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Abstract

We consider a mathematical model of cell-to-cell infection involving target cells, eclipse-phase cells, infectious cells, and dead cells. The infection mechanism is described by spatially discrete contacts in a continuous spatial domain, which reduces to a two-dimensional lattice dynamical system on a regular lattice. By reformulating the system as a nonlinear integral equation and applying a fixed-point argument, we establish the existence of planar traveling waves propagating in any given direction. Retaining the discrete spatial coupling, we incorporate distributed time delays through kernel functions to account for general eclipse-phase durations and prove the persistence of traveling waves. The framework also applies to nonlocal infection contacts. To further characterize the wave profiles, we approximate the one-dimensional spatial system by a reduced two-variable first-order ODE system. Phase-plane analysis then identifies heteroclinic orbits representing the wave profiles. Numerical simulations corroborate the theoretical results and illustrate the propagation of infection waves.

Keywords: Traveling wave, lattice dynamical system, nonlinear integral equation, integro-differential system, cell-to-cell transmission model

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1 Introduction

1.1 Cell-to-cell transmission model

In this paper, we consider a model of cell-to-cell infection governing the spatial spread of viral transmission and investigate its traveling wave solutions. Before formulating the model, we briefly outline the relevant biological background.

Viruses are pathogens capable of causing severe disease and, in some cases, death. Extensive research has been devoted to elucidating their life cycles, replication mechanisms, and routes of transmission. In experimental virology, the plaque assay is a classical and widely used in vitro technique for quantifying viral infectivity ([23, 28]). In this assay, a confluent monolayer of cultured host cells is inoculated with a virus-containing sample. As the infection spreads spatially from initially infected cells to adjacent uninfected ones, localized regions of cell lysis—termed plaques—develop. Conventional plaque assays typically use serial dilutions to obtain a dilution at which individual plaques can be clearly distinguished and counted, thereby determining the viral titer. Beyond plaque counting, quantitative analysis of plaque size and morphology provides additional information on viral spread and growth dynamics. Comparing these features across viral strains or drug concentrations provides crucial insights into viral fitness, replication kinetics, and therapeutic efficacy ([9, 13, 14, 26]). Because host cells in a plaque assay form a dense two-dimensional layer and direct cell-to-cell transmission predominantly occurs through physical contact between adjacent cells, the underlying spatial structure is naturally discrete. To capture these local cell-to-cell interactions, mathematical and agent-based models on spatial grids have gained increasing attention. In particular, Kim et al.([17]) and Kim et al.([18]) utilized such modeling frameworks to estimate the cell-entry efficiency of filoviruses.

Building upon these spatial considerations, we formulate a two-dimensional continuum model based on [18] to investigate the propagation of the infection front. The state variables denote the densities of target cells, cells in the eclipse phase, infectious cells, and dead cells, which are denoted by T , E , I , and D , respectively. Under the assumption that transmission occurs via contact with neighboring cells at a characteristic distance r , the governing equations on $\mathbb{R}^2$ are formulated as follows:

$$\begin{aligned} T_t &= -\beta T(x, t) \mathcal{K}[I(\cdot, t)](x) \\ E_t &= \beta T(x, t) \mathcal{K}[I(\cdot, t)](x) - \sigma E(x, t), \\ I_t &= \sigma E(x, t) - \delta I(x, t), \\ D_t &= \delta I(x, t), \end{aligned} \quad (x, t) = (x_1, x_2, t) \in \mathbb{R}^2 \times (t_0, \infty), \quad (1.1)$$

where

$$\begin{aligned} \mathcal{K}[I(\cdot, t)](x) &:= \frac{1}{4} \left[I(x_1 + r, x_2, t) + I(x_1 - r, x_2, t) \right. \\ &\quad \left. + I(x_1, x_2 + r, t) + I(x_1, x_2 - r, t) \right]. \end{aligned} \quad (1.2)$$

The positive parameters β , $1/\sigma$, and δ represent the infection rate, the duration of the eclipse phase, and the death rate of infected cells, respectively. Since the variable D does not appear in the first three equations of (1.1), it decouples from the remaining subsystem; we thus omit D from our subsequent analysis. In addition, note that $\mathcal{K}[I(t)] = I(t)$ whenever I is spatially uniform. A key feature of this model is that, although host cells are spatially stationary, the infection spreads through local cell-to-cell contact. Hereafter, we refer to (1.1) as a cell-to-cell transmission model.

We remark that for nonnegative and continuous initial data $(T(x, 0), E(x, 0), I(x, 0)) = (T_0(x), E_0(x), I_0(x))$, the system (1.1) generates a lattice dynamical system on the grid $\{x = (ir, jr) \in \mathbb{R}^2 : i, j \in \mathbb{Z}\} \cong \mathbb{Z}^2$ (see [6, 11]). A lattice dynamical system may seem more natural for modeling cell-to-cell transmission than the continuum formulation (1.1). However, system (1.1)

has a key advantage: it can be reformulated as a nonlinear integral equation, making the analysis much more tractable. Although traveling waves in spatially discrete systems have been extensively studied, including for bistable discrete diffusion equations [2, 5, 29] and monostable counterparts [3], system (1.1) is an epidemic-type model without standard diffusion and does not belong to these classes. Thus, a fundamentally different approach is required to establish traveling waves.

1.2 Existence theorem for traveling waves

To seek traveling wave solutions to (1.1), propagating along the direction $\mathbf{n} = (\cos \theta, \sin \theta)$, we introduce the moving coordinate

$$(T, E, I) = (U(z), V(z), W(z)), \quad z = \mathbf{n} \cdot x + ct, \quad \mathbf{n} = (n_1, n_2) = (\cos \theta, \sin \theta),$$

and plugging this ansatz into (1.1) yields

$$\begin{cases} c \frac{dU}{dz} = -\beta U(z) \mathcal{K}_\theta[W(\cdot)](z), \\ c \frac{dV}{dz} = \beta U(z) \mathcal{K}_\theta[W(\cdot)](z) - \sigma V(z), \\ c \frac{dW}{dz} = \sigma V(z) - \delta W(z), \end{cases} \quad (1.3)$$

where

$$\mathcal{K}_\theta[W(\cdot)](z) := \frac{1}{4} [W(z + r \cos \theta) + W(z - r \cos \theta) + W(z + r \sin \theta) + W(z - r \sin \theta)].$$

We impose the condition at infinity as

$$\lim_{z \rightarrow -\infty} (U(z), V(z), W(z)) = (U_0, 0, 0), \quad \lim_{z \rightarrow \infty} (U(z), V(z), W(z)) = (U_*, 0, 0). \quad (1.4)$$

Solutions to (1.3) with (1.4) present profile of traveling waves.

Regarding U_0 and U_* in (1.4), take $U_0 > \delta/\beta$ and let U_* be a positive number satisfying

$$U_* - (\delta/\beta) \log U_* = U_0 - (\delta/\beta) \log U_0, \quad 0 < U_* < \delta/\beta < U_0. \quad (1.5)$$

We note that if $U_0 > \delta/\beta$, then the reproduction number $R_0 := \beta U_0/\delta$, defined for the spatially uniform case in (1.1), is greater than 1 (see §2). We further define

$$\ell(\rho, c, \theta) := \frac{\beta \sigma U_0}{2(c\rho + \delta)(c\rho + \sigma)} [\cosh(\rho r \cos \theta) + \cosh(\rho r \sin \theta)] \quad (\rho \geq 0), \quad (1.6)$$

which is periodic in θ with the period $\pi/2$. Then, given $\rho > 0$, $0 \leq \theta < \pi/2$, there exists $c = c(\rho, \theta)$ satisfying $\ell(\rho, c(\rho, \theta), \theta) = 1$. We can show that $\partial c / \partial \rho = 0$ ($\rho > 0$) has a unique solution which gives a minimum of $c(\rho, \theta)$.

