2348]	[0, 0.2823]	[101.1734, 104.9277]	[0.0983, 0.1369]	[0, 0.2823]	[101.1734, 104.9278]
t_N^*	[0.3697, 0.5157]	[0, 0.2831]	[103.7509, 107.4853]	[0.1643, 0.2292]	[0, 0.2831]	[103.7509, 107.4853]	[0.0959, 0.1337]	[0, 0.2831]	[103.7509, 107.4853]

Table 7: MSE and PRE values of various existing and proposed estimators for different sample sizes under the gamma population.

Estimator	$n_{1N} = 100$			$n_{2N} = 200$			$n_{3N} = 300$		
	MSE	I _N	PRE	MSE	I _N	PRE	MSE	I _N	PRE
t_{1N}	[0.4028, 0.5811]	[0, 0.3068]	[100, 100]	[0.1790, 0.2583]	[0, 0.3070]	[100, 100]	[0.1044, 0.1506]	[0, 0.3068]	[100, 100]
t_{2N}	[0.3942, 0.5557]	[0, 0.2906]	[102.1816, 104.5708]	[0.1752, 0.2469]	[0, 0.2904]	[102.1689, 104.6173]	[0.1022, 0.1441]	[0, 0.2908]	[102.1526, 104.5108]
t_{3N}	[0.4293, 0.8295]	[0, 0.4825]	[93.8272, 70.0542]	[0.1908, 0.3687]	[0, 0.4825]	[93.8155, 70.0570]	[0.1113, 0.2150]	[0, 0.4823]	[93.8005, 70.0465]
t_{4N}	[0.3973, 0.5709]	[0, 0.3041]	[101.3843, 101.7867]	[0.1766, 0.2537]	[0, 0.3039]	[101.3590, 101.8132]	[0.1030, 0.1480]	[0, 0.3041]	[101.3592, 101.7568]
t_{5N}	[0.3977, 0.5633]	[0, 0.2940]	[101.2824, 103.1600]	[0.1767, 0.2504]	[0, 0.2943]	[101.3016, 103.1550]	[0.1031, 0.1460]	[0, 0.2938]	[101.2609, 103.1507]
t_{6N}	[0.3974, 0.5667]	[0, 0.2987]	[101.3588, 102.5410]	[0.1766, 0.2519]	[0, 0.2989]	[101.3590, 102.5407]	[0.1030, 0.1469]	[0, 0.2988]	[101.3592, 102.5187]
t_{7N}	[0.4138, 0.6609]	[0, 0.3739]	[97.3417, 87.9256]	[0.1839, 0.2938]	[0, 0.3741]	[97.3355, 87.9170]	[0.1073, 0.1714]	[0, 0.3740]	[97.2973, 87.8646]
t_{8N}	[0.3964, 0.5598]	[0, 0.2919]	[101.6145, 103.8049]	[0.1762, 0.2488]	[0, 0.2918]	[101.5891, 103.8183]	[0.1028, 0.1451]	[0, 0.2915]	[101.5564, 103.7905]
t_{9N}	[0.3989, 0.5703]	[0, 0.3005]	[100.9777, 101.8937]	[0.1773, 0.2534]	[0, 0.3003]	[100.9588, 101.9337]	[0.1034, 0.1478]	[0, 0.3004]	[100.9671, 101.8945]
t_{10N}	[0.3990, 0.6252]	[0, 0.3618]	[100.9524, 92.9463]	[0.1773, 0.2779]	[0, 0.3620]	[100.9588, 92.9471]	[0.1034, 0.1621]	[0, 0.3621]	[100.9671, 92.9056]
t_{11N}	[0.4007, 0.5764]	[0, 0.3048]	[100.5241, 100.8154]	[0.1781, 0.2562]	[0, 0.3048]	[100.5053, 100.8197]	[0.1039, 0.1494]	[0, 0.3046]	[100.4812, 100.8032]
t_{12N}	[0.3947, 0.5579]	[0, 0.2925]	[102.0522, 104.1585]	[0.1754, 0.2479]	[0, 0.2925]	[102.0525, 104.1952]	[0.1023, 0.1446]	[0, 0.2925]	[102.0528, 104.1494]
t_{13N}	[0.4009, 0.5756]	[0, 0.3035]	[100.4739, 100.9555]	[0.1782, 0.2558]	[0, 0.3034]	[100.4489, 100.9773]	[0.1039, 0.1492]	[0, 0.3036]	[100.4812, 100.9383]
t_N^*	[0.3941, 0.5508]	[0, 0.2845]	[102.2076, 105.5011]	[0.1752, 0.2448]	[0, 0.2843]	[102.1689, 105.5147]	[0.1022, 0.1428]	[0, 0.2843]	[102.1526, 105.4622]

