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Research Article

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Posted Date: October 13th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11208670/v1>

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Additional Declarations: No competing interests reported.

Open-Loop Nash Limits for Caputo Differential Games: Preceding-Node L1 Discretization

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Abstract

We study the relation between continuous-time and discrete-time Nash equilibria in a finite-horizon nonzero-sum differential game with Caputo memory. Preceding-node L1 time stepping replaces the continuous game by a sequence of finite games. We show that, under a uniform state bound, any vanishing-mesh sequence of increasingly accurate discrete Nash profiles has a subsequence whose controls converge weak-* and whose states converge uniformly to a genuine open-loop Nash equilibrium of the continuous game. The strategic issue is stronger than trajectory convergence: unilateral deviations must remain admissible and retain their payoff comparisons through the hereditary state law. A stopped unilateral-deviation recovery provides this passage, while exact discrete fractional summation by parts identifies the state limit. We also establish the discrete existence needed to populate the convergence theorem. For a fully specified nonlinear three-player game, certified full-box curvature yields a pure global Nash equilibrium on every sufficiently fine mesh. The result therefore links equilibria of the finite discrete-time games generated by the numerical dynamics to open-loop Nash equilibria of the underlying continuous-time hereditary game.

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MSC Classification: 49N70 , 91A23 , 34A08 , 65L20 , 65G20

1 Introduction

Consider a finite-horizon nonzero-sum differential game whose state law has Caputo memory. A standard computational route replaces that state law by a time-stepping recurrence and solves the finite game induced by the recurrence. This raises a game-theoretic question: as the mesh is refined, do Nash equilibria of those finite games approach a genuine equilibrium of the original game? For memory-dependent dynamics, convergence of trajectories alone does not answer it because every unilateral-deviation inequality must survive the same nonlocal limit.

Question. For the finite Nash games induced by a preceding-node L1 discretization of a finite-horizon nonzero-sum Caputo differential game, do vanishing-mesh discrete equilibria converge to genuine open-loop Nash equilibria of the continuous game?

Answer. Yes, along subsequences under a uniform state bound. More generally, if discrete ε_h -Nash profiles satisfy $\varepsilon_h \rightarrow 0$ and their nodal states are uniformly bounded, then a subsequence converges weak-* in controls and uniformly in states to a continuous open-loop Nash equilibrium. The proof preserves the full unilateral-deviation quantifier: every admissible L^∞ unilateral deviation is recovered on the meshes and carried through the nonlocal state law. For the concrete Set-A game below, the required state bounds and pure discrete Nash equilibria are themselves proved for every sufficiently fine mesh, so the abstract limit theorem applies to equilibria of the finite games that the L1 recurrence actually induces.

Time discretization therefore produces a sequence of dynamic games, not merely a sequence of state approximations. The strategic question is whether the equilibrium structure of those finite games remains faithful to the continuous game as the mesh vanishes. Hereditary dynamics make this link especially delicate: a unilateral change in one player's control changes the subsequent memory history, so convergence of equilibrium trajectories alone cannot preserve the Nash comparisons that define equilibrium.

The closest dynamic-game background concerns what happens when a continuous game is replaced by a computable finite or discrete-time problem. Bressan's tutorial uses time-discretized static games to motivate continuous-time solution concepts and gives a formal limiting analysis for the resulting Hamilton–Jacobi system ([Bressan 2011](#)). Cacace, Cristiani and Falcone approximate Nash solutions of infinite-horizon ODE games through Hamilton–Jacobi dynamic programming, while Nikooeinejad, Delavarkhalafi and Heydari compute open-loop Nash solutions through a Chebyshev pseudospectral reduction of the Pontryagin system ([Cacace et al. 2013](#); [Nikooeinejad et al. 2016](#)). A recent preprint by J. de Frutos, V. Gatón and J. Novo, *On Discrete-Time Approximations to Infinite Horizon Differential Games* (arXiv:2112.03153v3, 2026), proves that stationary Markov Nash equilibria of forward-Euler/rectangle-rule infinite-horizon ODE games are ε -Nash equilibria of the continuous game for small time steps. Our finite-horizon setting develops this discrete-to-continuous question for open-loop Nash equilibrium under a nonlocal Caputo state law and for finite games induced directly by the time-stepping recurrence.

Fractional-game and fractional-control work supplies the memory-side background. Gomoyunov proves the value of a two-player zero-sum Caputo game by approximation with first-order retarded games and constructs feedback procedures (Gomoyunov 2020). This places approximation squarely within fractional dynamic-game analysis. We develop the corresponding equilibrium-passage problem for nonzero-sum open-loop Nash play in the finite games generated by an L1 recurrence. A recent preprint by K. Buda, B. V. Rathish Kumar and A. Rath, *Nash Equilibrium of Bi-objective Optimal Control of Fractional Space-Time Parabolic PDE* (arXiv:2512.07327, 2025), formulates a two-control Caputo space-time fractional PDE Nash problem and uses L1 time discretization computationally; Amini, Nikooeinejad, Akrami and Heydari study infinite-horizon nonzero-sum fractional systems numerically (Amini et al. 2026). Sharp L1 analysis supplies the startup-sensitive numerical background used below (Kopteva 2019; Liao et al. 2019). The preceding-node evaluation adopted here is not a cosmetic implementation choice: node n is generated only from state and control information preceding that node, and this causality is used directly in the stopped first-exit and unilateral-deviation recovery arguments.

General equilibrium-approximation theory provides the conceptual framework for passing Nash inequalities between approximating games. Altman, Pourtallier, Haurie and Moresino formulate correspondences between approximating and limiting strategy spaces and give conditions under which exact or near equilibria transfer between the games (Altman et al. 2000). Gürkan and Pang develop direct and variational convergence conditions for approximate Nash problems, including multi-epiconvergence, while Bagh gives a variational-convergence framework under which cluster points of equilibria of approximating games remain equilibria of the limit game (Gürkan and Pang 2009; Bagh 2010). Dilmé studies limits of equilibria across finite discretizations of infinite normal-form games (Dilmé 2026). These results identify the kind of recovery and semicontinuity properties needed for equilibrium passage. Our task is to establish those properties from the structure of a nonlocal numerical game whose state operator, induced payoff functions and finite-dimensional strategy spaces all vary with the mesh.

The main contribution is a Nash-limit theorem that establishes when equilibrium is preserved in passing from the finite discrete-time games induced by the preceding-node L1 recurrence to the underlying continuous-time differential game. Control-affine weak-* compactness and exact discrete fractional summation by parts identify the nonlocal state limit. A stopped unilateral-deviation recovery then reconstructs every admissible L^∞ unilateral deviation on the meshes, so the full global Nash inequality passes to the Caputo limit. The abstract theorem assumes a uniform bound on the discrete states; the concrete Set-A analysis establishes such a bound uniformly over the admissible strategy box.

For the concrete nonlinear three-player game, the second contribution closes the existence side for a fully specified nonlinear three-state Caputo game using the Set-A baseline coefficient vector reported by George Lymperis and Lambros B. Drossos in the Research Square preprint *Common traits, different roles: structural position and non-linear dynamics in three-party bargaining* (Version 1, 2026; <https://doi.org/10.21203/rs.3.rs-10962752/v1>). Computer-assisted fractional-polynomial/Bernstein certificates

establish full-box strong convexity in each player's log-control variables, with certified margin $c_{\text{FB}} > 0.1941$. An endpoint-weighted defect estimate and a positive discrete fractional resolvent transfer the certified curvature across the L1 startup layer. Each player then has a unique global best response, Brouwer yields at least one pure global discrete Nash equilibrium on every sufficiently fine mesh, and the Nash-limit theorem produces a continuous open-loop Nash subsequential limit.

2 Continuous and discrete games

Fix $T > 0$, $0 < \nu < 1$, and players $\mathcal{I} = \{1, \dots, P\}$. Player i chooses

$$v_i \in L^\infty(0, T), \quad |v_i(t)| \leq \log m \quad \text{a.e.},$$

with $m > 1$. Set

$$z_i = e^{v_i} \in K_m := [m^{-1}, m].$$

The state is defined by

$$x(t) = x_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} F(x(s), z(s)) ds, \quad (1)$$

or equivalently by ${}^C D_t^\nu x = F(x, z)$ in the generalized Caputo sense. Throughout the abstract convergence argument,

$$F(x, z) = F_0(x) + \sum_{i=1}^P z_i H_i(x). \quad (2)$$

The control-affine structure is essential in the weak-* identification step. Throughout the abstract theory, F_0 and H_i are assumed continuous and locally Lipschitz in x , uniformly on compact state sets. Thus every bounded admissible control profile has a unique maximal Volterra solution; the maximal interval need not a priori equal $[0, T]$.

Remark 2.1 (Role of the exponential reparameterization). *The transformation $z_i = e^{v_i}$ exposes the affine control structure used in the weak-* limit. Once $x_h \rightarrow x$ uniformly, the factors $H_i(x_h)$ converge strongly and can be paired with $z_{i,h} \xrightarrow{*} z_i$ in (2). The strong-convexity argument is carried out directly in the log-control variables v , where the required curvature is certified on the full admissible box.*

Whenever a full-horizon solution is bounded, its forcing is bounded and the Volterra representation gives the standard fractional modulus

$$|x(t) - x(s)| \leq C|t - s|^\nu, \quad 0 \leq s, t \leq T.$$

Let

$$\ell_i(x) = \left(\frac{x_i - r_i}{s_i} \right)^2, \quad \psi_i(z_i) = \kappa_i (\log z_i)^2.$$

The continuous objective is

$$J_i(z) = \int_0^T [\ell_i(x(t)) + \psi_i(z_i(t))] dt. \quad (3)$$

Definition 2.2 (Extended full-horizon game). If an admissible profile fails to generate a state on the entire declared horizon, its cost is $+\infty$. An open-loop Nash equilibrium is a full-horizon admissible profile z^* satisfying

$$J_i(z_i^*, z_{-i}^*) \leq J_i(w_i, z_{-i}^*)$$

for every player i and every admissible $w_i \in L^\infty(0, T; K_m)$. Whenever global state existence holds on the entire admissible box, this coincides with the usual open-loop Nash notion. The extended convention assigns infinite cost precisely to profiles whose maximal Volterra state fails to reach the declared horizon, thereby keeping the abstract Nash comparison defined on the full admissible control box.

