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Regularity and reentry basins of low Earth orbits in the J_2 –solar radiation pressure problem

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Abstract

We numerically investigate the long-term dynamical structure of low Earth orbits (LEOs) using the Smaller Alignment Index (SALI), a fast numerical indicator of chaos, within a closed-form averaged model that incorporates the effects of solar radiation pressure and Earth’s oblateness. Our analysis reveals that the area-to-mass ratio is a key parameter governing the onset and extent of chaotic behavior in LEOs. We map the system’s chaotic regions, study the behavior of reentry trajectories and characterize their temporal laws over a timescale constrained by the **25**-year mitigation guideline. Within this physically relevant timescale, we show that most of the reentry trajectories exhibit regular motion. Reentry basins, constructed according to different mitigation guidelines up to **25** years, display fractal-like structures for less-stringent guidelines. The degree of this fractality is quantitatively assessed using the uncertainty exponent method. In most cases, for large area-to-mass ratios, reentry occurs on relatively short timescales (a few years) – short enough that no fractal behavior is observed in the basin boundaries. This numerical dynamical study offers insights into the development of dynamically informed deorbiting strategies.

Keywords: Solar radiation pressure, Chaos indicator, Deorbiting strategies, Fractality

1 Introduction

Exploiting – or “surfing” – the natural flow properties of a dynamical system instead of counteracting its dynamics is observed in real-world, living systems and shows up across several engineering disciplines. A few examples include the use of hydrodynamics properties in planktonic and fish navigation [1, 2], informed use of vortex dynamics in bio-inspired robotics for propulsion savings [3], near-optimal path planning using Lagrangian coherent structures in ocean engineering [4], and the increase or decrease of chaos in targeted mixing [5]. In the context of astrodynamics, examples include the modeling of natural transport phenomena through invariant sets [6], and the design of the Soviet Molniya constellation near the critical inclination to counteract the perigee drift [7], to name only a few.

Quite recently, it has been recognized that the use of natural disturbances and resonant phenomena might play a key role in managing the space environment to enable sustainable circumterrestrial operations, see, *e.g.*, [8–12] and the works cited therein. Understanding the effect of perturbations appears to be essential for end-of-life disposal (one of the most effective space debris mitigation measure) and monitoring of fragmentation events. Regarding the former, depending on the orbital regime, several perturbations of the idealized two-body problem can be exploited, such as the Earth’s non-spherical gravitational potential, the solar radiation pressure (SRP), the atmospheric drag, and the gravitational accelerations exerted by third bodies like the Moon and Sun.

This work focuses on the dynamical properties arising from the secular coupled effects of Earth’s oblateness and SRP, referred to hereafter as the J_2 -SRP problem. A key aspect of this framework is the dependence of the dynamics on the area-to-mass ratio (A/m) of the orbiting object. This idealized model stems from our desire to understand the behavior of a satellite equipped with a large solar sail deployed at the end of its operational life. The resulting high A/m , combined with carefully selected initial conditions near resonant configurations – orbital inclination and orientation with respect to the Sun – might enable the satellite to naturally reenter Earth’s atmosphere within a few years, without requiring active propulsion. This reentry is driven by a resonance-induced growth in orbital eccentricity [10, 13–15]. Current technologies in solar sailing lead to an area-to-mass ratio of about $5\text{ m}^2\text{kg}^{-1}$ [16]. In the present study, we consider area-to-mass values up to $10\text{ m}^2\text{kg}^{-1}$ to reflect large A/m devices [17] and account for possible future advancements [18].

Our work specifically differs from the recent works [13–15] based on a single-resonance hypothesis, used to explain the so-called “de-orbiting corridors” numerically found in [19]. Here we extend the analysis of the J_2 -SRP problem to investigate whether the overall dynamical picture changes when *all* resonant terms are included in the long-term model, with particular focus on large A/m devices. More specifically, we depart from the analytical endeavor and instead detect and portray numerically the location of chaos, its extent and geometry, associated timescale, and possible connection with reentry trajectories. The adopted J_2 -SRP model provides

a physically-motivated, reduced description of the long-term dynamics, especially for high area-to-mass-ratio objects, whose resonant structures have been extensively analyzed and shown to persist in higher-fidelity models [20–22]. It does not pretend to cover dynamical features under a full model complexity; yet and quite remarkably, this Hamiltonian construct has been shown to support the observed resonant evolution of the historical Echo I satellite [22]. Thus, in the regime and under the parameter sets investigated subsequently, the omission of drag is expected to have a limited impact on the dynamical structures investigated therein, allowing us to isolate and characterize properties induced by the eccentricity growth mechanism without the sophistication of higher-order geopotential or lunisolar perturbations. In this sense, this model once again provides a worthwhile testbed constituting a first step toward the broader goal of understanding the structures studied here in more comprehensive orbital models.

The complexity of the lifetime problem is typically discussed in the context of reentry date prediction and event location, due to uncertainties in atmospheric drag modeling [23]. Here, we point out a different kind of difficulty. In certain regions of the parameter-configuration space we show that determining whether an orbit will reenter due to the eccentricity growth mechanism or not is a difficult task due to the fractal boundaries of reentry basins. For such domains, we explicitly quantify how uncertainties in the initial state must be reduced in order to achieve a corresponding reduction in prediction error, depending on the chosen mitigation scenario.

The remainder of the paper is structured as follows:

- In Sec. 2, we present the averaged (secular) Hamiltonian model we base our analysis on. The model includes Earth’s J_2 perturbation and the SRP, both averaged in closed-form over the mean anomaly M of the satellite. The secular model is prone to resonate. We also describe the numerical tools used to explore its associated dynamics.
- In Sec. 3, we numerically explore the system, focusing on two observables: the Smaller Alignment Index (SALI), a well-established chaos indicator, and the orbital lifetime computed over fine meshes of initial conditions. The parameter A/m is allowed to take three distinct values in the set $\{1, 5, 10\} \text{ m}^2\text{kg}^{-1}$, which, within the context of this study and relative to one another, are regarded as low, intermediate, and high area-to-mass ratios, respectively. We present a cartography of regular and chaotic regions on a centennial timescale. We also find that the number of reentry orbits, at intermediate and high A/m values, follows an inverse-square power law in their lifetimes. The long-term stability of such orbits is then reconsidered over a more practical and physically relevant timescale, which we call the effective timescale, prescribed by the reentry dynamics that obey the 25-year guideline. While the long-term dynamics display widespread chaos, the vast majority of reentry orbits show a regular behavior over the effective timescale.
- In Sec. 4, we reveal that reentry basins may exhibit complex geometries depending on the targeted post-mission disposal guideline. The complexity is assessed qualitatively and quantitatively by computing uncertainty exponents. Our results reveal reentry

k	$\mathcal{T}_k(i)$	$\psi_k(\omega, \Omega, \lambda_\odot)$
1	$\frac{1}{4}(\cos \varepsilon + 1)(\cos i + 1)$	$\omega + \Omega - \lambda_\odot$
2	$-\frac{1}{4}(\cos \varepsilon + 1)(\cos i - 1)$	$-\omega + \Omega - \lambda_\odot$
3	$\frac{1}{2} \sin i \sin \varepsilon$	$\omega - \lambda_\odot$
4	$-\frac{1}{2} \sin i \sin \varepsilon$	$\omega + \lambda_\odot$
5	$-\frac{1}{4}(\cos \varepsilon - 1)(\cos i + 1)$	$\omega + \Omega + \lambda_\odot$
6	$\frac{1}{4}(\cos \varepsilon - 1)(\cos i - 1)$	$-\omega + \Omega + \lambda_\odot$

Table 1 The six harmonics governing the secular evolution of the Hamiltonian given in Eq. (1).

regions with clear fractal structures, highlighting that the robust design of reentry trajectories requires dynamically informed criteria.

- Finally, Sec. 5 summarizes our results, discusses perspectives, and closes the paper.

2 Secular Hamiltonian model and numerical tools

In this section we present the secular Hamiltonian and the numerical tools used in our study.

2.1 The secular Hamiltonian

We consider a perturbed Hamiltonian formulation of the two-body problem, incorporating the perturbations due to the Earth’s oblateness and SRP, as in [15]. In particular, we assume the so-called cannonball model, *i.e.*, that the Sun’s radiation is normal to the surface of the space object at all times, and we neglect the effects of the Earth’s albedo and shadowing. Since our focus is on the long-term evolution of the system, the Hamiltonian is averaged in closed form over the fast angle, namely the satellite’s mean anomaly M [24]. In terms of the Keplerian orbital elements $(a, e, i, \omega, \Omega, M)$, corresponding respectively to the semi-major axis, eccentricity, inclination, argument of perigee, longitude of the ascending node, and mean anomaly, the averaged Hamiltonian takes the form

$$\mathcal{H} = \frac{\mu r_E^2 J_2 (1 - 3 \cos^2 i)}{4a^3 (1 - e^2)^{3/2}} - \frac{3}{2} a e c_R P \frac{A}{m} \sum_{k=1}^6 \mathcal{T}_k(i) \cos \psi_k(\omega, \Omega, \lambda_\odot). \quad (1)$$

In Eq. (1), r_E denotes the Earth’s radius and J_2 its oblateness coefficient. The parameter c_R is the satellite’s reflectivity coefficient, A/m its area-to-mass ratio, and P the solar radiation pressure constant at 1 astronomical unit. Furthermore, we set $\mu = \mathcal{G} m_E$, where $\mathcal{G}$ is the gravitational constant and m_E the Earth’s mass. The $\mathcal{T}_k(i)$, for $k = 1, \dots, 6$, are the coefficients of the six SRP-induced harmonics listed in Table 1. The obliquity of the ecliptic is fixed at $\varepsilon = 23.4^\circ$, while the ecliptic longitude of the Sun evolves as $\lambda_\odot = \lambda_{\odot,0} + n_\odot t$, where $\lambda_{\odot,0}$ is determined from astronomical ephemerides and $n_\odot$ is the mean motion of the Sun at $2\pi/(86400 \times 365.25) \text{ s}^{-1}$.

