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Adaptive Boundary Control of a Euler-Bernoulli Overhead Bridge Subjected to Moving Vehicle Loads with Uncertain Boundary Stiffness

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ABSTRACT

Bridge structures are continuously subjected to dynamic loads induced by moving vehicles, resulting in structural vibrations that may reduce serviceability, safety and structural lifespan. These effects become more significant when uncertainties exist in the boundary support stiffness due to material degradation, environmental conditions or support flexibility. This paper presents an adaptive boundary control strategy for suppressing vibrations in a Euler-Bernoulli overhead bridge subjected to moving vehicle loads with uncertain boundary stiffness. The bridge is modelled as a distributed parameter system governed by a partial differential equation. A Galerkin model approximation is employed to transform the infinite-dimensional model into a finite-dimensional system suitable for control design. An adaptive controller is developed to estimate and compensate for the unknown boundary stiffness in real time, while Lyapunov stability theory is used to establish the stability of the closed-loop system. Numerical simulations demonstrate that the proposed control strategy effectively reduces bridge displacement and the velocity, achieves accurate estimation of the unknown stiffness parameter, minimizes estimation error, and guarantees the convergence of the Lyapunov energy function. The reconstructed beam deflection profiles further confirm the effectiveness of the proposed controller in mitigating structural vibrations under moving vehicle excitation. The proposed framework provides a reliable approach for enhancing bridge stability, durability, and structural integrity under uncertain boundary conditions.

KEYWORDS: Adaptive Boundary Control, Euler-Bernoulli, Overhead Bridge, Stiffness, Vehicle Load, Boundary Conditions, Structural Dynamics, Damping.

I. INTRODUCTION

Bridges are among the most important components of transportation infrastructure, providing safe and efficient movement of goods and people. During operation, bridge structures are continuously subjected to dynamic loads generated by moving vehicles. These loads induce vibrations that may adversely affect structural performance, passenger comfort, and long-term durability. Excessive vibration may accelerate fatigue damage, increase maintenance cost and in severe case, compromise structural safety. Consequently, the development of effective vibration control strategies has become an important area of research in structural dynamics and control engineering.

The Euler-Bernoulli beam theory has been widely adopted for modelling the transverse vibration of the bridge structures because it provides an accurate representation of the slender beam behavior while maintaining mathematical simplicity. In practical applications, however, bridge support rarely remains perfectly rigid throughout the service life of the structure. Factors such as the material deterioration, aging, environmental effects, foundation settlement, bearing deformation, and construction imperfections may introduce uncertainties into the boundary stiffness. These uncertainties significantly influence the dynamic response of the bridge and may reduce effectiveness of the conventional fixed parameter controllers.

Adaptive control has emerged as an effective technique for dealing with uncertain structural parameters because it continuously updates controller parameters in response to measured system behavior. Unlike conventional feedback controllers, adaptive control is capable of compensating for unknown or time-varying uncertainties while maintaining closed-loop stability. When combined with boundary control, adaptive strategies provide an efficient means of suppressing

vibrations without requiring distributed actuators along the beam.

Although several studies have investigated vibration control of Euler-Bernoulli beams under moving loads, relatively few have considered adaptive boundary control in the presence of uncertain boundary stiffness. This limitation motivates the present study. Pourzeynali *et al.*, [1] investigated the Comprehensive Study of Moving Load Identification on Bridge Structures using the explicit form of Newmark- β method, they used numerical method which includes a single simply supported bridge and a three-span continuous bridge, and the experimental study include a single-span simply supported by bridge installed by sensors. They also investigated the effect of factors such as the number of sensors, location of sensors roughness, measurement, noise sampling frequency and vehicle speed.

Cui *et al.*, [2] proposed an event-triggered adaptive control method for a rotating disk-beam system with time-varying boundary constraints. Their controller effectively handled parametric uncertainties while ensuring closed-loop stability through barrier Lyapunov techniques. Nevertheless, the study focused on rotating beam systems and did not investigate bridge structures subjected to moving vehicle loads.

You *et al.*, [3] studied the distributed bending stiffness estimation of bridges using adaptive inverse unit load method. They suggested that in bending stiffness to accurately predict two phase methodology named the adaptive inverse load method (AIULM) is developed for distributed the construction of bending stiffness from onboard deflection data. On the first phase, the bending stiffness was roughly estimated uniform discretization. A global stiffness smoothen method is applied to eliminate non-physical discontinuities of stiffness at inter-element boundaries, then the smooth result is applied to a second phase to define a node optimization function to adaptively adjust the node distribution. Sensors

Amadi and Nwaoburu [4] studied the numerical simulation of steady-state analysis to predict population size overtime using linear differential equation model. Dong *et al.*, [5] studied the vehicle bridge interaction element implementation in ABAQUS and verification. They introduced the ABAQUS through a user defined element (UEL). Implements a VBI element (vehicle-bridge coupling) in ABAQUS; highlights numerical strategies for coupled VBI simulation - helpful for simulating realistic PDE and vehicle. This theoretical approach has been widely applied in the analysis of bridge vibrations and problems involving moving loads Shi

et al., [6] carried out research on vehicle-bridge interaction model considering contact patch size. Improves VBI modelling by including contact-patch effects and motor-induced excitations they showed how refined contact modelling affects bridge response and controller design.