Our main result is on the existence of a planner traveling wave of (1.1), propagating in the direction $\mathbf{n} = (\cos \theta, \sin \theta)$ with monotone decreasing profile for T .

Theorem 1 *Assume $R_0 = \beta U_0/\delta > 1$. Then (1.1) possesses a traveling wave solution $(T, E, I) = (U(z), V(z), W(z))$, $z = x_1 \cos \theta + x_2 \sin \theta + ct$, for each $c > c^*(\theta)$, satisfying (1.4), where $c^*(\theta) = \inf_{0 \leq \rho} c(\rho, \theta)$ and $U(z)$ is monotone decreasing.*

We note that as seen in §3, $c^*(\theta)$ is also periodic with period $\pi/2$, and

$$\frac{dc^*}{d\theta} \begin{cases} < 0 & (0 < \theta < \pi/4), \\ > 0 & (\pi/4 < \theta < \pi/2), \end{cases} \quad \min_{0 \leq \theta < \pi/2} c^*(\theta) = c^*(\pi/4), \quad \max_{0 \leq \theta < \pi/2} c^*(\theta) = c^*(0). \quad (1.7)$$

This implies that there are traveling wave solutions propagating along the $\pi/4 \pmod{\pi/2}$ -degree direction having a speed lower than any other direction. Furthermore, as seen in Remark 4, $c^*(\theta)$ is monotone increasing in $r > 0$.

Remark 1 Theorem (1) has provided planar traveling wave solutions $(T, E, I)(x, t) = (U(z), V(z), W(z))$, where $z = \mathbf{n} \cdot x + ct$, $\mathbf{n} = (\cos \theta, \sin \theta)$. Restricting these profiles to the lattice $r\mathbb{Z}^2$ yields a traveling wave of the corresponding lattice dynamical system if n_1/n_2 is rational (or one component vanishes), since the projection of $r\mathbb{Z}^2$ onto $\mathbf{n}$ is then discrete (see [2]). For irrational directions, however, this projection is dense in $\mathbb{R}$, so the resulting incommensurate shifts do not yield a traveling wave of the lattice system ([2]).

Remark 2 In addition to the study for the existence of traveling waves, the asymptotic behavior of solutions to the Cauchy problem for (1.1) is also an important and interesting problem. However, we leave it for future work since this article focuses on the existence of traveling waves.

1.3 Model with distributed time delays for eclipse-phase duration

According to [18], we further consider the model taking the time delay into account by using a kernel function as

$$\begin{cases} T_t = -\beta T(x, t) \mathcal{K}[I(\cdot, t)](x), \\ E_t = \beta T(x, t) \mathcal{K}[I(\cdot, t)](x) - \sigma E(x, t), \\ I(x, t) = \int_0^\infty J(\xi) (\sigma E(x, t - \xi)) d\xi = \sigma \int_{-\infty}^t J(t - s) E(x, s) ds, \end{cases} \quad (1.8)$$

where $J(\xi)$ is a nonnegative function in $L^1(0, \infty)$. We note that the equation for $D(x, t)$ is omitted. In [18], for a specific J given by

$$J(\xi) = \sum_{j=1}^N \frac{\xi^{j-1}}{(j-1)!(\tau_I/N)^{j-1}} e^{-\frac{\xi}{\tau_I/N}},$$

where τ_I is the average duration of infectious phase and N the number of subdivided I compartment. By applying the chain trick with

$$I_j(x, t) = \int_0^\infty \frac{\xi^{j-1}}{(j-1)!(\tau_I/N)^{j-1}} e^{-\frac{\xi}{\tau_I/N}} \sigma E(x, t - \xi) d\xi,$$

they proposed the following system reduced from (1.8):

$$\begin{cases} T_t = -\beta T(x, t) \sum_{j=1}^N \mathcal{K}[I_j(\cdot, t)](x), & E_t = \beta T(x, t) \sum_{j=1}^N \mathcal{K}[I_j(\cdot, t)](x) - \sigma E(x, t), \\ (I_1)_t = \sigma E(x, t) - \frac{N}{\tau_I} I_1(x, t), & (I_j)_t = \frac{N}{\tau_I} I_{j-1}(x, t) - \frac{N}{\tau_I} I_j(x, t), \quad 2 \leq j \leq N, \end{cases}$$

though $\mathcal{K}$ is slightly different from the operator considered in [18], where cell-to-cell infection is assumed to occur along the radial direction. As a special case $N = 1$, that is, $J(\xi) = e^{-\delta \xi}$, $\delta =$

$1/\tau_I$, the system coincides with (1.1), without the equation for D . With the moving coordinate $z = \mathbf{n} \cdot x + ct$, putting $(U(z), V(z), W(z)) = (T(\mathbf{n} \cdot x + ct), E(\mathbf{n} \cdot x + ct), I(\mathbf{n} \cdot x + ct))$ in (1.8) yields

$$\begin{cases} c \frac{dU}{dz} = -\beta U(z) \mathcal{K}_\theta[W(\cdot)](z), \\ c \frac{dV}{dz} = \beta U(z) \mathcal{K}_\theta[W(\cdot)](z) - \sigma V(z), \\ W(z) = \sigma \int_0^\infty J(\xi) V(z - c\xi) d\xi = \frac{\sigma}{c} \int_{-\infty}^z J\left(\frac{z-s}{c}\right) V(s) ds. \end{cases} \quad (1.9)$$

Define

$$\tilde{\ell}(\rho, c, \theta) := \frac{\beta \sigma U_0}{2(c\rho + \sigma)} \left(\frac{1}{c} \int_0^\infty J(s/c) e^{-\rho s} ds \right) [\cosh(\rho r \cos \theta) + \cosh(\rho \sin \theta)], \quad (1.10)$$

and let $\tilde{c}(\rho, \theta)$ satisfy $\tilde{\ell}(\rho, \tilde{c}(\rho, \theta), \theta) = 1$, and define

$$\tilde{c}^*(\theta) := \inf_{\rho > 0} \tilde{c}(\rho, \theta).$$

Then the existence of traveling waves satisfying (1.9) is shown.

Theorem 2 *Let $J_\infty := \int_0^\infty J(\xi) d\xi$ and assume $\beta J_\infty U_0 > 1$. Then for each $c > \tilde{c}^*(\theta)$, there exists a traveling wave solution to (1.8), satisfying (1.9) with (1.4).*

This theorem is similarly proved as done for (1.3). Although Theorem 1 appears to be a special case of Theorem 2 with $J(\xi) = e^{-\delta \xi}$, the Cauchy problem for (1.1) is different from that for (1.8). Therefore, we do not regard Theorem 1 as a special case of Theorem 2. As a matter of fact, as seen in §3, we reformulate the Cauchy problem for (1.1) as a single integral equation, which makes it possible to analyze the behavior of solutions to the Cauchy problem for (1.1), although this is not pursued in the present article.

1.4 Model with nonlocal infection contact

As a closely related model, one may consider the following system of integro-differential equations:

$$\begin{cases} T_t = -\beta T(x, t) \bar{I}(x, t), \\ E_t = \beta T(x, t) \bar{I}(x, t) - \sigma E(x, t), \\ I_t = \sigma E(x, t) - \delta I(x, t), \end{cases} \quad (x, t) \in \mathbb{R}^n \times (t_0, \infty), \quad (1.11)$$

where

$$\bar{I}(x, t) := \int_{\mathbb{R}^n} k(x - \xi) I(\xi, t) d\xi = \int_{\mathbb{R}^n} k(\xi) I(x - \xi, t) d\xi,$$

and the convolution kernel k is assumed to be radially symmetric, i.e., $k(x) = k(|x|)$. This system provides a natural continuous analogue by choosing a kernel $k(|x|)$ concentrated near the average cell-to-cell distance r , under the assumption that local viral transmission is isotropic across all directions.