147 For the Hamiltonian formulation, it is convenient to introduce the canonical
 148 Delaunay action-angle variables (L, G, H, ℓ, g, h) , defined through

$$\begin{cases} L = \sqrt{\mu a}, & \ell = M, \\ G = L\sqrt{1 - e^2}, & g = \omega, \\ H = G \cos i, & h = \Omega. \end{cases} \quad (2)$$

149 The explicit time dependence introduced through the solar longitude $\lambda_\odot$ is removed by
 150 extending the phase space and introducing a dummy action J conjugate to a variable
 151 s such that $s(t) = n_\odot t + \lambda_{\odot,0}$. The resulting autonomous Hamiltonian, describing a
 152 system with 3 degrees of freedom (DoF) in the variables $\mathbf{x} = (\mathbf{p}, \mathbf{q})$ with $\mathbf{p} = (G, H, J)$
 153 and $\mathbf{q} = (g, h, s)$, is finally written as

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = h_0(\mathbf{p}) + h_1(\mathbf{p}, \mathbf{q}), \quad (3)$$

154 with

$$h_0(\mathbf{p}) = \frac{r_E^2 J_2 \mu^4 (G^2 - 3H^2)}{4G^5 L^3} + n_\odot J, \quad (4)$$

155 and

$$h_1(\mathbf{p}, \mathbf{q}) = -\frac{3c_R P L^2 \frac{A}{m}}{2\mu} \sqrt{1 - \frac{G^2}{L^2}} \sum_{k=1}^6 \mathcal{T}_k(i(L, G, H)) \cos \psi_k(g, h, s). \quad (5)$$

156 We note that since the angle ℓ is cyclic in Eq. (3), the corresponding action L , and
 157 consequently the semi-major axis a , is conserved. Thus, the Hamiltonian depends
 158 parametrically upon a and A/m .

159
 160 The equations of motion associated with the Hamiltonian in Eq. (3) are expressed
 161 in canonical form as

$$\dot{\mathbf{x}} = \mathcal{J} \nabla \mathcal{H}(\mathbf{x}), \quad (6)$$

162 where the matrix $\mathcal{J}$ is the standard symplectic matrix $\mathcal{J} = \begin{pmatrix} 0_3 & I_3 \\ -I_3 & 0_3 \end{pmatrix}$ with I_3 and
 163 0_3 denoting the 3×3 identity and zero matrices, respectively. The gradient of the
 164 Hamiltonian is written as $\nabla \mathcal{H}(\mathbf{x}) = (\partial_{\mathbf{p}} \mathcal{H}, \partial_{\mathbf{q}} \mathcal{H})$.

165
 166 Resonances strongly influence the dynamics associated with the equations of
 167 motion in Eq. (6). For a wavenumber $\mathbf{k} \in \mathbb{Z}^3$ labeling a specific resonance, the
 168 resonance condition takes the form

$$\mathbf{k} \cdot \varpi_0(\mathbf{p}) = 0, \quad (7)$$

where $\varpi_0(\mathbf{p}) = \partial_{\mathbf{p}} h_0(\mathbf{p})$ is the unperturbed frequency vector. This vector contains the frequency rates associated with $\dot{\omega}$, $\dot{\Omega}$ under the secular J_2 effect and the constant frequency $n_\odot$. The “resonant manifolds” associated to Eq. (7), defining a 2D surface in the space of the actions (L, G, H) – equivalently, (a, e, i) in orbital elements – have been portrayed in several instances for low values of $|\mathbf{k}|$, see *e.g.*, [13].

2.2 The Smaller Alignment Index method

To distinguish between regular and chaotic motion in the 6-dimensional phase space of the Hamiltonian of Eq. (3), we use the SALI method, first developed in [25]. This is a variational indicator based on the time evolution of the alignment of two deviation vectors from the studied orbit [26–28], which has been successfully applied to studies of various dynamical systems [29–36]. The evolution of a deviation vector $\mathbf{w}$ from a reference orbit is governed, at first order, by the variational equations

$$\dot{\mathbf{w}} = \mathcal{J}\nabla^2\mathcal{H}(\mathbf{x})\mathbf{w}. \quad (8)$$

To quantify the chaoticity of an orbit with initial condition $\mathbf{x}_0$, we compute the SALI by simultaneously integrating the equations of motion (Eq. (6)) and the variational equations (Eq. (8)) for two initially linearly independent deviation vectors, $\mathbf{w}_0^1$ and $\mathbf{w}_0^2$. The SALI at time t is defined as

$$\text{SALI}(\mathbf{x}_0, \mathbf{w}_0^1, \mathbf{w}_0^2, t) = \min \{ \|\hat{\mathbf{w}}^1(t) - \hat{\mathbf{w}}^2(t)\|, \|\hat{\mathbf{w}}^1(t) + \hat{\mathbf{w}}^2(t)\| \}, \quad (9)$$

where $\|\cdot\|$ denotes the Euclidean norm and the normalized deviation vectors are given by

$$\hat{\mathbf{w}}^i(t) = \frac{\mathbf{w}^i(t)}{\|\mathbf{w}^i(t)\|}, \quad i = 1, 2. \quad (10)$$

For brevity, we simply denote the index as $\text{SALI}(t)$.

In our study, the equations of motion and the corresponding variational equations are numerically integrated using the variable time-step integrator DOP853 [37]. The integrations are performed over timescales of up to 100 years, corresponding to approximately 10^5 orbital revolutions for the considered semi-major axis values. Throughout all simulations, the relative energy error typically remains below 10^{-8} ensuring the reliability of the numerical results. For the computation of the SALI, the initial deviation vectors are randomly drawn from a normal distribution and subsequently orthonormalized to unit length, so that $\text{SALI}(0) = \sqrt{2}$.

The distinction between regular and chaotic motion is efficiently captured by the SALI through its markedly different time evolution in the two cases. In particular, it has been shown in [27] that, after a short transient, the SALI for chaotic orbits

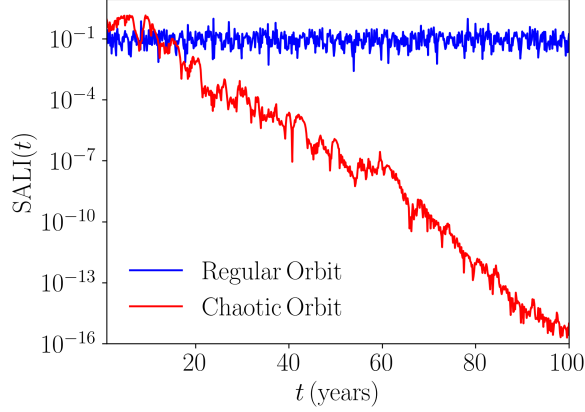


Fig. 1 Time evolution of the SALI for a representative regular orbit (blue curve) with initial condition $(a_0, e_0, i_0, \omega_0, \Omega_0) = (11\,100\text{km}, 0.05, 70^\circ, 0^\circ, 0^\circ)$ and a representative chaotic orbit (red curve) with initial condition $(a_0, e_0, i_0, \omega_0, \Omega_0) = (11\,100\text{km}, 0.05, 100^\circ, 0^\circ, 0^\circ)$, for the Hamiltonian system of Eq. (3) with $A/m = 10\,\text{m}^2\text{kg}^{-1}$. The SALI of the chaotic orbit decays to machine precision exponentially fast while the regular orbit exhibits bounded fluctuations about some positive constant.

201 exhibits an exponential decay of the form

$$\text{SALI}(t) \propto e^{-(\lambda_1 - \lambda_2)t}, \quad (11)$$

202 where $\lambda_1 \geq \lambda_2$ are the orbit's two largest Lyapunov exponents. This behavior
 203 originates from the tendency of deviation vectors to align along the most unstable
 204 direction, associated with the maximal Lyapunov exponent. As a consequence, the
 205 SALI rapidly tends to zero, providing a clear signature of chaotic dynamics. In con-
 206 trast, for regular motion in Hamiltonian systems with $N \geq 2$ DoF, deviation vectors
 207 eventually fall on the N -dimensional tangent space of the invariant torus. In this
 208 case, there is no mechanism enforcing their alignment along a single direction. As
 209 a result, the SALI does not decay to zero but instead fluctuates around a positive
 210 value, remaining practically constant on average [26, 27].

211
 212 This qualitative difference (exponential decay to zero for chaotic orbits versus
 213 bounded, non-zero fluctuations for regular ones) renders the SALI a particularly effi-
 214 cient and reliable indicator for distinguishing between ordered and chaotic dynamics.
 215 This behavior is demonstrated in Fig. 1 showing the time evolution of the SALI for two
 216 representative orbits. The chaotic orbit (red curve) exhibits the characteristic rapid
 217 exponential decay of the index, which reaches machine precision over the timescale
 218 considered. In contrast, the regular orbit (blue curve) displays bounded fluctuations
 219 of the SALI around a nearly constant positive value. These two examples highlight
 220 the clearly different temporal behavior of the index in chaotic and regular regimes.

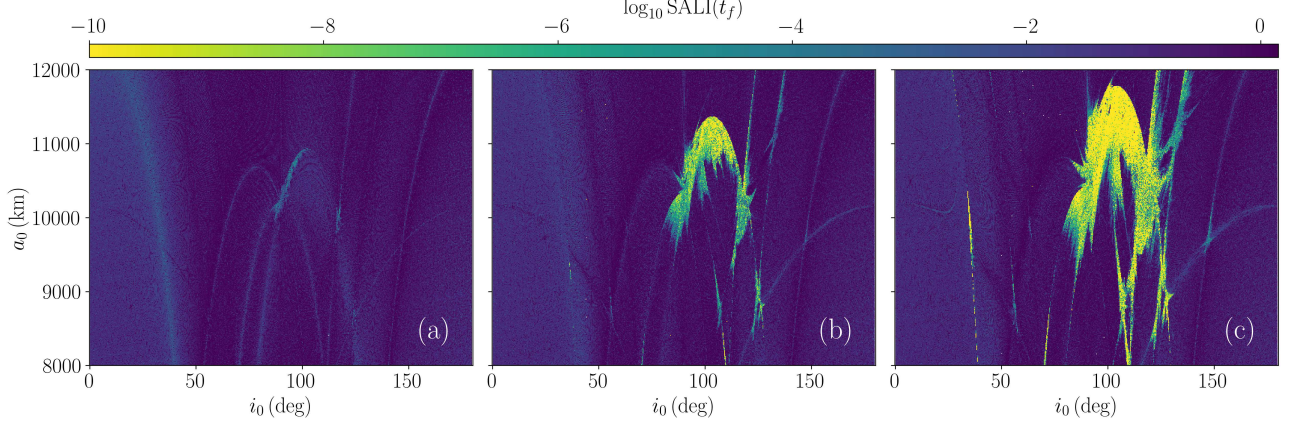


Fig. 2 Initial conditions in the $i - a$ space colored according to the $\log_{10}$ SALI value computed at $t_f = 100$ years, over the interval $a_0 \in [8\,000\text{km}, 12\,000\text{km}]$ and $i_0 \in (0^\circ, 180^\circ)$, with $e_0 = 0.05$ fixed and $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$, for (a) $A/m = 1\,\text{m}^2\text{kg}^{-1}$, (b) $A/m = 5\,\text{m}^2\text{kg}^{-1}$, and (c) $A/m = 10\,\text{m}^2\text{kg}^{-1}$. The presence of chaos increases with increasing A/m .