Krystic [7] studied the adaptive boundary control for unstable parabolic PDEs. he developed adaptive controllers for parabolic partial differential equations PDEs controlled from a boundary and containing unknown destabilizing parameters affecting the interior domain and they stand to be the first adaptive controllers for unstable PDEs without relative degree limitations, open-loop stability assumptions, or domain wide actuation. they also initiated an effort towards developing adaptive controllers for physical system such as fluid, thermal and chemical dynamics, where actuation can be only applied non-intrusively, the dynamics where unstable and the parameters. Wei *et al.*, [8] investigated the adaptive boundary control for a class of inhomogeneous Timoshenko beam equations with constraints (IBLF). They proposed Integral barrier Lyapunov function-based boundary control for inhomogeneous Timoshenko beam equation with boundary output constraints. An IBLF which depends explicitly on time, is employed to prevent the constraint violation. The adaption laws are designed to compensate for the system uncertainties. By using the proposed adaptive IBLF-based boundary control, the vibration of the Timoshenko beam system is suppressed greatly, without any discretization or simplification of the dynamics in the time and space. The robust adaptive boundary control of a flexible marine riser with vessel. Xia *et al.*, [9] investigated how vehicle motion dynamically interacts with the bridge vibrations and they emphasized the need to couple vehicle equation of motion ODEs with bridge PDE, dynamics putting the moving load into adaptive boundary control frame work, it enables real time suppression of vibration induced by traffic making sure that the structure or the bridge is safe for use. Ouyang [10] developed beams theories for the basis for PDE models used in modern vibration control design on moving loads. Yang *et al.*, [11] studied the interaction between moving vehicles and bridges using ordinary differential equations (ODEs) and partial differential equation (PDEs) for the bridge and he concluded that the accuracy of such models depends on the vehicle with the structural vibration of the bridges Abdelkarim and Saedpanah [12] published the stabilization of an axially moving Euler Bernoulli beam by an adaptive boundary control. They construct the control through a low-gain adaptive velocity feedback and prove that the designed control is able to stabilize exponentially the closed loop system Li *et al.*,

[13] Adaptive event-triggered output-feedback boundary control for a hyperbolic PDE-ODE coupled system.

Butcher [14] emphasized the use of Runge–Kutta methods for solving second-order structural ODEs efficiently. Corral *et al.*, [15] investigated adaptive structures in civil engineering and showed that structural adaptability can significantly reduce deformation and improve load-bearing efficiency under varying conditions.

Wang *et al.*, [16] proposed an adaptive boundary control strategy for an Euler–Bernoulli beam carrying an unknown payload. The authors designed an adaptive boundary controller together with an adaptive law to compensate for parameter uncertainty and established the asymptotic stability of the closed-loop system. Their simulation results demonstrated effective vibration suppression and accurate parameter estimation. However, the study considered uncertainty arising from payload variation rather than uncertain boundary stiffness under moving vehicle loading, which is the focus of the present

Ireksowardojo *et al.*, [17] proposed an adaptive tensioning system for high-speed railway bridges, where actuators are used to counteract external loads in real time. Their results showed that adaptive control strategies can significantly reduce displacement and improve structural performance, even under resonance conditions. Similarly, Gambarelli *et al.*, [18] investigated the behavior of adaptive concrete beams with integrated actuators using nonlinear finite element methods. Their findings demonstrated that adaptive mechanisms can effectively compensate for deformation and reduce structural stress, highlighting the importance of combining modeling and control strategies in modern structural design.

Zhou *et al.*, [19] reviewed recent developments in vehicle–bridge interaction and showed that the coupled dynamics between moving vehicles and bridge structures significantly influence bridge vibration responses. Their review highlighted the importance of considering vehicle speed, bridge flexibility, and structural properties in dynamic analysis. However, adaptive boundary control under uncertain boundary stiffness was not addressed

Sun *et al.*, [20] developed an event-triggered adaptive safe boundary control strategy for uncertain Euler–Bernoulli beam systems. Their approach simultaneously addressed unknown damping coefficients, bending stiffness, and boundary disturbances while guaranteeing system stability using Lyapunov analysis. Although their controller achieved

excellent vibration suppression, it did not consider the dynamic interaction between a moving vehicle and an overhead bridge, which is investigated in the present work.

The present study extends the work of Abdelkarim and saedpanah [12] for Euler–Bernoulli overhead bridge subjected to a moving vehicle load is modeled as a distributed parameter system governed by a partial differential equation. The bridge is assumed to possess uncertain boundary stiffness at one support, representing realistic variations in structural conditions. A Galerkin modal approximation is employed to obtain a reduced-order model suitable for controller design. An adaptive boundary control law is then developed to estimate and compensate for the unknown stiffness parameter in real time. The stability of the resulting closed-loop system is established using Lyapunov stability theory, while numerical simulations are performed to evaluate the effectiveness of the proposed controller.

Although recent studies have achieved significant progress in adaptive boundary control, event-triggered control, and vibration suppression of Euler–Bernoulli beam systems, limited attention has been given to adaptive boundary control of overhead bridge structures subjected to moving vehicle loads with uncertain boundary stiffness. Most existing studies focus on beam vibration, actuator faults, or unknown payloads without incorporating online estimation of uncertain boundary stiffness in bridge applications. Therefore, this study develops an adaptive boundary controller for an overhead bridge under moving vehicle loading, with stability established through Lyapunov analysis and validated using numerical simulations.

II. METHODOLOGY

Let consider an overhead bridge subjected to moving vehicle loads is modelled as an Euler–Bernoulli beam with uncertain boundary stiffness. the bridge is assumed to be slender, homogeneous and linearly elastic, while the vibrations is considered to be transverse with small deflections. The moving vehicle is represented as concentrated load travelling along the beam at a constant speed.

The transverse vibration of the bridge is governed by the Euler–Bernoulli beam equation.

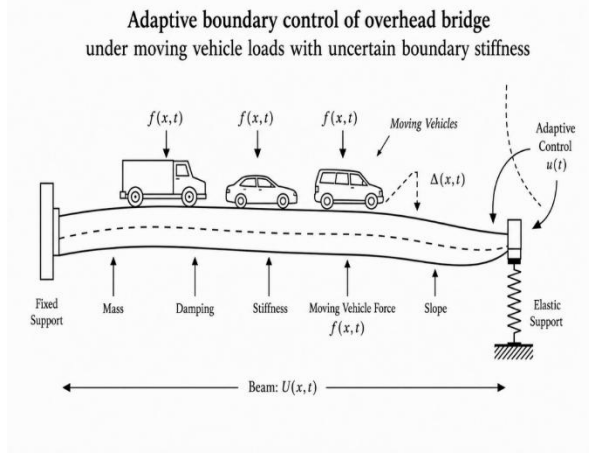


Figure 1: Schematic representation of the Euler-Bernoulli overhead bridge subjected to moving load with uncertain binary stiffness.

$$\mu \frac{\partial^2 u(x, t)}{\partial t^2} + \gamma \frac{\partial u(x, t)}{\partial t} + E \frac{\partial^4 u(x, t)}{\partial x^4} = f(x, t) \quad (1)$$

where $u(x, t)$ denotes the transverse displacement of the bridge, μ is the mass per unit length, γ is the damping coefficient, E represent flexural rigidity of the beam and $f(x, t)$ is the external moving vehicle load.

