Sharp Energy Decay and Indirect Stabilization for a Nonlinearly Coupled Wave–Diffusion System with Mixed Boundary Conditions

Alvaro H. Salas*[1]

[1]Department of Mathematics and Statistics, Universidad Nacional de Colombia, Sede Manizales, Colombia

Abstract

We study a wave–diffusion system on a bounded Lipschitz domain whose components satisfy the same mixed Dirichlet–Neumann boundary decomposition. The wave is semilinear, while dissipation acts only through the diffusion component; stabilization of the wave is therefore indirect. For a coercive mixed Laplacian and a defocusing subcritical source, we prove global finite-energy well-posedness, an exact energy identity, uniqueness, and asymptotic stabilization. For the linearized system we obtain a complete spectral description mode by mode and show that uniform exponential stability fails: the weakly damped wave eigenvalues satisfy

$$\Re s_n^{\pm} = -\frac{\kappa^2}{2\lambda_n} + O(\lambda_n^{-3/2}), \qquad \Im s_n^{\pm} = \pm\sqrt{\lambda_n} + O(\lambda_n^{-1}),$$

where λ_n are the mixed-Laplacian eigenvalues. A frequency-domain estimate then yields $\|(i\omega - \mathcal{A})^{-1}\| = O(|\omega|^2)$, and the Borichev–Tomilov theorem gives the sharp decay

$$E(t) \le \frac{C}{1+t}\, \|U_0\|^2_{\mathcal{D}(\mathcal{A})}\,.$$

The spectral asymptotics prove optimality of the exponent for general strong data. The result identifies the precise high-frequency obstruction to exponential decay and separates geometric coercivity, nonlinear well-posedness, and the diffusion-mediated stabilization mechanism. A one-dimensional mixed-boundary computation is included only as a reproducibility check of the spectral asymptotics.

Keywords. wave–diffusion system; indirect stabilization; mixed boundary conditions; polynomial stability; sharp energy decay; semilinear evolution equation.
2020 Mathematics Subject Classification. 35L70, 35K55, 35B40, 47D06, 93D20.

1 Introduction

Coupled hyperbolic–parabolic systems arise whenever a conservative or weakly dissipative mechanical field exchanges energy with a diffusive variable. Classical examples include thermoelasticity, viscoelastic relaxation, acoustic–thermal coupling, and structural systems with internal diffusion. A fundamental question is whether damping acting on only one component can stabilize the full coupled dynamics and, if so, at what rate. The answer

*Corresponding author: ahsalass@unal.edu.co. ORCID: 0000-0001-9343-6062.

depends not only on the presence of dissipation but also on the high-frequency transfer of energy between the conservative and dissipative channels.

This paper isolates that mechanism in a semilinear wave–diffusion model with mixed boundary conditions. Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be bounded, connected, and Lipschitz, with a measurable partition

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \qquad \Gamma_D \cap \Gamma_N = \varnothing,$$

and assume $|\Gamma_D|_{d-1} > 0$. We consider

$$\begin{cases} u_{tt} - \Delta u + f(u) + \kappa v = 0 & \text{in } \Omega \times (0,\infty), \\ v_t - \Delta v - \kappa u_t = 0 & \text{in } \Omega \times (0,\infty), \\ u = v = 0 & \text{on } \Gamma_D \times (0,\infty), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \Gamma_N \times (0,\infty), \\ (u, u_t, v)|_{t=0} = (u_0, u_1, v_0). \end{cases} \tag{1}$$

Here $\kappa \neq 0$ is the exchange coefficient. The wave equation contains no direct damping. Multiplying the first equation by u_t, the second by v, and adding produces an exact cancellation of the coupling terms; the sole dissipation is $\|\nabla v\|_{L^2}^2$. Thus any decay of the wave energy must be transmitted through the diffusive variable.

The mixed boundary decomposition is analytically significant. The positive-measure Dirichlet part yields a Poincaré inequality and a strictly positive first eigenvalue of the mixed Laplacian. At the same time, the spectral spacing and the parabolic resolvent at high frequency determine the rate at which the diffusion channel can absorb wave energy.

The main contributions are the following.

(i) Under a defocusing, energy-subcritical nonlinearity we establish global finite-energy well-posedness and the exact Lyapunov identity

$$E(t) + \int_0^t \|\nabla v(s)\|_2^2 \, ds = E(0).$$

(ii) We identify the largest invariant subset of the zero-dissipation set and deduce asymptotic stabilization for precompact trajectories; for strong solutions generated by bounded sets in the natural higher-order phase space, compactness follows from standard elliptic regularity.

(iii) For the linearized system we diagonalize with respect to the mixed Laplacian. Each spatial eigenvalue λ leads to the cubic characteristic polynomial

$$P_\lambda(s) = (s^2 + \lambda)(s + \lambda) + \kappa^2 s.$$

All roots lie in the open left half-plane, but the wave pair approaches the imaginary axis at the rate λ^{-1}.