This model is essentially same as the spatial SEIR model with nonlocal contamination by infection. Regarding the SIR model, there is a pioneer model by Kendall, taking into account the fact that susceptible individuals become infected through contact with infected individuals in their vicinity, and described spatial transmission by taking a spatial average around the infected

individuals ([15]). Then, the existence of traveling waves was also proved by Diekmann ([7]) for the one space-dimensional case ($n = 1$) through examining a nonlinear integral equation, called as the renewal equation, derived from the Kendall's SIR model. The readers refer to Ducasse ([8]) for the nice introduction to the renewal equation together with his significant contribution to the spatially inhomogeneous case.

Since the systems (1.1) and (1.11) involve the E variable, the results in [7] and [8] cannot be directly applied to our problem. However, inspired by [8], we successfully reformulate system (1.1) into a nonlinear integral equation. Applying a fixed-point argument for the monotone operator defined by this integral equation, following a similar approach to [7], we prove Theorem 1. Our approach can be extended to (1.11) for $k \in L^1(\mathbb{R})$ satisfying

$$\begin{cases} k(\xi) \geq 0, & k(-\xi) = k(\xi), & \int_{\mathbb{R}} k(\xi) d\xi = 1, \\ \exists \rho_* > 0 \text{ s.t. } & \int_{\mathbb{R}} k(\xi) e^{-\rho_* \xi} d\xi < \infty. \end{cases} \quad (1.12)$$

More precisely, defining

$$\hat{\ell}(\rho, c) := \frac{\beta \sigma U_0}{(c\rho + \delta)(c\rho + \sigma)} \int_{-\infty}^{\infty} k(\eta) e^{-\rho \eta} d\eta, \quad \rho \in [0, \rho_*], \quad c > 0, \quad (1.13)$$

we have $c = \hat{c}(\rho) > 0$ satisfying $\hat{\ell}(\rho, \hat{c}(\rho)) = 1$ ($0 < \rho \leq \rho_*$) and obtain the next result:

Theorem 3 *Assume (1.12) and $R_0 = \beta U_0 / \delta > 1$. Then (1.11) possesses a traveling wave solution $(T, E, I) = (U(z), V(z), W(z))$, $z = x + ct$, for each $c > \hat{c}^*$ satisfying (1.4), where $\hat{c}^* = \inf_{0 \leq \rho \leq \rho_*} \hat{c}(\rho)$ and $U(z)$ is monotone decreasing.*

As a related epidemic model, we mention the work of Tian and Yuan ([25]), who considered a spatially one-dimensional diffusive SEIR model where individuals diffuse randomly with nonlocal contamination as in (1.11), and proved the existence of traveling waves for their system. If all the diffusion coefficients vanish, their system formally reduces to (1.11) with $n = 1$. However, the presence of diffusion terms is essential to their argument, meaning their results cannot be directly applied to our problem. We also refer to [27], which studied the existence of traveling waves in a diffusive SEIR model with nonlocal delayed transmission and a general nonlinear incidence rate; there, too, the diffusion terms are indispensable for the proof.

Remark 3 Following the foundational work of Chen and Guo [3] on one-dimensional discrete monostable equations, traveling wave theory on lattices has been extensively developed, including extensions to time delays ([19]), multi-component interactions ([10]), and epidemic dynamics ([4]). However, most existing studies consider systems in which dispersal is governed by a standard discrete Laplacian and comparison arguments are readily applicable. In contrast, our system couples non-dispersing target cells through a planar contact operator $\mathcal{K}$ in a nonlinear mass-action form, resulting in a structurally distinct dispersion relation that precludes direct application of classical lattice frameworks.

1.5 Profile of the traveling waves

Although Theorem 1 guarantees the existence of the traveling waves with monotone $U(z)$ for $c > c^*(\theta)$, it does not provide a precise profile for $(V(z), W(z))$. In order to specify the profile of the traveling waves, instead of $\mathcal{K}_\theta$ we consider the one space-dimensional case, that is,

$$\mathcal{K}_1[W](z) := \frac{1}{2} [W(z+r) + W(z-r)]. \quad (1.14)$$

In classical lattice systems with discrete Laplacians, planar waves propagating along a principal axis (e.g., the x_1 -direction) naturally reduce to standard one-dimensional discrete diffusion equations, since the transverse differences vanish. In contrast, transmission in our setting is governed by the local average operator $\mathcal{K}$ defined in (1.2), and the resulting two-dimensional planar wave dynamics do not simply reduce to the one-dimensional nearest-neighbor contact model with $\mathcal{K}_1$ (e.g., compare $\mathcal{K}_\theta$ for $\theta = 0$).

Using the approximation

$$\mathcal{K}_1[W](z) = [W(z+r) + W(z-r) - 2W(z)]/2 + W(z) = \frac{r^2}{2} \left(\frac{d^2 W}{dz^2} + O(r^2) \right) + W(z),$$

we consider the approximate system given by

$$\begin{cases} c \frac{dU}{dz} = -\alpha U(z) \frac{d^2 W}{dz^2}(z) - \beta U(z)W(z), \\ c \frac{dV}{dz} = \alpha U(z) \frac{d^2 W}{dz^2}(z) + \beta U(z)W(z) - \sigma V(z), \\ c \frac{dW}{dz} = \sigma V(z) - \delta W(z), \end{cases} \quad (1.15)$$

where we put $\alpha = r^2 \beta / 2$. Regarding (1.15), we have the next result:

Theorem 4 *Assume $R_0 = \beta U_0 / \delta > 1$. Then the system of (1.15) with (1.4) has a positive solution provided that c satisfies*

$$c^2 - \alpha \sigma U_0 > 0, \quad c > 0, \quad (1.16)$$

with the profile that $U(z)$ is strictly monotone decreasing while $W(z)$ is unimodal, namely, it has a unique peak.

If we directly consider the dynamics of (1.15), we have to examine it on the 4-dimensional phase space. However, we find a reduction of (1.15) to a two component system of (U, W) and apply the phase plane analysis to examine the dynamics in the (U, W) -plane. As a result, for each c satisfying (1.16), we obtain a heteroclinic orbit corresponding to a traveling wave together with the profile of the solution as shown in Figure 5. However, it remains to prove how accurately the solutions obtained in Theorem 4 approximate the corresponding solutions to (1.3) with $\mathcal{K}_\theta = \mathcal{K}_1$.

1.6 Remark and summary of the rest of the article

We finally remark that the study for the diffusion processes in biology and epidemiology has a long history, for example to model plague epidemics in medieval Europe and the geographic spread of rabies in Europe and North America ([1, 20, 24]). Noble used a Kermack-McKendrick model ([16]) with a diffusion term added to each compartment to model the geographical spread of the plagues ([22]). Murray et al. constructed a model incorporating a diffusion term within the infected population to predict how rabies would spread if it entered the UK ([21]). As a mathematical aspect for such models, much attention has been paid to the existence of traveling waves since the work by Hosono and Ilyas ([12]). We emphasize that even if no direct diffusion appears as in the Kendall model or in our models, the propagation occurs and traveling waves emerge.

The rest of this article is organized as follows. In the next section we briefly examine the spatially homogeneous case in (1.11) and define the basic reproduction number R_0 . In §3, we first solve $\ell(\rho, c, \theta) = 1$ and prove (1.7). Then, inspired by the computation in the work [8] for the SIR model with spatial inhomogeneity, we convert the Cauchy problem of the system (1.1) into an integral equation. Through this formulation, we derive an integral equation for (1.3) with (1.4).