The moving vehicle load is modeled using the Dirac delta function as

$$f(x, t) = \sum_{v=1}^V P_v(t) \delta(x - x_v(t)) \quad (2)$$

Where $P_v(t)$ is the magnitude of the v -th point forces moving along the beam

$(x_v)(t)$ is the position of the v -th point force, x represents the spatial coordinates along the beam length, $\delta(\cdot)$ represent the Dirac delta function which models concentrated forces acting at instantaneous vehicle position. The bridge is assumed to be clamped at the left boundary, leading to the boundary conditions.

$$u(0, t) = 0 \quad (3)$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad (4)$$

These conditions indicate that the left end of the bridge experience neither displacement nor rotation.

At the right boundary, an uncertain elastic support together with adaptive boundary control is considered. The corresponding boundary conditions are;

$$E \frac{\partial^2 u}{\partial x^2}(L, t) = 0 \quad (5)$$

$$E \frac{\partial^3 u}{\partial x^3}(L, t) + \delta u(L, t) = u(t) \quad (6)$$

where δ denotes the unknown boundary stiffness parameter and $u(t)$ represent the adaptive boundary control inputs applied at the beam support. These boundary conditions characterize a beam whose stiffness is uncertain but actively compensated through boundary control.

The bridge is assumed to be initially at rest, giving the initial conditions

$$u(x, 0) = 0 \quad (7)$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \quad (8)$$

To facilitate the controller design, the beam displacement is approximated using the Galerkin modal expansion.

$$N(x, t) = u(x, t) \approx \sum_{n=1}^N q_n(t) \phi_n(x) \quad (9)$$

where $q_n(t)$ denotes the modal coordinate and $\phi_n(x)$ represents the corresponding assume mode shapes. Substituting the modal approximation into the governing equation and applying the Galerkin weighted residual procedure yield a finite-dimensional reduced-order model suitable for adaptive controller design and stability.

Adaptive Boundary Control Design

To facilitate the design of the adaptive boundary controller, the infinite- dimensional beam model is first reduced to a finite-dimensional system using Galerkin modal approximation. By substituting the assumed modal expansion into the governing partial differential equation and applying the Galerkin weighted residual method, the beam dynamics are represented by a reduced-order ordinary differential equation describing the dominant vibrations mode. This reduced model preserves the essential dynamic characteristic of the bridge while simplify the controller design.

The reduced- order dynamic equation is expressed as;

$$\mu_n \ddot{q}_n(t) + \gamma_n \dot{q}_n(t) + \delta_n q_n(t) = f(x, t) + u(t) \quad (10)$$

Adding a boundary control input to the governing equation to suppress the bridge vibration

$$u(t) = -k_1 q_n(t) - k_2 \dot{q}_n(t) \quad (11)$$

k_1 is the control gain (position), k_2 is the control gain (velocity), $q_n(t)$ represent the modal displacement, $f(x, t)$ represents the disturbance associated with the moving vehicle loads and the boundary uncertainty. The first feedback term reduces excessive bridge deflection, whereas the second term provides damping to attenuate vibrating.

Dividing equation (10) by u_n

$$\frac{\mu_n \ddot{q}_n}{u_n} + \frac{\gamma_n \dot{q}_n}{u_n} + \frac{\delta_n q_n}{u_n} = \frac{f(x, t)}{u_n} + \frac{u(t)}{u_n}$$

we let;

$$\bar{\gamma}_n = \frac{\gamma_n}{\mu_n}, \quad \bar{\delta}_n = \frac{\delta_n}{\mu_n}, \quad \bar{f}(x, t) = \frac{1}{\mu_n} f(x, t), \quad \bar{u}(t) = \frac{1}{\mu_n} u(t)$$

Then the equation becomes

$$\ddot{q}_n + \bar{\gamma}_n \dot{q}_n + \bar{\delta}_n q_n = \bar{f}(x, t) + \bar{u}(t) \quad (12)$$

Then estimation error parameter becomes

$$\bar{\delta}_n = \hat{\delta}_n + \tilde{\delta}_n \quad (13)$$

where $\hat{\delta}_n$ denotes the online estimate of the unknown boundary stiffness parameter. The adaptive term continuously compensates for the uncertainty in the boundary. “According to Lyapunov stability theory, if a positive definite function has a negative semi-definite derivative, then the system is stable”.

Substituting $\bar{u}(t)$ equation (11) into equation (12)

$$\ddot{q}_n + \bar{\gamma}_n \dot{q}_n + \bar{\delta}_n q_n = \bar{f}(x, t) - k_1 q_n - k_2 \dot{q}_n \quad (14)$$

$$\ddot{q}_n + (\bar{\gamma}_n \dot{q}_n + k_2 \dot{q}_n) + (\bar{\delta}_n q_n + k_1 q_n) = \bar{f}(x, t)$$

$$\ddot{q}_n + (\bar{\gamma}_n + k_2) \dot{q}_n + (\bar{\delta}_n + k_1) q_n = \bar{f}(x, t)$$

$$\ddot{q}_n = -(\bar{\gamma}_n + k_2) \dot{q}_n - (\bar{\delta}_n + k_1) q_n + \bar{f}(x, t) \quad (15)$$

Substituting equation (13) the uncertainty into equation (15)

$$\ddot{q}_n = -(\bar{\gamma}_n + k_2) \dot{q}_n - (\hat{\delta}_n + \tilde{\delta}_n + k_1) q_n - \bar{f}(x, t)$$

$$\ddot{q}_n = -(\bar{\gamma}_n + k_2) \dot{q}_n - \hat{\delta}_n q_n - \tilde{\delta}_n q_n - k_1 q_n - \bar{f}(x, t)$$

$$\ddot{q}_n = -(\bar{\gamma}_n + k_2) \dot{q}_n - (\hat{\delta}_n + k_1) q_n - \tilde{\delta}_n q_n - \bar{f}(x, t) \quad (16)$$

Substituting the state-space variable and the functions

$$\bar{\gamma}_n = \frac{\gamma_n}{\mu_n}, \quad \bar{\delta}_n = \frac{\delta_n}{\mu_n}, \quad \bar{f}(x, t) = \frac{1}{\mu_n} f(x, t),$$

$$\bar{u}(t) = \frac{1}{\mu_n} u(t)$$

In equation (16) as to get the dynamic system.

