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Transient Energy Exchange and Quantum Resources in a Driven Five-Level Kerr Optomechanical Model

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Abstract

We present a finite-time study of energy exchange, quantum correlations, work capacity, and phase sensitivity in a driven hybrid system composed of a five-level atom, a Kerr nonlinear optical mode, and a mechanical oscillator. The atom-field interaction is modulated periodically, while four local Lindblad channels describe two selected atomic transitions together with optical and mechanical damping. Rather than treating the dynamics as a completed heat-engine cycle, we analyze the system as a transient open quantum process and keep the different energetic and information-theoretic quantities clearly separated.

The initial optical and mechanical states are taken to be displaced squeezed thermal states, and the atomic subsystem is prepared in a coherent superposition carrying a relative phase. We follow the total energy, reservoir-associated currents, signed driving work, total and reduced entropies, atom-rest negativity, local mechanical ergotropy, and the quantum Fisher information associated with the initial atomic phase. Four parameter sets are considered in order to identify how the Kerr-plus-shift term, atom-field coupling strength, and atomic temperature bias influence the finite-time response. The calculations also track the first-law residual and include selected solver, time-step, and Hilbert-space checks to assess numerical sensitivity without claiming uniform convergence for every parameter set.

The results show that stronger atom-field coupling produces the largest sampled atom-rest negativity, but this increase in correlation does not lead to a comparable preservation of the reduced atomic phase quantum Fisher information. Within the employed finite truncation, removing the Kerr-plus-shift term changes the energy balance sufficiently to produce a small positive end-of-window signed work, whereas the other cases give negative final signed work. Because the zero- χ Hamiltonian has a formal high-occupation instability in the untruncated model and no dedicated enlarged-cutoff test was completed for that case, this small positive value is treated as a result conditional on the adopted finite truncation rather than as a converged engine-performance claim. The mechanical ergotropy remains a local state property and should not be identified with the work exchanged through the external modulation. Taken together, the results show that entanglement, metrological sensitivity, ergotropy, reservoir currents, and driving work respond differently to the same parameter changes. The conclusions therefore apply to the specified phenomenological finite-time model and should not be interpreted as evidence of a completed thermodynamic cycle or of a particular experimental implementation.

Keywords: Kerr optomechanics; five-level atom; open quantum systems; quantum entanglement; quantum Fisher information; ergotropy; finite-time energy exchange; Lindblad dynamics

I. INTRODUCTION

Hybrid quantum systems are attractive because they bring together degrees of freedom that respond to very different physical mechanisms. In cavity optomechanics, radiation pressure links an electromagnetic field to mechanical motion and provides a practical route to cooling, sensing, coherent state transfer, and the generation of nonclassical states [1–5]. When an atom and an optical nonlinearity are added, the dynamics become richer: excitation exchange, nonlinear phase evolution, mechanical displacement, dissipation, and external modulation can all act on comparable time scales. A meaningful analysis of such a system therefore requires both a clearly stated Hamiltonian and observables whose physical roles are not mixed together.

The quantum control of mechanical motion has progressed considerably over the last two decades. Theoretical work on dynamical backaction and sideband cooling established the conditions required to suppress mechanical occupation [6–8], and ground-state cooling was later demonstrated in microwave and optical platforms [9, 10]. Experiments with superconducting circuits have also achieved ground-state preparation and single-phonon control of a mechanical resonator [11], while strong optomechanical coupling and coherent exchange between optical and mechanical excitations have been observed [12, 13]. These results provide a solid physical background for hybrid optomechanical models, although the effective parameters used in a theoretical treatment still have to be interpreted with care.

Mechanical systems are also useful for studying nonclassical correlations. Radiation-pressure interactions can generate squeezed optical fields [14, 15], and entanglement has been demonstrated between mechanical motion and propagating microwave fields [16]. Remote and stabilized entanglement between mechanical oscillators has further shown that mechanical degrees of freedom can participate directly in distributed quantum states [17, 18]. Earlier theoretical proposals considered mirror–field entanglement and its signatures in optical output fields [19, 20]. These studies motivate the use of entanglement measures in the present

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model, while also emphasizing that the answer depends on the chosen bipartition. Here we focus on the atom versus the remaining optical–mechanical subsystem.

The atomic part of the model is based on the familiar physics of Jaynes–Cummings-type excitation exchange [21–24]. Hybrid atom–mechanical interfaces extend this idea to mechanical motion and have been explored in several experimental and theoretical settings [25–29]. We consider a generic five-level ladder with equal adjacent spacings and equal coherent coupling strengths. This choice is intended as an effective model rather than as a direct representation of a particular atom. A microscopic realization would require a separate discussion of selection rules, transition dipoles, reservoir engineering, and experimentally accessible parameter ranges.

Optical nonlinearity introduces a second important control mechanism. Kerr evolution produces photon-number-dependent phases and is known to generate squeezing, recurrences, and coherent-state superpositions [30, 31]. Related nonlinear effects appear in photon blockade and strongly interacting optical fields [32, 33], as well as in single-photon optomechanics where radiation pressure can generate an effective nonlinear optical response [34–36]. Engineered Kerr interactions have also been used to observe collapse-and-revival phenomena [37]. In the present Hamiltonian, the term proportional to n_f^2 contains both the normally ordered Kerr contribution and a linear photon-number shift. This distinction matters when the Kerr coefficient is set to zero.

The oscillator preparation provides another source of nonclassicality. Displaced and squeezed thermal states combine coherent excitation, quadrature squeezing, and thermal noise in a single controllable state. Mechanical amplification and reservoir engineering can enhance nonlinear interactions or produce strongly squeezed states [38–40]. Here, however, displacement and squeezing are used only to define the initial optical and mechanical states. Their stored energy is therefore part of the initial resource budget. This point is important whenever transient work or ergotropy is discussed, because a prepared nonequilibrium state can release energy even in the absence of a conventional heat-engine cycle.

