

Proof Assistants at the interplay of mathematical cultures: a semiotic-cultural analysis of proof in higher education

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This paper investigates how the use of Proof Assistants (PAs) in undergraduate mathematics teaching makes visible and reconfigures cultural practices of proof. Rather than evaluating whether PAs facilitate or hinder students' learning, we analyze them as cultural artifacts that embed specific epistemological commitments concerning, among others, validity, formalization, explicitness, and computational checking. The study is situated in a first-year university course in which students use Verbose Lean, a controlled natural language interface for Lean, to formalize proofs. The theoretical framework articulates Cultural Transposition with Lotman's notion of semiosphere in order to study proof as a cultural practice realized through systems of signs and validation. We focus on an episode in which two students work on the proof that if a real sequence tends to a strictly positive limit, then its terms are eventually positive. The analysis shows how students' activity moves across paper inscriptions, oral exchanges, Verbose Lean code, error messages, teacher's public correction, and the final proof text accepted by the PA. The results show that PAs do not merely add formal rigor to proof writing; they perturb existing proving practices by requiring students to make explicit objects, types, quantifiers, witnesses, local hypotheses, and inferential steps that may remain implicit in ordinary mathematical proof. We argue that PAs can act as epistemic revelators: they make observable implicit assumptions of proof, conceptualized here as "unthoughts", and provide a privileged site for studying continuities and discontinuities between cultures of proof in higher education.

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1. Proof assistants and cultural practices of proof in undergraduate education

Mathematical proof occupies a central position in university mathematics education, both as an object of knowledge and as a practice through which students are progressively introduced into the norms, forms of reasoning, and validation practices of advanced mathematics (Selden, 2012). This centrality becomes particularly critical at the transition from secondary to tertiary education. At this level, students are expected not only to understand mathematical statements, but also to produce and evaluate arguments according to new standards of explicitness, rigor, structure, and validation. For instance, Selden and Selden (2003) showed that undergraduate students may have difficulties in judging the validity of proof texts. This transition involves a change in the didactical contract between secondary school and university mathematics (Gueudet & Vandebrouck, 2022), and students may experience it as a crisis, especially regarding the "ways of thinking for mathematics" (Di Martino & Gregorio, 2019, p. 837).

These difficulties point to a broader issue. Several cultures co-exist about what proof is, what it is for, what its representations should be, how a proof is validated. This makes epistemology of proof a crucial point of discussion in undergraduate education research. As pointed out by

Hanna and Knipping (2020), recent developments at the interface between mathematics and computer science call for new approaches to proof teaching.

In this context, there is a growing interest in the use of Proof Assistants (PAs) in undergraduate mathematics teaching (e.g., Ouvrier-Bufferet, in press). Initially designed for mathematical, computational, and formal research, systems such as Lean and Coq/Rocq are now being adapted for educational use, including Verbose Lean and D $\exists$ Vduction, but their non-neutral use in the teaching and learning of proof remains underexplored. For Hanna et al. (2024), PAs do more than add formal rigor: they represent a completely novel teaching approach. In fact, their introduction requires a rethinking of proof itself and calls for a paradigmatic shift in which didactical, institutional, epistemological, formal, computational, and semiotic-cultural dimensions of proof are considered together. Thus, this contribution investigates what the use of PAs reveals about cultural practices of proof in undergraduate courses, exploring how these digital artifacts may redefine proof in higher education.

Therefore, this article has a twofold aim. Theoretically, it conceptualizes PAs as cultural artifacts that bring several practices of proof into contact (Section 2). We develop a framework articulating Cultural Transposition and Lotman's Semiosphere to investigate how PAs reconfigure the epistemology of proof (Section 3). Empirically, we operationalize this framework through the analysis of a didactical episode, in which two students work on a proof about limits of sequences involving Verbose Lean (hereafter named Verbose), demonstrating how PAs make the "unthoughts" of proof observable across sign systems, teacher mediation, and computational validation (Sections 4 and 5).

Our research question is: How does the use of a PA make visible and reconfigure cultural practices of proof?

In this article, proof is considered as a specific institutional form of mathematical argument, in which validity, explicitness, and modes of justification are regulated by historically situated mathematical and didactical practices.

2. Proof Assistants at the interplay of several cultures

2.1. An epistemological debate to open

Hanna et al. (2024) argue that Lean makes formalism central, but this formal rigor does not necessarily suppress students' creativity: students may explore multiple proof paths (including puzzle/maze processes) while being confronted with rigor in a more explicit way than in paper-and-pencil environments. They point out a mismatch between ordinary informal proofs and proofs in PAs, underlying the contrast between 'Cartesian' and 'Leibnizian' ideals of proofs. In doing so, they situate PAs within a broader epistemological and philosophical debate about what mathematical proof is, what it is for, and what counts as acceptable proof in different communities. We stress the fact that implementing PAs in undergraduate teaching brings more than these two ideals of proof and a translation between the three languages identified by Avigad (2019): informal mathematics, formal symbolic logic, Lean syntax.

2.2. "Ordinary" mathematical proof in undergraduate (without PAs)

Before considering PAs, it is necessary to recall that, in undergraduate mathematics, proof is already a heterogeneous object. At the secondary-tertiary transition, students encounter new expectations concerning rigor, explicitness, and validation. Moreover, university teaching does not present proof in a uniform way. Melhuish et al. (2022) show that lecturers combine formal

written results with spoken informal comments, heuristics, meta-commentary, and interactive moments, while often experiencing a tension between teaching syntactic skills and valuing semantic reasoning. Research on university teaching shows that lecturers' disciplinary identities influence their teaching practices (e.g., Bridoux et al., 2023). Thus, proof teaching may create tension between research-oriented mathematical practices and the institutional need to make proof teachable, understandable, and assessable.

Therefore, there are at least three intertwined situated practices of proof in the first year of undergraduate mathematics: a pre-existing practice in transition from secondary school, a university practice of ordinary mathematical proof (influenced by research practices), and an institutional-didactical practice in which proof is taught and regulated in the classrooms.

In this paper, culture is not understood primarily in an ethnic, geographical or national sense. We use the expression *cultural practices of proof* to designate situated ways of producing, representing, justifying, validating, and evaluating mathematical arguments. These practices are not fixed entities, nor are they reducible to the material supports through which they are carried out. They are historically and institutionally situated practices organized by meanings, norms, criteria of acceptability, degrees of explicitness, and forms of validation. This perspective is consistent with Geertz's interpretive conception of culture as "webs of significance" through which human action becomes meaningful (Geertz, 1973, p. 5), and with socio-cultural approaches in mathematics education that understand mathematical learning as participation in socially and historically organized practices (Bishop, 1991; Lerman, 2000).

2.3. Behind the formal proof visible in the interface of a PA

In recent work, comparing six PAs, Bartzia et al. (2026) show that differences in language, automation, feedback, validation of proof (optimizing teacher time) and proof visualization are not superficial interface issues, but didactically relevant properties allowing multiple teaching modalities. They emphasize that these features "are absent from usual practices in the teaching of proof and proving" (Bartzia et al., 2026, p. 6) and that PAs may help leverage programming skills to work on mathematical proof. They also underscore the results of previous research (Knobelsdorf et al., 2017): "[...] contrary to proof assistants, pen-and-paper as a learning tool does not "embody" enough knowledge about formal proofs to help students construct proofs" (Bartzia et al., 2026, p. 3).