Substituting $x_1 = q_n, x_2 = \dot{q}_n$

$$\dot{x}_1 = x_2 \quad (17)$$

$$\dot{x}_2 = \ddot{q}_n \quad (18)$$

Hence, equation (16) becomes;

$$\dot{x}_2 = \frac{-\gamma_n + k_2}{\mu_n} x_2 - \frac{\delta_n + k_1}{\mu_n} x_1 - \frac{\tilde{\delta}_n}{\mu_n} x_1 - \frac{1}{\mu_n} \bar{f}(x, t) \quad (19)$$

Lyapunov Stability Analysis

The stability of the proposed adaptive boundary control system is established using Lyapunov stability theory. A Lyapunov function is selected to represent the total energy of the controlled system together, the potential and kinetic energy together with the estimation error associated with the unknown boundary stiffness parameter. The Lyapunov candidate function represented.

$$V(t) = \frac{1}{2} \dot{q}_n^2 + \frac{1}{2} k_1 q_n^2 + \frac{1}{2\Gamma_n} \tilde{\delta}_n^2 \quad (19)$$

with a stabilizing spring k_1 where $\tilde{\delta}_n = \delta_n - \hat{\delta}_n$ is the boundary stiffness estimation error.

Be the Lyapunov function and Γ is the adaptation gain Differentiating V with respect to time t .

$$\dot{V} = \dot{q}_n \ddot{q}_n + k_1 q_n \dot{q}_n + \frac{1}{\Gamma_n} \tilde{\delta}_n \dot{\tilde{\delta}}_n \quad (20)$$

Substitute equation (20) $\ddot{q}_n$ into equation (16)

$$\dot{V} = \dot{q}_n [-(\bar{\gamma}_n + k_2) \dot{q}_n - (\hat{\delta}_n + k_1) q_n - \tilde{\delta}_n q_n - \bar{f}(x, t)] + k_1 q_n \dot{q}_n + \frac{1}{\Gamma_n} \tilde{\delta}_n \dot{\tilde{\delta}}_n \quad (21)$$

$$\dot{V} = -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - (\hat{\delta}_n + k_1) q_n \dot{q}_n - \tilde{\delta}_n q_n \dot{q}_n - \dot{q}_n \bar{f}(x, t) + k_1 q_n \dot{q}_n + \frac{1}{\Gamma_n} \tilde{\delta}_n \dot{\tilde{\delta}}_n$$

$$\dot{V} = -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - \hat{\delta}_n q_n \dot{q}_n - k_1 q_n \dot{q}_n - \tilde{\delta}_n q_n \dot{q}_n - \dot{q}_n \bar{f}(x, t) + k_1 q_n \dot{q}_n + \frac{1}{\Gamma_n} \tilde{\delta}_n \dot{\tilde{\delta}}_n$$

$$\dot{V} = -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - \hat{\delta}_n q_n \dot{q}_n - \tilde{\delta}_n q_n \dot{q}_n - \dot{q}_n \bar{f}(x, t) + \frac{1}{\Gamma_n} \tilde{\delta}_n \dot{\tilde{\delta}}_n \quad (22)$$

In order to cancel the uncertainty term from equation (22) we introduced the adaptive law since the true boundary stiffness is unknown, an adaptive parameter estimation law is incorporated into the controller. The estimation law is given by

$$\dot{\delta}_n = -\Gamma_n q_n(t) \dot{q}_n(t) \quad (23)$$

where $\Gamma_n > 0$ is the adaptation gain controlling the rate of the parameter estimation. This adaptive mechanism continuously updates the estimated stiffness using the measured modal displacement and velocity, thereby improving the robustness of the controller against uncertain boundary conditions.

The control strategy relies on the structural response of the bridge. Specifically, the modal displacement provides information on the magnitude of the bridge deflection, while the modal velocity indicates the rate at which the vibration changes with time. These response variables enable the controller to generate an appropriate boundary control action capable of suppressing oscillations and maintaining the structural stability despite variations in the boundary stiffness.

The proposed adaptive boundary control law combines feedback stabilization with online parameter estimation. Consequently, the controller not only mitigate structural vibration suppression and enhanced structural performance.

Then computing the error derivative by differentiating the error derivative

$$\dot{\delta}_n = \dot{\delta}_n - \dot{\delta}_n \quad (24)$$

Putting $\dot{\delta}_n = 0$

$$\dot{\delta}_n = -\dot{\delta}_n$$

$$\dot{\delta}_n = \Gamma_n q_n(t) \dot{q}_n(t)$$

$$\dot{\delta}_n = -(-\Gamma_n q_n \dot{q}_n)$$

$$\dot{\delta}_n = \Gamma_n q_n \dot{q}_n$$

$$\frac{1}{\Gamma_n} \dot{\delta}_n (\Gamma_n q_n \dot{q}_n) = \dot{\delta}_n q_n(t) \dot{q}_n(t) \quad (25)$$

Substitute (25) in equation (22)

$$\dot{V} = -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - \delta_n q_n \dot{q}_n - \tilde{\delta}_n q_n \dot{q}_n - \dot{q}_n \bar{f}(x, t) + \tilde{\delta}_n q_n \dot{q}_n \quad (26)$$

$$\dot{V} = -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - \delta_n q_n \dot{q}_n - \dot{q}_n \bar{f}(x, t) \quad (27)$$