The thermodynamic language used for driven quantum systems must also be chosen carefully. Quantum heat-engine theory has developed from early maser and quantum-cycle models [41, 42] to modern open-system descriptions of heat, work, entropy production, and finite-time operation [43–49]. In a time-dependent Hamiltonian, the explicit modulation contributes work, while dissipative channels exchange energy with the system. A periodic

Hamiltonian does not, by itself, guarantee that the density operator is periodic. For this reason, a finite observation window should not automatically be interpreted as a completed thermodynamic cycle.

Coherence and quantum correlations can nevertheless influence energetic behavior. Studies of coherent heat machines and noise-induced coherence have shown that quantum-state preparation may modify transport or power under suitable conditions [50–52]. Optomechanical work-extraction proposals and autonomous engine models similarly highlight the energetic role of mechanical motion [53, 54]. To quantify locally stored work capacity, we use mechanical ergotropy, defined relative to the bare mechanical Hamiltonian [55–57]. This quantity is distinct from the signed work exchanged with the external modulation and, in general, depends on which operations are allowed [58–60].

Entanglement and metrological sensitivity are treated as separate resources. Negativity is based on the partial transpose and detects non-positive-partial-transpose entanglement [61–63]. Quantum Fisher information, by contrast, measures the sensitivity of a quantum state to a specified parameter and provides the central quantity in quantum parameter-estimation bounds [64–68]. In this work the parameter is the relative phase initially encoded in the atomic superposition. The reduced atomic quantum Fisher information therefore measures how much information about that particular phase remains locally accessible during the open-system evolution.

Finally, the use of local dissipators requires a careful distinction between model definition and thermodynamic interpretation. The Gorini–Kossakowski–Lindblad–Sudarshan framework provides completely positive Markovian evolution under appropriate assumptions [69, 70], and Spohn’s inequality gives an entropy-production relation relative to suitable stationary reference states [71]. For an interacting system, however, a master equation built from local bare lowering operators does not generally coincide with a global thermal master equation. The differences between local and global descriptions are well known [72, 73]. We therefore use the local channels as a stated phenomenological model and separately report the full-Hamiltonian energy currents and the bare-Hamiltonian entropy-balance quantity associated with those channels.

Our aim is to examine how several quantum and energetic indicators evolve within one fully specified finite-time model. We compare four parameter sets that modify the Kerr-plus-shift term, the atom–field coupling strength, or the atomic temperature bias. For

each case we calculate the energy, signed driving work, reservoir currents, total and reduced entropies, atom–rest negativity, mechanical ergotropy, and atomic phase quantum Fisher information. Selected numerical consistency and discretization-sensitivity checks are reported. This approach allows the different resources to be compared without assuming in advance that stronger entanglement, larger ergotropy, improved metrological sensitivity, and positive work must occur together.

II. EFFECTIVE HAMILTONIAN AND ITS SCOPE

The basis ordering is atom, optical mode, mechanical mode. The atomic basis is ordered as $E_1, \dots, E_5$ from highest to lowest energy, with level energies (4.10, 3.05, 2.00, 0.95, -0.10). Adjacent levels are therefore separated by the common gap $\Delta = 1.05$, while the overall energy offset is only a reference choice. We set $\hbar = k_B = 1$ and scale all frequencies by a reference frequency ω_0 , chosen as $\omega_0 = 1$. Accordingly, the plotted time is the dimensionless quantity $\omega_0 t$, energies are expressed in units of $\hbar\omega_0$, and powers in units of $\hbar\omega_0^2$. The adjacent atomic spacing is therefore $\Delta = 1.05\omega_0$ rather than the definition of the frequency unit itself.

$$C = \sum_{j=1}^4 |E_j\rangle\langle E_{j+1}|, \quad H_A = \sum_{j=1}^5 \epsilon_j |E_j\rangle\langle E_j| \quad (1)$$

$$H(t) = H_0 + g_{\text{AF}} \cos(pt)V, \quad V = Ca + C^\dagger a^\dagger \quad (2)$$

$$H_0 = H_A + \omega_f n_f + \omega_m n_m + \chi n_f^2 - g_{\text{OM}} n_f (b + b^\dagger) \quad (3)$$

We work directly in the Schrödinger picture of the effective model. The interaction has the excitation-exchange structure familiar from Jaynes–Cummings physics [21]: C raises the atomic energy, while Ca and its Hermitian conjugate exchange one atomic excitation with one photon. The equal transition matrix elements are a simplifying assumption for a generic five-level ladder and should not be read as an exact spin-2 representation. The coherent Hamiltonian conserves $N_A + n_f$, where N_A has diagonal entries (4, 3, 2, 1, 0), even though the coupling strength is modulated in time. The choice $p = 1$ simply fixes the modulation frequency used in the calculation and is not intended as a claim about a specific experimental resonance.

$$\partial_t H = -p g_{\text{AF}} \sin(pt)V, \quad [H(t), N_A + n_f] = 0 \quad (4)$$

The radiation-pressure interaction couples the intracavity photon number to the mechanical displacement, in the usual form used in moving-boundary and cavity-optomechanical models [2, 74]. The overall sign follows from the chosen definition of the mechanical coordinate and has no independent physical significance.

The term proportional to n_f^2 represents an anharmonic Kerr response [30]. Because $n_f^2 = n_f(n_f - 1) + n_f$, changing χ modifies both the normally ordered nonlinear part and an effective linear photon-number shift. Consequently, Case 2 should be understood as removal of the combined Kerr-plus-shift contribution. Isolating only the nonlinear part would require keeping the linear term fixed while varying the coefficient of $n_f(n_f - 1)$, which is not done in the present comparison.

$$\chi n_f^2 = \chi n_f(n_f - 1) + \chi n_f \quad (5)$$

The relation between radiation-pressure coupling and an effective optical Kerr nonlinearity is discussed in Ref. [36]. Completing the mechanical square in the untruncated model gives the following identity:

$$\omega_m \left(b - \frac{g_{\text{OM}}}{\omega_m} n_f \right)^\dagger \left(b - \frac{g_{\text{OM}}}{\omega_m} n_f \right) + \omega_f n_f + \left(\chi - \frac{g_{\text{OM}}^2}{\omega_m} \right) n_f^2 \quad (6)$$

For the parameters used here, the coefficient g_{OM}^2/ω_m is 0.0182857. In the formal untruncated limit, setting $\chi = 0$ therefore makes the effective Hamiltonian unbounded from below. This is a known high-occupation limitation of the radiation-pressure approximation [75]; it does not imply that the finite-time, low-occupation numerical trajectory is itself unstable. We therefore treat Case 2 only as a finite-time dynamical comparison. A dedicated enlarged-cutoff test was not completed for this case, so conclusions that depend on its small positive signed work are stated conditionally on the adopted finite truncation.