The change $v_i \leftrightarrow z_i = e^{v_i}$ is a bijection and preserves states, costs and unilateral comparisons. The abstract Nash-limit theorem assumes that each effort density ψ_i is convex on K_m . For the logarithmic effort used here,

$$\psi_i''(z) = 2\kappa_i \frac{1 - \log z}{z^2},$$

so this holds whenever $1 < m \leq e$ (strictly for $m < e$); the production value is $m = 2$.

2.1 Finite game induced by the preceding-node L1 recurrence

Let $h = T/N$, $t_n = nh$, and define

$$b_r = \frac{(r+1)^{1-\nu} - r^{1-\nu}}{\Gamma(2-\nu)}, \quad r \geq 0.$$

The complete admissible discrete strategy box is

$$\mathcal{Z}_h := K_m^{PN},$$

with equivalent log-coordinate box $[-\log m, \log m]^{PN}$. For an intervalwise-constant discrete control z_h^j on $[t_j, t_{j+1})$, the induced recurrence is

$$x_h^0 = x_0, \quad h^{-\nu} \sum_{k=1}^n b_{n-k} (x_h^k - x_h^{k-1}) = F(x_h^{n-1}, z_h^{n-1}), \quad n = 1, \dots, N. \quad (4)$$

It is explicit because the right-hand side is evaluated at node $n-1$.

Remark 2.3 (Preceding-node causality). *To compute node n , the forcing uses (x_h^{n-1}, z_h^{n-1}) and the L1 memory uses only already-computed state increments. Consequently a stopped prefix through a first-exit node is determined entirely by the pre-exit data. This one-step lag is used repeatedly in the stopped compactness and non-circular certificate arguments.*

With trapezoidal weights $\tau_0 = \tau_N = 1/2$ and $\tau_n = 1$ otherwise, the exact discrete objective is

$$J_{i,h}(z_h) = h \sum_{n=0}^N \tau_n \ell_i(x_h^n) + h \sum_{j=0}^{N-1} \psi_i(z_{i,h}^j). \quad (5)$$

A global discrete ε_h -Nash profile satisfies

$$J_{i,h}(z_{i,h}, z_{-i,h}) \leq J_{i,h}(w_{i,h}, z_{-i,h}) + \varepsilon_h \quad (6)$$

for every admissible discrete unilateral deviation.

For continuous recovery we average in the *transformed* variable:

$$(\Pi_h z)^j = \frac{1}{h} \int_{t_j}^{t_{j+1}} z(t) dt.$$

Thus $\Pi_h z$ remains in K_m , and $\Pi_h z \rightarrow z$ strongly in $L^p(0, T)$ for every finite p . The recovery is defined by averaging z itself; the effort term is evaluated from $\log(\Pi_h z)$, so no commutation of averaging and logarithms is used.

3 Bounded L1 trajectories and Nash limits

3.1 Inverse kernel and compactness

Put $U_h^n = x_h^n - x_0$. Let $q = \{q_r\}_{r \geq 0}$ be the convolution inverse of the L1 value kernel.

Lemma 3.1 (Positive inverse L1 kernel). *The inverse weights satisfy*

$$q_r \geq 0, \quad q_{r+1} \leq q_r, \quad (7)$$

and

$$h^\nu \sum_{r=0}^{d-1} q_r \leq C_\nu (dh)^\nu, \quad C_\nu = \frac{1}{\Gamma(1+\nu)}. \quad (8)$$

Proof These are the complete-monotonicity-preserving inverse-kernel properties of the L1 value sequence; see (Li and Wang 2021, Secs. 2–3). The explicit constant in (8) is the standard fractional-integral mass bound for this inverse kernel. $\square$

Writing $g_h^n = F(x_h^{n-1}, z_h^{n-1})$, the recurrence has the exact inverse form

$$x_h^n - x_0 = h^\nu \sum_{j=1}^n q_{n-j} g_h^j. \quad (9)$$

Lemma 3.2 (Uniform fractional modulus). *If $\sup_h \max_n |x_h^n| \leq R$, then*

$$|x_h^n - x_h^m| \leq C |t_n - t_m|^\nu$$

uniformly in h .

Proof The state bound and compactness of the control box give $|g_h^j| \leq M$. Put

$$S_d := \sum_{r=0}^{d-1} q_r, \quad S_0 := 0,$$

and let $d = n - m > 0$. Subtracting the two inverse representations gives an old-history part and the last d forcing terms. Since q_r is nonnegative and decreasing,

$$\sum_{j=1}^m (q_{m-j} - q_{n-j}) = S_m + S_d - S_n \leq S_d,$$

whereas the last d terms contribute exactly S_d . Hence

$$|x_h^n - x_h^m| \leq 2Mh^\nu S_d \leq 2MC_\nu (dh)^\nu.$$

The piecewise-linear interpolants are therefore uniformly bounded and equicontinuous, while the preceding-node step interpolants differ from them by $O(h^\nu)$. $\square$

3.2 Exact summation by parts and state identification

For a grid function φ^n with $\varphi^N = 0$, define the right discrete operator

$$(\mathcal{D}_{T,h}^\nu \varphi)^k = h^{-\nu} \left[b_0 \varphi^k + \sum_{n=k+1}^N (b_{n-k} - b_{n-k-1}) \varphi^n \right], \quad 1 \leq k \leq N-1. \quad (10)$$

For $U_h^0 = 0$ one has the exact summation-by-parts identity

$$h \sum_{n=1}^N \varphi^n (\delta_h^\nu U_h)^n = h \sum_{k=1}^{N-1} U_h^k (\mathcal{D}_{T,h}^\nu \varphi)^k. \quad (11)$$

Indeed, after expanding the left side and interchanging the finite sums, the coefficient of U_h^k is

$$h^{1-\nu} \left[b_0 \varphi^k + \sum_{n=k+1}^N (b_{n-k} - b_{n-k-1}) \varphi^n \right],$$

which is exactly $h(\mathcal{D}_{T,h}^\nu \varphi)^k$. The terms in U_h^0 vanish by $U_h^0 = 0$, while the potential terminal term disappears because $\varphi^N = 0$. If $\varphi \in C_c^\infty(0, T)$ and $\varphi^n = \varphi(t_n)$, then φ vanishes in neighbourhoods of both endpoints, so $\varphi^N = 0$ is automatic for all sufficiently fine meshes and imposes no restriction on the distributional test class. The sampled support stays away from both endpoints. Writing $\tilde{\varphi}(s) = \varphi(T-s)$ turns $\mathcal{D}_{T,h}^\nu$ into the ordinary left L1 operator applied to the smooth reversed test function; the standard smooth-data L1 consistency estimate therefore gives uniform convergence on the sampled support to $D_{T-}^\nu \varphi$. This is precisely why the endpoint defect used later in the persistence proof does not obstruct the compactness theorem.

Theorem 3.3 (Compactness and consistency). *Let $h_k \downarrow 0$, and let (z_{h_k}, x_{h_k}) solve the exact recurrence (4). If*

$$\sup_k \max_n |x_{h_k}^n| < \infty,$$

then a subsequence satisfies

$$z_{h_k} \xrightarrow{*} z \quad \text{in } L^\infty(0, T),$$

and the piecewise-linear states converge uniformly to $x \in C^{0,\nu}([0, T])$. The limit is the Volterra state (1) driven by z .

Proof Weak-* compactness of the control box gives a weak-* convergent control subsequence. The piecewise-linear state interpolants are uniformly bounded and equicontinuous by lemma 3.2; Arzelà–Ascoli therefore gives a further subsequence converging uniformly on $[0, T]$. Apply (11) to $U_h = x_h - x_0$ and $\varphi \in C_c^\infty(0, T)$. Uniform state convergence and strong convergence of $\mathcal{D}_{T,h}^\nu \varphi$ pass the left side to the distributional Caputo pairing.

Let $\hat{x}_h$ denote the preceding-node step interpolant used by the recurrence. By lemma 3.2, $\|\hat{x}_h - x_h\|_{L^\infty} = O(h^\nu)$, so $F_0(\hat{x}_h)$ and $H_i(\hat{x}_h)$ converge uniformly to $F_0(x)$ and $H_i(x)$. Hence, for each i , the factor multiplying $z_{i,h}$ converges strongly in L^1 ; the weak-* convergence of $z_{i,h}$ therefore yields

$$\int_0^T z_{i,h} H_i(x_h) \varphi dt \longrightarrow \int_0^T z_i H_i(x) \varphi dt.$$

Thus the limit satisfies the generalized Caputo equation on $(0, T)$. Since $x_h^0 = x_0$ and the convergence is uniform, $x(0) = x_0$. Fractional integration of the distributional identity gives (1); equivalently one may invoke the generalized Caputo fundamental theorem of Li and Liu (2018). $\square$

Remark 3.4 (Additional strong state convergence). *The uniform convergence in theorem 3.3 immediately implies strong convergence of both the piecewise-linear and preceding-node state interpolants in $L^p(0, T)$ for every finite p . This is occasionally convenient in the cost passage but is not a separate compactness hypothesis.*

3.3 Recovery of arbitrary unilateral deviations

Throughout this subsection, $|\cdot|$ is the Euclidean norm on the state space and $B_R := \{x : |x| \leq R\}$.