Following an argument similar to that in [7], we prove the existence of a monotone increasing solution to the integral equation by constructing a monotone sequence that converges to a fixed point. We then complete the proof of Theorem 1 in §3.4. We next handle (1.8) in §4.1. As in §3, we directly convert the system (1.9) with (1.4) into an integral equation and derive (1.10). Once the integral equation is obtained, the existence of its fixed point can be proved in a similar manner to §3.3; we leave the details to the readers. For the proof of Theorem 3, in §4.2, after converting the Cauchy problem for (1.11) into an integral equation as done in §3, we derive an integral equation for the system that determines the traveling wave profile for (1.11). As in the previous section, we omit the existence proof of a fixed point, as the argument in §3.3 applies analogously. In §5, we present numerical simulations for the system (1.1) and illustrate the propagation of localized pattern of the infected population. The numerical results explicitly capture the traveling wave profile guaranteed by our theoretical findings. Finally, in §6, we reduce (1.15) to a two-dimensional dynamical system and apply the phase plane analysis to obtain a heteroclinic orbit, which yields the desired solution stated in Theorem 4. The profile of the traveling wave corresponding to the heteroclinic orbit is depicted in Figure 5, showing good agreement with the numerical results.

2 Spatially uniform problem

Corresponding to (1.1), we have the spatially uniform system of equations

$$\begin{cases} dT/dt = -\beta T(t)I(t), \\ dE/dt = \beta T(t)I(t) - \sigma E(t), \\ dI/dt = \sigma E(t) - \delta I(t), \end{cases} \quad (2.1)$$

which has a family of nonnegative equilibria $(T, E, I) = (T_0, 0, 0)$ ($\forall T_0 \geq 0$). We neglected D -equation since the system of (T, E, I) is closed.

Linearize (2.1) at $(T(0), I(0), E(0)) = (T_0, 0, 0)$ as

$$\begin{cases} dT/dt = -\beta T_0 I(t), \\ dE/dt = \beta T_0 I(t) - \sigma E(t), \\ dI/dt = \sigma E(t) - \delta I(t). \end{cases}$$

It is easy to see that if $T_0 > \delta/\beta$, then $(E(t), I(t))$ has an exponentially growth solution given by

$$\begin{pmatrix} E(t) \\ I(t) \end{pmatrix} = c \begin{pmatrix} \lambda + \delta \\ \sigma \end{pmatrix} e^{\lambda t}, \quad \lambda := \frac{1}{2} \left(-(\delta + \sigma) + \sqrt{(\delta + \sigma)^2 - 4\sigma(\delta - \beta T_0)} \right) > 0.$$

Thus, the reproduction number is defined by $R_0 = \beta T_0/\delta$, and the infection develops if $R_0 > 1$.

On the other hand, adding the three equations in (2.1) yields $d(T + E + I)/dt = -\delta I$. Using this, we eliminate I of the equation $(1/T)(dT/dt) = -\beta I$, to obtain

$$\frac{d}{dt} \left(T + E + I - \frac{\delta}{\beta} \log T \right) = 0.$$

Thus, we have

$$T(t) + E(t) + I(t) - \frac{\delta}{\beta} \log T(t) = T(t_0) + E(t_0) + I(t_0) - \frac{\delta}{\beta} \log T(t_0).$$

By taking $t_0 \rightarrow -\infty$, the solution emanating from $(T_0, 0, 0)$ satisfies

$$T(t) + E(t) + I(t) - \frac{\delta}{\beta} \log T(t) = T_0 - \frac{\delta}{\beta} \log T_0 \quad (-\infty < t < \infty).$$

We note that if $R_0 > 1$, i.e., $T_0 > \delta/\beta$, then there is $0 < T_* < T_0$ such that

$$T_* - \frac{\delta}{\beta} \log T_* = T_0 - \frac{\delta}{\beta} \log T_0. \quad (2.2)$$

This is verified by examining the function $y = x - (\delta/\beta) \log x$, which has a minimum at $x = \delta/\beta$. As a consequence, if $R_0 > 1$, there exists a solution $(T(t), E(t), I(t))$ to (2.1) satisfying

$$\lim_{t \rightarrow -\infty} (T(t), E(t), I(t)) = (T_0, 0, 0), \quad \lim_{t \rightarrow \infty} (T(t), E(t), I(t)) = (T_*, 0, 0).$$

We remark that we just set $U_0 = T_0$ and $U_* = T_*$ in the assumption in (1.5).

3 Proof of Theorem 1

3.1 Function $c^*(\theta)$

We first prove that $c^*(\theta)$ in Theorem 1 is uniquely determined and (1.7) is satisfied. Put

$$p(\rho, \theta) := \cosh(\rho r \cos \theta) + \cosh(\rho r \sin \theta).$$

Since $p(\rho, \theta + \pi/2) = p(\rho, \theta)$ and $p(\rho, \pi/2 - \theta) = p(\rho, \theta)$, it suffices to consider the equation $\ell(\rho, c, \theta) = 1$ for $\theta \in [0, \pi/2]$. In view of

$$\ell(\rho, 0, \theta) = \frac{\beta U_0}{\delta} \frac{p(\rho, \theta)}{2} > \frac{\beta U_0}{\delta} > 1, \quad \lim_{c \rightarrow \infty} \ell(\rho, c, \theta) = 0 \quad \text{and} \quad \frac{\partial \ell}{\partial c} < 0 \quad (0 < \rho),$$

we see that there exists $c = c(\rho, \theta)$ ($\rho > 0, 0 \leq \theta < \pi/2$) satisfying $\ell(\rho, c(\rho, \theta), \theta) = 1$.

Lemma 5 *If $(\partial c / \partial \rho)(\rho_0, \theta) = 0$ ($\rho_0 > 0$), then $(\partial^2 c / \partial \rho^2)(\rho_0, \theta) > 0$ holds. Hence, there exists a unique $\rho = \rho_m(\theta) > 0$ satisfying $\frac{\partial c}{\partial \rho}(\rho_m(\theta), \theta) = 0$. Furthermore, for $c^*(\theta) := c(\rho_m(\theta), \theta)$, (1.7) holds.*

Proof A direct computation provides $c(\rho, \theta)$ explicitly as

$$c(\rho, \theta) = \frac{1}{2\rho} \left\{ -(\delta + \sigma) + \sqrt{(\delta + \sigma)^2 + 4\delta\sigma(R_0 p(\rho, \theta)/2 - 1)} \right\} \quad (R_0 = \beta U_0 / \delta). \quad (3.1)$$

Indeed, $\ell(\rho, c, \theta) = 1$ implies

$$(c\rho)^2 + (\delta + \sigma)(c\rho) + \delta\sigma = \frac{\beta\sigma U_0}{2} p(\rho, \theta) = \frac{\delta\sigma R_0}{2} p(\rho, \theta),$$

which yields (3.1).

We examine $p(\rho, \theta)$. In view of

$$\frac{\partial p}{\partial \theta} = -\rho r [-\sin \theta \sinh(\rho r \cos \theta) + \cos \theta \sinh(\rho r \sin \theta)],$$

$\partial p / \partial \theta = 0$ implies

$$\theta = 0, \pi/4 \pmod{\pi/2}, \quad (3.2)$$

which follows from that

$$\frac{\sinh(\rho r \cos \theta)}{\rho r \cos \theta} = \frac{\sinh(\rho r \sin \theta)}{\rho r \sin \theta}$$

has no solution except for (3.2). By a simple computation, we can check

$$p(\rho, 0) = \cosh(\rho r) + 1 > p(\rho, \pi/4) = 2 \cosh \frac{\rho r}{\sqrt{2}} \quad (\rho > 0)$$

holds. As a result, for any $\rho > 0$,

$$\frac{\partial p}{\partial \theta} \begin{cases} < 0 & (0 < \theta < \pi/4), \\ > 0 & (\pi/4 < \theta < \pi/2), \end{cases}$$

and

$$\max_{0 \leq \theta < \pi/2} p(\rho, \theta) = p(\rho, 0), \quad \min_{0 \leq \theta < \pi/2} p(\rho, \theta) = p(\rho, \pi/4)$$

follows.