III. DISSIPATION, PREPARATION AND PARAMETERS

$$\dot{\rho} = -i[H(t), \rho] + \sum_{r=h,c,f,m} \mathcal{L}_r(\rho) \quad (7)$$

$$\mathcal{D}[L]\rho = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}, \quad \mathcal{L}_r = \mathcal{D}[L_{r,-}] + \mathcal{D}[L_{r,+}] \quad (8)$$

$$L_{r,-} = \sqrt{\gamma_r(\bar{n}_r + 1)}A_r, \quad L_{r,+} = \sqrt{\gamma_r\bar{n}_r}A_r^\dagger, \quad \bar{n}_r = (e^{\omega_r/T_r} - 1)^{-1} \quad (9)$$

The corresponding lowering operators are $A_h = |E_2\rangle\langle E_1|$, $A_c = |E_5\rangle\langle E_4|$, $A_f = a$, and $A_m = b$, with channel frequencies (1.05, 1.05, 1, 0.35). Identity operators on the unaffected subsystems are understood. Since the two selected atomic transitions have the same bare frequency, their designation as hot and cold channels requires independently engineered couplings. The damping rates are (0.025, 0.025, 0.018, 0.010). The optical and mechanical jumps are also defined using their bare frequencies. Thus the dissipative part of the model is phenomenological and local rather than a microscopic global or Floquet master equation for the full interacting Hamiltonian.

$$\rho(0) = |\psi_A\rangle\langle\psi_A| \otimes \rho_f \otimes \rho_m, \quad |\psi_A\rangle = \frac{|E_1\rangle + e^{i\phi}|E_5\rangle}{\sqrt{2}}, \quad \phi = \pi/4 \quad (10)$$

$$\rho_s = D_s(\alpha_s)S_s(r_se^{i\theta_s})\rho_{\text{th},s}(\bar{n}_{s,0})S_s^\dagger D_s^\dagger \quad (11)$$

$$D_s(\alpha) = e^{\alpha s^\dagger - \alpha^* s}, \quad S_s(\zeta) = e^{(\zeta^* s^2 - \zeta s^{\dagger 2})/2} \quad (12)$$

For each truncated oscillator, the thermal probabilities are proportional to q^n , where $q = \bar{n}/(1+\bar{n})$, and are normalized within the retained basis. Displacement is applied after squeezing in the order shown above. The optical preparation uses $(\alpha, r, \theta, \bar{n}_0) = (0.8, 0.15, 0, 0.05)$, while the mechanical mode uses (0.2, 0.10, 0, 0.20); both displacements are real. These parameters define the initial state used throughout this work. They are independent of the bath occupations, and the energetic cost of preparing these nonequilibrium states is not included in the subsequent dynamical energy balance.

TABLE I. Fixed parameters and parameter sets used in the four finite-time comparisons.

Quantity	Value
ω_f, ω_m, p	1.00, 0.35, 1.00
$g_{\text{OM}}; T_f, T_m$	0.08; 0.35, 0.45
Case 1: $(\chi, g_{\text{AF}}, T_h, T_c)$	(0.03, 0.18, 2.50, 0.35)
Case 2: $(\chi, g_{\text{AF}}, T_h, T_c)$	(0, 0.18, 2.50, 0.35)
Case 3: $(\chi, g_{\text{AF}}, T_h, T_c)$	(0.03, 0.30, 2.50, 0.35)
Case 4: $(\chi, g_{\text{AF}}, T_h, T_c)$	(0.03, 0.18, 4.00, 0.20)

IV. ENERGY, ENTROPY AND QUANTUM DIAGNOSTICS

$$U = \text{Tr}(\rho H), \quad J_r = \text{Tr}[H \mathcal{L}_r(\rho)], \quad P_{\text{drive}} = \text{Tr}(\rho \partial_t H) \quad (13)$$

$$Q_r(t) = \int_0^t J_r(s) ds, \quad W_{\text{out}}(t) = - \int_0^t P_{\text{drive}}(s) ds \quad (14)$$

$$R_{\text{FL}} = U(t) - U(0) - \sum_r Q_r(t) + W_{\text{out}}(t) \quad (15)$$

Our sign convention is that positive reservoir current carries energy into the modeled system, whereas positive W_{out} denotes work delivered by the system to the external modulation. The integrated reservoir exchanges and work are propagated with the density matrix at the same Runge–Kutta stages, avoiding reconstruction from the more widely spaced plotting points. We do not convert the hot-channel heat into an efficiency because the mechanical reservoir can also supply energy, the initial state already contains nonequilibrium resources, and periodic recurrence of the state has not been established. A positive W_{out} therefore denotes only transient signed work delivered to the modulation, not cyclic engine performance.

$$S = -\text{Tr}(\rho \log_2 \rho), \quad \sigma_H = \ln(2) \dot{S} - \sum_r J_r / T_r \quad (16)$$

The quantity σ_H is used as a full-Hamiltonian entropy-balance diagnostic. A second balance is obtained from the bare Hamiltonian $H_{\text{bare}} = H_A + \omega_f n_f + \omega_m n_m$ and the associated currents $J_r^{\text{bare}} = \text{Tr}[H_{\text{bare}} \mathcal{L}_r(\rho)]$. For the local channels defined above, the corresponding thermal reference states lead to the Spohn inequality $\sigma_{\text{bare}} \geq 0$ for full-rank states [71]. The difference between the two expressions is entirely due to the interaction-energy terms. This distinction is useful mathematically, but it does not account for any physical cost required to engineer the local reservoirs.