Lemma 3.5 (Stopped no-discrete-escape). *Fix a player i , discrete opponent controls converging weak- $*$ to z_{-i} , and an arbitrary admissible $w_i \in L^\infty(0, T; K_m)$ whose mixed continuous state exists on $[0, T]$. Drive the exact $L1$ recurrence by $\Pi_h w_i$ against the discrete opponents. The resulting mixed discrete states are uniformly bounded for all sufficiently fine meshes.*

Proof Let the continuous mixed state remain in a ball B_R , and choose a larger ball $B_{R+\delta}$. If the claim fails, select first exit nodes n_k . The recurrence is causal: only the prefix through n_k matters, so no post-exit state or control extension is used.

If $t_{n_k} \rightarrow 0$, the inverse representation and bounded pre-exit forcing give

$$|x_{h_k}^{n_k} - x_0| \leq C t_{n_k}^\nu \rightarrow 0,$$

contradicting first exit from the fixed larger ball. Otherwise a subsequence has $t_{n_k} \rightarrow \tau > 0$. Up to each first exit, preceding-node evaluation keeps all forcing evaluations inside a fixed compact set, so lemma 3.2 applies to the stopped prefixes. The proof of theorem 3.3, restricted to test functions supported in $(0, \tau)$, identifies the stopped limit with the unique mixed continuous state on every $[0, \tau - \varepsilon]$.

It remains to identify the exit node itself. For fixed $t < \tau$ and k large enough that $t < t_{n_k}$, the common Hölder modulus gives

$$|x_{h_k}^{n_k} - x(\tau)| \leq C |t_{n_k} - t|^\nu + |x_{h_k}(t) - x(t)| + |x(t) - x(\tau)|.$$

First let $k \rightarrow \infty$ and then $t \uparrow \tau$. The right side tends to zero, so $x_{h_k}^{n_k} \rightarrow x(\tau)$. The continuous mixed state lies in B_R at τ , hence strictly inside $B_{R+\delta}$, contradicting first exit. The same argument covers $\tau = T$. $\square$

Lemma 3.6 (Cost passage). *Suppose $z_h \xrightarrow{*} z$ in L^∞ and the associated state interpolants converge uniformly to x . Then the exact discrete tracking quadrature satisfies*

$$h \sum_{n=0}^N \tau_n \ell_i(x_h^n) \longrightarrow \int_0^T \ell_i(x(t)) dt,$$

and the exact discrete effort term satisfies

$$\int_0^T \psi_i(z_i(t)) dt \leq \liminf_{h \rightarrow 0} h \sum_{j=0}^{N-1} \psi_i(z_{i,h}^j).$$

If $w_i \in L^\infty(0, T; K_m)$ is fixed, the recovered controls $\Pi_h w_i$ satisfy

$$\Pi_h w_i \rightarrow w_i \quad \text{strongly in } L^p(0, T) \quad (1 \leq p < \infty),$$

and, whenever their mixed states converge uniformly,

$$J_{i,h}(\Pi_h w_i, z_{-i,h}) \longrightarrow J_i(w_i, z_{-i}).$$

Proof Let Q_h denote the trapezoidal functional. Uniform state convergence and continuity of ℓ_i give $\|\ell_i(x_h) - \ell_i(x)\|_\infty \rightarrow 0$, and hence

$$|Q_h[\ell_i(x_h)] - \int_0^T \ell_i(x) dt| \leq T \|\ell_i(x_h) - \ell_i(x)\|_\infty + |Q_h[\ell_i(x)] - \int_0^T \ell_i(x) dt| \rightarrow 0,$$

because $\ell_i(x(\cdot))$ is continuous. Since the discrete controls are represented by their piecewise-constant extensions,

$$h \sum_j \psi_i(z_{i,h}^j) = \int_0^T \psi_i(z_{i,h}(t)) dt.$$

Weak-* convergence on the bounded control box implies weak L^2 convergence: on the finite interval every L^2 test function belongs to L^1 , so it is an admissible L^∞ weak-* test. Convexity and continuity of ψ_i then give weak lower semicontinuity. For recovery, cell averages converge strongly in every finite L^p , remain in K_m , and ψ_i is Lipschitz on that compact interval, so the effort integrals converge strongly. This asymmetry is deliberate and essential: the incumbent sequence is only weakly compact and therefore passes through a liminf, whereas the unilateral competitor is a recovery sequence chosen by us and may be made strongly convergent. $\square$

Theorem 3.7 (Subsequential Nash limit). *Let $h_k \downarrow 0$. Suppose z_{h_k} are global discrete ε_{h_k} -Nash profiles with $\varepsilon_{h_k} \rightarrow 0$, and their nodal states have a common bound. Then a subsequence converges weak-* in controls and uniformly in states to an open-loop Nash equilibrium of the extended continuous game.*

Proof Apply theorem 3.3 to the incumbents. Fix player i and an admissible $w_i \in L^\infty(0, T; K_m)$ with finite extended cost. By lemma 3.5, the recovered mixed discrete states are uniformly bounded. Every subsequence therefore has a further subsequence converging uniformly; theorem 3.3 and uniqueness of the mixed Volterra state force every such cluster point to be the same state driven by (w_i, z_{-i}^*) . Hence the entire recovery-cost sequence has the unique cluster value $J_i(w_i, z_{-i}^*)$. Using lemma 3.6 in the discrete Nash inequality gives

$$J_i(z_i^*, z_{-i}^*) \leq \liminf_k J_{i,h_k}(z_{h_k}) \leq \liminf_k J_{i,h_k}(\Pi_h w_i, z_{-i,h_k}) = J_i(w_i, z_{-i}^*).$$

The limit incumbent has finite cost because its full-horizon state is bounded and the running costs are continuous on the compact state/control range. Any admissible deviation with infinite extended cost satisfies the Nash inequality immediately. Every finite-cost deviation is covered by the recovery argument above. Hence the Nash inequality holds over the full admissible unilateral strategy class. $\square$

Lemma 3.8 (Stopped compactness for complete profiles). *Let $h_k \downarrow 0$ and let z_{h_k} be arbitrary complete admissible discrete profiles. Suppose that, after a subsequence, $z_{h_k} \xrightarrow{*} z$, the continuous state driven by z exists on $[0, T]$ and remains in a ball B_R . Then the exact L^1 states cannot have first exits from any fixed larger ball $B_{R+\delta}$ for all sufficiently large k .*

Proof Assume first exits n_k exist. Before n_k all preceding-node forcing evaluations lie in the fixed compact ball $B_{R+\delta}$, so the stopped prefixes have the common modulus of lemma 3.2. The summation-by-parts argument of theorem 3.3 identifies every stopped subsequential limit with the continuous state driven by the weak-* limit control on $(0, \tau)$, where $t_{n_k} \rightarrow \tau$ after extraction. If $\tau = 0$, the inverse representation gives $|x_{h_k}^{n_k} - x_0| \leq C t_{n_k}^\nu \rightarrow 0$. If $\tau > 0$,

the endpoint argument from lemma 3.5 upgrades identification on compact subintervals to $x_{h_k}^{r_k} \rightarrow x(\tau)$. In either case the limit lies in B_R , contradicting first exit from $B_{R+\delta}$. $\square$

4 Concrete Set-A Caputo game

4.1 Game specification

We now instantiate the abstract theory in a three-state nonlinear Caputo game. The baseline numerical coefficient vector is Set A reported in the Lymperis–Drossos Research Square preprint identified in the Introduction. That preprint supplies only the Set-A coefficient values identified in table 1; the initial state, fractional order, horizon, strategic control parameterization, objectives, target, normalization and admissible control box are specified in the present paper.

For each player, $B_i^{\text{ref}} > 0$ denotes the baseline coefficient listed in table 1. Player i chooses a log-control v_i and the corresponding positive multiplier

$$z_i(t) = e^{v_i(t)} \in [1/2, 2], \quad \mathcal{B}_i(t) := B_i^{\text{ref}} z_i(t).$$

Thus $v_i = 0$ is the reference action $z_i = 1$, while positive and negative log-controls scale the deployed coefficient multiplicatively. The parameterization preserves positivity and makes the state law affine in z , exactly as required by the weak-* identification argument.

Quantity	Value	Status
$(C_i, Y_i)_{i=1}^3$	(0.7, 0.4); (0.6, 0.4); (0.5, 0.6)	from preprint above
(α, β, γ)	(0.5, 1.2, 0.5)	from preprint above
B^{ref}	(0.2, 0.2, 0.3)	from preprint above
x_0	(0.6, -0.3, 0.8)	specified here
r	(0, 0, 0)	specified here
s	(0.6, 0.3, 0.8)	specified here
(ν, T, κ_i, m)	(0.7, 4, 0.3, 2)	specified here

Table 1 Complete specification of the concrete Set-A Caputo game. The preprint identified in the text supplies only the Set-A coefficient values marked in the table; the initial condition and the strategic and fractional game specification are introduced here.