(iv) We prove the resolvent estimate $\|(i\omega - \mathcal{A})^{-1}\| \le C(1 + |\omega|^2)$ and infer the polynomial energy decay $E(t) = O(t^{-1})$ for data in $\mathcal{D}(\mathcal{A})$. The eigenvalue asymptotics show that this exponent cannot be improved uniformly on $\mathcal{D}(\mathcal{A})$.

The distinction between strong stability and uniform exponential stability is important. Energy dissipation alone does not determine a rate. Here high-frequency wave modes excite the diffusive variable only weakly: for $\lambda \gg 1$, the diffusion response is of order $\lambda^{-1/2}$ relative to the wave velocity, and the real parts of the wave eigenvalues are only of order $-\lambda^{-1}$. This mechanism is visible both analytically and numerically.

Our analysis is situated within the semigroup and frequency-domain theory of stabilization developed in, among many sources, [1–5]. Mixed-boundary elliptic operators and nonlinear evolution equations are treated using standard variational and monotonicity tools [6–9]. Recent work in nonlinear differential equations continues to emphasize well-posedness, mixed boundary geometry, stability, and asymptotic classification; see, for example, [14–16].

2 Functional setting and assumptions

Set

$$H = L^2(\Omega), \qquad V = \{w \in H^1(\Omega) : \operatorname{Tr} w = 0 \text{ on } \Gamma_D\}.$$

Because $|\Gamma_D|_{d-1} > 0$, there is $C_P > 0$ such that

$$\|w\|_{L^2} \le C_P \|\nabla w\|_{L^2}, \qquad w \in V. \tag{2}$$

Define the self-adjoint positive operator A associated with the form

$$a(w, z) = \int_\Omega \nabla w \cdot \nabla z \, dx, \qquad w, z \in V.$$

Thus $A = -\Delta$ with Dirichlet condition on Γ_D and homogeneous Neumann condition on Γ_N. Its resolvent is compact. We denote by

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \qquad \lambda_n \to \infty,$$

its eigenvalues, repeated according to multiplicity, and by $\{\phi_n\}$ an orthonormal basis of eigenfunctions in H.

We impose the following conditions on the wave nonlinearity.

Assumption 2.1. *The function $f \in C^1(\mathbb{R})$ satisfies $f(0) = 0$, and its primitive $F(s) = \int_0^s f(\xi)\, d\xi$ obeys*

$$F(s) \ge 0 \quad \text{for all } s \in \mathbb{R}.$$

Moreover, there are $C > 0$ and an exponent $p \ge 2$, energy-subcritical for $H^1(\Omega)$, such that

$$|f'(s)| \le C(1 + |s|^{p-2}).$$

For $d = 3$ we take $2 \le p \le 6$; for $d \le 2$, any finite p is allowed.

The natural phase space is

$$\mathcal{H} = V \times H \times H,$$

with norm

$$\|(u, p, v)\|_{\mathcal{H}}^2 = \left\|A^{1/2}u\right\|_2^2 + \|p\|_2^2 + \|v\|_2^2.$$

The nonlinear energy is

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left\|A^{1/2}u(t)\right\|_2^2 + \frac{1}{2}\|v(t)\|_2^2 + \int_\Omega F(u(t))\, dx. \tag{3}$$

3 Global well-posedness and dissipation

We first state the finite-energy result.

Theorem 3.1 (Global weak well-posedness)**.** *Assume 2.1. For every $(u_0, u_1, v_0) \in V \times H \times H$, problem (1) admits a unique global finite-energy solution satisfying*

$$u \in C([0,T]; V) \cap C^1([0,T]; H), \qquad v \in C([0,T]; H) \cap L^2(0,T; V)$$

for every $T > 0$. In addition,

$$E(t) + \int_0^t \left\| A^{1/2} v(s) \right\|_2^2 \, ds = E(0), \qquad t \geq 0. \tag{4}$$

If $(u_0, u_1, v_0) \in \mathcal{D}(A) \times V \times V$ and the corresponding Nemytzkii map has the standard local regularity induced by Assumption 2.1, then the solution is strong on every finite interval.