$$\sigma_{\text{bare}} = \ln(2) \dot{S} - \sum_r J_r^{\text{bare}} / T_r \geq 0 \quad (17)$$

$$\sigma_H - \sigma_{\text{bare}} = - \sum_r \text{Tr}[(H - H_{\text{bare}}) \mathcal{L}_r(\rho)] / T_r \quad (18)$$

Entropy derivatives are calculated directly from the master-equation generator and the spectral logarithm, retaining eigenvalues larger than 10^{-14} . Because the initial density operator is rank deficient, the entropy derivative can be singular at $t = 0$, so that point is omitted

from the rate plots. Small negative eigenvalues caused by finite numerical resolution are monitored rather than projected away. Because the fourth-order Runge–Kutta propagator is not positivity preserving, entropy-rate values and other spectral quantities close to this numerical scale are interpreted cautiously.

$$\mathcal{N} = \frac{\|\rho^{TA}\|_1 - 1}{2}, \quad \mathcal{W}_m = \text{Tr}(\rho_m H_m) - \sum_k r_k \varepsilon_k \quad (19)$$

Negativity is used to quantify non-positive-partial-transpose entanglement across the atom–rest partition [63]; a zero value should not be interpreted as a general proof of separability. Mechanical ergotropy is evaluated by arranging the eigenvalues r_k of the reduced mechanical state in decreasing order and the bare mechanical energies ε_k in increasing order [55]. It therefore measures the work that can be extracted from the reduced mechanical state by ideal local unitary operations relative to $H_m = \omega_m n_m$. It does not include changes of interaction energy that could accompany an actual decoupling or extraction protocol. We report both the absolute ergotropy and its change from the initial value.

$$F_\phi = 2 \sum_{\lambda_i + \lambda_j > \epsilon} \frac{|\langle i | \partial_\phi \rho_A | j \rangle|^2}{\lambda_i + \lambda_j}, \quad \epsilon = 10^{-12} \quad (20)$$

The unknown parameter is the relative phase prepared in the initial atomic superposition. Instead of estimating the derivative by subtracting two nearby simulations, we evolve $X = \partial_\phi \rho$ under the same linear generator as ρ , starting from the analytic derivative of the initial atomic state, and then trace out the two oscillators. This procedure gives an initial QFI of exactly one. For the exact completely positive dynamics, the subsequent evolution and partial trace are independent of ϕ , so monotonicity of the QFI under parameter-independent quantum channels implies $0 \leq F_\phi \leq 1$ [64]. In the finite-step calculation, the QFI is therefore interpreted together with the density-matrix positivity diagnostics. Non-monotonic revivals of the reduced quantity can occur because phase information can move between the atom and the traced-out degrees of freedom. The denominator threshold used in the spectral expression is stated explicitly above.

V. NUMERICAL PROCEDURE AND VALIDATION

The four main simulations span the interval $0 \leq t \leq 40$ and are propagated with the classical fourth-order Runge–Kutta method using a time step of 0.05. Observables are stored every 0.5 time units. The optical and mechanical Hilbert spaces contain 9 and 10 Fock states, respectively, so that the full atom–field–mechanical density operator has dimension 450. The density matrix and its phase derivative are evolved without renormalizing the trace and without applying a positivity projection. Hermiticity, trace preservation, the smallest density-matrix eigenvalue, and the first-law residual are monitored throughout the calculation. Additional checks include a halved time step for the strongest-coupling case and an independent integrating-factor implementation for Cases 1 and 2.

The truncated preparation was checked separately against higher-dimensional oscillator calculations. For the smaller truncations (5, 4), the adopted optical and mechanical preparations give trace-distance errors of 0.0155 and 0.00993, respectively, relative to the higher-dimensional references. Increasing the retained bases reduces these errors. This initial-state test alone is not sufficient, because the mechanical thermal channel can populate higher Fock levels during the evolution. For that reason, evolved-state cutoff comparisons were also performed for selected cases.

The dissipative right-hand side was compared with an independently assembled dense implementation at a smaller test dimension, giving agreement at the level of 7.9×10^{-17} . The instantaneous energy identity agrees to 4.9×10^{-17} in the same test. The initial mechanical ergotropy is 0.0189163551 in a 48-level calculation and agrees with the corresponding Gaussian-state expression to better than 7×10^{-18} . These tests confirm the internal consistency of the numerical implementation; they do not, by themselves, establish the microscopic validity of the chosen local dissipators.

The completed cutoff tests compare oscillator dimensions (9, 10) and (11, 12) for Cases 1 and 3, while the same-basis solver comparisons use the direct and integrating-factor implementations for Cases 1 and 2. The strongest-coupling case was also repeated with half the main time step. For each comparison, the maximum absolute difference at common output times is divided by $\max(1, \max |x_{\text{refined}}|)$ and evaluated separately for the energy, entropies, negativity, ergotropy, QFI, work, reservoir currents and heats, and the two entropy-balance rates. Table II lists the largest of these scaled differences. The values should be viewed as

TABLE II. Observed numerical-sensitivity indicators over the sampled trajectories. These values are not rigorous error bounds.

Case / test	Largest scaled change
1 / joint cutoff	1.30×10^{-4}
1 / independent integrator	1.22×10^{-4}
2 / independent integrator	1.20×10^{-4}
3 / joint cutoff	1.72×10^{-4}
3 / direct step halving	1.25×10^{-4}

TABLE III. End-of-window energy and quantum-resource diagnostics. The reported $\mathcal{N}_{\max}$ is the largest value sampled on the output grid.

Case	$U(40)$	$W_{\text{out}}(40)$	$\mathcal{N}_{\max}$	$F_{\phi}(40)$	$\mathcal{W}_m(40)$
1	2.2393	-0.00496	0.0844	0.0746	0.0057
2	2.2191	0.00472	0.0844	0.0732	0.0056
3	2.2199	-0.01479	0.1897	0.0729	0.0064
4	2.2432	-0.00597	0.0725	0.0183	0.0059

practical discretization-sensitivity indicators rather than rigorous error bounds.