With $\alpha, \beta, \gamma > 0$ denoting the three interaction coefficients, define the controlled Caputo state law by

$$\begin{aligned} {}^C D_t^\nu x_1 &= \alpha B_1^{\text{ref}} z_1 x_2 - (\alpha C_1 + \gamma Y_1) x_1 - Y_1 x_1 x_3, \\ {}^C D_t^\nu x_2 &= \beta x_1 - [C_2 + (1 + \gamma) Y_2] x_2 - B_2^{\text{ref}} z_2 x_1 x_3, \\ {}^C D_t^\nu x_3 &= B_3^{\text{ref}} z_3 x_1 x_2 - (C_3 + \gamma Y_3) x_3. \end{aligned}$$

Substituting the values in table 1 gives

$${}^C D_t^\nu x_1 = 0.1z_1x_2 - 0.55x_1 - 0.4x_1x_3, \quad (12a)$$

$${}^C D_t^\nu x_2 = 1.2x_1 - 1.2x_2 - 0.2z_2x_1x_3, \quad (12b)$$

$${}^C D_t^\nu x_3 = 0.3z_3x_1x_2 - 0.8x_3. \quad (12c)$$

For this game $r = (0, 0, 0)$ and $s = (0.6, 0.3, 0.8) = |x_0|$ componentwise, so the tracking term is normalized to the opening-state scale. With $\kappa_i = 0.3$ and $z_i = e^{v_i}$, each player's objective is

$$J_i(v) = \int_0^4 \left[\left(\frac{x_i(t)}{s_i} \right)^2 + 0.3v_i(t)^2 \right] dt, \quad |v_i(t)| \leq \log 2. \quad (13)$$

The target $r = 0$ is an analytically substantive part of the certified realization. The target enters the second-variation and reverse-density terms used in the full-box curvature bound; the certificate and the resulting fine-mesh existence theorem therefore apply to the zero-target game specified here. Equations (12), the common initial state x_0 , and (13) define the concrete continuous game completely. Its preceding-node L1 games are exactly the discrete games of section 2 applied to this specification.

4.2 State bounds

Proposition 4.1 (Continuous Set-A boundedness). *Every admissible control profile generates a unique state on $[0, T]$ that remains in*

$$K_A = [-0.61, 0.61] \times [-0.68, 0.68] \times [-0.32, 0.81]. \quad (14)$$

Proof Because the vector field is measurable in time through the control and locally Lipschitz in x uniformly on compact sets, the Volterra equation has a unique maximal local solution. Let d_j be the inward distance to a face of K_A . Direct multi-affine face algebra gives positive inward forcing margins. On the two x_1 faces one may use the constant relaxation rate

$$\mu_1 = 0.55 + 0.4(0.81) = 0.874;$$

indeed, on K_A , the natural x_1 relaxation rate is $\mu(t) = 0.55 + 0.4x_3(t) \leq 0.55 + 0.4(0.81) = 0.874 = \mu_1$, so before first exit the additional term $(\mu_1 - \mu(t))d_1$ is nonnegative. The x_2 and x_3 faces use the constant rates 1.2 and 0.8.

The finite face calculation may be summarized as follows; each margin is the minimum of a multi-affine expression over the corresponding face and control box, hence is attained at a vertex.

Face pair	constant rate	certified inward margin
$x_1 = -0.61$ or $x_1 = 0.61$	$\mu_1 = 0.874$	$\delta \geq 6071/50000$
$x_2 = -0.68$ or $x_2 = 0.68$	$\mu_2 = 1.2$	$\delta \geq 37/6250$
$x_3 = 0.81$	$\mu_3 = 0.8$	$\delta \geq 4989/12500$
$x_3 = -0.32$	$\mu_3 = 0.8$	$\delta \geq 89/12500$

Thus, up to a hypothetical first face hit,

$${}^C D_t^\nu d_j + \mu_j d_j \geq \delta_j > 0.$$

For each face the scalar comparison has the explicit variation-of-constants representation

$$d_j(t) = E_\nu(-\mu_j t^\nu) d_j(0) + \int_0^t (t-s)^{\nu-1} E_{\nu,\nu}(-\mu_j(t-s)^\nu) \rho_j(s) ds, \quad \rho_j(s) \geq \delta_j > 0.$$

Pollard's complete monotonicity theorem makes $E_\nu(-\lambda t^\nu)$ nonnegative and nonincreasing; moreover

$$t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu) = -\lambda^{-1} \frac{d}{dt} E_\nu(-\lambda t^\nu) \geq 0.$$

Thus both kernels are nonnegative (Pollard 1948), so every d_j stays positive. Hence no face can be reached on the maximal interval. The a priori bound prevents blow-up, so continuation gives existence through T . $\square$

Remark 4.2 (Role of K_A). *The rectangle K_A is a convenient global invariant set; the later computer-assisted argument uses a separate reachable enclosure. The later comparison box B_{FB} is chosen for the computer-assisted argument and may be tighter in some coordinates and looser in others; only containment of the certified reachable set is required.*

Proposition 4.3 (Uniform sufficiently-fine L1 state bound). *There exist $h_0 > 0$ and $R_A < \infty$ such that every admissible exact Set-A L1 trajectory with $h < h_0$ satisfies $\max_n |x_h^n| \leq R_A$.*

Proof Otherwise choose a vanishing-mesh sequence whose states leave every fixed larger ball. Weak-* compactness of the complete control box gives a subsequence $z_h \xrightarrow{*} z$. By proposition 4.1, the continuous state driven by z exists through T and remains in K_A . Applying lemma 3.8 to a fixed ball containing K_A contradicts the assumed escapes. $\square$

5 Full-box Set-A convexity certificate

The main application theorem uses logarithmic controls $v_i \in [-\log 2, \log 2]$. In these coordinates the effort Hessian is exactly

$$\frac{d^2}{dv_i^2}(\kappa_i v_i^2) = 2\kappa_i = 0.6. \quad (15)$$

The task is therefore to show that the state-mediated second variation is uniformly larger than -0.6 on the entire joint strategy box.

5.1 Second variation and reverse density

Fix a continuous admissible profile and first take a bounded direction $\eta \in L^\infty(0, T) \cap L^2(0, T)$, as occurs for differences of admissible log-controls. Let Y be the first state variation. With

$$A = F_x(x, z), \quad b_i = F_{v_i}(x, z),$$

it satisfies

$${}^C D_t^\nu Y = AY + b_i \eta, \quad Y(0) = 0. \quad (16)$$

For $f, g \in L^2(0, T)$ define the left and right fractional integrals I_{0+}^ν and I_{T-}^ν in the usual way. Fubini's theorem on the integrable triangular kernel gives the exact duality identity

$$\int_0^T g(t) \cdot (I_{0+}^\nu f)(t) dt = \int_0^T f(t) \cdot (I_{T-}^\nu g)(t) dt. \quad (17)$$

All boundary contributions in this formula are displayed explicitly.

Define the right reverse density p_i by the dual Volterra equation

$$p_i(t) = \frac{1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} \left[\frac{2x_i(s)}{s_i^2} e_i + A(s)^T p_i(s) \right] ds. \quad (18)$$

This Volterra-dual density is the continuous object associated with the second-variation identity. To see that identity explicitly, let R denote the second state variation and put

$$\mathcal{H} := F_{xx}[Y, Y] + 2F_{xv_i}[Y] \eta + F_{v_i v_i} \eta^2, \quad q_i := \frac{2x_i}{s_i^2} e_i.$$

Then

$$R = I_{0+}^\nu (AR + \mathcal{H}), \quad p_i = I_{T-}^\nu (q_i + A^T p_i).$$

Using (17) twice gives

$$\int_0^T (q_i + A^T p_i) \cdot R dt = \int_0^T p_i \cdot (AR + \mathcal{H}) dt,$$

so the common $p_i \cdot AR$ term cancels and

$$\int_0^T q_i \cdot R dt = \int_0^T p_i \cdot \mathcal{H} dt.$$

Substitution into the direct second derivative of the tracking cost gives

$$\begin{aligned} D^2 J_i(v)[\eta, \eta] &= 0.6 \|\eta\|_{L^2}^2 + \frac{2}{s_i^2} \|Y_i\|_{L^2}^2 \\ &+ \int_0^T p_i \cdot (F_{xx}[Y, Y] + 2F_{xv_i}[Y] \eta + F_{v_i v_i} \eta^2) dt. \end{aligned} \quad (19)$$

Because $z_i = e^{v_i}$,

$$F_{v_i} = z_i F_{z_i}, \quad F_{v_i v_i} = z_i F_{z_i} + z_i^2 F_{z_i z_i}. \quad (20)$$

Set A is affine in each z_i , so $F_{z_i z_i} = 0$ and therefore $F_{v_i v_i} = F_{v_i}$. The mixed derivative is $F_{xv_i} = z_i F_{xz_i}$. Thus, for Set A,

$$F_{xx}[Y, Y] = \begin{pmatrix} -0.8Y_1Y_3 \\ -0.4z_2Y_1Y_3 \\ 0.6z_3Y_1Y_2 \end{pmatrix}, \quad (21)$$

and

$$\begin{aligned} F_{xv_1}[Y] &= (0.1z_1Y_2, 0, 0)^T, & F_{v_1v_1} &= (0.1z_1x_2, 0, 0)^T, \\ F_{xv_2}[Y] &= (0, -0.2z_2(x_3Y_1 + x_1Y_3), 0)^T, & F_{v_2v_2} &= (0, -0.2z_2x_1x_3, 0)^T, \\ F_{xv_3}[Y] &= (0, 0, 0.3z_3(x_2Y_1 + x_1Y_2))^T, & F_{v_3v_3} &= (0, 0, 0.3z_3x_1x_2)^T. \end{aligned}$$

The nonnegative tracking-square term in (19) will be discarded in the lower bound. The estimates below are continuous in η under the L^2 norm, so the resulting quadratic-form inequality extends to all L^2 directions by density; no L^2 duality is being applied directly to an uncontrolled η^2 source.

Lemma 5.1 (Continuous and discrete Metzler comparison). *Let C be a constant Metzler matrix and let $A \in L^\infty(0, T; \mathbb{R}^{d \times d})$, $g \in L^\infty(0, T; \mathbb{R}^d)$ and $f \in L^\infty(0, T; \mathbb{R}_+^d)$. Suppose the Volterra solutions satisfy*

$${}^C D_t^\nu y = A(t)y + g(t), \quad y(0) = 0,$$

with $A_{jj}(t) \leq C_{jj}$, $|A_{jk}(t)| \leq C_{jk}$ for $j \neq k$, and $|g(t)| \leq f(t)$ componentwise. If

$${}^C D_t^\nu u = Cu + f, \quad u(0) = 0,$$

then $|y(t)| \leq u(t)$ componentwise. The analogous statement holds for the preceding-node value recurrence whenever $c_1 I + s_h C$ is entrywise nonnegative.

