

Online Resource 1: Technical Supplement to Open-Loop Nash Limits for Caputo Differential Games: Preceding-Node L1 Discretization

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Supplementary Information for *Dynamic Games and Applications*

1 Finite certificate specification

This supplement supports the zero-target Set-A Caputo game specified completely in the main paper. The cited Set-A source supplies only the baseline numerical coefficient values; the fractional dynamics, strategic controls, objective, admissible box and finite L1 games are defined in the main paper. This section records the certificate representation, certificate partitions, minimum slacks and arithmetic model. Reproducibility files for the computer-assisted curvature calculation are archived at Zenodo, DOI: [10.5281/zenodo.22924837](https://doi.org/10.5281/zenodo.22924837).

The finite certificate classes are summarized below. The authoritative representation is Bernstein/cellwise interval data; high-degree conversion to a power basis is diagnostic only.

Table 1: Finite certificate specification. “Cells” denotes the fixed interval partition. All stated slacks are lower bounds.

Object	Basis	Degree	Cells	Certified inequality / minimum slack
Nominal state q°	fractional poly- nomial in $(t/T)^\nu$	8	3072	$z^\circ = (1, 1, 1)$; outward residual/cell enclosure for the nominal residual, frozen upstream and used by all downstream objects
Full-box radius r	Bernstein in $(t/T)^\nu$	18	3072	componentwise ${}^C D_t^\nu r - C(q, r, z)r - u(q, z) \geq 0$; min slack $> 5.31 \times 10^{-4}$
Channel responses G_i	Bernstein in $(t/T)^\nu$	18	6144	${}^C D_t^\nu G_i - \overline{M}G_i - \beta_i \geq 0$; minima $> 4.29, 2.95, 4.12 \times 10^{-3}$ for P1–P3
P2 signed nominal correction	right Bernstein in $((T - t)/T)^\nu$	30	3072	predicate the nominal reverse-correction predicate; min slack $> 1.64 \times 10^{-3}$
P2 full-box reverse perturbation	right Bernstein in $((T - t)/T)^\nu$	30	6144	predicate the full-box reverse-perturbation predicate; min slack $> 8.25 \times 10^{-3}$
C4 coefficient envelopes	fixed interval cells	–	3072	predicate the C4 cell predicate, followed by directed evaluation of the loss formulas

A representative certificate inequality is the channel-response bound: for every certificate cell I_c , player i and component k , outward interval arithmetic gives an enclosure required to satisfy

$$\underline{\text{IA}}[{}^C D_t^\nu G_{i,k} - (\overline{M}G_i)_k - \beta_{i,k}](I_c) > 0. \quad (1)$$

The state-radius bound is equally explicit. If $q^- \leq q \leq q^+$ is the frozen nominal enclosure and $r^- \leq r \leq r^+$ the candidate radius on a cell, the interval computation forms the one-sided Metzler coefficients from $[q^- - r^+, q^+ + r^+]$, the control source $u(q, z)$ from the full control box, and requires componentwise

$$\underline{\text{IA}}[{}^C D_t^\nu r_j](I_c) - \overline{\text{IA}}[(C(q, r, z)r + u(q, z))_j](I_c) > 0. \quad (2)$$

Candidate-generation routines supply proposals; the certificate itself is determined by these outward inequalities, the exact rational weight checks, and the directed final arithmetic.

C4 coefficient envelopes. For the Set-A second-variation terms, write $p_i = (p_{i1}, p_{i2}, p_{i3})$ and define the certified full-box time suprema

$$\begin{aligned} A_{i,13} &= \|-0.8p_{i1} - 0.4z_2p_{i2}\|_{L^\infty}, & A_{i,12} &= \|0.6z_3p_{i3}\|_{L^\infty}, \\ D_{1,2} &= \|0.2z_1p_{11}\|_{L^\infty}, & E_1 &= \|0.1z_1p_{11}x_2\|_{L^\infty}, \\ D_{2,1} &= \|0.4z_2p_{22}x_3\|_{L^\infty}, & D_{2,3} &= \|0.4z_2p_{22}x_1\|_{L^\infty}, \\ E_2 &= \|0.2z_2p_{22}x_1x_3\|_{L^\infty}, & & \\ D_{3,1} &= \|0.6z_3p_{33}x_2\|_{L^\infty}, & D_{3,2} &= \|0.6z_3p_{33}x_1\|_{L^\infty}, \\ E_3 &= \|0.3z_3p_{33}x_1x_2\|_{L^\infty}. & & \end{aligned}$$

Each supremum is obtained by combining the certified state and signed-plus-correction reverse-density enclosures on the same fixed time cell before maximizing over cells. The main paper then combines these quantities with the certified global channel gains to obtain L_i^{xx} , L_i^{xv} and L_i^{vv} .