In fact, research on students' experiences and teaching practices with PAs (e.g., Avigad, 2019; Kerjean et al., 2022; Thoma & Iannone, 2022¹) suggests that PAs may make explicit aspects of proving that usually remain implicit, or are often performed automatically: the status of statements, the role of definitions, the management of quantifiers, and the implicitness tolerated in human-readable proofs. The difference between proof for a reader and proof for a machine, as well as the distinction between proving for understanding and proving for validation, are therefore crucial. The teacher-focused experiment of Ouvrier-Bufferet and Sisco (submitted) extends this discussion by showing that university teachers' practices with PAs are structured by tensions between paper-and-pencil proof and PA proof, semantic and syntactic work, formal validation and the functions of proof, as well as by unresolved issues concerning the conceptions of proof embedded in PAs. Therefore, PAs create mixed situations

¹ For a review of research, see Ouvrier-Bufferet & Sisco (submitted).

that involve much more than formalized and computational-interactive proof, encountering the “ordinary”/“classical” mathematical proof teaching.

2.4. Our contribution: towards a semiotic-cultural analysis of PAs

The literature reviewed above suggests that, when PAs are introduced in first-year university classrooms, at least five cultural practices of proof may become intertwined: students’ transitional practices from secondary school, ordinary university proof, institutional-didactical proof teaching, formalized proof, and the computational-interactive practice of the PA. These practices are distinguished analytically by the norms governing what must be made explicit, how a proof step is justified, and what counts as validation.

This complexity also points to a research need. In their review of proof-based university teaching, Melhuish et al. (2022) conclude that “there is still much to learn about the interaction between instructional practice and student outcomes and activity” (p. 18). Stylianides et al. (2024) similarly note that relatively few studies focus on “student-content issues in ‘technological environments’” (p. 53) or develop “novel theoretical frameworks and/or methodological approaches” capable of challenging existing perspectives (p. 55). The introduction of PAs into undergraduate teaching is therefore not only a new instructional setting; it also creates a research situation in which the cultural, epistemological, and semiotic dimensions of proof need to be studied together.

On this basis, PAs can be analyzed as perturbing artifacts. They do not replace existing proof practices; rather, they act as sites where cultural proof practices come into contact and hidden epistemological norms may become visible. As PAs increasingly enter mathematical and educational practices, the question is not whether they should be celebrated or rejected, but how their introduction may transform the cultural practices of proof already present in undergraduate mathematics. However, research still lacks a unified semiotic-cultural analysis of PAs as sites of cultural transposition. We assume that PAs make cultural practices of proof observable because they require translations across signs, proof structures, and validation systems. In this sense, PAs can act as epistemic revelators.

This is where a semiotic-cultural perspective becomes necessary. In this perspective, we present this framework in Section 3, and in Section 4 we focus on a didactical episode with Verbose, to show in Section 5 how the “unthoughts” of proof become visible in a culturally situated practice when transformed with a PA.

3. Theoretical framework: a semiotic-cultural approach

The introduction of PAs into undergraduate proof teaching raises an epistemological problem: as discussed above, PAs bring into the classroom specific criteria of validity, explicitness, formalization, and computational checking. However, these criteria do not appear in students’ activity as separate cultural practices of proof. In classroom situations, ordinary mathematical proof, institutional teaching practice, formalized proof, and computational checking are intertwined. They become observable through the ways in which students, teachers, and the PA produce, transform, reject, and stabilize mathematical signs.

A semiotic perspective is therefore necessary. The same sign, or “text²”, may circulate through several cultural practices of proof, acquiring different roles and conditions of legitimacy. The analytical problem is to reconstruct the transformations through which signs move across these practices.

Our framework articulates Cultural Transposition and Lotman’s Semiosphere. Cultural Transposition allows us to interpret encounters between practices organized by different norms and assumptions, and to analyze the “unthoughts” made visible (Mellone et al., 2019). Lotman’s Semiosphere (1990) conceptualizes the heterogeneous semiotic space in which these practices coexist, interact, and translate into one another. It is the theoretical tool that makes it possible to study relations between cultural practices of proof when they cannot be empirically separated.

3.1. Cultural transposition and the “unthoughts” of proof

Cultural Transposition was introduced in Mathematics Education to analyze what happens when educational practices originating in different cultural contexts are brought into contact, and local practices are reinterpreted through this encounter (Mellone et al., 2019). Whereas Didactic Transposition (Chevallard & Bosch, 2020) concerns the institutionally conditioned transformations through which bodies of knowledge are selected, reorganized, and made teachable, Cultural Transposition focuses on what becomes visible when practices shaped by different cultural assumptions and semiotic organizations encounter one another. Its purpose is not to compare cultures as fixed entities, nor to describe a simple transfer of methods from one context to another. Rather, it provides a way to analyze how the encounter with another practice can make visible the assumptions, values, and educational intentions that usually remain implicit within one’s own practice.

This perspective is relevant for the present study because the introduction of PAs into proof teaching and learning does not merely add a technical artifact to an existing classroom practice. It brings into contact cultural practices of proof organized by different norms of explicitness, formalization, and validation. The “otherness” involved here is therefore not primarily geographical or national, but epistemological, semiotic, and institutional. Ordinary university proof encounters formalized proof; institutional teaching practice encounters computational checking; students’ mathematical activity encounters a system that validates or rejects each step according to local formal constraints.

A central notion in Cultural Transposition is that of the *unthought*, drawn from Jullien’s work and developed in Mathematics Education to describe what remains unnoticed because it is taken for granted within a given cultural practice (Mellone et al., 2021). In Cultural Transposition, the unthought does not designate a simple absence of knowledge. It refers to what a practice does not need to thematize in order to function, precisely because its norms and assumptions are shared by the community that sustains it.

² Lotman’s semiotics of culture allows us to analyze this heterogeneity through a broad notion of “text”, understood as any meaningful semiotic formation circulating within a culture (Lotman, 1990). Students’ formulations, gestures, written inscriptions, diagrams, PA feedback messages, and proof fragments can therefore be treated as texts in Lotman’s sense.

In the case of proof, unthoughts are therefore not reducible to students' errors or misconceptions. They concern aspects of proving that may function implicitly in one cultural practice of proof but become evident when the same mathematical work must be recognized within another practice: for instance, the operational use of a definition, the role assigned to a mathematical object in the proof structure, or the kind of justification required for an inference.

We do not assume that a PA improves or worsens proof in general. Rather, we consider it as an artifact that may disturb (stabilized) proving practices and make some of their implicit assumptions observable.

3.2. The semiosphere: boundaries between cultures of proof and bridge-signs

If Cultural Transposition explains why encounters between cultural practices of proof are theoretically significant, Lotman's Semiosphere explains how such encounters can be analyzed when these practices are intertwined. For Lotman (1990), meaning cannot be produced by isolated signs or languages: semiosis requires a semiotic space – the Semiosphere – in which languages, texts, and cultural systems interact. This is particularly relevant for PA-mediated proof, because proof does not unfold in a single language or representational system. It involves natural language, mathematical symbolism, controlled natural language, PA proof states, feedback messages, paper inscriptions, blackboard writing, and institutional explanations.