For $\dot{V}(t) \leq 0$

Bounding the external force from equation (27) focusing on the disturbance $-\dot{q}_n \bar{f}(x, t)$

$$\dot{V} = -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - \delta_n q_n \dot{q}_n - \dot{q}_n \bar{f}(x, t)$$

$$|-\dot{q}_n \bar{f}(x, t)(t)| = |\dot{q}_n \bar{f}(x, t)| \quad (28)$$

we assume

$$|\bar{f}(x, t)(t)| \leq \bar{F}_{max}$$

$$|\dot{q}_n \bar{f}(x, t)(t)| \leq |\dot{q}_n| \cdot \bar{F}_{max}$$

Let apply the inequality

$$ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$$

where

$$a = |\dot{q}_n|$$

$$b = \bar{F}_{max}$$

$$|\dot{q}_n| \bar{F}_{max} \leq \frac{1}{2} \dot{q}_n^2 + \frac{1}{2} \bar{F}_{max}^2 \quad (29)$$

Substitute equation (29) into equation (27) and replacing $\dot{V}$ with;

$$-\dot{q}_n \bar{f}(x, t)(t) \leq \frac{1}{2} \dot{q}_n^2 + \frac{1}{2} \bar{F}_{max}^2$$

$$\dot{V} \leq -(\bar{\gamma}_n + k_2) \dot{q}_n^2 - \delta_n q_n \dot{q}_n + \frac{1}{2} \dot{q}_n^2 + \frac{1}{2} \bar{F}_{max}^2 \quad (30)$$

Collecting like terms from equation (30)

$$-(\bar{\gamma}_n + k_2) \dot{q}_n^2 + \frac{1}{2} \dot{q}_n^2 = -\left[(\bar{\gamma}_n + k_2) - \frac{1}{2}\right] \dot{q}_n^2 \quad (31)$$

$$\dot{V} \leq -\left[(\bar{\gamma}_n + k_2) - \frac{1}{2}\right] \dot{q}_n^2 - \delta_n q_n \dot{q}_n + \frac{1}{2} \bar{F}_{max}^2 \quad (32)$$

$$\dot{V} \leq -\left[(\bar{\gamma}_n + k_2) - \frac{1}{2}\right] \dot{q}_n^2 + \frac{1}{2} \bar{F}_{max}^2 \quad (33)$$

Converting in terms of x_1, x_2 to obtain the exact $|x|^2$
From the state-space variable equation (17) and (18)
Substituting into the lyapunov function

$$V = \frac{1}{2} x_n^2 + \frac{1}{2} k_1 x_n^2 + \frac{1}{2\Gamma_n} \delta_n^2 \quad (34)$$

Since $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Then the norm

$$\|x\|^2 = x_1^2 + x_2^2$$

Comparing V from equation (34) with $|x|^2$

Lower bound

$$V \geq \frac{1}{2} x_2^2 + \frac{1}{2} k_1 x_1^2 \quad (35)$$

$$V \geq \frac{1}{2} \min(1, k_1)(x_1^2 + x_2^2)$$

$$V \geq c_1 \|x\|^2$$

Upper bound

$$V \leq \frac{1}{2} \max(1, k_1)(x_1^2 + x_2^2) + \frac{1}{2\gamma_n} \tilde{\delta}_n^2$$

$$V \leq c_2 \|x\|^2 + c_3 \tilde{\delta}_n^2$$

Expressing $\dot{V}$ in the state form from equation (27)

$$\dot{V} = -(\gamma_n + k_2)x_2^2 - x_2 f(x, t) \quad (36)$$

Bounding the disturbance using

$$|\bar{f}(x, t)| \leq F_{max}$$

$$|x_2 \bar{f}(x, t)| \leq |x_2| F_{max}$$

Apply the inequality equation

$$|x_2| F_{max} \leq \frac{1}{2} x_2^2 + \frac{1}{2} F_{max}^2 \quad (37)$$

substitute equation (37) into equation (36)

$$\dot{V} \leq -\left[(\gamma_n + k_2) - \frac{1}{2}\right] x_2^2 + \frac{1}{2} x_2^2 + \frac{1}{2} F_{max}^2 \quad (38)$$

Converting equation (38) to norm form

Since $x_2^2 \leq \|x\|^2$

$$c = (\gamma_n + k_2) - \frac{1}{2} > 0$$

$$d = \frac{1}{2} F_{max}^2$$

Then,

$$\dot{V} \leq -c \|x\|^2 + d \quad (39)$$

Since the derivative of the Lyapunov function equation (39) is negative semi-definite, the total energy of the controlled bridge decreases with time. Consequently, all closed-loop signals remain bounded, while the bridge displacement, vibration velocity and parameter estimation error converge to stable values.

The Lyapunov analysis therefore demonstrates that the proposed adaptive boundary controller guarantees the stability of the overhead bridge despite the presence of uncertain boundary stiffness and moving vehicle excitation.

the system energy decreases (negative term)

the external force adds a bounded disturbance

Therefore, the system is UNIFORMLY ULTIMATE BOUNDED.

Computing the vibration region (Ultimate Bound Radius) from equation (39)

$$\dot{V} \leq -c \|x\|^2 + d \quad (40)$$

where

$$c = (\gamma_n + k_2) - \frac{1}{2} > 0$$

$$d = \frac{1}{2} F_{max}^2$$

Using Lyapunov bounds of

$$c_1 \|x\|^2 \leq V \leq c_2 \|x\|^2$$

where

$$c_1 = \frac{1}{2} \min(1, k_1)$$

$$c_2 = \frac{1}{2} \max(1, k_1)$$

At a steady state

$$\begin{aligned} \dot{V} &\leq 0 \\ -c \|x\|^2 + d &\leq 0 \end{aligned}$$

Solving for $\|x\|^2$

$$c \|x\|^2 \geq d$$

$$\|x\|^2 \geq \frac{d}{c}$$

Taking the square root

$$\|x\| \leq \sqrt{\frac{d}{c}} \quad (41)$$

$$\|x\| \leq \sqrt{\frac{\frac{1}{2} F_{max}^2}{(\gamma_n + k_2) - \frac{1}{2}}} \quad (42)$$

Equation (42) is the Ultimate Bounded Region the system settles inside $\|x\|$

Substituting the values into (42)

$$\gamma_n = 0.8, k_2 = 10, F_{max} = 5$$

$$c = (\gamma_n + k_2) - 0.5$$

$$c = 0.8 + 10 - 0.5 = 10.3$$

$$d = \frac{1}{2} F_{max}^2$$

$$d = \frac{1}{2}(5)^2 = 12.5$$

$$\|x\| \leq \sqrt{\frac{12.5}{10.3}}$$

$$\|x\| \leq \sqrt{1.2136}$$

$$\|x\| \leq 1.10$$

The vibration will never exceed a radius of about 1.10

The system settles inside bounded region

III. Results And Discussion

To evaluate the effectiveness of the proposed adaptive boundary control strategy, numerical simulations were carried out using the reduced-order model obtained through the Galerkin approximation. The simulations were performed under moving vehicle loads while considering uncertainty in the boundary stiffness. The controller parameters and adaptation gain were selected to ensure stable closed-loop performance. The effectiveness of the controller was assessed using the bridge displacement, modal velocity, control input, estimated boundary stiffness, stiffness estimation error, Lyapunov energy function and reconstructed full beam deflection.