Arithmetic model. The interval computations use IEEE-754 binary64 endpoints with explicit outward rounding rather than a process-wide rounding mode. In the Fortran implementation, interval products and affine combinations are expanded with `ieee_next_after` toward $-\infty$ or $+\infty$ after the relevant min/max operation. Bernstein subinterval ranges in the Python helper are obtained by de Casteljau subdivision and padded outward by repeated `nextafter` steps; the binding Player-2 C4 positive products/sums are rounded upward and the final curvature difference downward. Bernstein/cellwise intervals are the primary proof representation. High-degree Bernstein-to-power conversion is diagnostic only because binary64 conversion suffers catastrophic cancellation; it is not used as a proof representation.

2 Exact value form, graph transpose, and discrete Hessian

This appendix derives the reverse recurrence and the discrete Hessian identity directly from the preceding-node recurrence induced by the finite game and its exact finite graph transpose.

2.1 Exact value form and its variations

Multiply the preceding-node L1 recurrence by $s_h = \Gamma(2 - \nu)h^\nu$ and put

$$a_r = (r + 1)^{1-\nu} - r^{1-\nu}, \quad c_1 = 1 - a_1, \quad c_r = a_{r-1} - a_r \quad (r \geq 2).$$

Collecting equal nodal states gives the exact value form

$$x^n = a_{n-1}x^0 + \sum_{k=1}^{n-1} c_{n-k}x^k + s_h F(x^{n-1}, z^{n-1}), \quad 1 \leq n \leq N, \quad (3)$$

where the sum is empty for $n = 1$. Thus $x^1 = x^0 + s_h F(x^0, z^0)$ exactly. For $n \geq 2$ the sum may equivalently be written $c_1 x^{n-1} + \sum_{r=2}^{n-1} c_r x^{n-r}$, and

$$\sum_{k=1}^{n-1} c_{n-k} + a_{n-1} = 1.$$

Fix player i and perturb only the log-control,

$$v_i^j(\varepsilon) = v_i^j + \varepsilon \eta_j, \quad z_i^j(\varepsilon) = z_i^j e^{\varepsilon \eta_j}.$$

Therefore

$$\partial_{v_i} F = z_i \partial_{z_i} F, \quad \partial_{v_i v_i} F = z_i \partial_{z_i} F + z_i^2 \partial_{z_i z_i} F.$$

Since the controlled Set-A vector field of the main paper is affine in each z_i , $F_{z_i z_i} = 0$ and $F_{v_i v_i} = F_{v_i}$. Let

$$Y^n = \partial_\varepsilon x^n|_{\varepsilon=0}, \quad R^n = \partial_{\varepsilon\varepsilon} x^n|_{\varepsilon=0},$$

with $Y^0 = R^0 = 0$, and set

$$A^j = F_x(x^j, z^j), \quad b_i^j = F_{v_i}(x^j, z^j),$$

$$\mathcal{H}^j = F_{xx}^j[Y^j, Y^j] + 2F_{xv_i}^j[Y^j]\eta_j + F_{v_i v_i}^j \eta_j^2.$$

Differentiating (3) gives, for $1 \leq n \leq N$,

$$Y^n = (c_1 I + s_h A^{n-1})Y^{n-1} + \sum_{r=2}^{n-1} c_r Y^{n-r} + s_h b_i^{n-1} \eta_{n-1}, \quad (4)$$

$$R^n = (c_1 I + s_h A^{n-1})R^{n-1} + \sum_{r=2}^{n-1} c_r R^{n-r} + s_h \mathcal{H}^{n-1}. \quad (5)$$