Proof The Volterra equation gives $y_j \in I^\nu(L^\infty(0, T))$ because $A(\cdot)y(\cdot) + g(\cdot)$ is bounded on the finite horizon. For

$$V_\varepsilon(r) = \sqrt{r^2 + \varepsilon^2} - \varepsilon,$$

Gomoyunov's convex-chain estimate for functions in $I^\nu(L^\infty)$ (Gomoyunov 2018, Lemma 3.1) yields, almost everywhere,

$${}^C D_t^\nu V_\varepsilon(y_j) \leq V'_\varepsilon(y_j) {}^C D_t^\nu y_j.$$

The same inequality is available in the generalized-Caputo framework of Li and Liu (2018, Proposition 3.11). Passing $\varepsilon \downarrow 0$ in the corresponding Volterra inequality gives

$${}^C D_t^\nu |y| \leq C|y| + f$$

in the comparison sense. Thus the regularity used here is exactly the natural fractional-ODE class $I^\nu(L^\infty)$, which also contains the time-reversed right-density equations because their bounded Volterra forcing gives the same representation from the terminal endpoint.

Choose $\alpha \geq \max_j(-C_{jj})$. Then $C + \alpha I \geq 0$, and inversion of ${}^C D_t^\nu + \alpha$ uses the nonnegative kernel $t^{\nu-1}E_{\nu, \nu}(-\alpha t^\nu)$. Iteration of the resulting Volterra inequality preserves order and gives $|y| \leq u$.

For the discrete statement, if $B^n = c_1 I + s_h A^n$ has nonnegative diagonal entries, the hypotheses give $|B^n| \leq c_1 I + s_h C$ entrywise. Positivity of all L1 history coefficients then yields the claimed domination by induction in n . $\square$

5.2 Finite interval certificate

The certificate consists of a finite family of fractional-polynomial/Bernstein objects certified by outward-rounded interval bounds. The nominal profile is the zero log-control profile $v^\circ \equiv 0$, equivalently $z^\circ \equiv (1, 1, 1)$. Its degree-eight state candidate is denoted q° . The finite conditions are:

- (C1) **Reachable-state enclosure.** The degree-eight nominal candidate q° approximates the state driven by $z^\circ = (1, 1, 1)$. A nonnegative full-box radius supersolution around its frozen outward cell enclosure encloses every continuous state for $z \in [1/2, 2]^3$ inside

$$B_{\text{FB}} = [0.10, 0.65] \times [-0.36, 0.20] \times [0.10, 0.85], \quad (22)$$

with a strict cellwise first-exit buffer. The box B_{FB} serves as the comparison/bootstrap domain containing the certified reachable enclosure and may extend beyond K_A in nonbinding directions.

- (C2) **Positive full-box comparison system.** On $B_{\text{FB}} \times [1/2, 2]^3$, componentwise tangent magnitudes are dominated by the constant Metzler matrix

$$\overline{M} = \begin{pmatrix} -0.59 & 0.20 & 0.26 \\ 1.19 & -1.20 & 0.26 \\ 0.216 & 0.39 & -0.80 \end{pmatrix}, \quad (23)$$

where

$$F_x(x, z) = \begin{pmatrix} -0.55 - 0.4x_3 & 0.1z_1 & -0.4x_1 \\ 1.2 - 0.2z_2x_3 & -1.2 & -0.2z_2x_1 \\ 0.3z_3x_2 & 0.3z_3x_1 & -0.8 \end{pmatrix}. \quad (24)$$

Direct endpoint evaluation of (24) on $B_{\text{FB}} \times [1/2, 2]^3$ gives the diagonal and off-diagonal majorants displayed in (23); lemma 5.1 then supplies the componentwise positive comparison. The own-log-control source bounds are

$$\beta_1 = (0.072, 0, 0)^T, \quad \beta_2 = (0, 0.221, 0)^T, \quad \beta_3 = (0, 0, 0.1404)^T. \quad (25)$$

For the retuned rational weights

$$w_L = \left(1, \frac{9}{7}, 1\right)^T, \quad w_R = \left(1, \frac{7}{20}, \frac{12}{25}\right)^T,$$

direct rational arithmetic gives the stronger bounds

$$\overline{M}w_L \leq -\frac{13}{180}w_L, \quad \overline{M}^T w_R \leq -\frac{11}{160}w_R. \quad (26)$$

Thus both forward and reversed comparison recurrences have positive damped resolvents on the full reachable box.

- (C3) **Reverse-density and channel-response enclosures.** Right-sided Bernstein proof objects enclose p_i , including the full-box perturbation from the nominal reverse density. The signed nominal object retains the cancellation carried by its sign changes in the second variation. A separate nonnegative correction supersolution encloses its residual, preserving that information within an outward-rounded bound. For each own-control channel, a nonnegative left-sided Bernstein supersolution G_i satisfies

$${}^C D_t^\nu G_i \geq \overline{M} G_i + \beta_i, \quad G_i(0) = 0, \quad (27)$$

with

$$\sup_{0 \leq t \leq T} G_{i,k}(t) \leq \gamma_{ik}, \quad k = 1, 2, 3. \quad (28)$$

Positivity of the continuous resolvent and Young's inequality then give

$$\|Y_k\|_{L^2} \leq \gamma_{ik} \|\eta\|_{L^2}.$$

- (C4) **Coefficient envelopes and quadratic-form bounds.** The tangent enters this step exclusively through the global operator bound (28); the coefficient calculation is therefore separated from any pointwise tangent envelope.

On each fixed certificate cell, outward interval arithmetic combines the state, control and reverse-density enclosures and takes the scalar coefficient suprema $A_{i,13}, A_{i,12}, D_{i,k}, E_i$ associated with the terms in (19). Their explicit Set-A formulas are recorded in Online Resource 1; the comparison box B_{FB} supplies the bootstrap domain.

Cauchy–Schwarz together with (28) then gives the reproducible bounds

$$L_i^{xx} = A_{i,13} \gamma_{i1} \gamma_{i3} + A_{i,12} \gamma_{i1} \gamma_{i2}, \quad (29)$$

$$L_1^{xv} = D_{1,2} \gamma_{12},$$

$$L_2^{xv} = D_{2,1} \gamma_{21} + D_{2,3} \gamma_{23},$$

$$L_3^{xv} = D_{3,1} \gamma_{31} + D_{3,2} \gamma_{32}, \quad (30)$$

$$L_i^{vv} = E_i.$$

Consequently

$$\left| \int p_i \cdot F_{xx}[Y, Y] dt \right| \leq L_i^{xx} \|\eta\|_{L^2}^2, \quad (31)$$

$$\left| \int 2p_i \cdot F_{xv_i}[Y] \eta dt \right| \leq L_i^{xv} \|\eta\|_{L^2}^2, \quad (32)$$

$$\left| \int p_i \cdot F_{v_i v_i} \eta^2 dt \right| \leq L_i^{vv} \|\eta\|_{L^2}^2. \quad (33)$$

The cellwise envelopes act on (x, p, z) , while tangent information enters through the global $L^2 \rightarrow L^2$ gains γ_{ik} .

The proof implication is finite: nominal residual plus radius gives (C1), Jacobian/source extrema give (C2), signed reverse corrections and channel supersolutions give (C3), and fixed-cell coefficient envelopes with global channel gains give (C4). The load-bearing outward predicates are: First, the nominal object is tied to the fixed control $z^\circ = (1, 1, 1)$ through

$$\mathcal{R}_x(t) := {}^C D_t^\nu q^\circ(t) - F(q^\circ(t), z^\circ). \quad (34)$$

The accepted upstream nominal certificate supplies an outward cell enclosure $q^- \leq q^\circ \leq q^+$ together with an outward residual bound for (34); these frozen cells are the only nominal-state input used downstream. The full-box radius predicate is the one-sided inequality (39) below.

For the reverse objects, write $s = T - t$, let $A^\circ(s) = F_x(q^\circ(T - s), z^\circ)$ and define the nominal tracking source

$$\rho_i^\circ(s) := \frac{2[q^\circ(T - s)]_i}{s_i^2} e_i.$$

Let $\widehat{p}_i(s)$ be a signed right-density candidate expressed in the reversed variable. Its signed residual is

$$\mathcal{R}_{p_i}(s) := {}^C D_s^\nu \widehat{p}_i(s) - A^\circ(s)^T \widehat{p}_i(s) - \rho_i^\circ(s). \quad (35)$$

A nonnegative correction e_i^{corr} is accepted only if every certificate cell satisfies the componentwise outward predicate

$$\underline{\text{IA}} \left[{}^C D_s^\nu e_i^{\text{corr}} - \overline{M}^T e_i^{\text{corr}} - |\mathcal{R}_{p_i}| \right] (I_c) \geq 0. \quad (36)$$

Thus $p_i^\circ \in [\widehat{p}_i - e_i^{\text{corr}}, \widehat{p}_i + e_i^{\text{corr}}]$. For the full-box perturbation $q_i^{\text{fb}} \geq 0$, let $\mathcal{S}_i^{\text{fb}}$ be the componentwise interval majorant of

$$(A - A^\circ)^T p_i^\circ + \frac{2(x_i - [q^\circ]_i)}{s_i^2} e_i$$

over the certified state/control/reverse-density cells. The perturbation predicate is

$$\underline{\text{IA}} \left[{}^C D_s^\nu q_i^{\text{fb}} - \overline{M}^T q_i^{\text{fb}} - \mathcal{S}_i^{\text{fb}} \right] (I_c) \geq 0. \quad (37)$$

Consequently $|p_i - p_i^\circ| \leq q_i^{\text{fb}}$ componentwise. The signed-plus-correction structure retains the correlation needed by the binding curvature estimate.