Some larger-basis and additional independent-solver runs were not completed, so a uniform convergence certification for every case is not claimed. Across the four main simulations, the largest sampled trace error is 6.66×10^{-16} , the largest Hermiticity error is 4.07×10^{-16} , the smallest sampled density eigenvalue is -1.2×10^{-6} , and the largest absolute first-law residual is 6.45×10^{-10} . The trace, Hermiticity, and first-law diagnostics are small, but the minimum eigenvalue reveals a weak positivity violation at the 10^{-6} level, consistent with the use of a non-positivity-preserving finite-step integrator. The completed checks support the reported behavior for the tested cases, but they do not constitute a uniform convergence certification for all four parameter sets. Features whose magnitude is comparable to the numerical-resolution scale are therefore not assigned physical significance.

VI. FINITE-TIME RESULTS

The values in Table III are rounded to a precision appropriate for the numerical refinement tests described above. The listed maximum negativity is the largest value sampled on the output time grid and is not a separately optimized continuous-time maximum. All four rows in Table III correspond to the same explicitly specified initial preparation, so the differences can be attributed to the parameter changes within the adopted model.

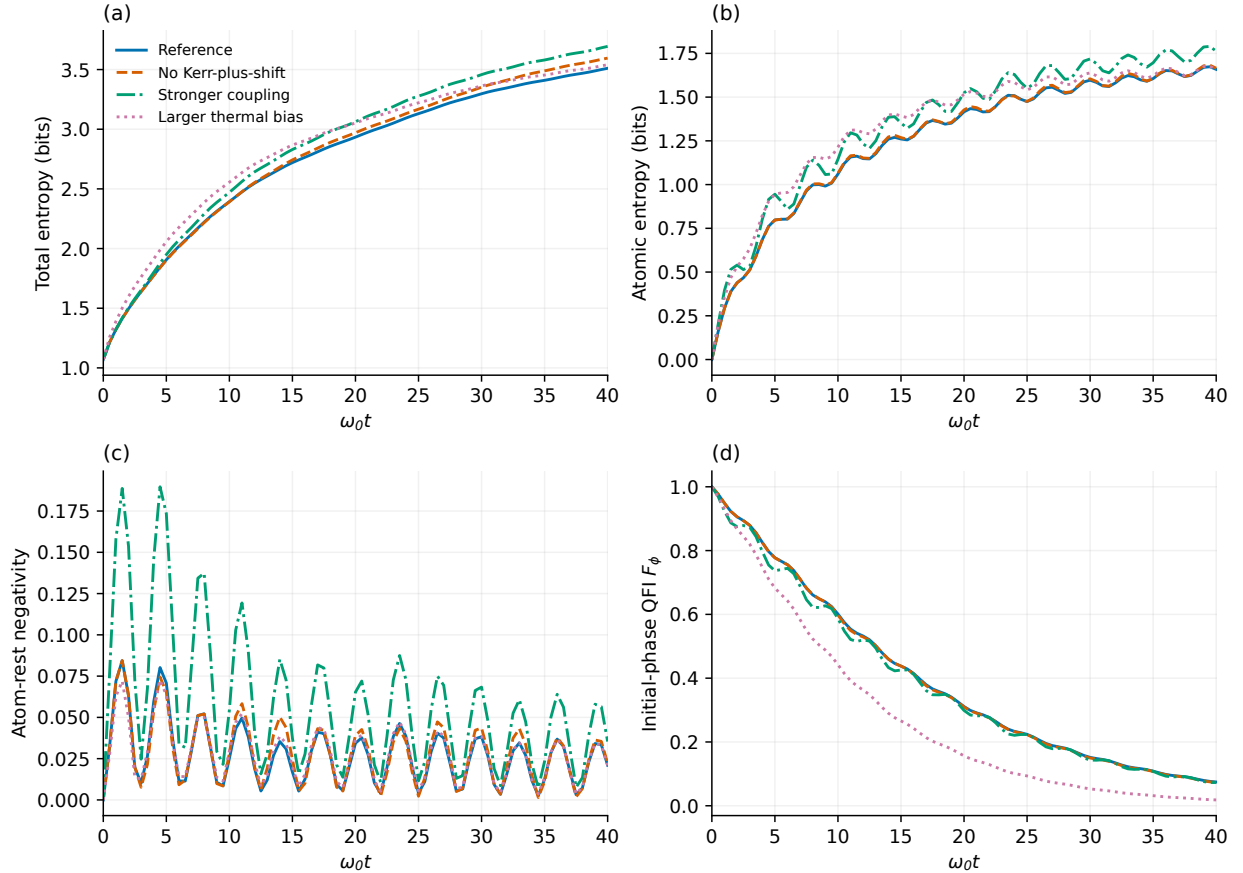


FIG. 1. Quantum diagnostics on common axes: total entropy, reduced atomic entropy, atom–rest negativity, and reduced atomic phase QFI. The initial product state has zero negativity and unit phase QFI. Atomic entropy is not an entanglement measure when the composite state is mixed.

Figure 1 shows that increasing the atom–field coupling has the clearest effect on the atom–rest entanglement. Case 3 reaches a sampled negativity of 0.1897, more than twice the reference value of 0.0844. The phase QFI behaves differently: the final values are 0.0729 for Case 3 and 0.0746 for the reference case. Thus a stronger build-up of atom–rest correlations

does not automatically preserve more information about the initially encoded atomic phase. The temperature-biased Case 4 suppresses the final QFI further, to 0.0183, indicating that the local metrological resource is especially sensitive to the altered dissipative balance.