Proof. Let $V_m = \operatorname{span}\{\phi_1, \dots, \phi_m\}$ and seek Galerkin approximations $u_m, v_m \in V_m$. The finite-dimensional system is locally solvable because the projected nonlinearity is locally Lipschitz. Testing the wave equation by $u_{m,t}$ and the diffusion equation by v_m gives

$$\frac{d}{dt}\left[\frac{1}{2}\|u_{m,t}\|_2^2 + \frac{1}{2}\left\|A^{1/2}u_m\right\|_2^2 + \frac{1}{2}\|v_m\|_2^2 + \int_\Omega F(u_m)\,dx\right] = -\left\|A^{1/2}v_m\right\|_2^2.$$

The coupling terms cancel exactly. Since $F \geq 0$, the estimate is coercive in $V \times H \times H$ and prevents finite-time blow-up of the Galerkin system. Hence

$$\{u_m\} \text{ is bounded in } L^\infty(0,T;V), \quad \{u_{m,t}\} \text{ in } L^\infty(0,T;H),$$

$$\{v_m\} \text{ in } L^\infty(0,T;H) \cap L^2(0,T;V).$$

The growth assumption and Sobolev embedding give the standard compactness needed to pass to the nonlinear term. One obtains a weak solution and the energy inequality. Applying the same argument to time-regularized approximations and using weak lower semicontinuity together with the equation recovers equality (4).

For uniqueness, let $(u^{(j)}, v^{(j)})$, $j = 1, 2$, be two solutions on a fixed finite interval. The difference satisfies an energy relation with source $f(u^{(1)}) - f(u^{(2)})$. On bounded energy balls the Nemytzkii map is locally Lipschitz from V into its dual in the stated subcritical range. Gronwall's lemma gives zero difference. Higher regularity follows by differentiating the Galerkin equations and using elliptic regularity for the mixed Laplacian. □

Remark 3.2. *No damping term is present in the wave equation. Identity (4) therefore displays genuine indirect dissipation: the wave can lose energy only through the exchange term κv and the parabolic dissipation in v.*

4 Asymptotic stabilization of the nonlinear system

The energy is nonincreasing, but strict decay requires identification of the invariant set where the dissipation vanishes.

Proposition 4.1 (Zero-dissipation invariant set)**.** *Assume $\kappa \neq 0$. Let a complete trajectory of (1) satisfy*

$$\left\|A^{1/2}v(t)\right\|_2 = 0 \qquad \textit{for every } t \in \mathbb{R}.$$

Then $v \equiv 0$, $u_t \equiv 0$, and every stationary state satisfies

$$Au + f(u) = 0.$$

If, in addition,

$$f(s)s \geq 0 \qquad \forall s \in \mathbb{R}, \tag{5}$$

then the only stationary state is $(u, u_t, v) = (0, 0, 0)$.

Proof. By the Poincaré inequality, $A^{1/2}v = 0$ implies $v = 0$. The second equation in (1) then yields $-\kappa u_t = 0$, so $u_t = 0$. The first equation gives $Au + f(u) = 0$. Testing by u and using (5),

$$\left\|A^{1/2}u\right\|_2^2 + \int_\Omega f(u)u\,dx = 0,$$

so both terms vanish and $u = 0$. □

Theorem 4.2 (Asymptotic stabilization)**.** *Assume 2.1, (5), and $\kappa \neq 0$. Every bounded trajectory of (1) that is precompact in $\mathcal{H}$ converges to the origin:*

$$\lim_{t\to\infty} \|(u(t), u_t(t), v(t))\|_{\mathcal{H}} = 0.$$

In particular, the conclusion holds for strong solutions whose orbit is bounded in a phase space compactly embedded in $\mathcal{H}$.

Proof. The energy E is a continuous Lyapunov functional and, by (4), $E'(t) = -\left\|A^{1/2}v(t)\right\|_2^2$ along strong trajectories. Precompactness guarantees a nonempty compact omega-limit set. LaSalle's invariance principle implies that the omega-limit set is contained in the largest invariant subset of $\{E' = 0\}$. Proposition 4.1 reduces this set to the origin. Hence the whole trajectory converges to zero. □

5 Linear generator and absence of exponential stability

We now set $f \equiv 0$. Let

$$U = (u, p, v)^\top, \qquad p = u_t.$$

Then $U_t = \mathcal{A}U$ on $\mathcal{H}$, where

$$\mathcal{A}\begin{pmatrix} u \\ p \\ v \end{pmatrix} = \begin{pmatrix} p \\ -Au - \kappa v \\ \kappa p - Av \end{pmatrix}, \tag{6}$$

with

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times V \times \mathcal{D}(A).$$

Proposition 5.1 (Generation and strong stability)**.** *The operator $\mathcal{A}$ generates a contraction semigroup $T(t)$ on $\mathcal{H}$. Moreover, $i\mathbb{R} \subset \rho(\mathcal{A})$, and $T(t)U_0 \to 0$ in $\mathcal{H}$ for every $U_0 \in \mathcal{H}$.*

Proof. Dissipativity follows from

$$\Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\left\|A^{1/2}v\right\|_2^2 \leq 0.$$