In view of (3.1), we see that $c(\rho, \theta)$ is inherited these properties from $p(\rho, \theta)$. In fact, by

$$\frac{\partial c}{\partial \theta}(\rho, \theta) = \frac{1}{2\rho} \frac{\delta \sigma R_0 (\partial p / \partial \theta)}{\sqrt{(\delta + \sigma)^2 + 4\delta \sigma (R_0 p(\rho, \theta)/2 - 1)}},$$

we have

$$\frac{\partial c}{\partial \theta}(\rho, 0) = \frac{\partial c}{\partial \theta}(\rho, \pi/4) = 0, \quad \frac{\partial c}{\partial \theta}(\rho, \theta) \begin{cases} < 0, & \theta \in (0, \pi/4), \\ > 0, & \theta \in (\pi/4, \pi/2), \end{cases} \quad (3.3)$$

and

$$\begin{aligned} & \max_{0 \leq \theta < \pi/2} c(\rho, \theta) = c(\rho, 0) \\ &= \frac{1}{2\rho} \left\{ -(\delta + \sigma) + \sqrt{(\delta + \sigma)^2 + 4\delta \sigma \left[\frac{R_0 (\cosh(\rho r) + 1)}{2} - 1 \right]} \right\} \\ &> \min_{0 \leq \theta < \pi/2} c(\rho, \theta) = c(\rho, \pi/4) \\ &= \frac{1}{2\rho} \left\{ -(\delta + \sigma) + \sqrt{(\delta + \sigma)^2 + 4\delta \sigma \left[R_0 \cosh \frac{\rho r}{\sqrt{2}} - 1 \right]} \right\}. \end{aligned} \quad (3.4)$$

We compute $\partial p / \partial \rho$. Put

$$D_\theta(\rho) := (\delta + \sigma)^2 + 4\delta \sigma (R_0 p(\rho, \theta)/2 - 1), \quad (n_1, n_2) = (\cos \theta, \sin \theta),$$

and compute

$$\begin{aligned} \frac{\partial c}{\partial \rho}(\rho, \theta) &= -\frac{1}{2\rho^2} \{ -(\delta + \sigma) + \sqrt{D_\theta(\rho)} \} + \frac{1}{2\rho} \frac{4\delta \sigma R_0 r [n_1 \sinh(\rho r n_1) + n_2 \sinh(\rho r n_2)]/2}{2\sqrt{D_\theta(\rho)}} \\ &= \frac{1}{2\rho^2 \sqrt{D_\theta(\rho)}} \left[(\delta + \sigma) \sqrt{D_\theta(\rho)} - D_\theta(\rho) + R_0 \delta \sigma \rho r [n_1 \sinh(\rho n_1 r) + n_2 \sinh(\rho n_2 r)] \right]. \end{aligned}$$

Hence, $\partial c / \partial \rho = 0$ implies

$$Q_\theta(\rho) := (\delta + \sigma) \sqrt{D_\theta(\rho)} - D_\theta(\rho) + R_0 \delta \sigma \rho r [n_1 \sinh(\rho n_1 r) + n_2 \sinh(\rho n_2 r)] = 0.$$

We let $(\partial c / \partial \rho)(\rho_0, \theta) = 0$, i.e., $Q_\theta(\rho_0) = 0$. Then,

$$\frac{\partial^2 c}{\partial \rho^2}(\rho_0, \theta) = \frac{\partial}{\partial \rho} \left(\frac{1}{2\rho^2 \sqrt{D_\theta(\rho)}} Q_\theta(\rho) \right) \Big|_{\rho=\rho_0} = \frac{1}{2\rho_0^2 \sqrt{D_\theta(\rho_0)}} \frac{dQ_\theta}{d\rho}(\rho_0).$$

Compute

$$\begin{aligned} \frac{dQ_\theta}{d\rho} &= \frac{(\delta + \sigma)}{2\sqrt{D_\theta(\rho)}} \frac{dD_\theta}{d\rho} - \frac{dD_\theta}{d\rho} \\ &\quad + \delta \sigma R_0 r [n_1 \sinh(\rho n_1 r) + n_2 \sinh(\rho n_2 r) + \rho n_1^2 r \cosh(\rho n_1 r) + \rho n_2^2 r \cosh(\rho n_2 r)] \\ &= \frac{(\delta + \sigma)}{2\sqrt{D_\theta(\rho)}} \frac{dD_\theta}{d\rho} - 2\delta \sigma R_0 r [n_1 \sinh(\rho n_1 r) + n_2 \sinh(\rho n_2 r)] \end{aligned}$$

$$\begin{aligned}
& + \delta \sigma R_0 r [n_1 \sinh(\rho n_1 r) + n_2 \sinh(\rho n_2 r) + \rho n_1^2 r \cosh(\rho n_1 r) + \rho n_2^2 r \cosh(\rho n_2 r)] \\
& = \frac{(\delta + \sigma) 2 \delta \sigma R_0 r [n_1 \sinh(\rho n_1 r) + n_2 \sinh(\rho n_2 r)]}{2 \sqrt{D_\theta(\rho)}} \\
& + \delta \sigma R_0 r [n_1 \{\rho n_1 r \cosh(\rho n_1 r) - \sinh(\rho n_1 r)\} + n_2 \{\rho n_2 r \cosh(\rho n_2 r) - \sinh(\rho n_2 r)\}].
\end{aligned}$$

In view of $x \cosh(x) - \sinh(x) > 0$ ($x > 0$), we obtain $(dQ_\theta/d\rho) > 0$, namely,

$$(\partial^2 c)/(\partial \rho^2)(\rho_0, \theta) > 0.$$

Hence, by the implicit function theorem, there exists $\rho_m(\theta)$ satisfying

$$\frac{\partial c}{\partial \rho}(\rho_m(\theta), \theta) = 0,$$

and $c^*(\theta) = \min_{\rho > 0} c(\rho, \theta) = c(\rho_m(\theta), \theta)$. Then

$$\frac{dc^*}{d\theta} = \frac{\partial c}{\partial \rho}(\rho_m(\theta), \theta) \frac{d\rho_m}{d\theta} + \frac{\partial c}{\partial \theta}(\rho_m(\theta), \theta) = \frac{\partial c}{\partial \theta}(\rho_m(\theta), \theta).$$

In view of (3.3), we obtain (1.7). We conclude the proof of Lemma 5. $\square$

Remark 4 Although we have not discussed above how $c^*(\theta)$ depends on the parameter r , it is easy to see from (3.1) and $p(\theta, \rho)$ that $c(\rho, \theta)$ is monotone increasing in $r > 0$, and so is $c^*(\theta)$.

3.2 Derivation of an integral equation

We put

$$(T(x, t_0), E(x, t_0), I(x, t_0)) = (T_0(x), E_0(x), I_0(x)).$$

For $\mathcal{K}$ defined in (1.2), namely,

$$\mathcal{K}[f(\cdot)](x) = \frac{1}{4} [f(x_1 + r, x_2) + f(x_1 - r, x_2) + f(x_1, x_2 + r) + f(x_1, x_2 - r)] \quad (x = (x_1, x_2)).$$

Because of

$$\mathcal{K}[f_1](x) - \mathcal{K}[f_2](x) \geq 0 \quad (x \in \mathbb{R}) \quad \text{if} \quad f_1(x) \geq f_2(x) \quad (x \in \mathbb{R}),$$

$\mathcal{K}$ is a monotone operator. Furthermore,

$$\begin{aligned}
\int_a^b \mathcal{K}[g(\cdot, t)](x) dt &= \mathcal{K} \left[\int_a^b g(\cdot, t) dt \right] (x) \\
&= \int_a^b g(x_1 + r, x_2, t) dt + \int_a^b g(x_1 - r, x_2, t) dt \\
&\quad + \int_a^b g(x_1, x_2 + r, t) dt + \int_a^b g(x_1, x_2 - r, t) dt
\end{aligned}$$

holds.