Using the space variable

$$x_1 = q_n, x_2 = \dot{q}_n$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-\gamma_n + k_2}{\mu_n} x_2 - \frac{\hat{\delta}_n + k_1}{\mu_n} x_1 - \frac{\tilde{\delta}_n}{\mu_n} x_1 - \frac{1}{\mu_n} f(x, t)$$

Adaptive law

$$\dot{\hat{\delta}}_n = -\Gamma_n x_1 x_2$$

Estimated Error

$$\tilde{\delta}_n = \delta_n - \hat{\delta}_n$$

Control input

$$\bar{u}(t) = -k_1 q_n - k_2 \dot{q}_n$$

Full beam deflection

$$N(x, t) = q_n(t)x^2(3L - x)$$

Table1: Simulation Parameters

Parameters	Description	Value
μ_n	Modal mass	1kg
γ_n	Damping coefficient	0.8
δ_n	Stiffness parameters	20
k_1	Control gain(position)	25
k_2	Control gain (velocity)	10
Γ_n	Adaptation gain	15
$\hat{\delta}_n(0)$	Initial estimate	10
$q_n(0)$	Initial displacement	0
$\dot{q}_n(0)$	Initial velocity	0
$F(x, t)$	External force	$5\sin(3t)$
Simulation time	Time interval	0-20s

Modal Displacement vs Time

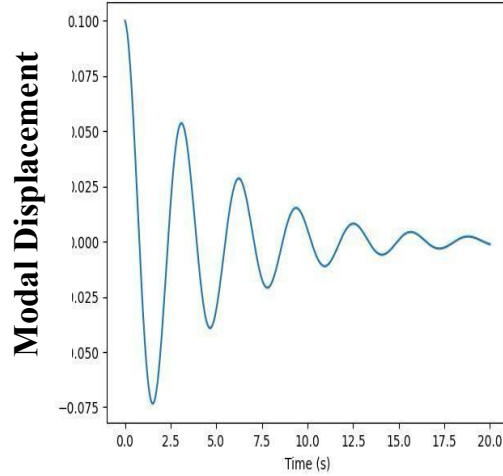


figure 2: Modal displacement with different variation of $\mu_n = 1.0$, $\gamma_n = 0.8$, $\delta_n = 20.0$, $k_1 = 25.0$, $k_2 = 10.0$, $\Gamma_n = 15.0$

The Modal Displacement response of the controlled bridge under moving vehicle loads excitation.

The displacement response, as shown in figure 2 illustrate the time evolution of the modal coordinate $q_n(t)$. The system initially exhibits oscillatory behavior due to applied external excitation and initial conditions. However, as time progresses, the amplitude of oscillation decreases and settle within a small neighborhood around zero.

This behavior indicates that the displacement remains bounded for all time and eventually converges to a small region near the equilibrium. This is consistent with the concept of uniform ultimate boundedness,

which guarantees that the system states do not diverge but remain within a bound despite external disturbances and parameter uncertainties. Zhou et al. (2021) presented a comprehensive review of vehicle–bridge interaction models and concluded that the coupled dynamics between a moving vehicle and a bridge significantly influence the structural response of bridge systems. Their review highlighted that factors such as vehicle speed, bridge flexibility, damping, and structural properties strongly affect bridge vibration characteristics. However, their work concentrated on dynamic analysis and did not consider adaptive boundary control for vibration suppression under uncertain boundary conditions.

Modal Velocity vs Time

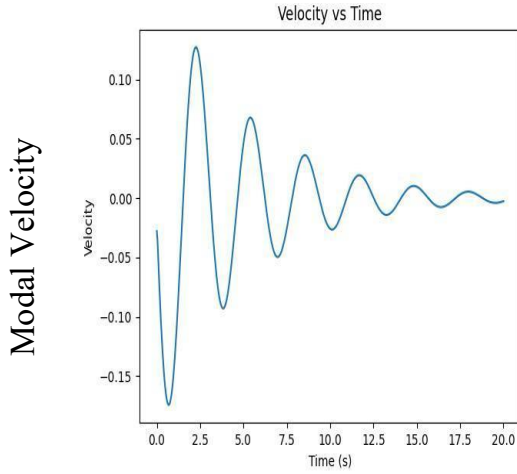


Figure 3: Modal velocity response with different variation of $\mu_n = 1.0$ $\gamma_n = 0.8$, $\delta_n = 20.0$, $k_1 = 25.0$, $k_2 = 10.0$, $\Gamma_n = 15.0$

Figure 3: Modal Velocity(m/s) vs Time(s) $\dot{q}_n(t)$ vs t

Modal Velocity response of the controlled bridge. The velocity $\dot{q}_n(t)$ response presented in figure 3 shows a decaying oscillatory pattern. The initial high velocity corresponds to the transient response of the system, which the amplitude diminishes steadily overtime, which remains within a bounded region and approaches a small neighborhood around zero as time increases.