The range condition for $I - \mathcal{A}$ reduces, after eliminating p and v, to a coercive elliptic variational problem, so the Lumer–Phillips theorem applies. To exclude imaginary eigenvalues, suppose $\mathcal{A}U = i\omega U$. Taking the real part of the energy inner product gives $v = 0$. The third component then gives $\kappa p = 0$, hence $p = 0$; the first gives $i\omega u = 0$, and the second yields $Au = 0$. Thus $U = 0$. Compactness of the mixed-Laplacian resolvent implies compactness of the resolvent of $\mathcal{A}$, so the spectrum on the imaginary axis is empty. Strong stability follows from the Arendt–Batty–Lyubich–Vu theorem. □

Because $A\phi_n = \lambda_n \phi_n$, each eigenspace of A is invariant under $\mathcal{A}$. Writing

$$u = q\phi_n, \qquad p = r\phi_n, \qquad v = z\phi_n,$$

leads to

$$\frac{d}{dt}\begin{pmatrix} q \\ r \\ z \end{pmatrix} = M_{\lambda_n}\begin{pmatrix} q \\ r \\ z \end{pmatrix}, \qquad M_\lambda = \begin{pmatrix} 0 & 1 & 0 \\ -\lambda & 0 & -\kappa \\ 0 & \kappa & -\lambda \end{pmatrix}.$$

The characteristic polynomial is

$$P_\lambda(s) = \det(sI - M_\lambda) = (s^2 + \lambda)(s + \lambda) + \kappa^2 s = s^3 + \lambda s^2 + (\lambda + \kappa^2)s + \lambda^2. \tag{7}$$

Lemma 5.2 (Modal stability)**.** *For every $\lambda > 0$, all three roots of P_λ lie in the open left half-plane.*

Proof. The coefficients of $s^3 + a_1 s^2 + a_2 s + a_3$ are

$$a_1 = \lambda, \qquad a_2 = \lambda + \kappa^2, \qquad a_3 = \lambda^2.$$

The cubic Routh–Hurwitz criterion requires $a_j > 0$ and $a_1 a_2 > a_3$. Here

$$a_1 a_2 - a_3 = \lambda\kappa^2 > 0.$$

□

Theorem 5.3 (High-frequency spectral asymptotics)**.** *As $\lambda \to \infty$, the roots of (7) consist of one parabolic root*

$$s_\lambda^0 = -\lambda + \frac{\kappa^2}{\lambda} + O(\lambda^{-2}), \tag{8}$$

and a complex-conjugate wave pair

$$s_\lambda^\pm = \pm i\sqrt{\lambda} - \frac{\kappa^2}{2\lambda} \pm i\frac{\kappa^2}{2\lambda^{3/2}} + O(\lambda^{-2}). \tag{9}$$

Consequently, $T(t)$ is not uniformly exponentially stable.

Proof. For the parabolic root, write $s = -\lambda + \eta$ in (7). Since

$$P_\lambda(-\lambda + \eta) = \eta((\lambda - \eta)^2 + \lambda) - \kappa^2(\lambda - \eta),$$

a consistent expansion $\eta = O(\lambda^{-1})$ gives $\eta = \kappa^2\lambda^{-1} + O(\lambda^{-2})$.

For the positive wave root set $s = i\sqrt{\lambda} + \delta$, $|\delta| = o(\sqrt{\lambda})$. Since

$$P_\lambda(i\sqrt{\lambda}) = i\kappa^2\sqrt{\lambda},$$

and

$$P'_\lambda(i\sqrt{\lambda}) = 2i\lambda^{3/2} - 2\lambda + \kappa^2,$$

one Newton step yields

$$\delta = -\frac{i\kappa^2\sqrt{\lambda}}{2i\lambda^{3/2} - 2\lambda + \kappa^2} + O(\lambda^{-2}),$$

which expands as

$$\delta = -\frac{\kappa^2}{2\lambda} + i\frac{\kappa^2}{2\lambda^{3/2}} + O(\lambda^{-2}).$$

The negative wave root is its complex conjugate. Since $\Re s_\lambda^\pm \uparrow 0$, the spectral bound cannot be separated uniformly from the imaginary axis; exponential stability is impossible. □

Figure 1 illustrates the asymptotic law for the one-dimensional mixed problem $u(0) = 0$, $u_x(1) = 0$, for which $\lambda_n = ((n + 1/2)\pi)^2$.