Finally, for every C4 coefficient expression Φ defined above and every fixed certificate cell I_c , interval arithmetic produces a directed enclosure $\text{IA}_c[\Phi]$ and a nonnegative scalar $M_{\Phi, c}$ with

$$\text{IA}_c[\Phi] \subset [-M_{\Phi, c}, M_{\Phi, c}], \quad M_\Phi := \max_c M_{\Phi, c}. \quad (38)$$

The quantities $A_{i,13}, A_{i,12}, D_{i,k}, E_i$ are precisely these M_Φ values. Only the coefficient fields (x, p, z) are paired cellwise; the tangent enters afterwards through the global channel gains γ_{ik} .

The exact finite interval predicates are the nominal residual (34), the reverse-correction and reverse-perturbation predicates (36)–(37), and the cellwise C4 enclosure (38). For the state radius the interval computation requires, on every fixed cell,

$$\underline{\text{IA}}[{}^C D_t^\nu r_j](I_c) - \overline{\text{IA}}[(C(q, r, z)r + u(q, z))_j](I_c) > 0. \quad (39)$$

Online Resource 1 records basis degrees, cell counts, minimum slacks and the directed-arithmetic implementation. The certificate is determined by the finite outward predicates above, the exact rational checks and the directed final arithmetic.

Proposition 5.2 (Computer-assisted full-box certificate). *Conditions (C1)–(C4) hold for the zero-target Set-A game specified in section 4. The certified values are*

	<i>Player 1</i>	<i>Player 2</i>	<i>Player 3</i>
γ_{i1}	0.132959	0.083176	0.073943
γ_{i2}	0.116137	0.241134	0.089334
γ_{i3}	0.066195	0.102292	0.185373
L_i^{xx}	< 0.02680	< 0.05762	< 0.01815
L_i^{xv}	< 0.11681	< 0.14763	< 0.14420
L_i^{vv}	< 0.15352	< 0.20061	< 0.17351
$0.6 - L_i^{xx} - L_i^{xv} - L_i^{vv}$	> 0.3028	> 0.1941	> 0.2641

The full-box state-radius maxima are

$$(0.022964, 0.025553, 0.013515),$$

the minimum outward state-radius residual slack is $> 5.31 \times 10^{-4}$, and the minimum cellwise distance of the certified reachable enclosure from ∂B_{FB} is

$$d_{\text{FB}} > 0.0389. \quad (40)$$

All channel-response residual slacks are positive; the smallest is 2.95×10^{-3} . For the binding Player 2 reverse-density chain, the signed-nominal correction has minimum outward slack $> 1.64 \times 10^{-3}$ and the full-box perturbation certificate has minimum outward slack $> 8.25 \times 10^{-3}$.

Proof The finite computation evaluates predicates (C1)–(C4) on the fixed Bernstein/cell representations, using exact rational arithmetic for the comparison weights and directed interval arithmetic for residuals and the final loss algebra. Online Resource 1 records the representation, cell counts, slacks and arithmetic model. $\square$

Combining (19) with proposition 5.2 and discarding the positive tracking-square term gives

$$D^2 J_i(v)[\eta, \eta] \geq c_i \|\eta\|_{L^2}^2, \quad \min_i c_i \geq c_{\text{FB}} := 0.1941. \quad (41)$$

Thus the continuous own-player objectives are strongly convex in the log chart on the full joint strategy box.

6 Transfer across the L1 startup layer

The continuous certificate is finite, and its natural basis has the familiar L1 startup defect at the fractional endpoints. This startup loss is consistent with sharp analyses of the L1 method on weakly singular solutions; see, e.g., (Kopteva 2019; Liao et al. 2019). For $q(t) = t^\nu$,

$${}^C D_t^\nu q = \Gamma(1 + \nu), \quad (\delta_h^\nu q)^1 = \frac{1}{\Gamma(2 - \nu)},$$

and for $\nu = 0.7$ their difference is approximately 0.205604, independently of h .

Lemma 6.1 (Endpoint-weighted defect). *For any fixed polynomial Q and*

$$q(t) = Q((t/T)^\nu),$$

there is $C_Q < \infty$ such that

$$|(\delta_h^\nu q)^n - {}^C D_t^\nu q(t_n)| \leq C_Q n^{-\nu}, \quad 1 \leq n \leq N. \quad (42)$$

The mirrored statement holds for $Q(((T - t)/T)^\nu)$ after reversing the grid index.

Proof By linearity it is enough to take $q(t) = t^\beta$, $\beta = m\nu$. The L1 derivative is the Caputo derivative of the piecewise-linear interpolant $I_h q$:

$$(\delta_h^\nu q)^n = \frac{1}{\Gamma(1 - \nu)} \int_0^{t_n} (t_n - s)^{-\nu} (I_h q)'(s) ds.$$

Hence the defect is the same integral with $(I_h q)' - q'$ in the integrand.

For $m = 1$ ($\beta = \nu$), the first cell is treated by the rescaling $s = hy$; its contribution is bounded by $C n^{-\nu}$. On cell $[t_{j-1}, t_j]$, $j \geq 2$, the mean-value theorem gives

$$|(I_h q)'(s) - q'(s)| \leq Ch t_{j-1}^{\nu-2}.$$

Therefore the remaining contribution is bounded by

$$C \sum_{j=2}^n j^{\nu-2} (n - j + 1)^{-\nu}.$$

For $j \leq n/2$ the second factor is $O(n^{-\nu})$ and the first-factor series is summable; for $j > n/2$ the first factor is $O(n^{\nu-2})$ while $\sum_{r \leq n/2} r^{-\nu} = O(n^{1-\nu})$. Thus the two pieces are $O(n^{-\nu})$ and $O(n^{-1})$, respectively.

For $m \geq 2$, $\beta > 1$ and the cellwise slope error is $Ch t_{j-1}^{\beta-2}$. After nondimensionalizing the convolution one obtains

$$Ch^{\beta-\nu} \sum_{j=2}^n j^{\beta-2} (n-j+1)^{-\nu}.$$

Splitting at $n/2$ gives $Ch^{\beta-\nu} n^{\beta-1-\nu}$. Since $nh \leq T$, this is bounded by $CT^{\beta-\nu} n^{-1}$, which is sharper than $Cn^{-\nu}$. The first cell is no larger. Only finitely many monomials occur in Q , so their constants may be summed. The right-endpoint statement follows after the reflection $t \mapsto T-t$. $\square$

Because the accepted certificate graph contains only finitely many fractional-polynomial proof objects, let $\mathcal{Q}$ denote this finite family and define

$$C_{\mathcal{Q}} := \max_{Q \in \mathcal{Q}} C_Q < \infty. \quad (43)$$

Thus all endpoint-defect estimates used below are uniform over the complete certificate family; profile dependence enters only through separately bounded coefficient perturbations on the fixed compact proof domains.

Let

$$a_k = (k+1)^{1-\nu} - k^{1-\nu}, \quad s_h = \Gamma(2-\nu)h^\nu.$$

Concavity gives

$$a_{n-1} \geq (1-\nu)n^{-\nu}. \quad (44)$$

On the enlarged full-box comparison domain take a Metzler majorant $\overline{M}$ with positive weights w satisfying $\overline{M}w \leq -\sigma w$ for some $\sigma > 0$. We write

$$\|y\|_w := \max_{1 \leq j \leq 3} \frac{|y_j|}{w_j}.$$

This is the weighted max norm adapted to the positive comparison system: the inequality $\overline{M}w \leq -\sigma w$ supplies an explicit scalar damping rate in this norm.

Lemma 6.2 (Positive endpoint resolvent). *Assume $e^0 = 0$ and that a nonnegative error sequence is dominated componentwise by the exact preceding-node comparison recurrence*

$$e^n \leq (c_1 I + s_h \overline{M})e^{n-1} + \sum_{r=2}^{n-1} c_r e^{n-r} + s_h r^n, \quad n \geq 1, \quad (45)$$

where $c_1 = 1 - a_1$, $c_r = a_{r-1} - a_r > 0$, and $\overline{M}w \leq -\sigma w$ for a positive vector w . If

$$0 \leq r^n \leq (An^{-\nu} + Bh^\nu)w,$$

then, for all sufficiently small h , there is a nonnegative supersolution E^n with $e^n \leq E^n$ and

$$\max_n \|E^n\|_w \leq h^\nu \left[\frac{\Gamma(2-\nu)}{1-\nu} A + \frac{B}{\sigma} \right]. \quad (46)$$

Proof Set $E^0 = 0$ and, for $n \geq 1$,

$$E^n = M_{A,B} h^\nu w, \quad M_{A,B} := \frac{\Gamma(2-\nu)}{1-\nu} A + \frac{B}{\sigma}.$$

At $n = 1$ the comparison recurrence is $e^1 \leq s_h r^1$. The choice above dominates this first step once

$$h^\nu \leq \frac{1}{\sigma \Gamma(2-\nu)},$$

because $M_{A,B} \geq \Gamma(2-\nu)A + B/\sigma$. For $n \geq 2$, $\sum_{r=1}^{n-1} c_r = 1 - a_{n-1}$, so substitution of the constant positive sequence leaves the residual

$$a_{n-1} E^n - s_h \overline{M} E^n - s_h r^n.$$

The endpoint part is nonnegative because (44) gives

$$a_{n-1} E^n \geq (1-\nu) n^{-\nu} h^\nu \frac{\Gamma(2-\nu)}{1-\nu} A w = s_h A n^{-\nu} w.$$

For the smooth source, $-\overline{M} w \geq \sigma w$ gives

$$-s_h \overline{M} E^n \geq s_h \sigma M_{A,B} h^\nu w \geq s_h B h^\nu w.$$

Thus E is a supersolution of (45). Since the off-diagonal entries of $\overline{M}$ are nonnegative, a sufficient explicit condition for $M_h := c_1 I + s_h \overline{M}$ to be entrywise nonnegative is

$$s_h \leq \min_j \frac{c_1}{-M_{jj}} = \frac{c_1}{1.2}. \quad (47)$$

Under this condition $c_r \geq 0$ and $M_h \geq 0$, so induction from $e^0 = E^0 = 0$ yields $0 \leq e^n \leq E^n$ for every n . This proves (46). In the channel-response node $\overline{M}$ and β_i are constant, so the coefficient-lag term is absent and $B = 0$. $\square$

6.1 Discrete channel gains: positive matrix kernel and Young bound

The channel gains require one additional discrete step beyond the pointwise proof-object transfer.