We write (1.1) again, without D equation, as

$$\begin{cases} T_t(x, t) = -\beta T(x, t) \mathcal{K}[I(\cdot, t)](x), \\ E_t(x, t) = \beta T(x, t) \mathcal{K}[I(\cdot, t)](x) - \sigma E(x, t), \\ I_t(x, t) = \sigma E(x, t) - \delta I(x, t), \end{cases} \quad (x, t) \in \mathbb{R}^2 \times (t_0, \infty). \quad (3.5)$$

We convert (3.5) into an integral equation. Following [7], we put

$$\frac{T(x, t)}{T_0(x)} = e^{-\beta u(x, t)}, \quad \text{i.e.,} \quad u(x, t) = -\frac{1}{\beta} \log \frac{T(x, t)}{T_0(x)}. \quad (3.6)$$

Then, the first equation of (3.5) implies

$$u(x, t) = \int_{t_0}^t \mathcal{K}[I(\cdot, \tau)](x) d\tau. \quad (3.7)$$

From the third equation in (3.5),

$$I(x, t) = e^{-\delta(t-t_0)} I_0(x) + \sigma \int_0^{t-t_0} e^{-\delta s} E(x, t-s) ds,$$

hence,

$$\mathcal{K}[I(\cdot, \tau)](x) = e^{-\delta(\tau-t_0)} \mathcal{K}[I_0](x) + \sigma \int_0^{\tau-t_0} e^{-\delta s} \mathcal{K}[E(\cdot, \tau-s)](x) ds. \quad (3.8)$$

Plugging this into (3.7) yields

$$\begin{aligned} u(x, t) &= a(t) \mathcal{K}[I_0](x) + \sigma \int_{t_0}^t \left\{ \int_0^{\tau-t_0} e^{-\delta s} \mathcal{K}[E(\cdot, \tau-s)](x) ds \right\} d\tau \\ &= a(t) \mathcal{K}[I_0](x) + \sigma \int_0^{t-t_0} \left\{ \int_{t_0+s}^t e^{-\delta s} \mathcal{K}[E(\cdot, \tau-s)](x) d\tau \right\} ds \\ &= a(t) \mathcal{K}[I_0](x) + \sigma \int_0^{t-t_0} e^{-\delta s} \left\{ \int_{t_0}^{t-s} \mathcal{K}[E(\cdot, \eta)](x) d\eta \right\} ds, \end{aligned} \quad (3.9)$$

where we used Fubini's theorem and put

$$a(t) := \int_{t_0}^t e^{-\delta(s-t_0)} ds = \frac{1 - e^{-\delta(t-t_0)}}{\delta}. \quad (3.10)$$

Meanwhile, in view of $E_t + \sigma E = -T_t$, we have

$$\mathcal{K}[E(\cdot, t)](x) = e^{-\sigma(t-t_0)} \mathcal{K}[E_0](x) - \int_0^{t-t_0} e^{-\sigma\tau} \mathcal{K}[T_t(\cdot, t-\tau)](x) d\tau.$$

Inserting this into (3.9) yields

$$\begin{aligned} u(x, t) &= a(t) \mathcal{K}[I_0](x) + \sigma \int_0^{t-t_0} e^{-\delta s} \left(\int_{t_0}^{t-s} e^{-\sigma(\eta-t_0)} \mathcal{K}[E_0](x) d\eta \right) ds \\ &\quad - \sigma \int_0^{t-t_0} e^{-\delta s} \left\{ \int_{t_0}^{t-s} \left(\int_0^{\eta-t_0} e^{-\sigma\tau} \mathcal{K}[T_t(\cdot, \eta-\tau)](x) d\tau \right) d\eta \right\} ds \\ &= a(t) \mathcal{K}[I_0](x) + b(t) \mathcal{K}[E_0](x) \\ &\quad - \sigma \int_0^{t-t_0} e^{-\delta s} \left\{ \int_0^{t-s-t_0} \left(\int_{t_0+\tau}^{t-s} e^{-\sigma\tau} \mathcal{K}[T_t(\cdot, \eta-\tau)](x) d\eta \right) d\tau \right\} ds \\ &= a(t) \mathcal{K}[I_0](x) + b(t) \mathcal{K}[E_0](x) \\ &\quad - \sigma \int_0^{t-t_0} e^{-\delta s} \left\{ \int_0^{t-s-t_0} e^{-\sigma\tau} \mathcal{K}[T(\cdot, t-s-\tau) - T_0(\cdot)](x) d\tau \right\} ds \\ &= a(t) \mathcal{K}[I_0](x) + b(t) \mathcal{K}[E_0](x) \\ &\quad + \sigma \int_0^{t-t_0} e^{-\delta s} \left\{ \int_0^{t-s-t_0} e^{-\sigma\tau} \mathcal{K} \left[T_0(\cdot) \left(1 - e^{-\beta u(\cdot, t-s-\tau)} \right) \right] (x) d\tau \right\} ds, \end{aligned} \quad (3.11)$$

where $b(t)$ is defined by

$$\begin{aligned}
b(t) &:= \sigma \int_0^{t-t_0} e^{-\delta s} \left(\int_{t_0}^{t-s} e^{-\sigma(\eta-t_0)} d\eta \right) ds \\
&= \int_0^{t-t_0} e^{-\delta s} (1 - e^{-\sigma(t-s-t_0)}) ds \\
&= \begin{cases} \frac{1 - e^{-\delta(t-t_0)}}{\delta} + \frac{e^{-\sigma(t-t_0)} - e^{-\delta(t-t_0)}}{\sigma - \delta} & (\delta \neq \sigma), \\ \frac{1 - e^{-\delta(t-t_0)}}{\delta} - (t-t_0)e^{-\delta(t-t_0)} & (\delta = \sigma). \end{cases}
\end{aligned} \tag{3.12}$$

Consequently, solutions to (3.5) satisfy (3.11).

In reverse, by a solution $u(x, t)$, we can obtain a solution to (3.5) through (3.6). Indeed, since (3.11) allows the expression

$$\begin{aligned}
u(x, t) &= a(t)\mathcal{K}[I_0](x) + b(t)\mathcal{K}[E_0](x) \\
&\quad + \sigma \int_0^{t-t_0} e^{-\delta s} \left\{ \int_0^{t-s-t_0} e^{-\sigma(t-s-t_0-\tau)} \mathcal{K} \left[T_0(\cdot) \left(1 - e^{-\beta u(\cdot, \tau+t_0)} \right) \right] (x) d\tau \right\} ds,
\end{aligned}$$

once we obtain a continuous solution $u(x, t)$ to (3.11), we see that the solution is C^1 in t . Hence, we can trace the above computation process in reverse. In fact, define $T(x, t)$ by

$$T(x, t) = T_0(x) e^{-\beta u(x, t)},$$

and let $E(x, t)$ and $I(x, t)$ be the solutions to

$$E_t + \sigma E = -T_t, \quad E(x, t_0) = E_0(x),$$

and

$$I_t = -\delta I - T_t - E_t, \quad I(x, t_0) = I_0(x),$$

respectively. Then we obtain (3.9) from (3.11) and the first equation of (3.6) through (3.8). As a result, we obtain (3.5).

3.3 Traveling waves of the cell-to-cell transmission model

We convert the system (1.3) with (1.4) into an integral equation through (3.11).