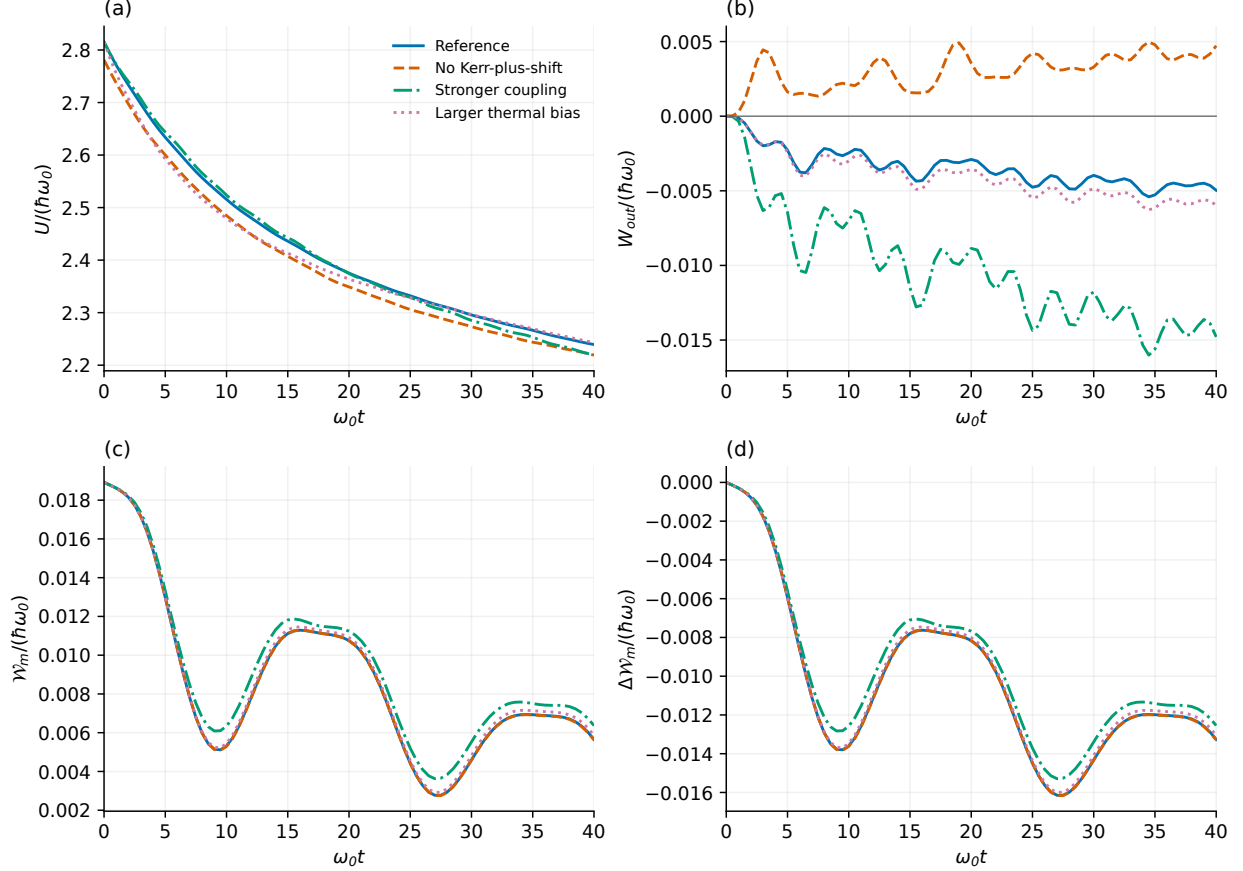


FIG. 2. Energy, signed work, absolute local mechanical ergotropy, and its change from the initial state. Signed work is shown on a separate axis so that small positive or negative intervals are not obscured by larger energy transfers. The ergotropy change removes the initial resource baseline.

The mechanical mode initially carries an ergotropy of approximately 0.018916 because the displaced squeezed thermal preparation is already out of equilibrium. Plotting both the absolute ergotropy and its change from the initial value makes it possible to distinguish genuine transient charging from simple depletion of the prepared resource. This local work capacity is not the same as the work exchanged with the external modulation. Connecting the two would require a concrete extraction protocol and an accounting of any coupling or decoupling cost.

The sign of the accumulated work also depends on the parameter set. A positive value

means that, over the chosen time window, more energy has been transferred from the system to the modulation than in the opposite direction. Because the initial state is a prepared nonequilibrium state and no cyclic recurrence condition is imposed, this finite-time sign should not be interpreted as a heat-engine efficiency statement.

Within the employed finite truncation, Case 2 is the only case with positive final signed work, reaching 0.004722 at $t = 40$ after a sampled maximum of 0.004971. This shows that the calculated signed work need not remain negative over the chosen observation window. However, because the $\chi = 0$ Hamiltonian is formally unbounded from below in the untruncated radiation-pressure model and Case 2 was not subjected to the enlarged-cutoff test used for Cases 1 and 3, this small positive value should be regarded as conditional on the adopted truncation. Moreover, setting $\chi = 0$ removes both the nonlinear Kerr contribution and its associated linear shift, so the change cannot be attributed uniquely to photon–photon nonlinearity.

TABLE IV. Integrated full-Hamiltonian reservoir currents at $t = 40$. Positive values denote net energy transfer into the modeled system.

Case	Q_h	Q_c	Q_f	Q_m
1	-0.309035	0.010787	-0.350054	0.067401
2	-0.306378	0.010412	-0.328691	0.067484
3	-0.296697	-0.002249	-0.377525	0.066395
4	-0.285160	-0.004585	-0.355507	0.067237

The reservoir currents reveal an important consequence of the chosen initial atomic state. At $t = 0$ the upper state E_1 is populated while E_2 is not, so the channel labeled hot initially drives downward relaxation and its current is negative. The corresponding bare-Hamiltonian values are -0.0382705 for $T_h = 2.5$ and -0.0568493 for $T_h = 4$. For the initial product state, the interaction contribution to this current vanishes at the initial instant. Increasing the nominal hot temperature therefore strengthens the early relaxation process rather than guaranteeing immediate heat absorption from that channel.