3 Stability and orbital lifetimes

In this section we discuss stability results in the long-term, the distribution of reentry orbits and the stability over an effective timescale.

3.1 Long-term stability

Using the SALI indicator, we perform a stability analysis of the secular Hamiltonian in Eq. (3) to reveal the geometry and extent of possible chaotic structures. To this end, taking into account that the phase space of the autonomous Hamiltonian has dimension 6, we compute dynamical maps over 2-dimensional slices of fine Cartesian meshes of initial conditions. From the computation of several dynamical maps parameterized by the angular variables, we noticed that the observed structures do not significantly change for different values of the system's initial phases ω_0 , Ω_0 and $\lambda_{\odot,0}$, as maps presented in Fig. A1 in Appendix A denote. This allows us, in what follows, to present results with specific fixed values of initial angles, namely set to $\Omega_0 = 0^\circ$, $\omega_0 = 0^\circ$, and $\lambda_{\odot,0} = 0^\circ$. Thus, we have 3 remaining coordinates, (G, H, J) , and 2 free parameters (a and A/m). The action J has no actual dynamical meaning, leading to $\binom{4}{2} = 6$ possible different coordinate planes. Here, we specifically focus on results in the $i - a$ space by freezing the initial eccentricity to $e_0 = 0.05$, underlining that the resonant manifolds of Eq. (7) do not strongly depend on the choice of e_0 for values $e_0 \in \{0.001, 0.005, 0.01, 0.05\}$ (see Fig A2 in Appendix A). The only parameter left, A/m , takes a value in the set $\{1, 5, 10\}\,\text{m}^2\text{kg}^{-1}$.

Fig. 2 shows three dynamical maps of the region $i_0 \in (0^\circ, 180^\circ)$ with an initial semi-major axis covering the range $a_0 \in [8\,000\text{km}, 12\,000\text{km}]$ for increased values of the parameter A/m . For $A/m = 1\,\text{m}^2\text{kg}^{-1}$ (panel (a)), the vast majority of orbits are

regular and do not show a chaotic signature on the considered timescale. We note, however, that SALI delineates several thin structures related to the resonances of Table 1. Increasing the value of A/m to $5 \text{ m}^2\text{kg}^{-1}$ (panel (b)), we observe the appearance of an inverted U-shaped, yellow-colored, region corresponding to connected chaotic seas, located around $i_0 \sim 100^\circ$ and $a_0 \sim 11\,000\text{km}$. Thinner chaotic structures emanate from the main sea along resonant lines. The chaotic structures grow further with the increase of A/m to $10 \text{ m}^2\text{kg}^{-1}$ (panel (c)). Several chaotic filamentary structures emerge at higher and lower altitudes, predominantly along the resonant lines and across resonant junctions. In particular, highly-inclined orbits, with $i_0 \sim 110^\circ$, might now be chaotic at lower semi-major axis values, *e.g.*, at $a_0 \sim 8\,000\text{km}$, and connect to the large chaotic sea at larger a values.

For the sake of completeness, Fig. B3 in Appendix B and Fig. C5 in Appendix C complement the cartography by presenting dynamical maps in two other slices of the phase space, namely $i - e$ and $e - a$ space, respectively. Even in these spaces, we see that the presence of chaos grows with the increase of A/m .

3.2 Orbital lifetimes

Connections between chaotic transport and orbital lifetimes in Earth's orbits have been reported in many instances as the result of the eccentricity growth mechanism, especially in medium Earth orbits, see *e.g.*, [8, 38, 39]. In the following, an orbit is said to be a reentry orbit over the time window $[0, t_f]$ if it reaches its critical eccentricity $e_\star$ at time $t_\star \leq t_f$, where t_f is set to 100 years. The critical eccentricity $e_\star$ is derived from the perigee altitude condition

$$r_\pi = a_0(1 - e_\star) \geq r_E + \delta, \quad (12)$$

where δ is set to 120km. Solving for $e_\star$, one finds that

$$e_\star = 1 - \frac{r_E + \delta}{a_0}. \quad (13)$$

In the following, we investigate the connection between chaos and orbital lifetime, restricting our analysis to reentry trajectories with orbital lifetimes below or equal to 25 years, in accordance with standard mitigation guidelines [40, 41]. For this set of initial conditions, we further distinguish orbits achieving reentry within 5 years, reflecting more stringent targets currently advocated [42–44].

Fig. 3 shows the distribution of reentry orbits in the $i - a$ space, classified according to the two mitigation scenarios (5 and 25 years), for $A/m = 5$ and $10 \text{ m}^2\text{kg}^{-1}$. The case where $A/m = 1 \text{ m}^2\text{kg}^{-1}$ is omitted due to the absence of significant structures, as not many orbits reenter within 25 years for this value of A/m . We observe that the distribution of the reentry orbits in Fig. 3 is overall reminiscent of the chaotic regions depicted in Fig. 2, particularly the inverted U-shaped structure at large a_0 and i_0 values. Nevertheless, this correspondence is not one-to-one. The most

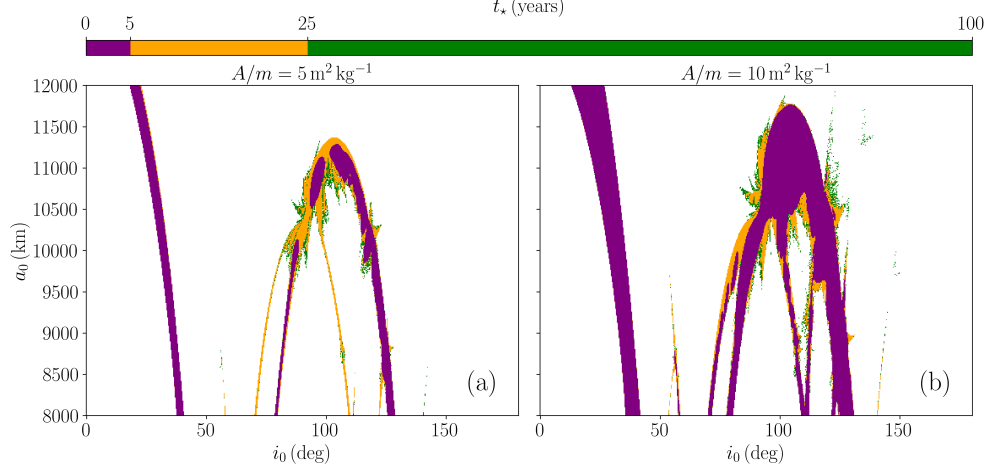


Fig. 3 Identification of initial conditions in the $i - a$ space consistent with the 5-year (violet) and 25-year (orange) mitigation guidelines, for $e_0 = 0.05$ and $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$, with (a) $A/m = 5 \text{ m}^2 \text{ kg}^{-1}$ and (b) $A/m = 10 \text{ m}^2 \text{ kg}^{-1}$. Initial conditions corresponding to orbits with reentry times $25 < t_* \leq 100$ years are colored in green.

noticeable example is the well delineated reentry corridor emanating from $i_0 \sim 40^\circ$ which corresponds to $\psi_1 = 0$. Interestingly enough, most of the reentry orbits comply with the stringent 5-year rule. Orbits that fail to satisfy the 5-year criterion but still comply with the 25-year rule are predominantly found at large values of a_0 , and tend to lie mostly near the boundaries of the reentry domain. Lifetime maps shown in Fig. 3 for the $i - a$ space are also shown for the $i - e$ and $a - e$ spaces in Fig. B4 and Fig. C6 in Appendices B and C respectively.

We further characterize the reentry orbits by looking at the number N_* of reentry orbits at time t_* . We find that N_* closely follows an inverse-square power law,

$$N_* \propto t_*^\gamma, \quad \gamma \sim -2, \quad (14)$$

as shown in Fig. 4. Both fits have a determination coefficient of $R^2 > 0.96$ and small residuals. Table 2 extends this analysis to various e_0 values and reports the corresponding exponents γ . This consistent value, $\gamma \sim -2$, provides a clear functional form that can be useful for predictions.

3.3 Effective stability

In subsection 3.1, regularity and chaoticity were assessed at a rather large time set to 100 years. In this section, we instead use the SALI to characterize the nature of an orbit at a more practically relevant time τ , referred to as the *effective time*. Over the

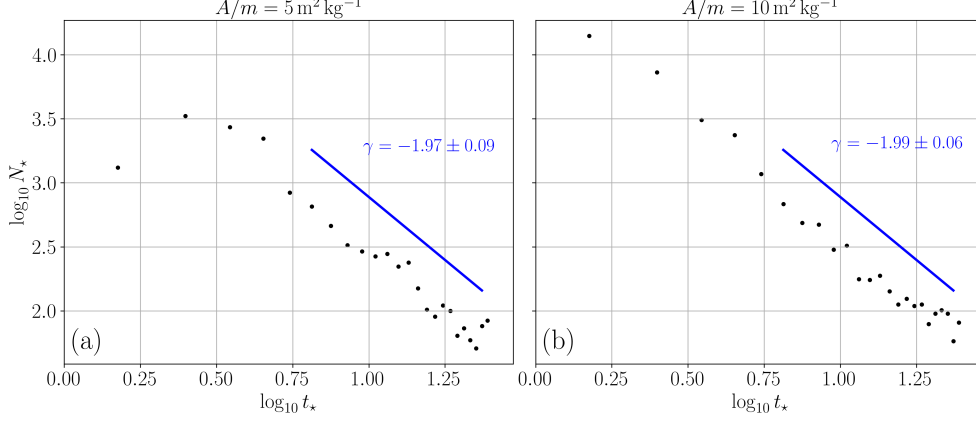


Fig. 4 The number of initial conditions N_* from Fig. 3 leading to orbits that reenter in the time window of 25 years as a function of the reentry time t_* , for (a) $A/m = 5 \text{ m}^2 \text{ kg}^{-1}$ and (b) $A/m = 10 \text{ m}^2 \text{ kg}^{-1}$. The results follow power laws with exponents γ very close to $\gamma = -2$.