The Semiosphere is not homogeneous. Its boundaries are not merely lines of separation, but sites where translation and transformation take place. Lotman describes the boundary as “bilingual and polylingual” and as “a mechanism for translating texts of an alien semiotics into ‘our’ language” (Lotman, 1990, pp. 136-137). In a later English version of this text, Lotman and Clark (2005) also give the example of boundary figures who “belong to two worlds” and “operate as a kind of interpreter” (p. 211). This point is essential for our study: what matters at a boundary is not only that two systems meet, but that something must function between them, making translation possible while preserving traces of difference.

In PA-mediated proof, mathematical signs often have to become readable in another system before they can function there. A recalled definition may become an operational tool; an ordinary mathematical formulation may become a quantified structure; a plausible calculation may become a locally justified inference; an expression meaningful in students' reasoning may be rejected by the PA because it does not occupy the expected formal role. These transformations are precisely what make the Semiosphere useful for our analysis: it allows us to study how cultural practices of proof interact through signs, rather than treating them as separate domains.

To describe this process, we introduce the construct of *bridge-sign*, our term for an analytical function within the semiosphere, grounded in Lotman's account of boundary translation and of figures that operate as interpreters between semiotic worlds (Lotman & Clark, 2005; Manolino, 2021, 2024). A bridge-sign is a mathematical object whose circulation across semiotic boundaries makes both continuity and discontinuity between cultural practices of proof observable. It is not defined in advance by its mathematical content. It is identified empirically when a sign changes role, status, or acceptability as it moves across systems of practice.

The bridge-sign therefore specifies, for the analysis of proof, the Lotmanian function of boundary translation. It is not simply a connector between already stabilized meanings. It is a sign through which the work of translation becomes visible: what appears self-evident in one cultural practice of proof may become problematic when the same sign has to function in another. In this sense, bridge-signs also connect the Semiosphere with Cultural Transposition, because they make the unthoughts of proof observable as transformations required for a mathematical text to become readable and valid across different systems of signs and validation.

The articulation of Cultural Transposition and Semiosphere therefore gives the analysis both its object and its method. Cultural Transposition directs attention to the unthoughts made visible when cultural practices of proof come into contact. The Semiosphere makes it possible to follow the semiotic transformations through which this contact becomes observable. The aim of our study is thus not only to describe what is done with a PA, but to reconstruct how mathematical signs change status while a proof text becomes progressively recognizable as valid by the PA.

4. Methodology and context of the study

4.1. The experiment and Verbose Lean

The data analyzed in this paper come from the APPAM³ project, in a first-year university course entitled "Logic and Computer-Assisted Proofs". The course (practical class) is taught in the second semester of a selective dual-degree undergraduate program in mathematics and computer science in France. The teacher-researcher has a specific profile: he is a mathematician who uses Lean in part of his research and develops Verbose, a controlled natural language interface for Lean, for educational purposes. Massot (n.d.) presents Verbose as providing "tactics and commands for Lean in a very controlled natural language" and specifies that its aim is to make Lean code easier to transfer to traditional paper proof.

Verbose aims to bring the writing of formalized proof closer to ordinary mathematical discourse. It allows students to write proof steps in a controlled natural language while maintaining Lean's formal checking. Verbose combines a global view of a proof text with a local view of the proof state, including goals and context. This makes it a particularly relevant environment for our study, because it is located at the boundary between ordinary proof writing and machine-checked proof.

Online Resource A provides a contextual description of the Verbose environment, the analyzed exercise, the French proof commands visible in the screenshots, and the role of feedback messages in students' proving activity. A general view of the Verbose interface is provided in Online Resource A, Figure A1.

4.2. The mathematical task

The analyzed task concerns the following statement: if a real sequence u tends to a strictly positive limit l , then $u_n > 0$ for sufficiently large n . In Verbose, the exercise is formulated with

³ APPAM (Les assistants de preuve pour les apprentissages mathématiques Diagnostiquer, analyser, concevoir, expérimenter, évaluer) (Proof assistants in mathematics learning - Appraising, Analyzing, Designing, Experimenting, Assessing): <https://anr.fr/Project-ANR-23-CE38-0009>

data, given by the teacher, $(u : \mathbb{N} \rightarrow \mathbb{R})$ and $(l : \mathbb{R})$, the hypothesis $h1 : l > 0$, and the conclusion $(u \text{ converges to } l) \rightarrow \text{exists } N, \text{ for all } n \geq N, u\ n > 0$.

The task is a classical formalization exercise on mathematical content already familiar to the students. They had encountered sequences in secondary school and again during the first semester of university. Thus, the interest of the task does not lie in introducing a new mathematical object. Its interest lies in the required coordination between known content and proof structure: using the definition of convergence, choosing a suitable positive ε , obtaining a witness N , instantiating a universal property, and deriving positivity from an absolute-value inequality.

The task is institutionally classical and clearly oriented toward formalization. However, students' activity includes exploratory phases: they try expressions, use paper, delete formulations, interpret PA feedback, and reformulate their proof. The analysis focuses on how this multimodal activity is progressively transformed into a proof text accepted by the PA and institutionally recognizable as formal mathematical proof.

4.3. Data

The broader data corpus consists of video and audio recordings of class sessions on sequences, convergence, and limits, together with recordings of pairs of students working during the sessions. For this paper, we focus on one episode involving two students, S. and M., working on the exercise described above.

The analyzed data include the video of students' work, an external audio recording, the transcript of their interaction, the final Verbose proof text, and the video and transcript of teacher's public correction of the same exercise at the blackboard. The teacher's correction is important because it runs partly in parallel with students' work. It is not treated as a direct intervention on the pair, but as an institutional semiotic resource circulating in the classroom.

4.4. Unit of Analysis and Operational analysis grid

The data analysis is qualitative and semiotic. A unit of analysis (UA) is an episode of semiotic-cultural transposition. An episode is a segment of activity in which a mathematical "text" must be transformed in order to move from one cultural practice of proof or validation system to another.

An episode begins when a problem of transition between practices becomes observable. Indicators of the beginning of a unit include hesitation, erasure, change of medium, movement from paper to screen, an error message, a verbal reformulation, the appearance of a new mathematical object, a change in PA goal state, or the emergence of a parallel teacher intervention. A UA ends when the sign or proof fragment is stabilized, abandoned, rejected, or transformed into a new problem.

Each unit (UA) is analyzed through the same operational grid: temporal delimitation; multimodal observables; mathematical object at stake; cultural practice activated; semiotic boundary or translation; validation authority; local outcome; and contribution to the research question. Multimodal observables include students' words, written notes, gestures, paper inscriptions, writing in the PA, error messages, goal states, and any parallel teacher discourse.

The segmentation of the episode was based on the detailed reconstruction of students' work and of the parallel teacher correction. Through this reconstruction, we identified seven UAs. They are not equal time intervals, nor successive lines of the final proof. They correspond to moments in which the status of a mathematical text changes, or in which a translation difficulty between cultural practices of proof becomes observable.