Figure 4 separates the two entropy-balance conventions and the modulation power. The non-negative bare-Hamiltonian quantity is the one directly connected with the stationary reference states of the stipulated local channels. The full-Hamiltonian quantity includes

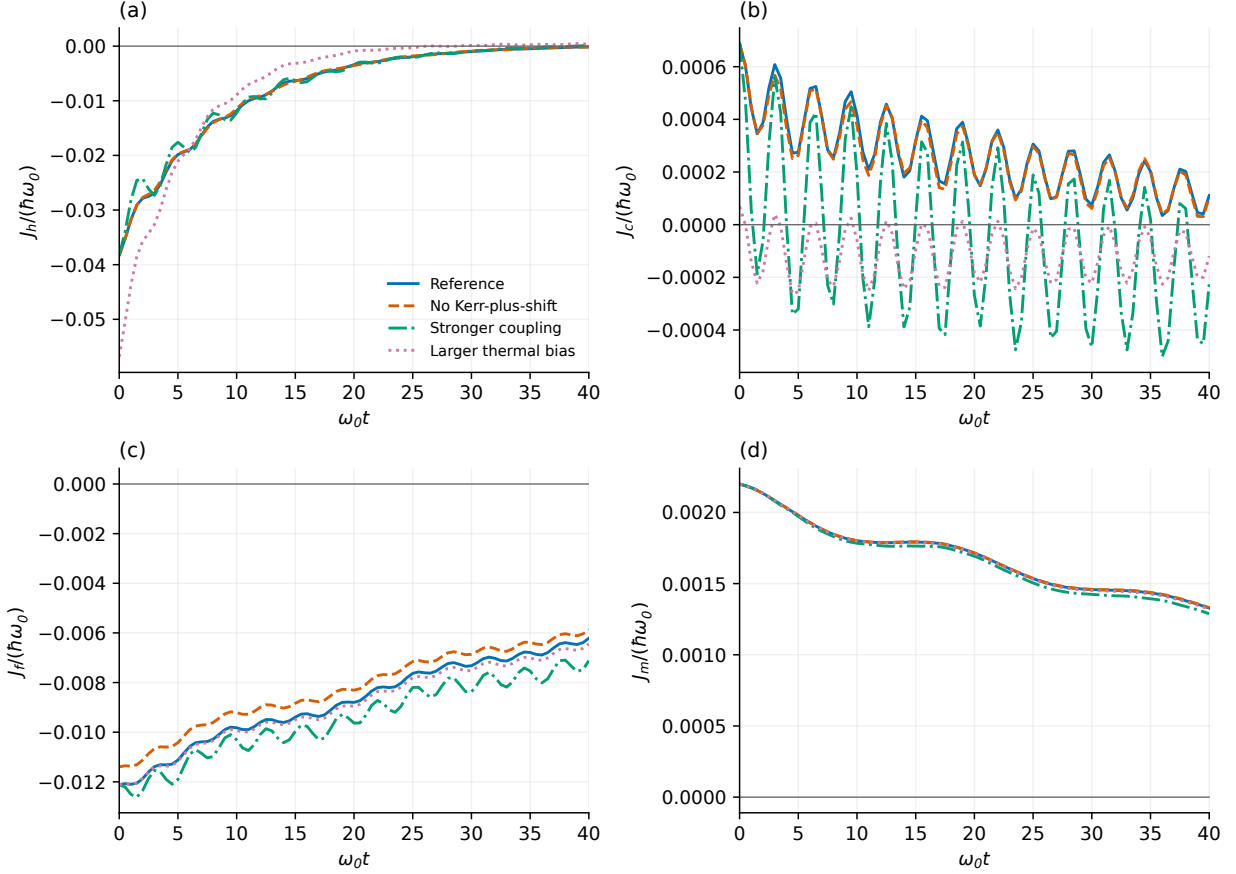


FIG. 3. Full-Hamiltonian reservoir-associated currents. Positive values correspond to energy entering the modeled system. The panels use common case styles but represent different reservoirs; the channel labeled “hot” need not heat the prepared state at every time.

interaction-energy contributions and is therefore used only as a diagnostic. The very small first-law residual confirms that the numerical bookkeeping of energy is internally consistent. Since the evolution is not shown to be cyclic and more than one reservoir can supply energy, no Carnot-type efficiency or positive-only conversion ratio is assigned to these transient trajectories.

VII. LIMITS AND CONCLUSION

We have examined a driven five-level atom coupled simultaneously to a Kerr nonlinear optical mode and a mechanical oscillator, with dissipation represented by four local Lindblad channels. The purpose of the analysis was not to force the dynamics into a heat-engine in-