	$A/m = 5 \text{ m}^2 \text{ kg}^{-1}$	$A/m = 10 \text{ m}^2 \text{ kg}^{-1}$
e_0	γ	γ
0.001	-2.02 ± 0.07	-2.03 ± 0.06
0.005	-1.98 ± 0.07	-2.04 ± 0.05
0.01	-1.99 ± 0.07	-2.05 ± 0.05
0.05	-1.97 ± 0.09	-1.99 ± 0.06

Table 2 Power law exponents γ (Eq. (14)) as a function of e_0 . All numerical fittings have a determination coefficient of $R^2 > 0.96$.

300 set of reentry trajectories that comply with the 25-year rule, $\mathcal{R}_{25}$, where

$$\mathcal{R}_{25} = \{\mathbf{x}_0 \mid t_*(\mathbf{x}_0) \leq 25 [\text{years}]\}, \quad (15)$$

301 we define the effective time τ of an initial condition $\mathbf{x}_0$ as

$$\begin{aligned} \tau : \mathcal{R}_{25} &\rightarrow (0, 25] \\ \mathbf{x}_0 &\mapsto \tau(\mathbf{x}_0) = t_*. \end{aligned} \quad (16)$$

302 Thus, the effective time has an upper bound of 25 years over the timescale of interest.
303 Moreover, if a reentry occurs earlier, the orbital behavior is characterized over the
304 reentry timescale. By recording the value of the SALI at time τ , we can classify
305 the orbits as regular or chaotic at the more practically effective time governed by
306 the reentry dynamics. We underline that this approach is facilitated by the intrinsic
307 design of the SALI, which monitors the instantaneous alignment of deviation vectors,
308 reflecting in this way the local nature of the orbit, contrarily to, for example, the

maximal Lyapunov exponent which averages the growth of tangent vectors over time.

The composite plot of Fig. 5 shows that most of the orbits previously identified in Fig. 3(b) are regular over the effective time interval. More specifically, among all reentry orbits, 99.7% for $A/m = 5 \text{ m}^2\text{kg}^{-1}$ and 99.8% for $A/m = 10 \text{ m}^2\text{kg}^{-1}$ have $\log_{10}(\text{SALI}(\tau)) \geq -3$. In addition to the SALI dynamical map evaluated at the effective time (panel (a)), the time evolutions of the SALI of some representative orbits (panels (b), (c) and (d)) are also shown. In panels (b), (c) and (d) the reentry time is denoted by a dashed vertical line. Panels (b) and (c) display the prototypical behavior of the index for regular motion (see Fig. 1). Although the orbit in panel (b) has a longer reentry time than that of panel (c), both orbits are regular up to their respective reentry times. In contrast, the orbit in panel (d) exhibits chaotic behavior at the time of reentry (approximately, 24.7 years). This orbit is one of the very few effectively chaotic orbits present in panel (a), as further confirmed by panel (e), which shows the distribution of $\log_{10} \text{SALI}(\tau)$ values in colors corresponding to panel (a). The scarcity of values below $\log_{10} \text{SALI}(\tau) = -5$ further supports the predominance of regular behavior over the effective timescale. Thus, the results of Fig. 5 clearly demonstrate that over the effective timescale, the reentry dynamics of almost all orbits are regular.

To further illustrate this point, Fig. 6 shows the long-term evolution (up to 50 years) of the eccentricity of two nominal chaotic orbits. In gray, the evolutions of a cluster of 50 orbits near to the respective nominal orbits are shown. These neighboring initial conditions are drawn from a two-dimensional normal distribution and further constrained to lie within 1° and 1 km of the corresponding nominal orbit. Despite the chaotic nature of both nominal orbits, we observe a clear divergence of the nearby orbits from the nominal one, occurring at approximately 10 years for the blue orbit in panel (a) and around 3 years for the red orbit in panel (b). Nevertheless, in the case of panel (a), the nominal orbit remains effectively regular up to its reentry time, with no appreciable divergence of nearby orbits prior to this event. Importantly, this evolution complies with the 5-year rule. For this case, the separation of nearby orbits becomes evident only after the crossing of the critical eccentricity (indicated by dotted horizontal lines in both panels of Fig. 6). This is an example of an eventually chaotic orbit which behaves regularly on the effective timescale. In contrast, the nominal orbit in panel (b) exhibits a much earlier divergence of neighboring orbits, revealing its chaotic character well before reentry. We note that this orbit does not satisfy the 25-year mitigation requirement and therefore does not appear in Fig. 5, as it reaches the critical eccentricity at approximately 40 years.

4 Smooth and fractal boundaries of reentry basins

While the previous sections have focused on the regularity of transport and the identification of ordered and chaotic regions, the present section explores the geometry of reentry basins.

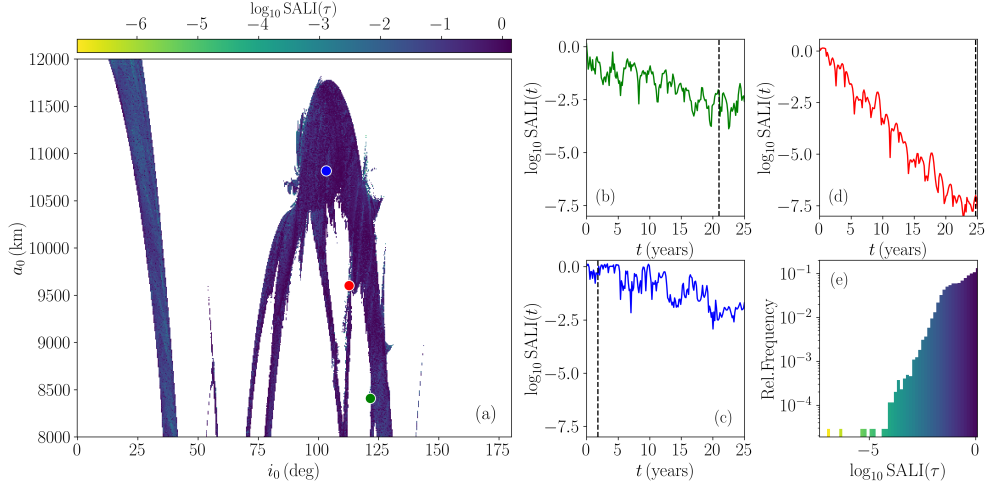


Fig. 5 (a) Stability of the reentry orbits of Fig. 3(b) at the effective time τ (Eq. (16)) for $A/m = 10 \text{ m}^2 \text{ kg}^{-1}$. Over the reentry process, almost all orbits are regular, as the coloring of initial conditions according to their $\log_{10} \text{SALI}(\tau)$ indicates. Specific initial conditions are denoted by green, blue and red dots. The time evolution of $\text{SALI}(t)$ of these three orbits are respectively shown in panels (b), (c) and (d). Reentry times are marked by the dashed vertical lines in these panels. The orbits considered in (b) and (c) behave regularly until their reentry time with the one in (b) requiring more time to reenter. The orbit considered in (d) is a chaotic orbit which reenters at 24.7 years. In (e) we see the distribution of the $\log_{10} \text{SALI}(\tau)$ values for the orbits displayed in (a).

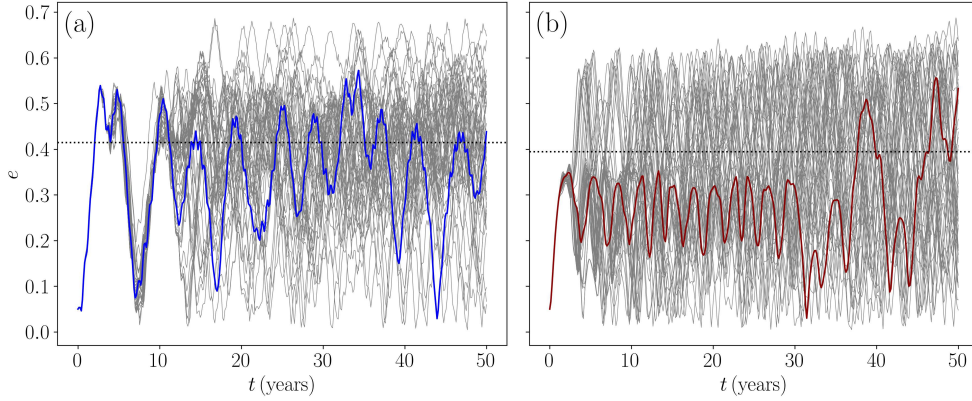


Fig. 6 Evolution of the eccentricity of 50 nearby randomly generated initial conditions (shown in gray) in a neighborhood of a nominal orbit (blue or red). The nominal orbit in (a) corresponds to the initial condition $(a_0, e_0, i_0, \omega_0, \Omega_0) = (11100 \text{ km}, 0.05, 100^\circ, 0^\circ, 0^\circ)$ (blue) and in (b) to $(a_0, e_0, i_0, \omega_0, \Omega_0) = (10727.98 \text{ km}, 0.05, 118.36^\circ, 0^\circ, 0^\circ)$ (red). The dotted horizontal line corresponds to the value of the respective nominal orbit's critical eccentricity e_* (Eq. (13)). All orbits in (a) and (b) have $A/m = 10 \text{ m}^2 \text{ kg}^{-1}$.

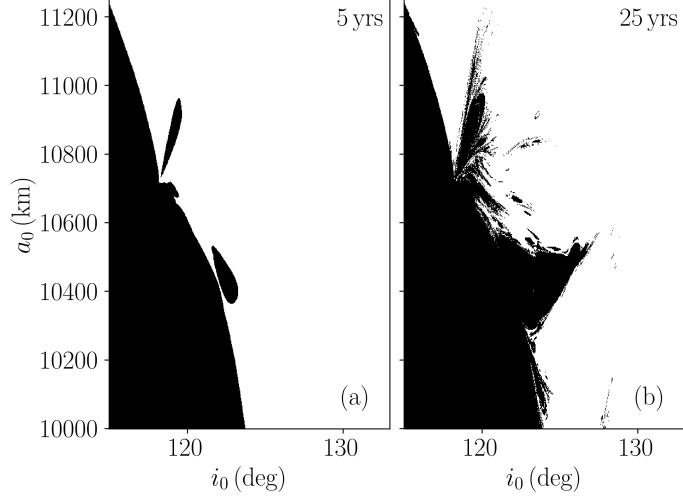


Fig. 7 Distribution of initial conditions of reentry orbits (black points) complying with the (a) 5-year and (b) 25-year mitigation guideline, over $a_0 \in [10\,000\text{km}, 11\,250\text{km}]$ and $i_0 \in [115^\circ, 133^\circ]$ with $e_0 = 0.05$ and $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$, for $A/m = 10\text{ m}^2\text{kg}^{-1}$. The reentry basin boundaries exhibit increasing structural complexity depending on the mitigation scenario. A visual inspection of the boundaries suggests fractality, something which is confirmed and quantified through the computation of uncertainty exponents.