Table 8: MSE and PRE values of various existing and proposed estimators for different sample sizes under the lognormal population.

Estimator	$n_{1N} = 100$			$n_{2N} = 200$			$n_{3N} = 300$		
	MSE	IN	PRE	MSE	IN	PRE	MSE	IN	PRE
t_{1N}	[0.4070, 1.3767]	[0, 0.7044]	[100, 100]	[0.1809, 0.6119]	[0, 0.7044]	[100, 100]	[0.1055, 0.3569]	[0, 0.7044]	[100, 100]
t_{2N}	[0.3947, 0.5659]	[0, 0.3025]	[103.1163, 243.2762]	[0.1754, 0.2515]	[0, 0.3026]	[103.1357, 243.3002]	[0.1023, 0.1467]	[0, 0.3027]	[103.1281, 243.2856]
t_{3N}	[0.4380, 4.4308]	[0, 0.9011]	[92.9224, 31.0711]	[0.1940, 1.9693]	[0, 0.9015]	[93.2474, 31.0720]	[0.1136, 1.1487]	[0, 0.9011]	[92.8697, 31.0699]
t_{4N}	[0.3938, 1.2317]	[0, 0.6803]	[103.3520, 111.7723]	[0.1750, 0.5474]	[0, 0.6803]	[103.3714, 111.7830]	[0.1021, 0.3193]	[0, 0.6802]	[103.3301, 111.7758]
t_{5N}	[0.3949, 0.7974]	[0, 0.5048]	[103.0641, 172.6486]	[0.1755, 0.3544]	[0, 0.5048]	[103.0769, 172.6580]	[0.1024, 0.2067]	[0, 0.5046]	[103.0273, 172.6657]
t_{6N}	[0.3939, 0.7846]	[0, 0.4980]	[103.3257, 175.4652]	[0.1751, 0.3487]	[0, 0.4978]	[103.3124, 175.4804]	[0.1021, 0.2034]	[0, 0.4980]	[103.3301, 175.4671]
t_{7N}	[0.4206, 2.5953]	[0, 0.8379]	[96.7665, 53.0459]	[0.1869, 1.1535]	[0, 0.8380]	[96.7897, 53.0472]	[0.1090, 0.6729]	[0, 0.8380]	[96.7890, 53.0391]
t_{8N}	[0.3937, 0.7400]	[0, 0.4680]	[103.3782, 186.0405]	[0.1749, 0.3289]	[0, 0.4682]	[103.4305, 186.0444]	[0.1021, 0.1919]	[0, 0.4680]	[103.331, 185.9823]
t_{9N}	[0.3992, 1.0152]	[0, 0.6068]	[101.9539, 135.6087]	[0.1774, 0.4512]	[0, 0.6068]	[101.9729, 135.6161]	[0.1035, 0.2632]	[0, 0.6068]	[101.9324, 135.6003]
t_{10N}	[0.3965, 0.9206]	[0, 0.5693]	[102.6482, 149.5438]	[0.1762, 0.4092]	[0, 0.5694]	[102.6674, 149.5357]	[0.1028, 0.2387]	[0, 0.5693]	[102.6265, 149.5182]
t_{11N}	[0.4051, 1.3749]	[0, 0.7054]	[100.4690, 100.1309]	[0.1801, 0.6111]	[0, 0.7053]	[100.4442, 100.1309]	[0.1050, 0.3565]	[0, 0.7055]	[100.4762, 100.1122]
t_{12N}	[0.3945, 0.5659]	[0, 0.3029]	[103.1686, 243.2762]	[0.1753, 0.2515]	[0, 0.3030]	[103.1945, 243.3002]	[0.1023, 0.1467]	[0, 0.3027]	[103.1281, 243.2856]
t_{13N}	[0.4055, 1.3609]	[0, 0.7020]	[100.3699, 101.1610]	[0.1802, 0.6048]	[0, 0.7021]	[100.3885, 101.1739]	[0.1051, 0.3528]	[0, 0.7021]	[100.3806, 101.1621]
t_N^*	[0.3932, 0.5532]	[0, 0.2892]	[103.5097, 248.8612]	[0.1748, 0.2459]	[0, 0.2891]	[103.4897, 248.8410]	[0.1019, 0.1434]	[0, 0.2894]	[103.539, 248.8842]