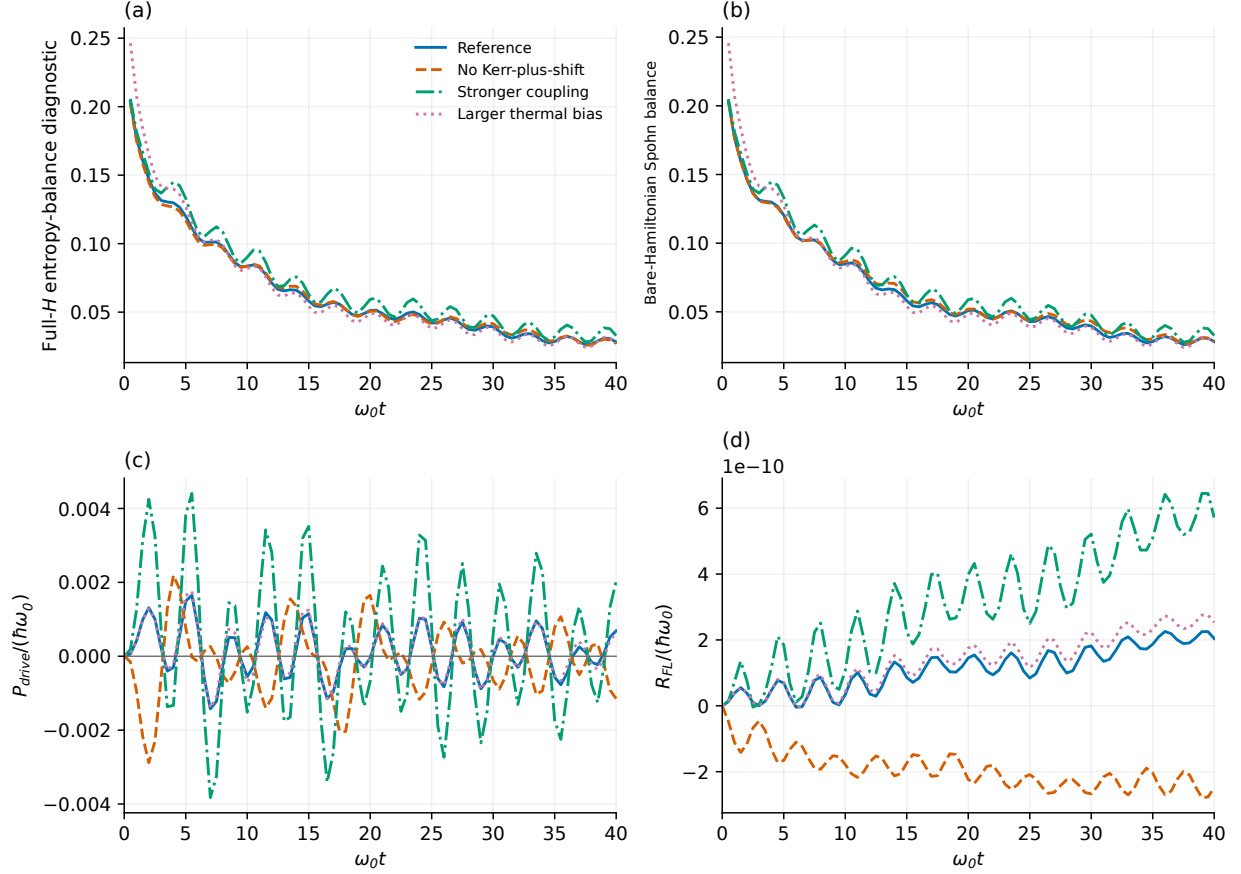


FIG. 4. Full-Hamiltonian entropy-balance diagnostic, bare-Hamiltonian Spohn balance, modulation power, and numerical first-law residual. The two entropy balances use different energy operators. The singular initial rate is omitted. The residual tests numerical energy bookkeeping for the adopted equation; it does not establish the microscopic validity of the local reservoir model.

terpretation, but to understand how several physically different quantities evolve during the same finite time interval. By specifying the initial displaced squeezed thermal states explicitly and keeping the full-Hamiltonian energy currents separate from the bare-Hamiltonian entropy balance, the calculation provides a consistent basis for comparing work, ergotropy, entanglement, entropy, and phase sensitivity.

The numerical results make clear that these quantities do not respond in the same way to parameter changes. Increasing the atom–field coupling from the reference value to the stronger Case 3 produces the largest sampled atom–rest negativity, rising from 0.0844 to 0.1897. The final atomic phase QFI, however, remains close to the reference value and does not show a comparable improvement. This indicates that a stronger build-up of global corre-

lations does not necessarily protect the particular phase information that remains accessible in the reduced atomic state. Case 4, which has the larger atomic temperature bias, has an even smaller final QFI, showing that the encoded phase is especially sensitive to the altered dissipative environment.

The energetic observables tell a similarly nuanced story. The mechanical mode starts with nonzero ergotropy because the initial state is displaced and squeezed, so part of its work capacity is already present at $t = 0$. Changes in that ergotropy should therefore be discussed relative to the initial baseline. The externally exchanged work is a separate quantity determined by the time-dependent modulation. Among the four cases, the zero- χ comparison produces a small positive final signed work within the employed finite truncation, whereas the other three cases end with negative signed work. Because removing χ changes both the normally ordered Kerr interaction and a linear photon-number shift, this behavior cannot be assigned to a pure Kerr effect alone. In addition, the absence of a dedicated enlarged-cutoff test for Case 2 means that the small positive value should not be treated as a converged high-occupation prediction.

The reservoir currents also illustrate why labels such as “hot” and “cold” should not be interpreted only from their temperatures. The initially populated upper atomic level causes the nominal hot channel to carry energy out of the system at early times. In addition, the mechanical reservoir can inject energy over part of the evolution. These features, together with the nonequilibrium initial preparation, prevent a simple identification of one absorbed heat input from which an engine efficiency could be calculated. The simulated interval covers about 6.37 modulation periods, but the density operator is not shown to return to its initial state at a phase-matched time. The present results are therefore best described as transient open-system energy exchange rather than as a completed thermodynamic cycle.

The model has several limitations that should remain explicit. The local dissipators are phenomenological and are not derived from a microscopic global or Floquet treatment of the interacting Hamiltonian. The five-level ladder uses equal spacings and equal coherent transition strengths and is not tied to a particular atomic species. The external modulation is prescribed without including the physical cost of generating it, and the preparation cost of the displaced squeezed thermal states is also outside the simulated energy balance. In addition, the formal zero-Kerr Hamiltonian requires caution at high optical occupation because of the radiation-pressure-induced quadratic contribution. Numerically, the completed

cutoff, solver, and step-size checks support the reported behavior in the cases to which they were applied. Nevertheless, the fourth-order Runge–Kutta propagation produces a weak positivity violation at the 10^{-6} level, and the validation is not uniform across all four parameter sets. A future study should therefore include positivity-preserving propagation and enlarged-cutoff tests for every case, especially the zero- χ comparison.

Within these limits, the main conclusion is that energy transport and quantum-resource dynamics in this hybrid system are strongly parameter dependent but are not interchangeable. Entanglement, local phase sensitivity, mechanical ergotropy, reservoir currents, and signed driving work each capture a different aspect of the evolution. The strongest coupling enhances detected atom–rest entanglement, the altered thermal bias strongly reduces the retained phase information, and the removal of the Kerr-plus-shift term changes the calculated transient work balance within the adopted truncation. These observations provide a clearer picture of how nonlinear optical dynamics, atomic coupling, mechanical motion, and dissipation compete in a multilevel hybrid quantum system, while leaving cyclic engine operation and experimental realization as separate questions for future study.

DATA AVAILABILITY

No data are available for this theoretical study.

CODE AVAILABILITY

No code is available.

AUTHOR CONTRIBUTIONS

S. Jamal Anwar contributed to the conception of the study, model development, analysis, interpretation of the results, and preparation of the manuscript. M. Ibrahim and S. Abdul Khaliq contributed to the scientific discussion, interpretation of the results, and review of the manuscript. All authors reviewed and approved the final manuscript.

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Not applicable. This study is theoretical and does not involve human participants, animals, or biological samples.

CONSENT TO PARTICIPATE

Not applicable.

CONSENT FOR PUBLICATION

Not applicable.

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