In particular, within this episode, the sign $l/2$ emerges as a semiotic thread across the classroom semiosphere. Its role is not stable. The same expression circulates through students' oral discussion and paper exploration, is rejected by Verbose when placed in the wrong formal role, is later legitimized in the teacher's blackboard correction, and finally reappears in students' Verbose proof as a formally productive parameter. $l/2$ was identified as a bridge-sign because it repeatedly crosses semiotic boundaries while changing role and mode of validation, thereby making continuities and discontinuities between cultural practices of proof observable.

The guiding question for each UA is therefore: what transformation must a "text" undergo in order to move from one cultural practice of proof to another, and which authority validates, rejects, or stabilizes this transformation? Table 1 summarizes the seven UAs and the semiotic-cultural problem associated with each.

Table 1. An overview of the analysis grid.

Unit	Video duration	Why it starts here	Why it ends here	Semiotic-cultural problem
UA1 - Paper/screen exploration of the definition of a limit The mathematical meaning circulates between paper and software without formal stabilization	approx. 00:00–09:20	The students start with 'Assume h_p : u converges to l ' and work on paper with absolute value, epsilon and inequalities.	It ends when they return to the software with a new object, $l/2$, and propose it as a candidate.	Transition from free symbolic exploration on paper to controlled formulation in the PA.
UA2 - First occurrence of $l/2$ as an incorrect candidate A new mathematical object, $l/2$, appears, and the PA redefines its role via a type error	09:24–09:35	The following appears on screen: Let's prove that $l/2$ works.	It ends with the error message: $l/2$ is real but the system expects a natural number.	$l/2$ is recognized as a mathematically useful symbol, but placed in the wrong formal role.
UA3 - Return to the rappel and pre-correction block The difficulty of transforming a written definition into a test move	approx. 10:30–12:13	The students try again to copy/activate the reminder from the limit definition.	Ends with the start of the teacher's collective correction.	The definition of a limit is still treated as text to be copied, not as a framework to be specialized.
UA4 - Teacher's public correction as a parallel institutional resource Introduction of a new institutional element into the classroom's semiosphere, without	parallel to the demonstration, starting at 12:13	The teacher begins the correction, clarifies the goal and introduces a diagram (Figure 2).	It ends when, during the correction, the focus shifts from understanding the goal to specialization with $l/2$.	It is not an intervention on the group: it is a public institutional discourse that can become an available semiotic resource.

direct intervention in the pair's work				
UA5 - Students' attempt with l and not yet with $l/2$ Students construct the correct logical structure but with a parameter that is still insufficient	14:08–19:07	The students construct a more structured proof: N is convenient, Let $n \geq N$, use of the universal hypothesis.	It ends as soon as $l/2$ is reinserted into the specialization row.	The logical structure of the proof progresses, but the parameter is still l , not $l/2$; there remains a mismatch between the strategy and the necessary inequality.
UA6 - Re-functionalisation of $l/2$ as epsilon/tolerance The crucial transformation of $l/2$ takes place: from a useful but poorly positioned sign to a formally relevant resource	19:08–19:25	$l/2 > 0$ appears in the line 'As u tends towards $l \dots$ '; shortly afterwards, the universal property is also rewritten as $\leq l/2$.	It ends when $l/2$ has entered the context and the problem shifts from the choice of parameter to the justification of the inferences.	$l/2$ changes status: no longer a candidate for N , but a parameter that produces N from the definition of the limit.
UA7 - Stabilization of the final algebraic chain The problem is no longer to choose the parameter, but to make the inference chain locally validated by the PA	approx. 19:25–end of exercise	After entering $l/2$, students work on the steps $-(l/2) \leq u_n - l$, $l/2 \leq u_n$, $u_n > 0$.	Ends with the final acceptance of the proof.	Transition from plausible calculation to inference locally accepted by the PA.

5. Experimental results: the cultural transposition of the bridge-sign

This section presents the seven UAs as successive moments in the semiotic-cultural transposition of $l/2$. The aim is not to describe students' work as a linear progression toward the correct proof, but to reconstruct how mathematical meaning, proof roles, and validation authority are reorganized across the episode.

The final accepted proof in Verbose has the following structure. The students assume that u tends to l ; they use convergence with $l/2 > 0$ to obtain a natural number N such that for every $n \geq N$, $|u_n - l| \leq l/2$; they show that this N works; they fix $n \geq N$; they instantiate the universal property; they derive $-(l/2) \leq u_n - l$, then $l/2 \leq u_n$, and finally $u_n > 0$.

The interest of the episode lies in the fact that this final text is not produced linearly. It emerges through a sequence of transformations across paper, oral discussion, Verbose feedback, the teacher's public correction, and repeated reformulations in the PA. The analysis refers to selected elements of the Verbose interface and of the teacher's blackboard correction.

Across the seven UAs, validation is not located in a single authority. It is distributed among students' exploratory reasoning, paper inscriptions, the recalled definition of convergence, Verbose feedback, the teacher's public correction, ordinary mathematical norms, and the final computational acceptance by the PA. The analysis therefore specifies, for each UA, which authority validates, rejects, legitimizes, or stabilizes the mathematical text at stake.

5.1. UA1: Paper/screen exploration of the definition of a limit

At the beginning, the students have written `Assume hyp: u converges to l` in Verbose and work mainly on paper. They manipulate absolute-value inequalities and attempt to relate $|u_n - l| \leq \varepsilon$ to the positivity of u_n (Figure 1). They also try to introduce an arbitrary epsilon in PA and to copy the recalled definition of convergence.

This unit shows a first boundary between a symbolic exploratory practice and a formalized proof practice. On paper, signs can remain provisional; students can try, delete, and leave roles implicit. In Verbose, however, each introduced object must be locally justified by the current goal. The system rejects attempts such as introducing an epsilon when the proof state does not require it. The recalled definition is available as a text, but it is not yet used as an operational tool.

The unthought made visible here concerns the use of definitions. In ordinary mathematical activity, students may treat a known definition as a piece of text to be recalled. In a formalized proof, the definition must be instantiated at the right moment and with the right object so that it produces new elements in the local context.

In terms of validation authority, the work is not yet stabilized. Paper allows provisional exploration and locally negotiated reasoning between students, whereas Verbose rejects moves that do not correspond to the current proof state. The first tension is therefore between exploratory validation and local computational validation.

The image shows a piece of paper with handwritten mathematical work. The text includes:

- $|u_n - l| \leq \varepsilon$ (with some corrections)
- $a > b \Rightarrow \exists \varepsilon > 0 \text{ tq } a = b + \varepsilon$ appliqué à P . $\exists \varepsilon > 0 \text{ tq } P = \varepsilon$
- $|u_n - l| \leq \varepsilon \Leftrightarrow -\varepsilon \leq u_n - l \leq \varepsilon$
- Comme $P > 0$ on a $N \text{ tq } \forall n \geq N$
- $|u_n - P| = |P - u_n| \leq |P| + |u_n|$
- $|u_n - P| \leq |u_n| + |P| = |u_n| + P$
- $|u_n - P| \leq \varepsilon \Leftrightarrow -\varepsilon \leq u_n - P \leq \varepsilon \Leftrightarrow -\varepsilon - P \leq u_n \leq \varepsilon - P$
- $\Leftrightarrow -\varepsilon - P \leq u_n \leq \varepsilon - P$
- $\Leftrightarrow -\varepsilon - P \leq u_n \leq \varepsilon - P$
- $\varepsilon > 0 \text{ or } -\varepsilon - P > 0$

Figure 1. Students' paper work during the initial exploration of the definition of convergence.