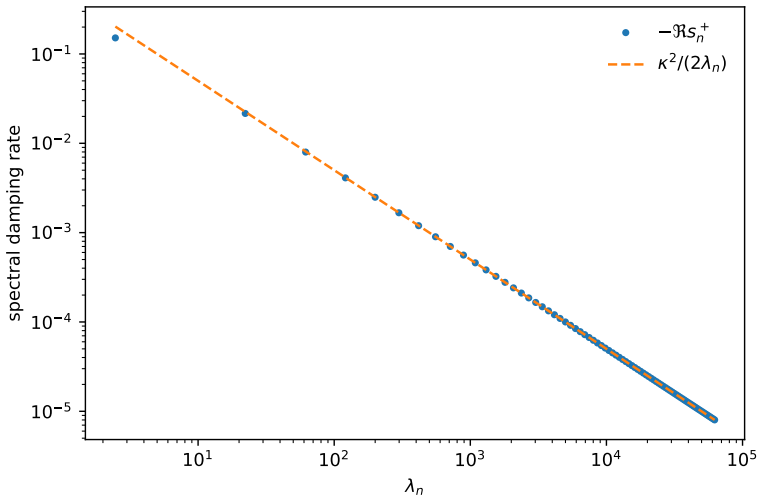


Figure 1: Modal damping rate for $\kappa = 1$ and the one-dimensional mixed Laplacian. The computed quantity $-\Re s_n^+$ approaches $\kappa^2/(2\lambda_n)$, confirming the high-frequency expansion (9).

6 Resolvent growth

The spectral asymptotics supply the lower-order obstruction. We next prove the matching upper resolvent estimate.

Theorem 6.1 (Quadratic resolvent bound)**.** *There exists $C > 0$ such that*

$$\left\|(i\omega - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \le C(1 + |\omega|^2), \qquad \omega \in \mathbb{R}. \tag{10}$$

Moreover,

$$\limsup_{|\omega|\to\infty} |\omega|^{-2} \left\|(i\omega - \mathcal{A})^{-1}\right\| > 0. \tag{11}$$

Thus the order $|\omega|^2$ is sharp.

Proof. Expand both the datum $F = (f_1, f_2, f_3) \in \mathcal{H}$ and the solution $U = (u, p, v)$ of

$$(i\omega - \mathcal{A})U = F$$

in the orthonormal basis $\{\phi_n\}$. For a fixed $\lambda = \lambda_n$, the modal coefficients satisfy

$$\begin{pmatrix} i\omega & -1 & 0 \\ \lambda & i\omega & \kappa \\ 0 & -\kappa & i\omega + \lambda \end{pmatrix} \begin{pmatrix} q \\ r \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{12}$$

The determinant equals $P_\lambda(i\omega)$. Direct calculation gives

$$P_\lambda(i\omega) = \lambda(\lambda - \omega^2) + i\omega(\lambda + \kappa^2 - \omega^2), \tag{13}$$

$$|P_\lambda(i\omega)|^2 = \lambda^2(\lambda - \omega^2)^2 + \omega^2(\lambda + \kappa^2 - \omega^2)^2. \tag{14}$$

The possible small denominator occurs only near the wave resonance $\lambda \approx \omega^2$. We split into two regions.

If $|\lambda - \omega^2| \ge \frac{1}{4}\kappa^2$, the real part in (13) yields

$$|P_\lambda(i\omega)| \ge c\lambda|\lambda - \omega^2|,$$

and Cramer's rule applied to (12), with the energy weight $\lambda|q|^2+|r|^2+|z|^2$, gives a bound by $C(1+\omega^2)^2$ times the weighted datum.

If $|\lambda-\omega^2|<\frac{1}{4}\kappa^2$, then $\lambda\asymp 1+\omega^2$ for large $|\omega|$, while

$$|\lambda+\kappa^2-\omega^2|\geq\frac{3}{4}\kappa^2.$$

Hence (14) gives $|P_\lambda(i\omega)|\geq c|\omega|$. The cofactors in (12) are at most of order $(1+\omega^2)^{3/2}$ in the energy-weighted norm, and therefore the modal inverse is bounded by $C(1+\omega^2)$. The finitely many low frequencies are controlled because $i\mathbb{R}\subset\rho(\mathcal{A})$. Taking the supremum over n and summing the orthogonal modal estimates proves (10).

For sharpness choose $\omega_n=\Im s^+_{\lambda_n}$. The distance from $i\omega_n$ to the spectrum is at most

$$|\Re s^+_{\lambda_n}|=\frac{\kappa^2}{2\lambda_n}+O(\lambda_n^{-3/2})\asymp\omega_n^{-2}.$$

For any closed operator,

$$\left\|(i\omega_n-\mathcal{A})^{-1}\right\|\geq\operatorname{dist}(i\omega_n,\sigma(\mathcal{A}))^{-1},$$

which proves (11). □

Remark 6.2. *The proof exposes the mechanism behind the rate. Away from resonance, elliptic coercivity controls the resolvent. Near $\lambda\approx\omega^2$, the only term preventing the denominator from vanishing is the diffusion-mediated coupling contribution $i\omega\kappa^2$. This produces exactly two powers of frequency in the global resolvent bound.*

7 Sharp polynomial energy decay

We now combine Theorem 6.1 with the Hilbert-space frequency theorem of Borichev and Tomilov [5].