The boundaries of reentry basins develop in a complex way, depending on the mitigation guideline chosen. An example is provided in Fig. 7 for the 5-year and 25-year scenarios, where initial conditions leading to reentry orbits are colored in black. In the more restrictive 5-year case, the basin boundaries appear relatively smooth. In contrast, for the 25-year scenario, the boundaries become considerably more structured and complex, showing multiple intricate patterns on small scales. This property indicates that even tiny variations in the initial conditions might influence whether an orbit leads to reentry or not. To quantify and characterize this complex transition witnessed over several domains, we employ the uncertainty exponent method, often traced back to [45], and subsequently applied in various dynamical systems contexts [46–48]. As our results will demonstrate, distinct fractal-like structures emerge in the basin boundaries depending on the mitigation scenario. In the following, we continue our focus on domains within the $i - a$ space, considering the case $A/m = 10\text{ m}^2\text{kg}^{-1}$, with $e_0 = 0.05$ and $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0$ as before.

The uncertainty method compares labels assigned by a given observable to two nearby initial conditions, namely $\mathbf{x}_0 \pm \epsilon$, where ϵ represents a small perturbation of the considered initial condition $\mathbf{x}_0$. In our case, the observable assigns a binary label to each initial condition depending on whether the corresponding orbit satisfies the prescribed mitigation guideline. The initial condition $\mathbf{x}_0$ is said to be ϵ -uncertain if the two perturbed initial conditions $\mathbf{x}_0 + \epsilon$ and $\mathbf{x}_0 - \epsilon$ receive different labels under the action of this observable.

Over discretized domains $\mathcal{D}$ of initial conditions in the $i - a$ space containing substantial reentry boundaries, we computed the uncertainty fraction

$$f(\epsilon) = \frac{N_{\text{uncertain}}(\epsilon)}{N}, \quad (17)$$

over a range of ϵ values, setting $A/m = 10 \text{ m}^2\text{kg}^{-1}$ throughout. In this investigation, we perturb the initial condition only in the i -direction, so that ϵ has a nonzero component only in i ; we therefore denote it simply by ϵ . The values of ϵ used to generate the best fit lines in Fig. 8 consist of 13 logarithmically spaced points between $(5.4 \times 10^{-4})^\circ$ and 0.13° . The quantities $N_{\text{uncertain}}(\epsilon)$ and N denote, respectively, the number of ϵ -uncertain initial conditions and the total number of sampled initial conditions in $\mathcal{D}$. As established in [45], the uncertainty fraction $f(\epsilon)$ scales as

$$f(\epsilon) \propto \epsilon^\alpha, \quad (18)$$

where α is called the uncertainty exponent. For smooth boundaries $\alpha = 1$, whilst $0 < \alpha < 1$ indicates fractal basin boundaries.

This power law scaling has important practical implications for reentry predictions when $\alpha \neq 1$. For illustrative purposes, let us consider the case $\alpha = 1/2$. A reduction of the initial condition uncertainty by a factor 100 then leads to a reduction in the fraction of uncertain points by a factor of only 10. In other words, substantial improvements in the precision of the initial conditions translate into limited gains in the reliability of the predicted label. This situation worsens for smaller α , *i.e.*, for increasingly fractal boundaries.

The composite plot in Fig. 8 illustrates how the uncertain regions (*i.e.*, the sets of uncertain points) evolve as the mitigation guideline becomes progressively less restrictive. For pedagogical purposes, we consider scenarios corresponding to 5-, 10-, 15-, 20- and 25-year thresholds. Comparing Fig. 7 and Fig. 8, we observe a strong correlation between the uncertain points computed from the boundary of the reentry set and the points lying on the boundary itself, at least for the cases shown in panels (b), (c) and (d) of Fig. 8, which correspond to $\epsilon \approx 0.034^\circ$. From the results of Fig. 8 it is evident that, in the 5-year case the set of uncertain points forms a smooth curve. In contrast, in the 10-year, and especially the 25-year scenario, the set of uncertain points develops a significantly more intricate structure. This qualitative observation is confirmed quantitatively through the estimation of the associated uncertainty exponents α . The values of α reveal a sharp transition in the power laws, ranging from smooth basin boundaries (with α close to 1) to strongly fractal ones (reaching $\alpha = 0.393$) as seen in panel (a) of Fig. 8. We note that the linear fits in that panel have R^2 coefficients ranging from 0.993 to 0.999. Overall, in the considered $i - a$ domain, the variation in uncertainty exponents demonstrates that compliance with the 25-year reentry guideline requires significantly higher precision in the initial conditions than satisfying the 5-year criterion. While this result is consistent with intuition, the present analysis

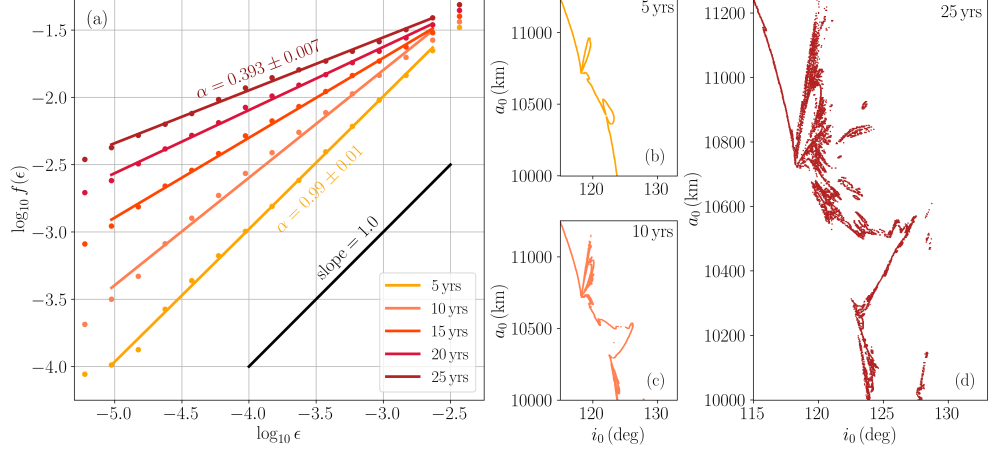


Fig. 8 (a) Slopes of the linear fits reveal the uncertainty exponents α (Eq. (18)) computed for a series of mitigation guidelines for domain $\mathcal{D}$ (see Fig. 7 and Table 3). Fractality becomes progressively more pronounced for less restrictive mitigation scenarios, with an uncertainty exponent close to $\alpha = 1$ for the 5-year case, decreasing to $\alpha = 0.393$ for the 25-year scenario, confirming the emergence of fractal basin boundaries. The corresponding transition is qualitatively reflected in the structure of the set of uncertain points presented in (b), (c), and (d) for the 5-, 10- and 25-year scenarios respectively, which range from smooth to increasingly complex patterns. The uncertain points shown in (b), (c) and (d) are computed using $\epsilon \approx 0.034^\circ$.

provides a quantitative and rigorous characterization of this dependence.

We repeated the same computation and analysis over several domains $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, and obtained qualitatively and quantitatively similar results. Although the values of α vary from one domain to another, the presence of fractal basin boundaries remains clearly pronounced, with exponents $\alpha < 1$ in all cases for the 25-year scenario. The definitions of the considered domains $\{\mathcal{D}_i\}_{i=1}^3$, together with the corresponding results, are reported in Table 3. The evolution of the reentry sets for the extreme 5- and 25-year scenarios, along with their uncertainty exponents as a function of time, are shown in Fig. 9.

To further assess the robustness of our results, we extended the analysis to the parameter space $A/m - a$ for frozen metrical and angular variables. The view offers a substantially finer sampling in A/m , compared with the three values $\{1, 5, 10\} \text{m}^2 \text{kg}^{-1}$ coarsely considered so far, providing a more detailed view of the evolution of the uncertain sets across the parameter space. The transition from smooth to increasingly intricate structures of reentry basins, reflected in the uncertainty exponent, is clearly not a consequence of the chosen A/m used previously. The results of Fig. 10 therefore provide an additional confirmation of the properties of the reentry basins identified above.

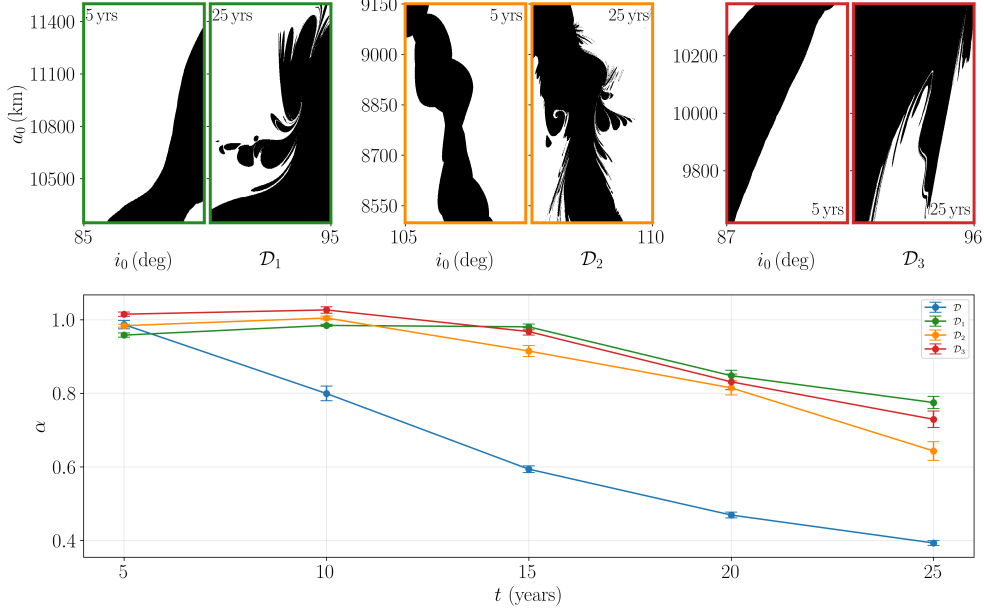


Fig. 9 Upper row: Reentry basins denoted by black points for the domains $\{\mathcal{D}_i\}_{i=1}^3$ of Table 3. Lower panel: Uncertainty exponent α (Eq. (18)) as a function of the years considered in the various mitigation scenarios, for the domains reported in Table 3. The basins exhibit a clear fractality for the 25-year mitigation scenario.