The inscriptions show provisional manipulations of absolute-value inequalities before the proof structure is stabilized in Verbose.

5.2. UA2: First occurrence of $l/2$ as an incorrect candidate

After working on paper, the students discuss the need to choose a quantity smaller than l and formulate $l/2$ as a possible choice. They then type `Montrons que  $l/2$  convient` (Let's prove that $l/2$ works). Verbose rejects the step with a type mismatch: $l/2$ has type `real`, whereas the system expects a natural number. This type mismatch is documented in Online Resource A, Figure A2.

This is a crucial moment. Mathematically, the quantity is relevant: choosing a positive epsilon smaller than l is a standard way to derive eventual positivity. The error does not concern the global mathematical pertinence of the expression. It concerns its role in the proof. In the sentence `Montrons que ... convient` (Let's prove that ... works), the expression is placed in the position of the witness N , which must be a natural number.

The PA therefore separates two aspects that may remain fused in exploratory reasoning: mathematical usefulness and formal role. It makes visible that a sign cannot be validated only because it is mathematically promising; it must be located in the correct position of the proof structure and have the expected type.

Here, validation authority shifts sharply to Verbose. Students' discussion gives mathematical legitimacy to $l/2$ as a promising quantity, but PA rejects it because it is placed in the role of N ⁴. This type mismatch makes visible a distinction between mathematical relevance and formal role.

5.3. UA3: Return to the rappel and pre-correction block

After the type mismatch, the students return to the recall of the definition of convergence. They attempt to copy or activate the general form involving `forall  $\varepsilon > 0$ , exists  $N$ , forall  $n \geq N$  ...`. These attempts fail because the students still treat the definition as a text to be inserted rather than as a schema to be specialized through the hypothesis `u converge to l`.

This unit reveals an important unthought of ordinary proof: saying "we use the definition of convergence" compresses several operations. In the PA, this phrase must be unpacked. One must identify the relevant hypothesis, provide a positive real number, obtain a witness N , and introduce the resulting universal property into the context.

The boundary here is between definition-as-recalled-text and definition-as-operator. Verbose makes this boundary visible because it validates only the latter.

The validation authority is distributed between the reminder, the students, and the PA. The reminder provides a legitimate mathematical reference, but it does not by itself validate a proof move. Verbose requires the definition to be activated through the convergence hypothesis and a positive parameter.

5.4. UA4: Teacher's public correction as a parallel institutional resource

While the students are working in Verbose, the teacher begins a public correction at the blackboard. This unit does not correspond to a direct intervention in the students' proof text; rather, it introduces an institutional mathematical discourse that becomes available in the classroom semiosphere. The correction coordinates several semiotic resources.

The teacher first clarifies the goal of the exercise through the real-line diagram (Figure 2).

⁴ Shortly after the rejection, the students themselves formulate the distinction between a numerical quantity and the rank to be produced:

"Mais non ! Non, on veut le rang."

"Le rang c'est sûrement pas l sur 2."

[Translation: "But no! No, we want the rank." "The rank is surely not $l/2$."]

“Il faut vraiment avoir compris la définition de u tend vers l .”

“ l sur 2, je vais le mettre sur mon dessin.”

“Et donc, si je fais une fenêtre de taille l sur 2...”

[Translation: “One really has to have understood the definition of u converges to l .”

“ $l/2$, I am going to put it on my diagram.” “And so, if I make a window of size $l/2$...”]

The diagram functions as a shared visual text: it represents the positivity of l , the target region to the right of 0, and the need to choose a “window” around l that remains inside this target region. In this discourse, the window becomes a semiotic resource through which the teacher connects the semantic meaning of convergence with the choice of a suitable positive quantity. The later blackboard specialization of convergence with $l/2$ is documented in Online Resource A, Figure A3.

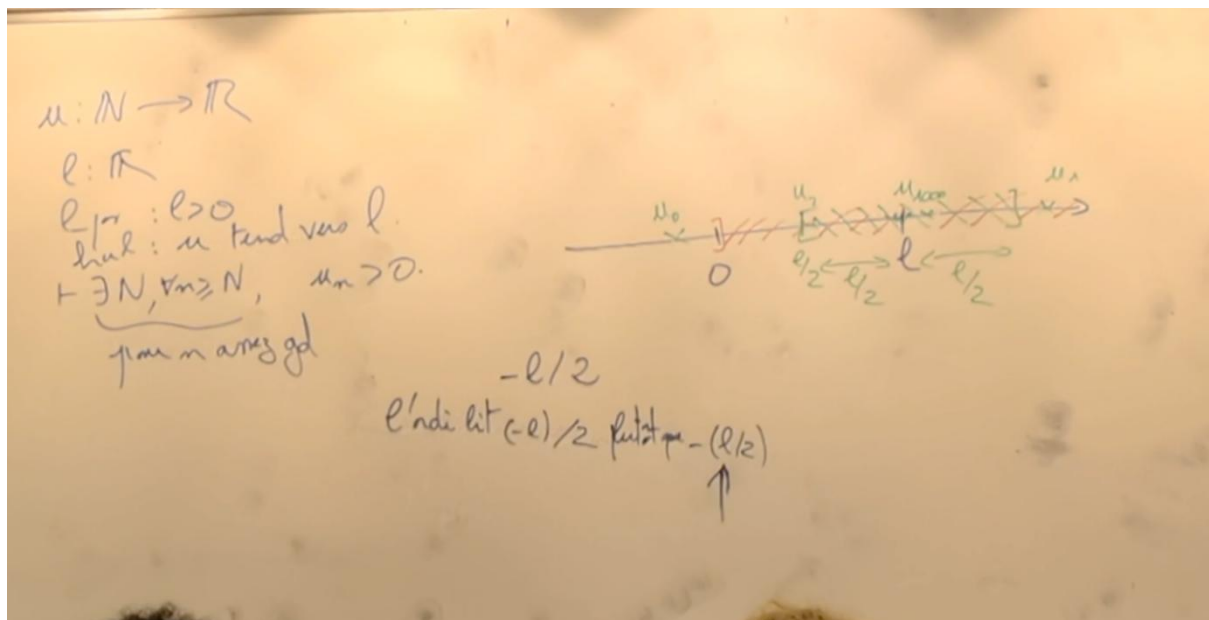


Figure 2. Teacher’s diagram on the real line during the public correction. The diagram supports the coordination of the positivity of l , the choice of a positive quantity smaller than l , and the eventual positivity of u_n .

The teacher’s discourse explicitly marks the role of $l/2$ in this coordination. After representing the “window” on the diagram, he states:

“ l sur 2 joue deux rôles, comme taille de fenêtre, et puis aussi, comme une borne à gauche. [...] On a le dessin, c’est très bien, et pour finir, il faut faire la démonstration qui va avec le dessin. En étant précis, en mettant les choses dans le bon ordre, et jusqu’à ce que l’ordinateur et moi soyons contents de ce que vous avez écrit. Donc maintenant, votre job, c’est de raconter le dessin à l’ordinateur.”