Estimated Stiffness Error

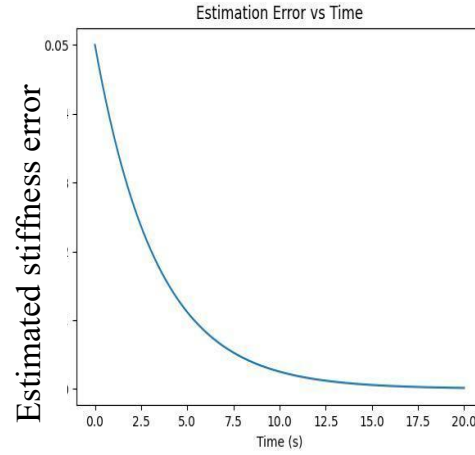


Figure 4: Stiffness Error(N/m) vs Time(s) with different variation of $\mu_n = 1.0$ $\gamma_n = 0.8$, $\delta_n = 20.0$, $k_1 = 25.0$, $k_2 = 10.0$, $\Gamma_n = 15.0$

The Convergence of the Boundary Stiffness Error in figure 4 illustrates the time evolution of the estimation error $\tilde{\delta}_n = \delta_n - \hat{\delta}_n$ defined as the difference between the true stiffness parameter and its estimate. The plots show that the estimation error starts from a nonzero initial value and decrease over time, eventually remaining within a small bounded region.

This bounded convergence indicates that the adaptive law successfully reduces the estimation error and maintains it within acceptable limits. The results confirm the uncertainty in the stiffness parameter is effectively handled and the system maintains stability in accordance with the principle of uniform ultimate boundedness. Wang et al, (2023) investigated a robust boundary control strategy for Euler–Bernoulli beam systems subjected to unknown distributed disturbances. Their proposed disturbance-observer-based controller effectively suppressed beam vibrations and guaranteed the stability of the closed-loop system. Nevertheless, the study did not address adaptive estimation of uncertain boundary stiffness or the influence of moving vehicle loads, which form the major focus of the present work.

Estimated Parameter Response

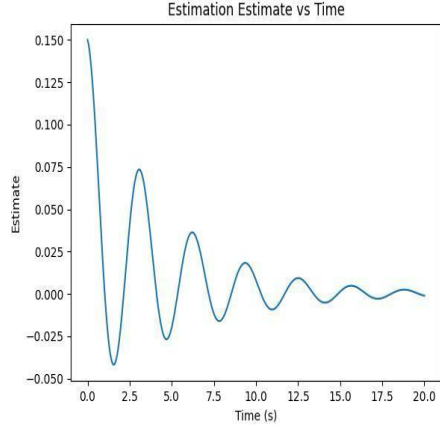


Figure 5: Modal velocity response with different variation of $\mu_n = 1.0$ $\gamma_n = 0.8$, $\delta_n = 20.0$, $k_1 = 25.0$, $k_2 = 10.0$, $\Gamma_n = 15.0$

The Estimated Parameter Response of the evolution of the stiffness $\hat{\delta}(t)$ shown in figure 5 demonstrates the performance of the adaptive law. Starting from an initial guess, the estimate undergoes oscillations and gradually settles within a bounded region. This indicates that the adaptive mechanism is effectively updating the parameter in response to system dynamics. Although the estimate may not converge exactly to the through value, its bounded behavior is sufficient to guarantee system stability under the concept of Uniform Ultimate Boundedness (UUB).

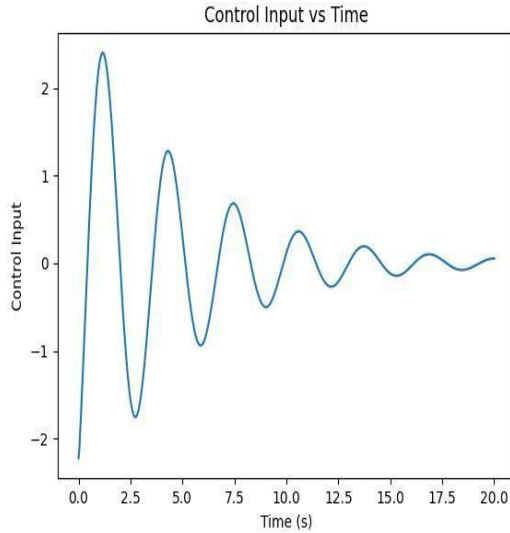


Figure 6: Modal velocity response with different variation of $\mu_n = 1.0$ $\gamma_n = 0.8$, $\delta_n = 20.0$, $k_1 = 25.0$, $k_2 = 10.0$, $\Gamma_n = 15.0$

The Adaptive Boundary Control Input generated during the vibration suppression process in figure 5 illustrate the control inputs $\bar{u}(t)$ applied in the system. The control inputs exhibit relatively high oscillations at the initial stage as the controller actively counteracts the system disturbance. As the system stabilizes, the magnitude of the control inputs decreases and remains within a bounded range.

Since the control input is a function of the system states, its boundedness further confirms that the states themselves are bounded. This behavior supports the conclusion that the overall system satisfies uniform ultimate boundedness and the control strategy is both effective and efficient

Wang et al. (2025) proposed an adaptive boundary control strategy for an Euler–Bernoulli beam carrying an unknown payload. An adaptive estimation law was incorporated into the controller to compensate for parameter uncertainty, while Lyapunov stability theory was employed to prove the asymptotic stability of the closed-loop system. Their numerical simulations demonstrated that the proposed controller effectively reduced structural vibrations and accurately estimated the unknown parameter. However, the study focused on payload uncertainty and did not consider the influence of uncertain boundary stiffness or moving vehicle loads on bridge structures, which are investigated in the present study.

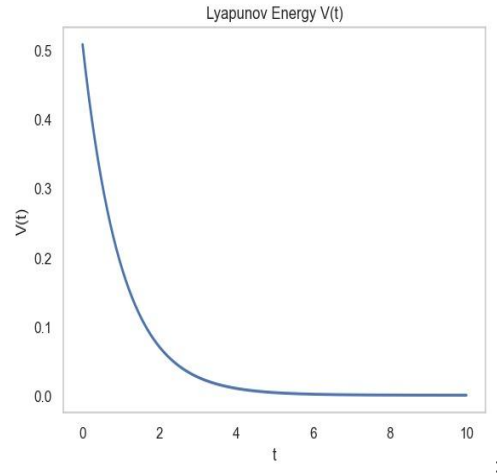


Figure 7: Modal velocity response with different variation of $\mu_n = 1.0$ $\gamma_n = 0.8$, $\delta_n = 20.0$, $k_1 = 25.0$, $k_2 = 10.0$, $\Gamma_n = 15.0$.