5 Conclusions

Based on an averaged model incorporating Earth's J_2 oblateness and SRP, the present study complemented and supplemented several aspects of the problem. The main contributions and conclusions are summarized as follows:

1. The secular Hamiltonian has 3 DoF and depends parametrically on two quantities: the secularly invariant semi-major axis a and the area-to-mass ratio A/m . To visualize the onset and extent of chaos within this high-dimensional space, the SALI chaos indicator was used, focusing primarily on sections in the $i - a$ space (with results in the $e - i$ and $a - e$ spaces reported in Appendix B and C, respectively) for several A/m values. We showed numerically that the extent of chaotic motion increases with increasing A/m , while the overall dynamical structures remained practically unchanged for different initial phases of the system (see Appendix A).
2. For $A/m \geq 5 \text{ m}^2 \text{ kg}^{-1}$, the number of orbits reaching their critical eccentricity within 25 years follows an inverse-square power law with respect to the corresponding reentry time. Moreover, for sufficiently large values of A/m , most reentry orbits reach their critical eccentricity within only a few years.
3. Despite the potential for long-term chaotic behavior, the majority of orbits that reach their critical eccentricity within 25 years exhibit regular dynamics until their reentry.

	α of $\mathcal{D}$	α of $\mathcal{D}_1$	α of $\mathcal{D}_2$	α of $\mathcal{D}_3$
5 years	0.99 ± 0.01	0.958 ± 0.006	0.984 ± 0.004	1.015 ± 0.006
10 years	0.8 ± 0.02	0.986 ± 0.002	1.005 ± 0.005	1.027 ± 0.009
15 years	0.594 ± 0.009	0.981 ± 0.007	0.92 ± 0.01	0.97 ± 0.01
20 years	0.469 ± 0.008	0.85 ± 0.01	0.81 ± 0.02	0.83 ± 0.02
25 years	0.393 ± 0.007	0.77 ± 0.02	0.64 ± 0.03	0.73 ± 0.02

Table 3 Uncertainty exponents α (Eq. (18)) computed over various domains in the $i - a$ space according to various mitigation guidelines. We namely consider the domains $\mathcal{D} = [115^\circ, 133^\circ] \times [10\,000\text{km}, 11\,250\text{km}]$ (Fig. 7), together with $\mathcal{D}_1 = [85^\circ, 95^\circ] \times [10\,250\text{km}, 11\,500\text{km}]$, $\mathcal{D}_2 = [100^\circ, 110^\circ] \times [8\,500\text{km}, 10\,000\text{km}]$, and $\mathcal{D}_3 = [87^\circ, 96^\circ] \times [9\,620\text{km}, 10\,380\text{km}]$ (Fig. 9), all having $e_0 = 0.05$, $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$ and $A/m = 10\,\text{m}^2\text{kg}^{-1}$. In general, for mitigation timescales above 10 years the uncertainty exponents are significantly smaller than 1, indicating the fractal nature of the reentry basin boundaries.

4. By varying the mitigation guideline from 5 to 25 years, we found that the associated basins of reentry orbits develop increasingly pronounced fractal-like structures. This fractality was quantified using the uncertainty exponent method, with uncertain points corresponding to the boundaries of the reentry basins. As fractality increases, so does the precision required in specifying initial conditions. Hence, from an operational point of view, it is easier to target the 5-year reentry basin as opposed to the 25-year basin. This suggests that the enforcement of a lower mitigation guideline is, from the dynamical perspective, advantageous for satellite mission design and disposal strategies.

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Declarations

Not Applicable

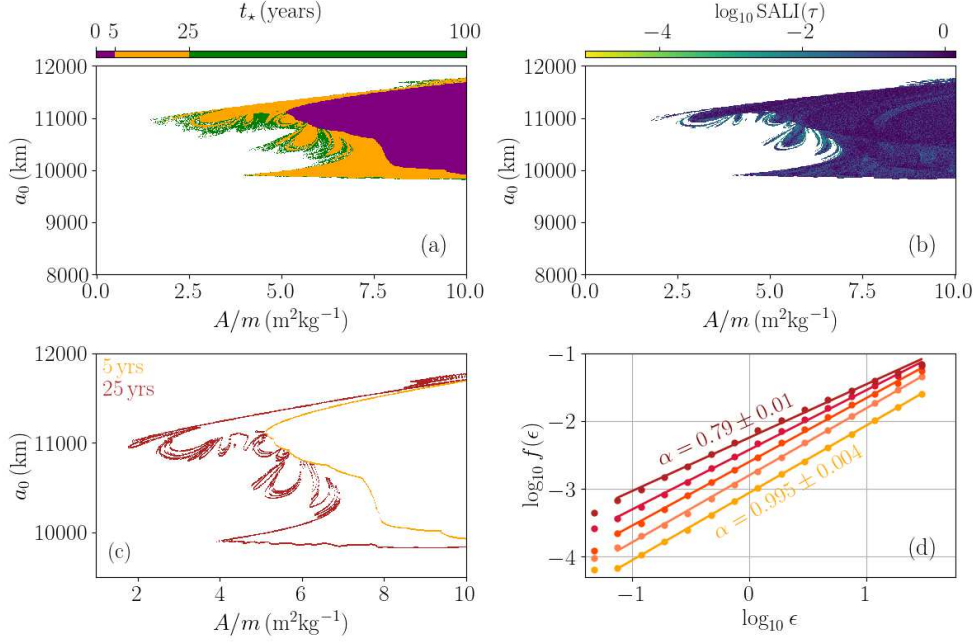


Fig. 10 Panel (a) identifies initial conditions complying with the 5-year (violet), 25-year (orange) mitigation guidelines, for initial conditions in the parameter space $A/m - a$ where $A/m \in [0.001 \text{ m}^2 \text{ kg}^{-1}, 10 \text{ m}^2 \text{ kg}^{-1}]$ and $a_0 \in [8000 \text{ km}, 12000 \text{ km}]$, for $e_0 = 0.05$, $i_0 = 100^\circ$, and $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$. Initial conditions corresponding to orbits with reentry times $25 < t_* \leq 100$ years are colored in green. Panel (b) shows the stability of the reentry orbits at the effective time τ (Eq.(16)) for the same initial conditions as in (a). Once again, evidenced by the predominance of high $\log_{10} \text{SALI}(\tau)$ values, all orbits are regular at least up until reentry. Panel (c) maps the uncertain points calculated from a perturbation in a_0 at $\epsilon \approx 4.68 \text{ km}$ for the 5- and 25-year mitigation guidelines, showing the transition from smooth to fractal-like boundaries qualitatively. Panel (d) reports the slopes of the linear fits which reveal the uncertainty exponents α (Eq.(18)) of the boundaries in the parameter space, for the same domain as in panel (c), with colors as those in Fig. 8, for the boundaries corresponding to the 5-, 10-, 15-, 20-, 25-year mitigation guidelines. For less strict mitigation guidelines, fractality is more pronounced, as indicated by the $\alpha < 1$ value for the 25-year guideline and $\alpha \approx 1$ for the 5-year guideline.

Appendix A Additional results in the $i - a$ space

A.1 Sensitivity of the maps with respect to the initial phases

Fig. A1 shows several dynamical maps based on the SALI values at time $t_f = 100$ years for $e_0 = 0.05$ when the initial phases of the system are varied, for $A/m \in \{1, 5, 10\} \text{ m}^2 \text{ kg}^{-1}$. The dynamical structures (large chaotic seas or filamentary structures) are not significantly influenced by the initial phases of the system. Due to this lack of sensitivity, the initial phases can be set to any values without notable changes in the results, thereby justifying the choice made in the main text to set all initial phases to zero, namely $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$.

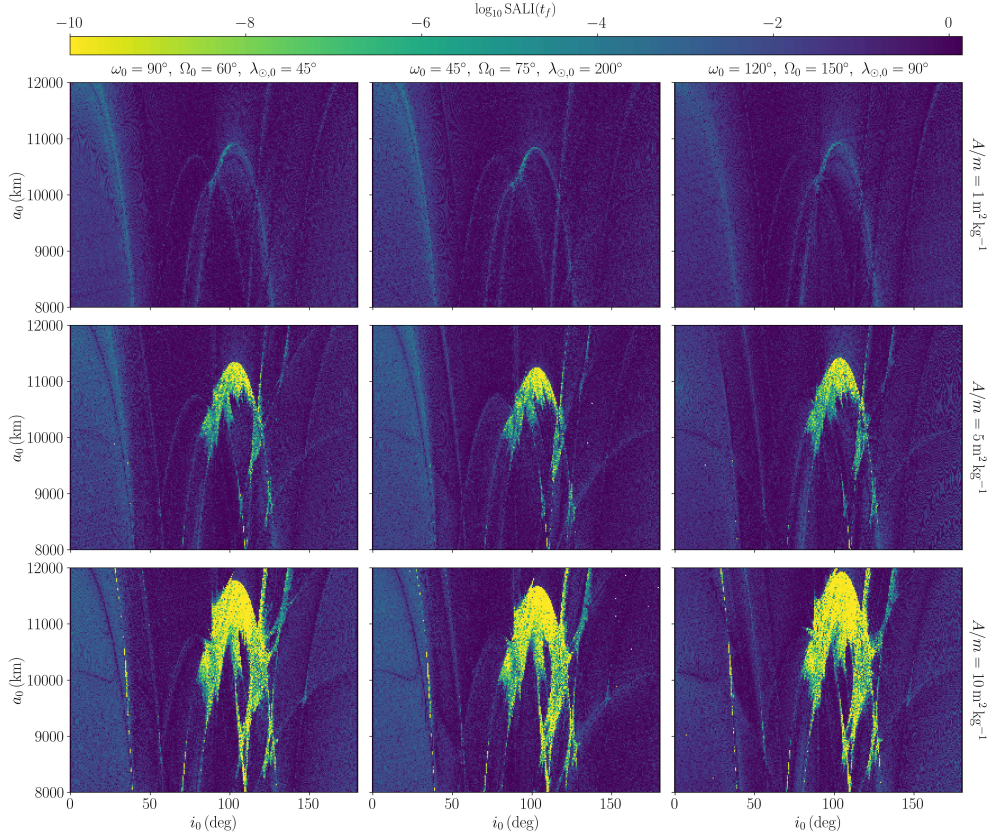


Fig. A1 Dynamical maps when the phases of the system are varied. Initial conditions are colored according to the $\log_{10}\text{SALI}$ value computed at $t_f = 100$ years, over the interval $a_0 \in [8\,000\text{km}, 12\,000\text{km}]$ and $i_0 \in (0^\circ, 180^\circ)$, with $e_0 = 0.05$ fixed and $\omega_0 = \Omega_0 = \lambda_{\odot,0}$ varied according to the column headings. Each row corresponds to $A/m \in \{1, 5, 10\} \text{ m}^2\text{kg}^{-1}$. The observed dynamical structures do not significantly depend upon the phases of the system.