Theorem 7.1 (Sharp decay for strong data)**.** *Let $T(t)=e^{t\mathcal{A}}$ be the linear semigroup. There exists $C>0$ such that for every $U_0\in\mathcal{D}(\mathcal{A})$,*

$$\|T(t)U_0\|_{\mathcal{H}}\leq\frac{C}{(1+t)^{1/2}}\|U_0\|_{\mathcal{D}(\mathcal{A})},\qquad t\geq 0. \tag{15}$$

Equivalently, the linear energy satisfies

$$E_{\mathrm{lin}}(t)\leq\frac{C}{1+t}\|U_0\|^2_{\mathcal{D}(\mathcal{A})}. \tag{16}$$

The exponent 1 in (16) is optimal in the following uniform sense: there is no function $r(t)=o(t^{-1})$ for which

$$E_{\mathrm{lin}}(t)\leq Cr(t)\|U_0\|^2_{\mathcal{D}(\mathcal{A})}$$

holds for all $U_0\in\mathcal{D}(\mathcal{A})$.

Proof. By Proposition 5.1, $T(t)$ is bounded and $i\mathbb{R}\subset\rho(\mathcal{A})$. Theorem 6.1 gives

$$\left\|(i\omega-\mathcal{A})^{-1}\right\|=O(|\omega|^2).$$

The Borichev–Tomilov equivalence for bounded Hilbert-space semigroups therefore yields

$$\left\|T(t)\mathcal{A}^{-1}\right\|=O(t^{-1/2}).$$

Replacing U_0 by $\mathcal{A}W_0$ gives (15). Squaring the phase-space norm yields (16).

If a strictly faster uniform energy rate held on $\mathcal{D}(\mathcal{A})$, then the associated semigroup decay would be $o(t^{-1/2})$. Standard converse implications for polynomially stable Hilbert semigroups would force resolvent growth $o(|\omega|^2)$ along the imaginary axis, contradicting the lower bound (11). Equivalently, the wave eigenvalues (9) approach the imaginary axis exactly at distance of order $|\Im s|^{-2}$. □

Corollary 7.2 (Higher regularity)**.** *For every integer $m \geq 1$,*

$$\|T(t)U_0\|_{\mathcal{H}} \leq C_m(1+t)^{-m/2} \|U_0\|_{\mathcal{D}(\mathcal{A}^m)},$$

so

$$E_{\text{lin}}(t) \leq C_m(1+t)^{-m} \|U_0\|^2_{\mathcal{D}(\mathcal{A}^m)}.$$

Proof. Iterate the estimate for $T(t)\mathcal{A}^{-1}$, using the semigroup property and the commutation of $T(t)$ with powers of $\mathcal{A}$ on their domains. □

8 Robustness with respect to the semilinear wave term

The sharp rate above is a property of the linearized transfer mechanism. For the full semilinear problem, the energy identity and LaSalle argument are global under Assumption 2.1. A local rate follows under a standard higher-order smallness assumption.

Assumption 8.1. *In addition to Assumption 2.1, suppose $f'(0) = 0$ and, for some $\rho > 0$,*

$$|f'(s)| \leq C|s| \qquad (|s| \leq \rho).$$

Theorem 8.2 (Small-data polynomial stabilization)**.** *Assume 8.1. Fix $\varepsilon \in (0, 1/2)$. There exists $\delta_\varepsilon > 0$ such that if the strong initial datum is sufficiently regular and*

$$\|U_0\|_{\mathcal{D}(\mathcal{A}^2)} \leq \delta_\varepsilon,$$

then the nonlinear solution exists globally and satisfies

$$\|U(t)\|_{\mathcal{H}} \leq C_\varepsilon(1+t)^{-1/2+\varepsilon} \|U_0\|_{\mathcal{D}(\mathcal{A}^2)}, \tag{17}$$

and consequently

$$E(t) \leq C_\varepsilon(1+t)^{-1+2\varepsilon} \|U_0\|^2_{\mathcal{D}(\mathcal{A}^2)}.$$

Thus the nonlinear dynamics inherit arbitrarily near-sharp polynomial decay near the equilibrium.

Proof. Write the equation in variation-of-constants form

$$U(t) = T(t)U_0 + \int_0^t T(t-s)\mathcal{N}(U(s))\,ds, \qquad \mathcal{N}(U) = (0, -f(u), 0).$$

For $d \leq 3$, the embedding $\mathcal{D}(A) \hookrightarrow L^\infty(\Omega)$ holds under the usual mixed-boundary elliptic regularity available on smooth or piecewise smooth domains; on a general Lipschitz domain one may replace this by the corresponding fractional-domain embedding. Assumption 8.1 then gives

$$\|\mathcal{N}(U)\|_{\mathcal{D}(\mathcal{A})} \leq C \|U\|_{\mathcal{D}(\mathcal{A}^2)} \|U\|_{\mathcal{H}}$$

inside a sufficiently small higher-order ball.