[Translation: “ $l/2$ plays two roles, as the size of the window and also as a left bound. [...] We have the drawing, that is very good, and finally we have to produce the proof that goes with the drawing. Being precise, putting things in the right order, and until the computer and I are happy with what you have written. So now your job is to tell the drawing to the computer.”]

This extract is important because it shows that, in the teacher’s practice, several cultural practices of proof coexist. The teacher does not oppose heuristic visualization, semantic interpretation of convergence, ordinary mathematical explanation, ordered proof writing, and

computational validation; he articulates them in a single discourse. The “window” operates, within his discourse, as a sign that links the diagram, the definition of convergence, the choice of $l/2$, and the proof text expected by Verbose.

In terms of validation authority, this unit is institutional and public. The teacher legitimizes a way of reading the goal, selecting $l/2$, specializing convergence, and organizing the proof. This authority differs from computational validation: it does not validate the students’ code, but it produces a public semiotic resource that can support later reformulations. At this stage, however, the analysis does not yet show that this resource has been appropriated by the students; it only shows that the teacher’s correction makes available an institutional articulation between the diagram, the proof strategy, and the PA’s expected order.

5.5. UA5: Students’ attempt with l and not yet with $l/2$

After the beginning of the public correction, the students produce a more recognizable proof structure in Verbose. They obtain an N , write `Let’s prove that N works`, introduce $n \geq N$, and try to instantiate the universal property. However, they still use l as the parameter in the convergence definition, not $l/2$.

The proof structure advances, but the quantitative choice remains insufficient. From $|u_n - l| \leq l$, one can obtain at most $u_n \geq 0$, not $u_n > 0$ in the non-strict inequality regime used by the system. Thus, students have partially translated the proof into PA’s syntax, but the mathematical parameter has not yet been stabilized.

This unit shows that cultural transposition is not a single passage from informal to formal proof. It is a gradual and sometimes misaligned process. Students may appropriate the structure of formal proof while still failing to stabilize a sign in the right mathematical function.

The validation authority is now mixed. Verbose validates part of the formal structure, such as obtaining N and introducing $n \geq N$, but it blocks the inferential chain because the parameter l is insufficient for strict positivity. The teacher’s correction is already available in the classroom semiosphere, but it has not yet been incorporated into students’ Verbose text.

5.6. UA6: Re-functionalization of $l/2$ as epsilon/tolerance

Later, the previously rejected quantity $l/2$ reappears in students’ Verbose text, this time in the line where the convergence hypothesis is specialized with a positive real number. It is no longer placed in the role of the witness N ; it is now used as the epsilon that allows the students to obtain a suitable N (Figure 3).

At this point, the analysis can identify $l/2$ as a bridge-sign. It has crossed several boundaries: it first appeared in students’ exploratory reasoning as a promising quantity; it was rejected by the PA when placed in the wrong role; it was publicly legitimized in the teacher’s correction as an appropriate quantity for specializing convergence; and it now becomes formally productive in the PA.

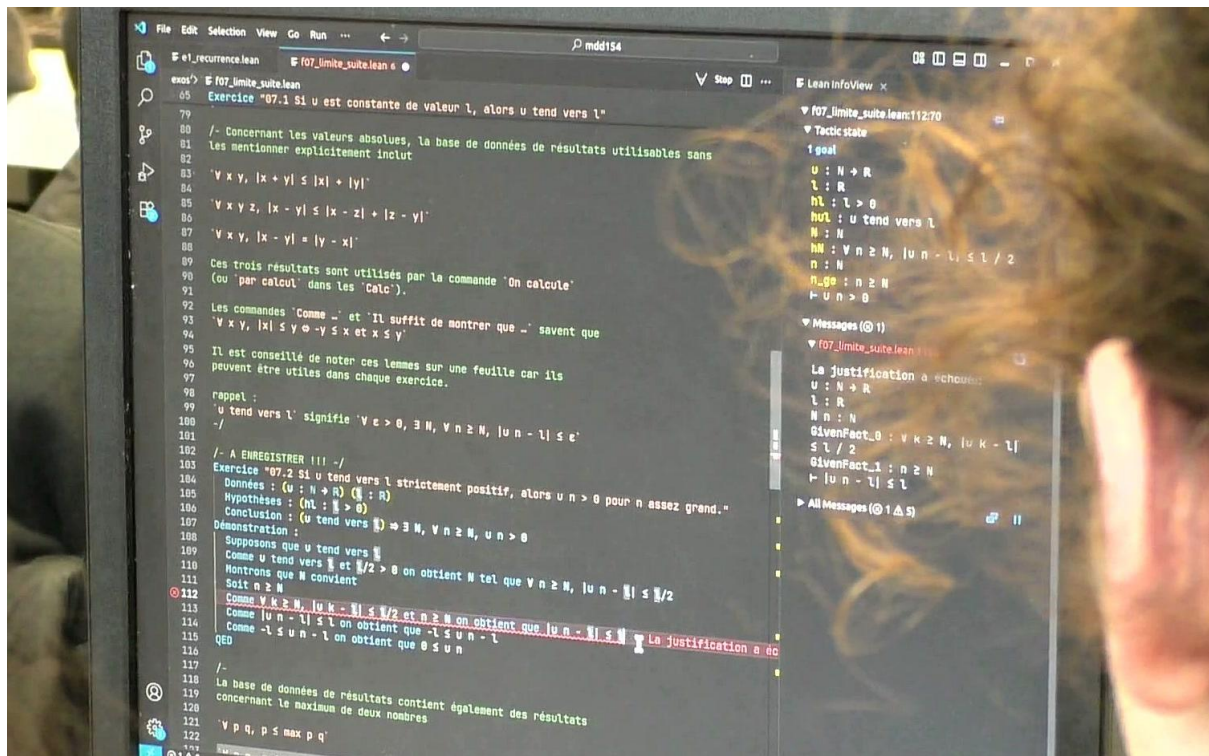


Figure 3. Refunctionalization of $l/2$ in students' Verbose proof. The expression is no longer used as a candidate for the witness N , but as the positive real parameter with which the convergence hypothesis is specialized.

This refunctionalization does not prove direct causal influence from the teacher's correction. However, the temporal and semiotic co-presence is strong. Public correction functions as an institutional resource available in the classroom semiosphere, while the PA requires students to relocate the sign in a formally acceptable role.

This is the central reorganization of validation authority. The teacher's correction has legitimized $l/2$ institutionally as the appropriate value for specializing convergence. Verbose now validates $l/2$ computationally, but only because the students have relocated it into the correct formal role. The same expression thus moves from rejected object to validated parameter.

5.7. UA7: Stabilization of the final algebraic chain

Once the quantity is used in the correct role, the remaining difficulty concerns the algebraic chain from the absolute-value inequality to positivity. Students must transform a humanly plausible calculation into a sequence of locally justified steps accepted by Verbose: from $|u_n - l| \leq l/2$ to $-(l/2) \leq u_n - l$, then to $l/2 \leq u_n$, and finally to $u_n > 0$.