Let $c_1 = 1 - a_1$ and $c_r = a_{r-1} - a_r > 0$ for $r \geq 2$. For sufficiently small h ,

$$M_h := c_1 I + s_h \overline{M}$$

is entrywise nonnegative. Define nonnegative matrix impulse coefficients

$$K_0 = I, \quad K_m = M_h K_{m-1} + \sum_{r=2}^m c_r K_{m-r}, \quad m \geq 1. \quad (48)$$

Induction gives $K_m \geq 0$ entrywise.

Lemma 6.3 (Discrete componentwise channel gain). *Let Y^n be the exact discrete first variation for player i in direction $\eta = (\eta_0, \dots, \eta_{N-1})$, and assume that the discrete*

state nodes on the interval under consideration lie in B_{FB} . This hypothesis will be supplied non-circularly by the first-exit argument below. Write

$$\|\eta\|_{\ell_h^2}^2 = h \sum_{j=0}^{N-1} |\eta_j|^2, \quad \|Y_k\|_{\ell_h^2(1:N)}^2 = h \sum_{n=1}^N |Y_k^n|^2.$$

Then

$$\|Y_k\|_{\ell_h^2(1:N)} \leq \gamma_{ik,h} \|\eta\|_{\ell_h^2}, \quad \gamma_{ik,h} := s_h \sum_{m=0}^{N-1} e_k^T K_m \beta_i. \quad (49)$$

Moreover

$$\gamma_{ik,h} \leq \gamma_{ik} + Ch^\nu \quad (50)$$

with a constant independent of the discrete control profile.

Proof Up to the state-box bootstrap, lemma 5.1 gives $|b_i^j| \leq \beta_i$ and, under (47),

$$|c_1 I + s_h A^j| \leq c_1 I + s_h \overline{M} = M_h$$

entrywise. Taking componentwise absolute values in the exact tangent recurrence therefore gives a positive comparison recurrence. Its solution is

$$y^n = s_h \sum_{j=0}^{n-1} K_{n-1-j} \beta_i |\eta_j|, \quad |Y^n| \leq y^n.$$

For component k , apply the scalar discrete Young inequality to the nonnegative scalar kernel $\kappa_m = s_h e_k^T K_m \beta_i$ and the sequence $|\eta_j|$. With the h -weighted norm the common factor $\sqrt{h}$ cancels on both sides, yielding

$$\|y_k\|_{\ell_h^2(1:N)} \leq \left(s_h \sum_{m=0}^{N-1} e_k^T K_m \beta_i \right) \|\eta\|_{\ell_h^2},$$

which proves (49). Since $Y^0 = 0$, the Hessian norm over the indices $j = 0, \dots, N-1$ is no larger than the displayed $1:N$ norm.

Let $u_{i,h}^n$ solve the same positive comparison recurrence with unit scalar forcing. By (48),

$$(u_{i,h}^N)_k = s_h \sum_{m=0}^{N-1} e_k^T K_m \beta_i = \gamma_{ik,h}.$$

The certified G_i in (27) is a fixed finite Bernstein supersolution for this unit-forcing response. Sampling G_i on the L1 grid creates the endpoint defect of lemma 6.1; that defect is absorbed by lemma 6.2. Here $\overline{M}$ and β_i are constant, so there is no coefficient-lag source and the Bh^ν term in (46) is absent at this node. Hence

$$u_{i,h}^n \leq G_i(t_n) + O(h^\nu)$$

uniformly in n , and at $n = N$,

$$\gamma_{ik,h} \leq \sup_{0 \leq t \leq T} G_{i,k}(t) + O(h^\nu) \leq \gamma_{ik} + O(h^\nu).$$

Because $K_m \geq 0$, the exact unit-forcing response is nondecreasing in n and its terminal component is its discrete-time maximum. The continuous certificate uses the safe quantity $\gamma_{ik} = \sup_t G_{i,k}(t)$, while the discrete operator norm is obtained from the exact positive matrix kernel and discrete Young inequality. $\square$

6.2 Reverse sweep and finite certificate graph

Write the L1 value form with $c_1 = 1 - a_1$ and $c_r = a_{r-1} - a_r$ for $r \geq 2$. Put

$$d_i^j := \frac{2x_i^j}{s_i^2} e_i, \quad g^j := h\tau_j d_i^j.$$

If $A^j = F_x(x_h^j, z_h^j)$ and λ^j is the raw reverse accumulator of the exact computational graph, then set $\lambda^N = g^N = h d_i^N / 2$ and, for $j = N - 1, \dots, 1$, define

$$\lambda^j = g^j + (c_1 I + s_h (A^j)^T) \lambda^{j+1} + \sum_{r=2}^{N-j} c_r \lambda^{j+r}, \quad (51)$$

where empty sums vanish. Supplement A, subsection “Exact graph transpose” and its following “Gradient density and exact Hessian” subsection derive the coefficient collection in (51), including the terminal trapezoidal half-weight, and obtain the normalization

$$\pi^j = \frac{s_h}{h} \lambda^{j+1}. \quad (52)$$

The terminal half-weight is a localized $O(h)$ source effect in the reversed recurrence. Since $0 < \nu < 1$, it is smaller than the $O(h^\nu)$ endpoint-transfer scale for $h \leq 1$ and is absorbed by the same resolvent estimate. After index reversal, (51) is again a preceding-node value recurrence with transposed Jacobian, so lemmas 6.1 and 6.2 apply to the right-sided Bernstein proof objects. The density π^j is therefore the normalized exact transpose of the finite L1 computational graph.

The certificate dependency graph is finite and acyclic. In topological order, the nominal state feeds the full-box state enclosure; that enclosure feeds both the signed reverse-density certificates and the unit channel responses; the unit responses determine the discrete channel norms; and the state/reverse-density coefficient fields together with those norms feed the final quadratic-form bounds.

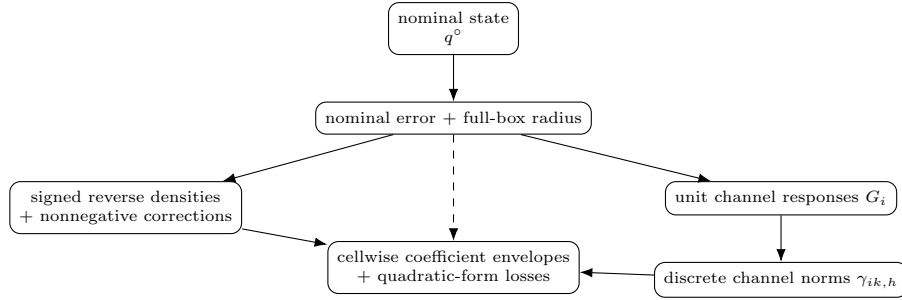


Fig. 1 Load-bearing certificate dependency graph. Solid arrows show direct proof-object dependencies; the dashed arrow records the state enclosure entering the final coefficient fields. Pointwise proof objects transfer through the endpoint resolvent, while the channel-norm node additionally uses the positive matrix kernel and discrete Young inequality.

Proposition 6.4 (Finite certificate-graph transfer). *There are $C < \infty$ and $h_1 > 0$ such that, for every $h < h_1$, every pointwise continuous certificate enclosure in the graph above has a discrete counterpart enlarged by at most Ch^ν componentwise, and the channel gains satisfy*

$$\gamma_{ik,h} \leq \gamma_{ik} + Ch^\nu.$$

The constants are uniform over the complete admissible discrete control box as long as the discrete state remains in B_{FB} .

Proof Order the finite graph topologically. At a pointwise node, the proof object is a fixed fractional polynomial/Bernstein function. Sampling it creates the endpoint defect of lemma 6.1. Every algebraic coefficient map at that node is evaluated on a compact product of the certified state box, the fixed control box $[1/2, 2]^3$, and the already-certified upstream enclosure ranges. These maps are smooth on an open neighbourhood of that compact product, hence possess finite Lipschitz constants there. Consequently, if all upstream discrete enclosures differ from their continuous counterparts by at most $C_{\text{up}}h^\nu$, the coefficient perturbation at the current node is bounded by $LC_{\text{up}}h^\nu$, with L independent of the mesh, node number and admissible control profile. Together with the sampled endpoint defect, the residual source therefore has the form

$$An^{-\nu} + Bh^\nu$$

with profile-independent constants. The appropriate forward or reversed positive comparison recurrence then gives an $O(h^\nu)$ nonnegative correction by lemma 6.2. Recursively assigning to each graph node the resulting finite transfer constant makes the uniformity explicit; finiteness and acyclicity of the graph ensure that this recursion terminates.

The channel-norm node is handled differently. The fixed unit-forcing channel response transfers by the preceding pointwise argument, whereas the operator norm itself is obtained from the exact positive matrix kernel and discrete Young inequality in lemma 6.3. Hence (50) follows. Because the graph has finitely many nodes, the finitely many constants produced in this induction may be replaced by their maximum, yielding one common C and h_1 . In particular, this constant is independent of the mesh, node number and admissible discrete profile. $\square$

Lemma 6.5 (Non-circular discrete state bootstrap). *There exists $h_{\text{box}} > 0$ such that every admissible exact Set-A L1 trajectory satisfies*

$$x_h^n \in B_{\text{FB}}, \quad n = 0, \dots, N,$$

whenever $h < h_{\text{box}}$.