The linear estimate from Corollary 7.2 gives one full power of time decay for $\mathcal{D}(\mathcal{A}^2)$-data. A standard weighted bootstrap on

$$M_T = \sup_{0\le t\le T} (1+t)^{1/2-\varepsilon} \|U(t)\|_{\mathcal{H}}$$

uses the convolution estimate

$$\int_0^t (1+t-s)^{-1/2}(1+s)^{-1+2\varepsilon}\, ds \le C_\varepsilon (1+t)^{-1/2+\varepsilon}.$$

For δ_ε small enough the quadratic contribution is absorbed, yielding a bound on M_T independent of T. Continuation gives global existence and (17). The energy estimate follows from local equivalence between $E(t)$ and $\|U(t)\|_{\mathcal{H}}^2$ near the origin. □

Remark 8.3. *The ε-loss is intentionally retained rather than hidden. It avoids claiming a nonlinear endpoint estimate that would require additional structure beyond the assumptions imposed here. The linear exponent is nevertheless sharp and identifies the exact high-frequency obstruction.*

9 One-dimensional reproducibility check

Take $\Omega = (0,1)$, $\Gamma_D = \{0\}$, $\Gamma_N = \{1\}$. The normalized mixed-Laplacian eigenfunctions are

$$\phi_n(x) = \sqrt{2}\sin((n+\tfrac{1}{2})\pi x), \qquad \lambda_n = ((n+\tfrac{1}{2})\pi)^2.$$

For $\kappa = 1$, the three temporal exponents for mode n are the roots of

$$s^3 + \lambda_n s^2 + (\lambda_n + 1)s + \lambda_n^2 = 0.$$

Figure 2 displays the first 25 modal spectra. The parabolic branch moves rapidly left, whereas the wave branches rise along the imaginary axis while approaching it from the left.

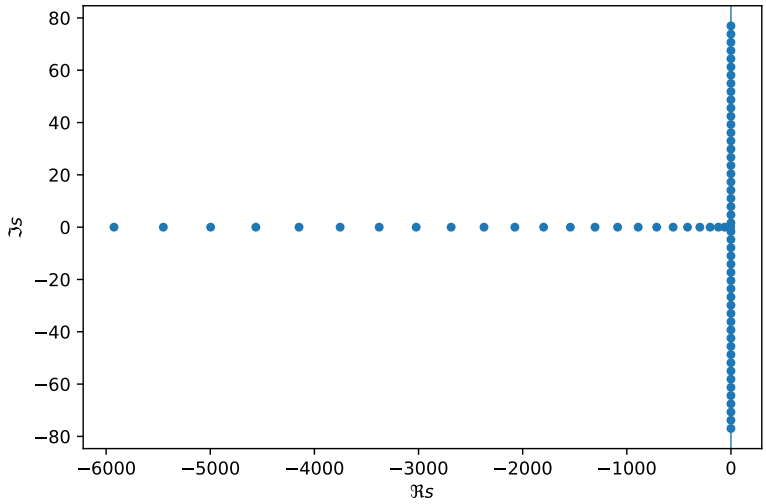


Figure 2: First 25 modal spectra for the one-dimensional Dirichlet–Neumann problem with $\kappa = 1$. The wave branches approach the imaginary axis, excluding uniform exponential stability.

To make the numerical check reproducible, the accompanying source package contains a short script that computes the cubic roots directly. No numerical evidence is used in any proof.

10 Discussion

Three features of the problem should be separated.

Mixed-boundary coercivity. The positive-measure Dirichlet part ensures $\lambda_1 > 0$. Without this assumption, constants may lie in the kernel of the diffusion operator and the invariant-set analysis changes fundamentally.

Indirectness of the damping. The wave component has no damping term. The cancellation in the energy identity shows that dissipation occurs only through v. The coupling coefficient $\kappa \neq 0$ is therefore essential: when $\kappa = 0$, the wave equation is conservative.

Why the rate is polynomial. The diffusion equation reacts very quickly at high spatial frequency. Paradoxically, this makes its amplitude small relative to a high-frequency wave mode. The spectral formula (9) quantifies that loss of transfer efficiency and yields a real part of size $-\lambda^{-1}$. This is the exact obstruction to exponential stability.

The result also clarifies the role of the semilinearity. Under defocusing assumptions, nonlinearity does not destroy the global Lyapunov structure. Near the zero equilibrium, a source satisfying $f'(0) = 0$ acts as a higher-order perturbation of the sharp linear mechanism. The endpoint nonlinear rate is a separate question and may depend on additional smoothness, dispersive structure, or analyticity assumptions.