Put

$$\phi(z) = -\frac{1}{\beta} \log \frac{U(z)}{U_0}, \quad \text{i.e.,} \quad \frac{U(z)}{U_0} = e^{-\beta \phi(z)}. \tag{3.13}$$

Then, the conditions (1.4) and (1.5) imply

$$\lim_{z \rightarrow -\infty} \phi(z) = 0, \quad \lim_{z \rightarrow \infty} \phi(z) = \phi^* := \frac{1}{\beta} \log \frac{U_0}{U_*} = \frac{1}{\delta} (U_0 - U_*). \tag{3.14}$$

Recall

$$\mathcal{K}_\theta[f](z) = \frac{1}{4} \left[f(z + n_1 r) + f(z - n_1 r) + f(z + n_2 r) + f(z - n_2 r) \right], \quad (n_1, n_2) = (\cos \theta, \sin \theta).$$

We now derive the integral equation for ϕ of (3.13) from (3.11). In view of

$$\lim_{t_0 \rightarrow -\infty} (T(x, t), E(x, t), I(x, t)) = (U_0, 0, 0), \quad \lim_{t_0 \rightarrow -\infty} a(t) = 1/\delta, \quad \lim_{t_0 \rightarrow -\infty} b(t) = 1/\delta,$$

and

$$\mathcal{K}[I_0](x) = 0, \quad \mathcal{K}[E_0](x) = 0, \quad T_0(x) = U_0,$$

as $t_0 \rightarrow -\infty$, (3.11) becomes

$$u(x, t) = \sigma \int_0^\infty e^{-\delta s} \left(\int_0^\infty e^{-\sigma \tau} \mathcal{K}[U_0(1 - e^{-\beta u(\cdot, t-s-\tau)})](x) d\tau \right) ds.$$

For the traveling wave solution, considering

$$\phi(z) = -\frac{1}{\beta} \log \frac{U(z)}{U_0} = -\frac{1}{\beta} \log \frac{T(\mathbf{n} \cdot x + ct)}{T_0} = u(\mathbf{n} \cdot x + ct), \quad z = \mathbf{n} \cdot x + ct,$$

and the translation invariance, $u(\mathbf{n} \cdot x + \xi, t) = u(\mathbf{n} \cdot x, t + \xi/c)$, we obtain the integral equation

$$\begin{aligned} \phi(z) &= \sigma \int_0^\infty \frac{1}{c} e^{-\frac{\delta}{c}s} \left(\int_0^\infty \frac{1}{c} e^{-\frac{\sigma}{c}\tau} \mathcal{K}[U_0(1 - e^{-\beta u(\cdot, t-s-\tau/c)})](x) d\tau \right) ds \\ &= \frac{\sigma}{c^2} \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[U_0(1 - e^{-\beta \phi(\cdot-s-\tau)})](z) d\tau \right) ds, \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} &\mathcal{K}_\theta[U_0(1 - e^{-\beta \phi(\cdot-s-\tau)})](z) \\ &= U_0 - \frac{U_0}{4} \left[e^{-\beta \phi(z+n_1 r-s-\tau)} + e^{-\beta \phi(z-n_1 r-s-\tau)} + e^{-\beta \phi(z+n_2 r-s-\tau)} + e^{-\beta \phi(z-n_2 r-s-\tau)} \right]. \end{aligned}$$

Define

$$H^c(s, \tau) := \frac{\delta \sigma}{c^2} e^{-\frac{\delta}{c}s} e^{-\frac{\sigma}{c}\tau}, \quad G(w) := \frac{1}{\delta} U_0 (1 - e^{-\beta w}), \quad (3.16)$$

and write (3.15) as

$$\phi(z) = \mathcal{F}_\theta[\phi](z) := \int_0^\infty \int_0^\infty H^c(s, \tau) \mathcal{K}_\theta[G(\phi(\cdot-s-\tau))](z) d\tau ds \quad (3.17)$$

$$= \int_0^\infty \int_{-\infty}^z H^c(s, z-\xi) \mathcal{K}_\theta[G(\phi(\cdot-s))](\xi) d\xi ds. \quad (3.18)$$

We see from (3.18) that if there is a continuous solution $\phi(z)$ satisfying (3.17), then it is of class C^1 . It is notified that

$$\begin{aligned} \int_0^\infty \int_0^\infty H^c(s, \tau) ds d\tau &= 1, \quad G(0) = 0, \quad G' = dG/dw > 0, \\ G'(0) &= \frac{\beta}{\delta} U_0 > 1, \quad G(\phi^*) = \frac{1}{\delta} U_0 (1 - U_*/U_0) = \phi^*. \end{aligned} \quad (3.19)$$

Furthermore, using

$$G(w) = G(w) - G(0) = \left(\int_0^1 G'(sw) ds \right) w,$$

$$\begin{aligned}
G(w) - G'(0)w &= \left(\int_0^1 G'(sw)ds - G'(0) \right) w \\
&= \int_0^1 \left(\int_0^1 s(dG/dw^2)(\tau sw) ds \right) d\tau \cdot w^2,
\end{aligned}$$

we have

$$\begin{aligned}
G(w) &> w \quad (0 < w < \phi^*), \\
G'(0)w - A_0w^2 &\leq G(w) \leq G'(0)w \quad (w \geq 0),
\end{aligned} \tag{3.20}$$

where $A_0 = \max_{w \geq 0} |G''(w)|/2 = U_0\beta^2/(2\delta)$.

From the above property, we easily see $\mathcal{F}_\theta[0] = 0$ and $\mathcal{F}_\theta[\phi^*] = \phi^*$. We can also check that $\mathcal{F}_\theta$ is a continuous map of

$$Y_0 := \{w \in C^0(\mathbb{R}) : w \geq 0, \lim_{z \rightarrow -\infty} w(z) = 0, \lim_{z \rightarrow \infty} w(z) = \phi^*\}$$

into itself with the uniform norm. Indeed,

$$\begin{aligned}
\lim_{z \rightarrow -\infty} \mathcal{K}_\theta[G(w(\cdot - s - \tau))](z) &= \mathcal{K}_\theta[G(0)](z) = 0, \\
\lim_{z \rightarrow \infty} \mathcal{K}_\theta[G(w(\cdot - s - \tau))](z) &= \mathcal{K}_\theta[G(\phi^*)](z) = \phi^*.
\end{aligned}$$

Furthermore, $\mathcal{K}_\theta$ is monotone operator satisfying $\mathcal{K}_\theta[w_1] \geq \mathcal{K}_\theta[w_2]$ ($w_1 \geq w_2$), and for an increasing function w , we have

$$\mathcal{K}_\theta[w](z_1) - \mathcal{K}_\theta[w](z_2) > 0 \quad (z_1 > z_2). \tag{3.21}$$

The linearized problem of (3.17) around $\phi = 0$ is given by

$$\psi(z) = \int_0^\infty \int_0^\infty H^c(s, \tau) G'(0) \mathcal{K}_\theta[\psi(\cdot - s - \tau)](z) d\tau ds. \tag{3.22}$$

We look for a solution $\psi = e^{\rho z}$ to (3.22). Plugging it into (3.22) yields

$$e^{\rho z} = \int_0^\infty \int_0^\infty H^c(s, \tau) G'(0) \mathcal{K}_\theta[e^{\rho(\cdot - s - \tau)}](z) d\tau ds,$$

hence, in view of

$$\begin{aligned}
\mathcal{K}_\theta[e^{\rho(\cdot - s - \tau)}](z) &= \frac{1}{4} [e^{\rho(z+n_1r-s-\tau)} + e^{\rho(z-n_1r-s-\tau)} + e^{\rho(z+n_2r-s-\tau)} + e^{\rho(z-n_2r-s-\tau)}] \\
&= e^{\rho(z-s-\tau)} \frac{\cosh(\rho n_1) + \cosh(\rho n_2)}{2},
\end{aligned}$$

we have

$$\begin{aligned}
1 &= G'(0) \left(\int_0^\infty \int_0^\infty \frac{\delta\sigma}{c^2} e^{-\frac{\delta}{c}s} e^{-\frac{\sigma}{c}\tau} e^{-\rho(s+\tau)} d\tau ds \right) \frac{1}{2} [\cosh(\rho n_1) + \cosh(\rho n_2)] \\
&= \frac{\beta}{\delta} U_0 \cdot \frac{\delta\sigma}{c^2} \left(\rho + \frac{\delta}{c} \right)^{-1} \left(\rho + \frac{\sigma}{c} \right)^{-1} \frac{1}{2} [\cosh(\rho n_1) + \cosh(\rho n_2)] \\
&= \frac{\beta\sigma U_0}{2(c\rho + \delta)(c\rho + \sigma)} [\cosh(\rho n_1) + \cosh(\rho n_2)] = \ell(\rho, c, \theta),
\end{aligned}$$

which provides (1.6).