This final unit shows the specificity of computational validation. It is not enough that the conclusion be mathematically evident to the students or to the teacher. The proof must be textualized as a sequence of acceptable local inferences in the PA. The final QED marks the stabilization of the proof text, but this stabilization is distributed across several resources: students' activity, paper exploration, teacher correction, mathematical definitions, and PA validation.

In the final unit, validation authority becomes computationally decisive but remains semiotically distributed. Verbose validates the proof only when each algebraic step is locally justified. Yet this final validation relies on previous resources: students' exploratory work, the definition of convergence, the teacher's public correction, and the stabilized role of $l/2$. The final QED is therefore the endpoint of a distributed process of validation. The final accepted Verbose proof is shown in Online Resource A, Figure A4.

6. Discussion

The analysis suggests that Verbose does not simply add a computational layer to an already stabilized practice of proof: the PA acts as a perturbing artifact, requiring students to make their reasoning readable within another system of proof. Here, assumptions usually left implicit in ordinary proving practices become observable. In this sense, the analyzed episode is not only an example of students' difficulty with formalism; it is a case in which formalism becomes the site where cultural practices of proof encounter one another.

The trajectory of the bridge-sign $l/2$ illustrates that a mathematical sign's function is not stable across different cultural practices of proof. This trajectory should not be interpreted as a chronological before-and-after effect of PA use: exploratory, institutional-didactical, formalized, and computational practices are intertwined within the same episode. Indeed, in students' reasoning, $l/2$ is mathematically meaningful: the PA's rejection does not invalidate $l/2$ as a mathematical idea, but invalidates the role assigned to it at that moment in the formalized proof structure. Legitimized in the teacher's institutional discourse, $l/2$ becomes computationally valid only when refunctionalized as the positive real parameter used to specialize the convergence hypothesis, rather than as the natural-number witness N . The bridge-sign therefore reveals both continuity in the same mathematical strategy, and discontinuity in the conditions of its validation across practices.

A complementary phenomenon appears in the teacher's use of the "window", which functions as a bridge-sign coordinating semantic oral meaning, visual diagram, and proof in Verbose. In the teacher's discourse, this movement is not a rupture but a coexistence and a coordination of several cultural practices of proof. However, as Melhuish et al. (2022) note regarding advanced mathematics lectures, spoken explanations, heuristics, and meta-commentary may not be reconstructed by students. Our episode extends this issue to PA-mediated proof: what is semiotically coordinated in the teacher's discourse may remain only partially permeable for students working inside Verbose. Thus, the "window" as a teacher's bridge-sign opens an interpretive perspective: semiotic continuity in institutional discourse may coexist with cultural non-alignment in students' activity, thereby highlighting the teaching relevance of paying explicit attention to the bridging function of such signs.

Furthermore, the episode highlights a redistribution of validation resulting from the coexistence of several cultural practices of proof in the classroom semiosphere. The operational work compressed in a familiar phrase such as "using the definition of convergence" must be unpacked in Verbose. In this sense, the PA functions as a debugging environment for proof, where students test formulations and receive feedback on the role that each sign occupies in the current proof state. From a semiotic-cultural perspective, this debugging is not merely syntactic: it requires a specific mediation activity within the educational system. It reveals how validation is redistributed between mathematical meaning, cultural norms, institutional discourse, computational checking, etc. This specifies the contribution of real-time feedback

discussed by Hanna et al. (2024) and Bartzia et al. (2026): computational validation supports the reorganization of the proof text, but it does not by itself validate the mathematical meaning attributed to each step.

From this semiotic-cultural perspective, well-documented difficulties about proof at the secondary-tertiary transition, such as managing quantifiers and definitions operationally (e.g., Gueudet & Vandebrouck, 2022; Selden, 2012), are reframed, not only as individual cognitive errors or lack of understanding, but as ruptures at the boundaries between cultural practices of proof.

Moreover, while transformations in signs, proof roles and validation systems are observable, students' underlying semantic understanding remains only partially and indirectly accessible through their formulations, hesitations, paper-and-pencil work (mainly as a support for calculations), oral interactions and responses to feedback. The PA is not only a place where a final proof is written, but also the main space in which students test ideas and progressively stabilize the proof structure. Acting within the PA centralizes the testing of formal structures and of semiotic controls, but for students, it does not make explicit the relations between the teacher's semantic work and the formal order required by the PA. Nor does it guarantee that students actively reflect on the required translations between semantic and syntactic dimensions (as suggested by previous research; e.g., Thoma & Iannone, 2022), and between cultural practices of proof. These translations are fragile: the bridge-signs do not automatically become shared resources between teacher and students.

7. Conclusion and perspectives

By conceptualizing PAs as epistemic revelators, this study demonstrates how, in the analyzed episode, these digital artifacts can make visible and reconfigure cultural practices of proof in higher education. The proving process is not a duality between informal and formal reasoning, but a progressive, complex set of translations across different cultural practices and semiotic systems. The trajectory of $l/2$ showed how one mathematical expression can circulate across cultural practices of proof while changing role and conditions of legitimacy. The operationalization of the bridge-sign construct (e.g., $l/2$) allowed us to make some of these translations visible, revealing both continuities (e.g., students' ideas, teacher's explanation, formalized proof) and discontinuities (e.g., between being mathematically relevant and formally valid in PA).

The analysis also showed that the PA reconfigures proof practices by redistributing validation. In the episode, the final QED is decisive, but it is not the only source of legitimacy. Validity emerges through a process in which students' provisional reasoning, institutional mathematical discourse, and the PA's local checking interact without coinciding. The PA therefore does not simply validate a finished proof; it participates in the cultural and semiotic transformation through which a proof becomes acceptable. This view of the reconfiguration and redistribution of validation enriches existing literature on proof teaching and learning with PAs (e.g., Hanna & Yan, 2021; Thoma & Iannone, 2022). Accordingly, the study contributes to research on cultural practices in mathematical argument by showing how a proof's legitimacy is reconfigured when it must become readable and valid across human and computational systems of validation.

Theoretically, articulating Cultural Transposition (Mellone et al., 2021) with Lotman's (1990) Semiosphere answers the need for novel frameworks to analyze student-content interactions in technological environments, as called for by Stylianides et al. (2024). This dual lens prevents the artificial separation of proof cultures, allowing researchers to identify unthoughts and to analyze the circulation and transformation of signs across boundaries.

Looking forward, the non-alignment observed in the classroom semiosphere underscores a broader structural challenge: the coexistence of several cultural practices of proof does not guarantee their mutual permeability for students, even in PA-mediated activity. The PA therefore makes visible not only the need for translation between cultural practices of proof, but also the conditions under which such translations remain partial. Future research should extend this methodological analysis to other mathematical domains, other PAs and institutional contexts, over longer periods, specifically investigating the conditions under which bridge-signs become shared resources between teachers and students. Finally, as PAs increasingly enter undergraduate education, the research focus must shift from evaluating their efficiency to understanding how they fundamentally reshape the epistemology and the teachability of mathematical proof.

Supplementary Information

Online Resource A. Contextual and visual documentation of the Verbose environment, the analyzed task, PA feedback, teacher correction, and the final accepted proof.