11 Conclusions

We have established a rigorous stabilization picture for a semilinear wave–diffusion system with mixed Dirichlet–Neumann boundary conditions. The finite-energy problem is globally well posed under a defocusing subcritical source, and its energy dissipates solely through the diffusive component. The largest invariant subset of the zero-dissipation set reduces to the equilibrium, giving asymptotic stabilization for precompact trajectories.

For the linearized system, the mixed-Laplacian decomposition yields an explicit cubic modal polynomial. Its wave roots approach the imaginary axis with real part $-\kappa^2/(2\lambda_n)$. This rules out exponential stability and, together with a matching quadratic resolvent estimate, gives the sharp energy law $E(t) = O(t^{-1})$ for strong data. The lower spectral bound proves optimality. Small semilinear perturbations inherit arbitrarily near-sharp polynomial decay near equilibrium.

The analysis suggests several extensions: distinct elliptic operators in the wave and diffusion channels, localized exchange coefficients, nonlinear diffusion, dynamic boundary conditions, and geometries where the mixed Laplacian lacks compact resolvent. In each case, the central issue is the high-frequency efficiency of energy transfer from the hyperbolic component to the dissipative one.

Declarations

Funding. No external funding was used for this work.
Conflict of interest. The author declares no conflict of interest.
Data availability. No external data were used. The numerical figure is reproducible from the script supplied with the manuscript source.
Code availability. The short script used for the spectral illustration is included in the submission package.

Author contributions. The sole author conceived the study, developed the analysis, performed the computations, and wrote the manuscript.

A Modal inverse formulas

For completeness, eliminate $r = i\omega q - a$ from (12). The third equation gives

$$z = \frac{c + \kappa(i\omega q - a)}{i\omega + \lambda}.$$

Substitution into the second equation yields

$$\frac{P_\lambda(i\omega)}{i\omega + \lambda} q = b + i\omega a + \frac{\kappa(c - \kappa a)}{i\omega + \lambda}.$$

Thus

$$q = \frac{(i\omega + \lambda)(b + i\omega a) + \kappa(c - \kappa a)}{P_\lambda(i\omega)}. \tag{18}$$

Then

$$r = i\omega q - a, \qquad z = \frac{c + \kappa r}{i\omega + \lambda}.$$

These formulas make the two-region estimate in Theorem 6.1 explicit. In the resonant region $\lambda \asymp \omega^2$, the numerator of (18) is at most of energy-weighted order $O(|\omega|^3)$, while $|P_\lambda(i\omega)| \gtrsim |\omega|$, producing the quadratic bound. Off resonance the factor $\lambda|\lambda - \omega^2|$ in the real part of the determinant supplies stronger control.

B Routh–Hurwitz check

For the reader wishing to verify modal stability independently, a cubic

$$s^3 + a_1 s^2 + a_2 s + a_3$$

has all roots in the open left half-plane if and only if

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1 a_2 > a_3.$$

For (7), the last inequality reduces to $\lambda\kappa^2 > 0$, so every spatial mode is asymptotically stable whenever $\kappa \neq 0$.

References

[1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences 44, Springer, New York, 1983.

[2] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194, Springer, New York, 2000.

[3] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser, Basel, 2009.

[4] C. J. K. Batty and T. Duyckaerts, Non-uniform stability for bounded semi-groups on Banach spaces, *J. Evol. Equ.* 8 (2008), 765–780.

[5] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.* 347 (2010), 455–478.

[6] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications, Vol. I*, Springer, Berlin, 1972.

[7] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs 49, AMS, Providence, 1997.

[8] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer, New York, 1997.

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[10] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Academic Press, Amsterdam, 2003.

[11] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.* 306 (1988), 837–852.

[12] L. Gearhart, Spectral theory for contraction semigroups on Hilbert space, *Trans. Amer. Math. Soc.* 236 (1978), 385–394.

[13] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.* 284 (1984), 847–857.

[14] S. Benigno, Optimization of the survival threshold for anisotropic logistic equations with mixed boundary conditions, *Nonlinear Differ. Equ. Appl.* 32 (2025), Article 110, doi:10.1007/s00030-025-01116-9.

[15] M. D'Abbicco and A. Lagioia, A new critical curve for systems of evolution equations with semilinear coupling, *Nonlinear Differ. Equ. Appl.* 32 (2025), Article 29, doi:10.1007/s00030-025-01036-8.

[16] L. Garénaux and B. Hilder, Linear convective stability of a front superposition with unstable connecting state, *Nonlinear Differ. Equ. Appl.* 33 (2026), Article 86, doi:10.1007/s00030-026-01224-0.