A.2 Additional maps when e_0 is varied

In order to witness the changes (if any) to the chaotic structures present in the phase space when varying the value of e_0 , Fig. A2 shows several SALI maps computed at time $t_f = 100$ years for $e_0 \in \{0.001, 0.005, 0.01, 0.05\}$. As in the case of the initial phases shown in Fig. A1, variations in e_0 cause barely discernible differences in the chaotic seas and filaments across the corresponding maps, for all considered A/m values, justifying the choice of $e_0 = 0.05$ throughout the main body of our investigation.

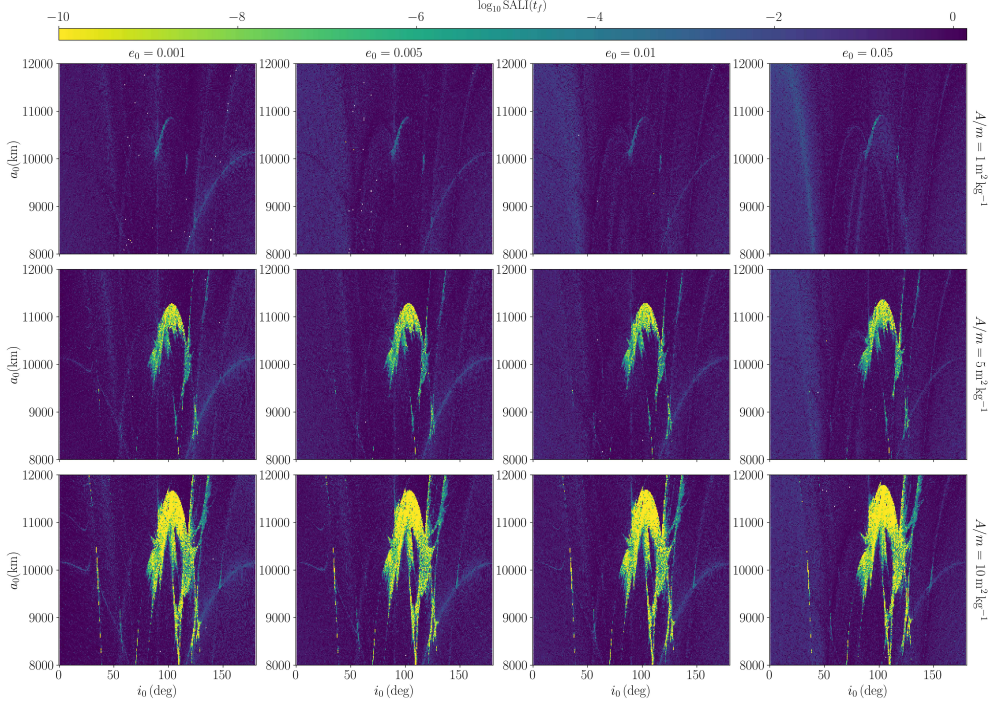


Fig. A2 Dynamical maps when e_0 is varied. Initial conditions are colored according to the $\log_{10}$ SALI value computed at $t_f = 100$ years, over the interval $a_0 \in [8\,000\text{km}, 12\,000\text{km}]$ and $i_0 \in (0^\circ, 180^\circ)$, with $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$ fixed and $e_0 \in \{0.001, 0.005, 0.01, 0.05\}$ per column. Each row corresponds to $A/m \in \{1, 5, 10\} \text{ m}^2\text{kg}^{-1}$. The observed dynamical structures of the system do not show a significant dependence on the value of e_0 within the considered range of initial eccentricities.

Appendix B Stability and lifetime maps in the $i - e$ space

To provide a more holistic visualization of the entire 6-dimensional phase space of the Hamiltonian in Eq. (3), another slice of this phase space, namely $i - e$ space, is considered here. Mapping the SALI at $t_f = 100$ years and the orbital lifetime respectively, Fig. B3 and Fig. B4 show initial conditions chosen over the range $i_0 \in (0^\circ, 180^\circ)$ and $e_0 \in (0, 1)$, with a_0 taking on three different values, namely $a_0 \in \{8\,000, 9\,500, 11\,000\} \text{ km}$, for $A/m \in \{1, 5, 10\} \text{ m}^2\text{kg}^{-1}$. A white sheen is placed over a large upper portion of each map to identify initial conditions corresponding to an initial eccentricity e_0 exceeding the associated critical eccentricity e_* . These orbits are nonphysical; we plot them, however, for completeness' sake. As with the $i - a$ space in the main manuscript, the extent of chaos in the $i - e$ space increases with increasing A/m , with the largest amount of chaos observed for $a_0 = 11\,000\text{km}$ and $A/m = 10 \text{ m}^2\text{kg}^{-1}$.

Turning to the orbital lifetime maps of Fig. B4, while keeping in mind the SALI maps of Fig. B3, we see a correspondence (albeit not one-to-one, due to the broad band

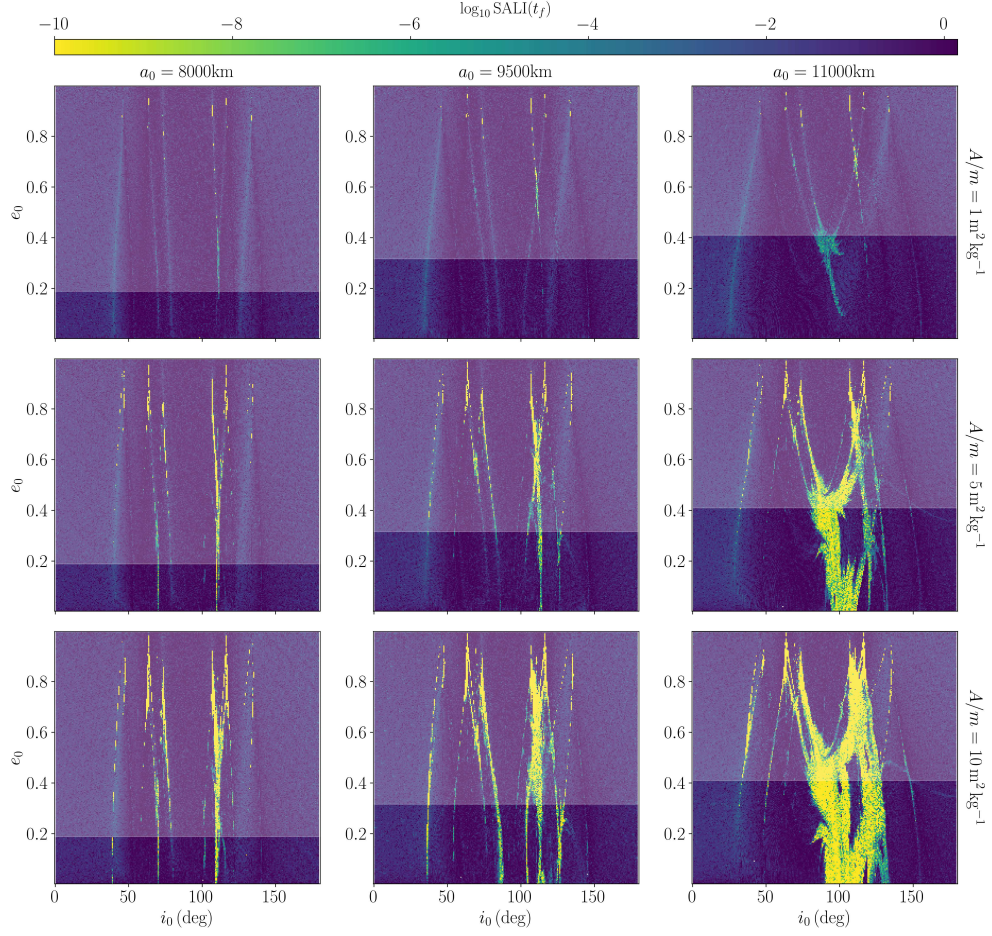


Fig. B3 Initial conditions colored according to the $\log_{10}\text{SALI}$ value computed at $t_f = 100$ years, over the interval $i_0 \in (0^\circ, 180^\circ)$ and $e_0 \in (0, 1)$, with $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$ fixed and $a_0 \in \{8\,000, 9\,500, 11\,000\}$ km per column. Each row corresponds to $A/m \in \{1, 5, 10\} \text{ m}^2\text{kg}^{-1}$. Non-physical initial conditions are masked using a white sheen.

502 of reentry orbits around $i_0 \sim 35^\circ$ to $i_0 \sim 40^\circ$) between the orbits that reenter within 25
 503 years and the chaotic regions revealed by the SALI. We see, in fact, that most orbits
 504 reenter within 5 years, apart from the nonphysical orbits which naturally have $t_\star = 0$.