Figure 7: Lyapunov Energy(J) vs Time(s) $V(t)$ vs t

The Evolution of the Lyapunov Energy function demonstrating the stability of the closed-loop-system in figure 7 shows that the time evolution of the Lyapunov $V(T)$ which represents the total energy of

the system, including both mechanical energy and parameter estimation error. It is observed that the Lyapunov function is positive initially and decreases monotonically over time, eventually approaching a small value.

The bounded and decreasing nature of $V(T)$ confirms that the system satisfies the Lyapunov stability conditions, where the time derivative of the Lyapunov function is negative semi-definite. This guarantees that all system states, including displacement, velocity, and parameter estimation error, remains bounded for all time. Sun et al. (2025) developed an event-triggered adaptive safe boundary control strategy for uncertain Euler–Bernoulli beam systems. Their approach simultaneously addressed unknown damping coefficients, bending stiffness, and boundary disturbances while guaranteeing system stability using Lyapunov analysis. Although their controller achieved excellent vibration suppression, it did not consider the dynamic interaction between a moving vehicle and an overhead bridge, which is investigated in the present work.

Therefore, the Lyapunov energy plots provide strong evidence that the system satisfies uniform ultimate boundedness, ensuring that system trajectories converge to a small bounded region around equilibrium.

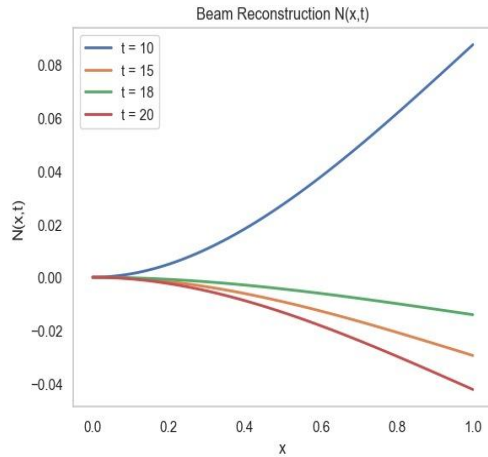


Figure 8: Full Beam Deflection(m) $N(x, t)$ vs x (m) at different fixed times

The Full Beam Deflection profiles at different time instants under adaptive boundary control.

The different colored curves represent the beam deflection at various time instances. It is observed that the beam undergoes oscillatory motion, as indicated by the alternating positive and negative deflections at

different times. The magnitude of the deflection initially increases and then gradually decreases over time, demonstrating the damping effect introduced by the adaptive boundary control.

The progression of the curves from larger amplitudes to smaller ones indicates that the vibration energy of the system is being dissipated, leading to stabilization of the beam. This confirms that the control strategy effectively suppresses vibrations and improves the structural response of the system. Cui et al. (2025) proposed an event-triggered adaptive control method for a rotating disk–beam system with time-varying boundary constraints. Their controller effectively handled parametric uncertainties while ensuring closed-loop stability through barrier Lyapunov techniques. Nevertheless, the study focused on rotating beam systems and did not investigate bridge structures subjected to moving vehicle loads.

IV. CONCLUSION

This paper presented an adaptive boundary control strategy for suppressing vibration in a Euler–Bernoulli overhead bridge subjected to moving vehicle loads with uncertain boundary stiffness. The bridge dynamics were modeled using partial differential equation and reduced to a finite-dimensional system through the Galerkin approximation. An adaptive boundary controller was developed to estimate the compensate for the unknown boundary stiffness in real time, while Lyapunov stability theory was employed to established the stability of the closed-loop system.

The numerical simulation demonstrated that the proposed controller effectively attenuated the bridge vibrations by reducing the modal displacement and modal velocity. Furthermore, the adaptive estimation law accurately identified the unknown boundary stiffness, leading to a progressive reduction in the estimation error. The decreasing Lyapunov energy function verified the theoretical stability of the controlled system, while the reconstructed beam deflection profiles confirmed the controller’s ability to suppress the structural vibrations along the entire bridge span.

Overall, the proposed adaptive boundary control framework provides an effective and robust approach for mitigating bridge vibrations under moving vehicle excitation and uncertain boundary condition. The finding contributes to the development of intelligent vibration control strategies capable of enhancing the safety, durability and serviceability of modern bridge structures.

Future work may extend the proposed approach by incorporating nonlinear beam effects, and

experimental validation on laboratory-scale bridge models.

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DATA AVAILABILITY

Yes, The data set generated or analyzed in the current study was made available by the corresponding author based on reasonable request.

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ETHICS, CONSENT TO PARTICIPATE, AND CONSENT TO PUBLISH

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Nomenclature

Symbol	Description	SI Unit
x	Spatial coordinate along the bridge	M
L	Length	M
t	Time	S
$u(x, t)$	Transverse displacement of the bridge	M
$u(\dot{x}, t)$	Transverse velocity of the bridge	m/s
$\ddot{u}(x, t)$	Transverse acceleration of the bridge	m/s ²
μ_n	Modal mass	Kg
EI	Flexural rigidity of the bridge	N. m ²
E	Young modulus	Pa(N/m ²)
I	Second moment of area	m ⁴
γ	Structural damping coefficient	N.s/m
$f(x, t)$	External moving loads	N
P	Magnitude of the moving load vehicle	N
v	Vehicle speed	m/s
$\delta(x - x_v(t))$	Dirac delta function	-
$\phi_n(x)$	Assumed mode shape (spatial Eigen function)	-
$q_n(t)$	Modal displacement	M

$\ddot{q}_n(t)$	Modal acceleration	m/s ²
$\dot{q}_n(t)$	Modal velocity	m/s
ζ	Modal damping ratio	N.s/m
$u(t)$	Adaptive boundary control input	N
δ_n	Unknown boundary stiffness parameter	N/m
$\hat{\delta}_n$	Estimated boundary stiffness	N/m
$\tilde{\delta}_n$	Boundary stiffness error	N/m
Γ_n	Adaptation gain	-
$N(x, t)$	Reconstructed full beam deflection	M
$V(t)$	Lyapunov function	J
$\dot{V}(t)$	Time derivative of the Lyapunov function	J/s
	Number of mode	-

n