Statements and Declarations

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Competing interests The authors have no relevant financial or non-financial interests to disclose.

Ethics approval No formal approval was obtained from a research ethics committee. The study was conducted in accordance with the applicable institutional and national requirements for research involving human participants and personal data. The authors consulted the data protection officers of the participating universities and prepared a data management plan addressing the requirements of the General Data Protection Regulation (GDPR) for the APPAM project. These consultations concerned data protection and privacy compliance.

Consent to participate Written informed consent was obtained from all participants included in the study for the processing of written data and for audio and/or video recording. Participation was voluntary, and participants could withdraw their consent at any time.

Consent for publication Participants were informed that no directly identifying information would appear in publications and consented to the use of fully anonymized excerpts for research publication.

Data availability The underlying data are not publicly available because the audiovisual materials contain potentially identifying information and participant consent does not cover public sharing. Access is restricted to authorized APPAM researchers for scientific research

compatible with the original purpose of the study. Fully anonymized excerpts supporting the analysis are included in the article and Online Resource A.

References

- Avigad, J. (2019). Learning logic and proof with an interactive theorem prover. In G. Hanna, D. A. Reid, & M. de Villiers (Eds.), *Proof technology in mathematics research and teaching* (pp. 277–290). Springer. https://doi.org/10.1007/978-3-030-28483-1_13
- Bartzia, E.-I., Beffara, E., Meyer, A., & Narboux, J. (2026). Proof assistants for undergraduate mathematics education: elements of an a priori analysis. *International Journal of Mathematical Education in Science and Technology*, 1–44. <https://doi.org/10.1080/0020739X.2026.2632264>
- Bishop, A. J. (1991). *Mathematical enculturation: A cultural perspective on mathematics education*. Kluwer Academic Publishers. <https://doi.org/10.1007/978-94-009-2657-8>
- Bridoux, S., Grenier-Boley, N., & Leininger-Frézal, C. (2023). *Les recherches en pédagogie universitaire [Research in higher education pedagogy]*. ISTE Editions.
- Chevallard, Y., & Bosch, M. (2020). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 214–218). Springer. https://doi.org/10.1007/978-3-030-15789-0_48
- Di Martino, P., & Gregorio, F. (2019). The mathematical crisis in Secondary-Tertiary Transition. *International Journal of Science and Mathematics Education*, 17(4), 825–843. <https://doi.org/10.1007/s10763-018-9894-y>
- Geertz, C. (1973). *The interpretation of cultures*. Basic Books.
- Gueudet, G., & Vandebrouck, F. (2022). Transition secondaire-supérieur : Ce que nous apprend la recherche en didactique des mathématiques [The secondary-tertiary transition: What research in mathematics education tells us]. *EpiDEMES*, 1. <https://doi.org/10.46298/epidemes-7486>
- Hanna, G., & Knipping, C. (2020). Proof in mathematics education, 1980-2020: an overview. *Journal of Educational Research in Mathematics, Special Issue*, 1–13. <https://doi.org/10.29275/jerm.2020.08.sp.1.1>
- Hanna, G., Larvor, B., & Yan, X. K. (2024). Using the proof assistant Lean in undergraduate mathematics classrooms. *ZDM Mathematics Education*, 56(7), 1517–1529. <https://doi.org/10.1007/s11858-024-01577-9>
- Hanna, G., & Yan, X. K. (2021). Opening a discussion on teaching proof with automated theorem provers. *For the Learning of Mathematics*, 41(3), 42–46. <https://www.jstor.org/stable/27091220>
- Kerjean, M., Le Roux, F., Massot, P., Mayero, M., Mesnil, Z., Modeste, S., Narboux, J., & Rousselin, P. (2022). Utilisation des assistants de preuves pour l'enseignement en L1 [Using proof assistants for teaching in Licence 1]. *La Gazette de la Société Mathématique de France*, 174, 35–55. <https://hal.science/hal-03979238v1>
- Knobelsdorf, M., Frede, C., Böhne, S., & Kreitz, C. (2017). Theorem provers as a learning tool in theory of computation. In *Proceedings of the 2017 ACM Conference on International Computing Education Research (ICER '17)* (pp. 83–92). Association for Computing Machinery. <https://doi.org/10.1145/3105726.3106184>

- Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Ablex.
- Lotman, Y. M. (1990). *Universe of the mind: A semiotic theory of culture* (A. Shukman, Trans.). I. B. Tauris.
- Lotman, J., & Clark, W. (2005). On the semiosphere. *Sign Systems Studies*, 33(1), 205–229. <https://doi.org/10.12697/SSS.2005.33.1.09>
- Manolino, C. (2021). The Semiosphere Lens to look at Lesson Study practices in their Cultural Context: a case study. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 214–221). Khon Kaen, Thailand: PME.
- Manolino, C. (2024). Cultural semiotics for mathematical discourses. *Semiotica*, 2024(259), 61–77. <https://doi.org/10.1515/sem-2023-0030>
- Massot, P. (n.d.). *Verbose Lean 4* [Computer software]. GitHub. <https://github.com/PatrickMassot/verbose-lean4> <https://github.com/PatrickMassot/verbose-lean4/blob/master/Verbose/English/Examples.lean>
- Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, 67, 100986. <https://doi.org/10.1016/j.jmathb.2022.100986>
- Mellone, M., Pacelli, T., & Liljedahl, P. (2021). Cultural transposition of a thinking classroom: To conceive possible unthoughts in mathematical problem solving activity. *ZDM Mathematics Education*, 53, 785–798. <https://doi.org/10.1007/s11858-021-01256-z>
- Mellone, M., Ramploud, A., Di Paola, B., & Martignone, F. (2019). Cultural transposition: Italian didactic experiences inspired by Chinese and Russian perspectives on whole number arithmetic. *ZDM Mathematics Education*, 51(1), 199–212. <https://doi.org/10.1007/s11858-018-0992-7>
- Ouvrier-Bufferet, C. (in press). Teaching and learning proof in mathematics at university: New perspectives in education with proof assistants? In A. J. Soares, E. Bernard, A. Maffia, C. Ouvrier-Bufferet, P. Rosário, & F. Salvarani (Eds.), *Inclusive Mathematics Education based on Digital Learning*. Springer.
- Ouvrier-Bufferet, C., & Sisco, C. (submitted). University mathematics teachers' practices with Proof Assistants: A synthesis of existing research, preliminary findings, and theoretical framework for future investigation. In *SI IJRUME "University teachers' practices, knowledge, and training"*.
- Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 391–422). Springer. https://doi.org/10.1007/978-94-007-2129-6_17
- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4–36. <https://doi.org/10.2307/30034698>
- Stylianides, G. J., Stylianides, A. J., & Moutsios-Rentzos, A. (2024). Proof and proving in school and university mathematics education research: a systematic review. *ZDM Mathematics Education*, 56(1), 47–59. <https://doi.org/10.1007/s11858-023-01518-y>
- Thoma, A., & Iannone, P. (2022). Learning about Proof with the Theorem Prover LEAN: the Abundant Numbers Task. *International Journal of Research in Undergraduate Mathematics Education*, 8(1), 64–93. <https://doi.org/10.1007/s40753-021-00140-1>

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