505 Appendix C Stability and lifetime maps in the $e - a$ 506 space

507 Continuing the exploration of the high-dimensional phase space of the Hamiltonian
 508 in Eq. (3), Fig. C5 and Fig. C6 present maps of the SALI at $t_f = 100$ years and the
 509 orbital lifetime, respectively, over initial conditions in the $e - a$ space. The scans are

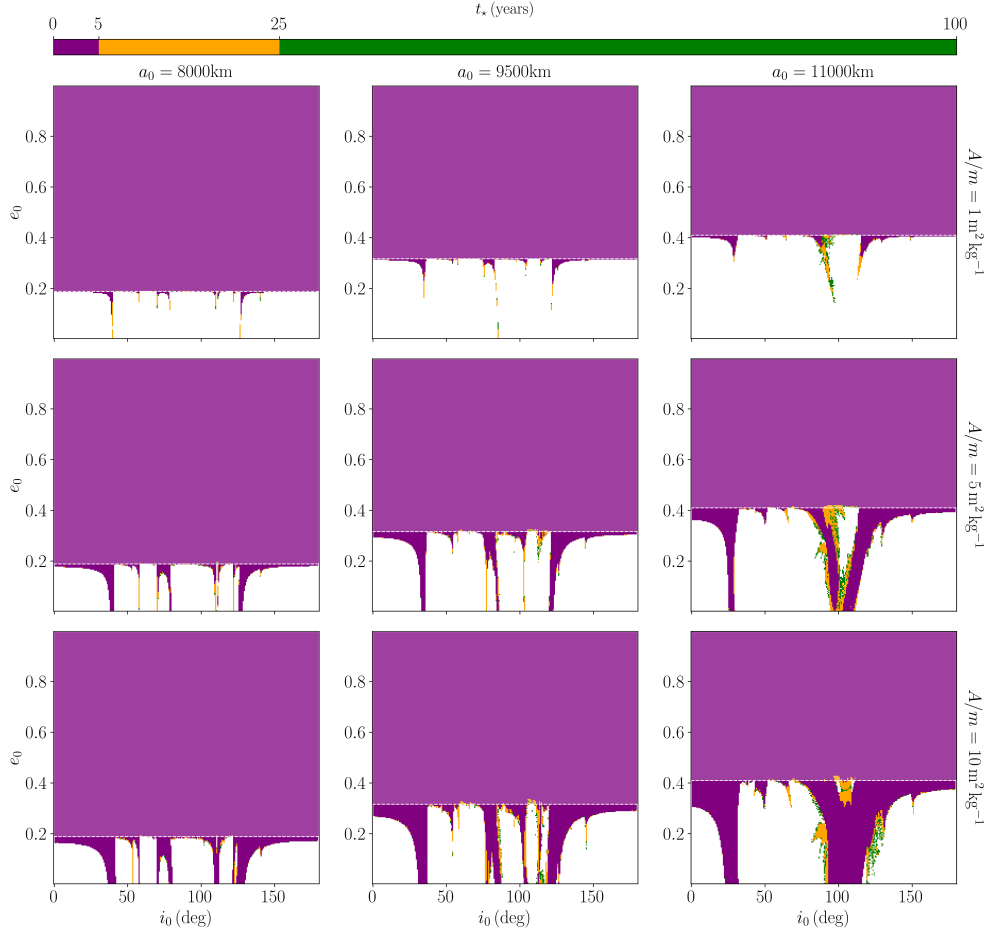


Fig. B4 Initial conditions colored according to their orbital lifetime t_* , over the interval $i_0 \in (0^\circ, 180^\circ)$ and $e_0 \in (0, 1)$, with $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$ fixed and $a_0 \in \{8\,000, 9\,500, 11\,000\}$ km per column. Each row corresponds to $A/m \in \{1, 5, 10\} \text{ m}^2 \text{kg}^{-1}$.

performed for $e_0 \in (0, 1)$, $a_0 \in [8\,000\text{km}, 12\,000\text{km}]$ and $i_0 \in \{50^\circ, 100^\circ, 150^\circ\}$, with $A/m \in \{1, 5, 10\} \text{ m}^2 \text{kg}^{-1}$. As before, nonphysical orbits are masked by a white sheen.

The $e-a$ space again follows the trend of the $i-a$ and $i-e$ spaces, with the extent of chaotic regions increasing with A/m . The most pronounced chaotic behavior in this space occurs for $i_0 \sim 100^\circ$, as expected from the prominent inverted U-shaped chaotic sea observed around $i_0 \sim 100^\circ$ in Fig. 2. Furthermore, a qualitative correspondence can be identified between the chaotic regions in Fig. C5 and the reentry structures in Fig. C6.

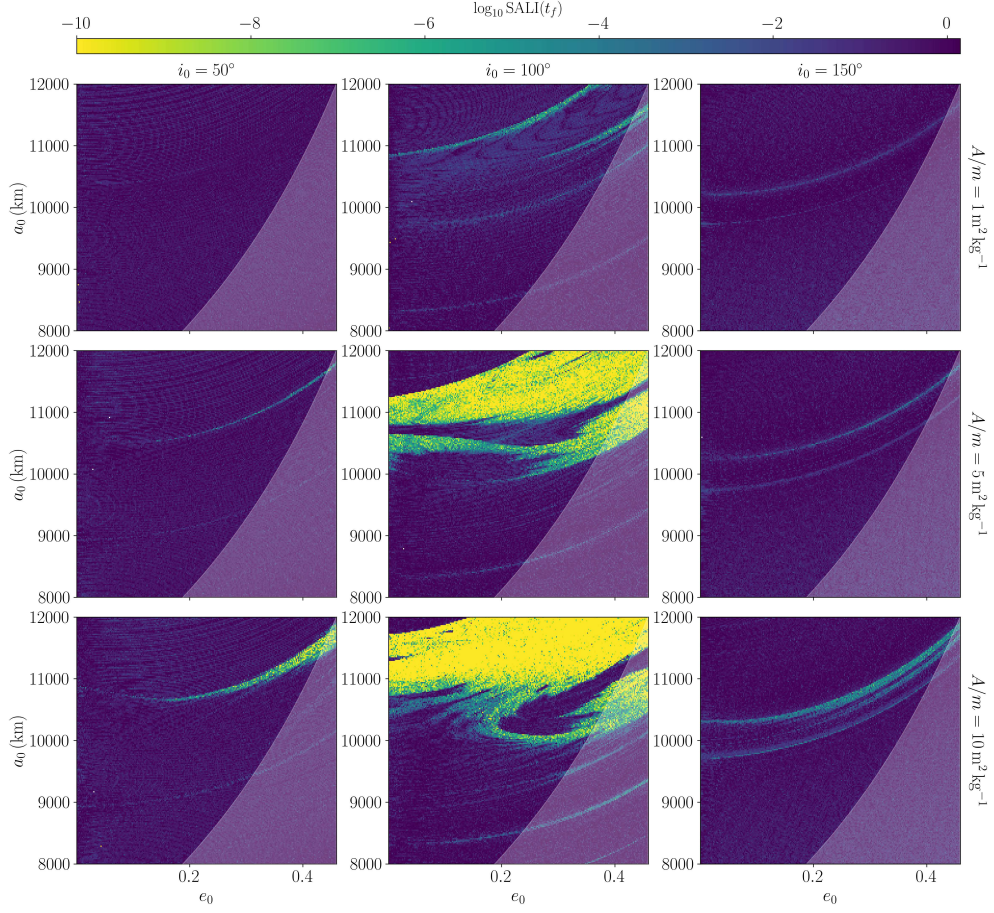


Fig. C5 Initial conditions colored according to the $\log_{10}$ SALI value computed at $t_f = 100$ years, over the interval $e_0 \in (0, 1)$ and $a_0 \in [8000\text{km}, 12000\text{km}]$, with $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$ fixed and $i_0 \in \{50^\circ, 100^\circ, 150^\circ\}$ per column. Each row corresponds to $A/m \in \{1, 5, 10\} \text{ m}^2 \text{ kg}^{-1}$. Nonphysical initial conditions are masked using a white sheen.

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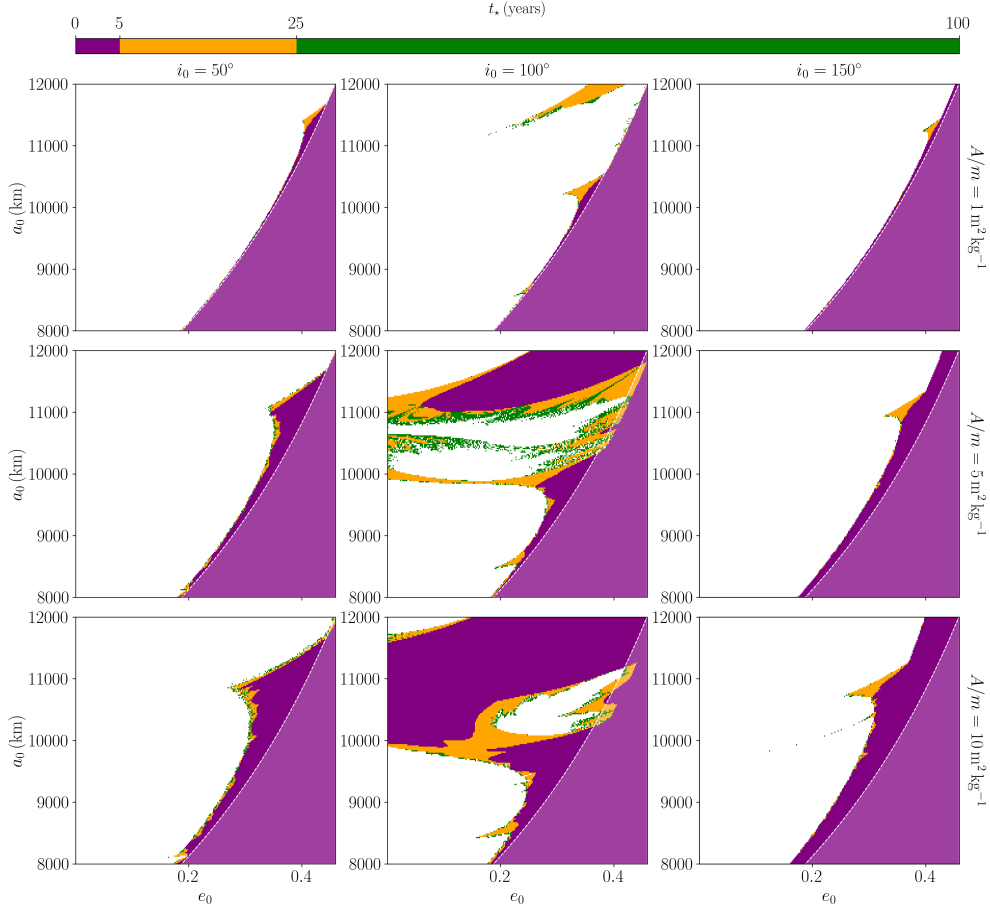


Fig. C6 Initial conditions colored according to their orbital lifetime t_* , over the interval $e_0 \in (0, 1)$ and $a_0 \in [8\,000\text{km}, 12\,000\text{km}]$, with $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0^\circ$ fixed and $i_0 \in \{50^\circ, 100^\circ, 150^\circ\}$ per column. Each row corresponds to $A/m \in \{1, 5, 10\} \text{ m}^2\text{kg}^{-1}$.

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