2.2 Exact graph transpose

The tracking part of the direct second derivative is

$$h \sum_{n=0}^N \tau_n \left[\frac{2(Y_i^n)^2}{s_i^2} + d_i^n \cdot R^n \right], \quad d_i^n := \frac{2x_i^n}{s_i^2} e_i.$$

Because $R^0 = 0$, only nodes $1, \dots, N$ enter the linear-in- R term. Define

$$g^j := h \tau_j d_i^j, \quad 1 \leq j \leq N.$$

For compactness write $B^j = c_1 I + s_h A^j$. Rearranging (5),

$$R^n - B^{n-1} R^{n-1} - \sum_{r=2}^{n-1} c_r R^{n-r} = s_h \mathcal{H}^{n-1}. \quad (6)$$

Multiply the n th identity by a vector λ^n and sum from $n = 1$ to N . The coefficient multiplying a fixed R^j , $1 \leq j \leq N$, is obtained as follows:

- $+\lambda^j$ comes from the leading R^j in the $n = j$ equation;
- $-(B^j)^T \lambda^{j+1}$ comes from the one-step term in the $n = j + 1$ equation;
- for every $r \geq 2$ with $j + r \leq N$, the history term in the $n = j + r$ equation contributes $-c_r \lambda^{j+r}$.

Hence

$$\begin{aligned} \sum_{n=1}^N \lambda^n \cdot \left(R^n - B^{n-1} R^{n-1} - \sum_{r=2}^{n-1} c_r R^{n-r} \right) \\ = \sum_{j=1}^N R^j \cdot \left[\lambda^j - (B^j)^T \lambda^{j+1} - \sum_{r=2}^{N-j} c_r \lambda^{j+r} \right], \end{aligned} \quad (7)$$

where terms containing λ^{N+1} are absent and empty sums are zero. This is the exact finite-history transpose; no limiting integration-by-parts argument is used.

Choose the backward variables so that the bracket in (7) equals the tracking source g^j . The terminal coefficient gives

$$\lambda^N = g^N = \frac{h}{2} d_i^N.$$

For $j = N - 1, \dots, 1$ the remaining coefficients give

$$\lambda^j = g^j + (c_1 I + s_h (A^j)^T) \lambda^{j+1} + \sum_{r=2}^{N-j} c_r \lambda^{j+r}. \quad (8)$$

Equation (8), together with the terminal equation, is exactly the reverse recurrence stated in the main paper. Substituting this choice into (7) and using (6) yields

$$h \sum_{n=0}^N \tau_n d_i^n \cdot R^n = s_h \sum_{n=1}^N \lambda^n \cdot \mathcal{H}^{n-1} = s_h \sum_{j=0}^{N-1} \lambda^{j+1} \cdot \mathcal{H}^j. \quad (9)$$

The terminal half-weight is present because $g^N = h d_i^N / 2$; it is a localized $O(h)$ source contribution, hence smaller than the $O(h^\nu)$ endpoint-transfer scale for $0 < \nu < 1$ and $h \leq 1$. The fixed condition $R^0 = 0$ leaves the initial endpoint without a boundary contribution.

2.3 Gradient density and exact Hessian

Define the density that multiplies the nodewise nonlinear second variation by

$$\pi^j := \frac{s_h}{h} \lambda^{j+1}, \quad j = 0, \dots, N - 1.$$

Then (9) becomes

$$h \sum_{n=0}^N \tau_n d_i^n \cdot R^n = h \sum_{j=0}^{N-1} \pi^j \cdot \mathcal{H}^j.$$

The direct second derivative of the complete discrete objective is

$$D^2 J_{i,h}[\eta, \eta] = 0.6h \sum_{j=0}^{N-1} \eta_j^2 + h \sum_{n=0}^N \tau_n \frac{2(Y_i^n)^2}{s_i^2} + h \sum_{n=0}^N \tau_n d_i^n \cdot R^n.$$

Eliminating the last term with the exact transpose identity gives the discrete Hessian identity stated in the main paper. Thus the formula includes the full L1 memory, preceding-node evaluation, trapezoidal endpoint weights and the log-coordinate chain rule of the finite computational graph.