Proof Choose $h_{\text{box}} \leq h_1$ so that the state-enclosure inflation supplied by proposition 6.4 is smaller than $d_{\text{FB}}/2$. Suppose, inductively, that $x_h^0, \dots, x_h^{n-1} \in B_{\text{FB}}$ and that n is the first alleged exit. Because the scheme is preceding-node, every coefficient used to produce node n is evaluated at nodes no later than $n-1$ and therefore lies in the comparison domain. Hence the transferred state certificate is valid at node n itself. Its continuous enclosure is at distance at least d_{FB} from ∂B_{FB} , while the discrete inflation is $< d_{\text{FB}}/2$; thus x_h^n lies strictly inside B_{FB} , contradicting first exit. The induction starts at the given $x_0 \in B_{\text{FB}}$. $\square$

6.3 Exact discrete Hessian and curvature persistence

Let $\eta = (\eta_0, \dots, \eta_{N-1})$ be an own-player log-control direction and Y^n the exact discrete first variation. Throughout this section

$$\|\eta\|_{\ell_h^2}^2 := h \sum_{j=0}^{N-1} |\eta_j|^2,$$

which is the discrete L^2 norm induced by the piecewise-constant control interpolation. Differentiating the explicit state graph and eliminating the second state variation with the exact reverse sweep gives

$$\begin{aligned} D^2 J_{i,h}[\eta, \eta] &= 0.6h \sum_{j=0}^{N-1} \eta_j^2 + h \sum_{n=0}^N \tau_n \frac{2(Y_i^n)^2}{s_i^2} \\ &\quad + h \sum_{j=0}^{N-1} \pi^j \cdot (F_{xx}^j[Y^j, Y^j] + 2F_{xv_i}^j[Y^j] \eta_j + F_{v_i v_i}^j \eta_j^2). \end{aligned} \quad (53)$$

The identity is derived directly in the log chart, keeping the curvature calculation in the coordinates of the theorem. The raw Euclidean Hessian eigenvalues scale like h . The mesh-independent quantity is the normalized quadratic form relative to $h \sum_j \eta_j^2$.

Theorem 6.6 (Full-box curvature persistence). *There exists $h_{\text{cvx}} > 0$ such that for every $h < h_{\text{cvx}}$, every player, and every opponent profile in the complete admissible discrete box, the exact discrete objective is strongly convex in that player's own log-control vector.*

Proof By lemma 6.5 and proposition 6.4, the pointwise state and reverse-density factors in the last line of (53) differ from their continuous certified enclosures by $O(h^\nu)$, while lemma 6.3 supplies the required ℓ_h^2 tangent bounds $\gamma_{ik,h} \leq \gamma_{ik} + O(h^\nu)$. The coefficient suprema in (C4) are maxima of finitely many interval-polynomial coefficient maps, and the loss formulas are finite products and sums. On the compact certified ranges set

$$A_* := \max_i \max\{A_{i,13}, A_{i,12}\}, \quad D_* := \max_{i,k} D_{i,k}, \quad \gamma_* := \max_{i,k} \gamma_{ik}.$$

If every scalar envelope and channel gain is perturbed by at most ε , direct expansion gives, for example,

$$|\Delta L_i^{xx}| \leq (2\gamma_*^2 + 4A_*\gamma_* + 2A_*)\varepsilon + O(\varepsilon^2),$$

and analogous finite bounds for L_i^{xv} and L_i^{vv} using D_* and the compact state/reverse-density ranges. Hence there is a finite constant C_{loss} , independent of h , such that all three loss perturbations are at most $C_{\text{loss}}\varepsilon$ for sufficiently small ε . Taking $\varepsilon = O(h^\nu)$ from proposition 6.4, there is $C < \infty$ such that, uniformly over the complete joint discrete strategy box,

$$D^2 J_{i,h}[\eta, \eta] \geq h(c_i - Ch^\nu) \sum_{j=0}^{N-1} \eta_j^2. \quad (54)$$

Since $\min_i c_i > 0.1941$, choose $h_{\text{cvx}} > 0$ so that $Ch^\nu < \min_i c_i$ whenever $h < h_{\text{cvx}}$. Then (54) has a strictly positive quadratic coefficient uniformly over the complete joint discrete strategy box. Thus every own-player discrete objective is strongly convex on the full own log-control box for every $h < h_{\text{cvx}}$. $\square$

7 Discrete equilibrium existence and continuous limit

Lemma 7.1 (Unique full-box best responses). *For every sufficiently fine Set-A mesh and every fixed opponent profile, the own-player discrete objective has a unique global minimizer on $[-\log 2, \log 2]^N$.*

Proof The feasible box is compact, so continuity gives existence. By theorem 6.6, the objective is strongly convex in the own log-control vector on the entire box, hence the global minimizer is unique. $\square$

Theorem 7.2 (Pure discrete Nash equilibria on the sufficiently-fine tail). *There exists $h_* > 0$ such that every exact Set-A preceding-node L1 game with $h < h_*$ has a pure global Nash equilibrium.*

Proof Take $h_* \leq h_{\text{cvx}}$. The own log-control box $[-\log 2, \log 2]^N$ is compact and convex. For fixed opponents, lemma 7.1 gives a unique global own minimizer. Joint continuity of the explicit recurrence and (5), together with the constant compact feasible box, lets Berge's maximum theorem promote the resulting single-valued best-response correspondence to a continuous map in the opponents. The product best-response map is a continuous self-map of the compact convex joint box, so Brouwer's theorem gives a fixed point. Because each component is the unique global full-box minimizer, the fixed point is a pure global Nash equilibrium. $\square$

Corollary 7.3 (Continuous Set-A Nash limit). *Let $h_k \downarrow 0$ be any sequence of sufficiently fine uniform meshes, and choose one exact pure Nash equilibrium on each mesh. A subsequence converges weak-* in controls and uniformly in states to a continuous open-loop Nash equilibrium of the Set-A Caputo game.*

Proof The common discrete state bound follows from proposition 4.3. The discrete equilibria satisfy (6) with $\varepsilon_{h_k} = 0$. Apply theorem 3.7. $\square$

Remark 7.4 (Theorem scope). *The concrete result is a sufficiently-fine-mesh theorem for the zero-target Set-A game specified in section 4. It gives pure global Nash existence for every sufficiently fine preceding-node L1 game and subsequential convergence of any vanishing-mesh equilibrium sequence to a continuous open-loop Nash equilibrium.*

Taken together, theorem 7.2 and corollary 7.3 connect equilibrium existence in the finite discrete-time games to equilibrium in the continuous hereditary game. The finite games are therefore more than state approximations: on sufficiently fine meshes they possess global Nash equilibria, and every vanishing-mesh selection of such equilibria

has a subsequence converging to an open-loop Nash equilibrium of the continuous strategic problem.

8 Supplementary information and reproducibility

Online Resource 1 is the Technical Supplement. It records the finite certificate representation, certificate partitions and minimum slacks, the exact graph-transpose derivation of the reverse recurrence, the discrete Hessian identity, and the arithmetic model used for the interval/Bernstein certificate.

Online Resource 2 contains reproducibility files for the computer-assisted curvature bound: the computational source and the frozen certificate data underlying the reported quantities. The same materials are archived at Zenodo ([Lympers and Drossos 2026](#)). They make the computer-assisted part of the argument inspectable and reproducible; the mathematical claim remains the finite collection of inequalities stated in proposition 5.2.

9 Discussion and conclusion

A time discretization of a differential game creates another strategic game, not only an approximation of its state equation. State convergence by itself therefore does not establish that the discrete and continuous games have compatible equilibrium behavior. The relevant object is the unilateral comparison that defines Nash equilibrium. With hereditary dynamics this comparison is intrinsically nonlocal: a player's deviation changes the subsequent memory history as well as the contemporaneous trajectory.

For the preceding-node L1 games studied here, that strategic structure survives the vanishing-mesh passage. Under a uniform state bound, increasingly accurate discrete Nash profiles admit subsequences converging to continuous open-loop Nash equilibria. The stopped unilateral-deviation construction preserves the admissible unilateral comparisons, while weak-* state identification and discrete fractional summation by parts carry the hereditary state law to the limit. The result thus links equilibrium of the finite discrete-time games generated by the numerical dynamics to equilibrium of the continuous-time differential game.

The Set-A analysis shows that this link can be populated by equilibria whose existence is established inside the discrete games. Certified full-box curvature gives unique global best responses on every sufficiently fine mesh, and the resulting continuous best-response map has a pure global Nash fixed point. Any vanishing-mesh selection of these discrete equilibria then has a subsequence converging to a continuous open-loop Nash equilibrium. This closes both sides of the approximation: existence in the finite games and strategic validity of their continuous limit.

The proof also separates the ingredients needed for extensions. A different hereditary discretization need not reproduce the L1 formulas used here, but it must provide a reliable state-limit mechanism and a way to recover unilateral deviations through its discrete memory history. Likewise, the concrete curvature certificate is one route to discrete equilibrium existence rather than part of the abstract Nash-limit theorem. The broader implication is that discretizations of dynamic games should be assessed

at the level of strategic deviations and equilibrium, not only at the level of state trajectories. The framework developed here provides one route for doing so when the dynamics retain memory.

Statements and Declarations

Funding. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Competing interests. The authors have no relevant financial or non-financial interests to disclose.

Data and code availability. No empirical datasets were used. The Technical Supplement, computational source, and frozen certificate data supporting the computer-assisted curvature calculation are archived at Zenodo (Lymperis and Drossos 2026). The record is identified by DOI [10.5281/zenodo.22924837](https://doi.org/10.5281/zenodo.22924837).

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