8. Discussion

In Table 2, the MSE and PRE values of existing and proposed estimators for Dataset 1 has been shown. According to the table, it is clear that the proposed estimator has the lowest MSE, falling within the range **[94.3941, 105.8851]** which indicates minimal estimation error and high reliability for the given dataset. The corresponding PRE value of the proposed estimator is **[558.1478, 547.0938]** which is higher than that of the existing ones. Fig. 1 and 2 illustrates the MSE values and PRE values of the proposed estimator in comparison with the existing estimators for Dataset 1. Further highlighting that the proposed estimator is the most efficient estimator.

For Dataset 2, Table 3 illustrates that the proposed estimator achieves the minimum MSE value ranging within **[171.9685, 180.2257]**, indicating that it has minimal estimation error as compared to all other existing estimators. The higher PRE value of the proposed estimator ranges between **[117.9384, 115.8614]**, proves that it is more efficient than the existing estimator. In Fig. 3 and 4, the MSE and PRE analysis for Dataset 2 clearly indicates that the proposed estimator performs better than the existing ones.

Table 4 indicates that, for Dataset 3, the proposed estimator yields the smallest MSE range of **[1.8429, 8.6795]**, demonstrating that it is superior and more efficient than the other estimators. The PRE value for the proposed estimator, lying within the range of **[160.0714, 184.1081]**, is higher than that of the existing ones, indicating its enhanced efficiency. Fig. 5 and 6 displays the comparative MSE and PRE outcomes for Dataset 3, highlighting the superior effectiveness of the proposed estimator compared to existing estimators.

For Dataset 4, Table 5 illustrates that the proposed estimator achieves the minimum MSE value ranging within **[0.0613, 0.2239]**, indicating that it has minimal estimation error as compared to all other existing estimators. The higher PRE value of the proposed estimator ranges between **[1462.2243, 1230.3490]**, proves that it is more efficient than the existing estimator. In Fig. 7 and 8, the MSE and PRE analysis for Dataset 2 clearly indicates that the proposed estimator performs better than the existing ones.

9. Conclusion

This paper presents a neutrosophic almost unbiased estimator for estimating the neutrosophic population mean $\bar{Y}_N$, formulated through the use of available auxiliary information. To examine the accuracy of the proposed estimator family, its bias and MSE were computed within the neutrosophic sampling framework using a first-order approximation. The efficiency of our suggested estimator was evaluated relative to existing estimators through data obtained from a real-world. Through the findings of both empirical and simulation studies, we obtained strong support indicating that our proposed estimator surpasses the existing estimators within the neutrosophic sampling framework. It is important to emphasize that neutrosophic estimators offer the greatest advantage in estimating the population mean when the study variable exhibits uncertainty or nondeterminism. Based on both empirical (Tables 2, 3, 4 and 5) and simulation (Tables 6, 7 and 8) results, we conclude that the proposed estimators effectively utilize auxiliary information to estimate the finite population mean, even in the presence of unclear, vague, ambiguous, indeterminate, or interval-based data. The neutrosophic framework has wide-ranging applications, particularly in fields like environmental monitoring, forecasting under uncertainty, and complex system evaluation.

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