Theorem 6 Assume $\beta U_0/\delta > 1$. If there exists $\rho_0 > 0$ satisfying $\ell(\rho_0, c, \theta) < 1$ for a positive c , then there exists a fixed-point $\tilde{\phi}(z)$ of $\mathcal{F}_\theta$ defined in (3.17) and $\tilde{\phi}(z)$ is a monotone increasing function satisfying $\tilde{\phi}(-\infty) = 0, \tilde{\phi}(\infty) = \phi^*$.

Proof We arrange the argument as in the proof of [7, Theorem 6.1]. For simplicity of notation, throughout the proof we omit c and θ in $\ell(\rho, c, \theta)$, namely, simply write $\ell(\rho)$. Since $\ell(0) = \beta U_0/\delta > 0$, there is $\rho_1 \in (0, \rho_0)$ such that

$$\ell(\rho_1) = 1 \quad \text{and} \quad \ell(\rho) < 1 \quad (\rho_1 < \rho \leq \rho_0).$$

Define

$$\bar{\psi}(z) = \begin{cases} \phi^* e^{\rho_1 z}, & -\infty < z < 0, \\ \phi^*, & 0 \leq z < \infty, \end{cases}, \quad \underline{\psi}(z) = \max\{\phi^* e^{\rho_1 z} - B_1 e^{\rho_2 z}, 0\}, \quad (3.23)$$

where we take

$$B_1 > \phi^*, \quad 0 < \rho_2 \leq \rho_1 + \min\{\rho_1, \rho_0 - \rho_1\}, \quad (3.24)$$

so that $\phi^* e^{\rho_1 z} - B_1 e^{\rho_2 z}$ vanishes on $(-\infty, 0)$ and $\ell(\rho_2) < 1$. Then we easily see $\underline{\psi}(z) < \bar{\psi}(z)$ $(-\infty < z < \infty)$.

By $\bar{\psi}(z) \leq \phi^*$ $(-\infty < z < \infty)$,

$$\begin{aligned} \mathcal{F}_\theta[\bar{\psi}](z) &= \int_0^\infty \int_0^\infty H^c(s, \tau) \mathcal{K}_\theta[G(\bar{\psi}(\cdot - s - \tau))](z) \, d\tau ds \\ &\leq \int_0^\infty \int_0^\infty H^c(s, \tau) \mathcal{K}_\theta[G(\phi^*)](z) \, d\tau ds = \phi^*. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{F}_\theta[\bar{\psi}](z) &= \int_0^\infty \int_0^\infty H^c(s, \tau) \mathcal{K}_\theta[G'(0)\bar{\psi}(\cdot - s - \tau)](z) \, d\tau ds \\ &\leq \int_0^\infty \int_0^\infty H^c(s, \tau) G'(0) \mathcal{K}_\theta[\phi^* e^{\rho_1(\cdot - s - \tau)}](z) \, d\tau ds \\ &= \int_0^\infty \int_0^\infty H^c(s, \tau) e^{-\rho_1(s+\tau)} G'(0) \frac{1}{2} [\cosh(\rho_1 r n_1) + \cosh(\rho_2 r n_2)] \phi^* e^{\rho_1 z} \\ &= \ell(\rho_1) \phi^* e^{\rho_1 z} = \phi^* e^{\rho_1 z}. \end{aligned}$$

Hence, $\mathcal{F}_\theta[\bar{\psi}](z) \leq \bar{\psi}(z)$. By the monotonicity of $G(w)$ and (3.21), we easily verify that $\mathcal{F}_\theta[\bar{\psi}](z)$ is monotone increasing and satisfies

$$\lim_{z \rightarrow -\infty} \mathcal{F}_\theta[\bar{\psi}](z) = 0, \quad \lim_{z \rightarrow \infty} \mathcal{F}_\theta[\bar{\psi}](z) = \phi^*.$$

We inductively define a sequence $\{\psi_j\}_{j=0,1,\dots}$ as

$$\psi_0 = \bar{\psi}, \quad \psi_{j+1} = \mathcal{F}_\theta[\psi_j] \quad (j = 0, 1, \dots).$$

Since $\mathcal{F}_\theta$ is monotone operator satisfying $\mathcal{F}_\theta[w_2] \leq \mathcal{F}_\theta[w_1]$ $(w_2 \leq w_1)$, we have $\psi_0 \geq \psi_1 \geq \dots \geq \psi_j \geq \psi_{j+1}, \dots$. We can also show by induction that each ψ_j is monotone increasing and satisfies

$$\psi_j(z) > 0, \quad \lim_{z \rightarrow -\infty} \psi_j(z) = 0, \quad \lim_{z \rightarrow \infty} \psi_j(z) = \phi^*.$$

We show $\mathcal{F}_\theta[\underline{\psi}] \geq \underline{\psi}$ for suitable $\theta > 0$ and B_1 . In view of $\rho_2 \in (\rho_1, 2\rho_1]$,

$$(\underline{\psi}(z))^2 \leq (\bar{\psi}(z))^2 = \begin{cases} (\phi^*)^2 e^{2\rho_1 z} & (-\infty < z < 0) \\ (\phi^*)^2 & (0 \leq z < \infty) \end{cases} \leq (\phi^*)^2 e^{\rho_2 z}. \quad (3.25)$$

In addition to $\mathcal{F}_\theta[\underline{\psi}](z) \geq 0$, by (3.20) and (3.25),

$$\mathcal{F}_\theta[\underline{\psi}](z) \geq \int_0^\infty \int_0^\infty H^c(s, \tau) \mathcal{K}_\theta[G'(0)\underline{\psi}(\cdot - s - \tau) - A_0(\underline{\psi}(\cdot - s - \tau))^2](z) \, d\tau ds$$

$$\begin{aligned}
&\geq \int_0^\infty \int_0^\infty H^c(s, \tau) G'(0) \mathcal{K}_\theta[\phi^* e^{\rho_1(\cdot-s-\tau)} - B_1 e^{\rho_2(\cdot-s-\tau)}](z) d\tau ds \\
&\quad - \int_0^\infty \int_0^\infty H^c(s, \tau) A_0 \mathcal{K}_\theta[(\phi^*)^2 e^{\rho_2(\cdot-s-\tau)}](z) d\tau ds \\
&= \ell(\rho_1) \phi^* e^{\rho_1 z} - \ell(\rho_2) B_1 e^{\rho_2 z} - \frac{A_0 \ell(\rho_2)}{G'(0)} (\phi^*)^2 e^{\rho_2 z} \\
&= \phi^* e^{\rho_1 z} - B_1 e^{\rho_2 z} + B_1 e^{\rho_2 z} \left\{ 1 - \ell(\rho_2) - \frac{A_0 (\phi^*)^2}{B_1 G'(0)} \ell(\rho_2) \right\} \\
&\geq \phi^* e^{\rho_1 z} - B_1 e^{\rho_2 z} + B_1 e^{\rho_2 z} \left\{ 1 - \ell(\rho_2) - \frac{A_0 (\phi^*)^2}{B_1 G'(0)} \right\}.
\end{aligned}$$

Recall $\ell(\rho_2) < 1$. In view of (3.24), if B_1 satisfies

$$B_1 > \frac{A_0 (\phi^*)^2}{G'(0)(1 - \ell(\rho_2))},$$

then $\mathcal{F}_\theta[\underline{\psi}](z) \geq \underline{\psi}(z)$ holds.

Since $\psi_0 = \bar{\psi}(z) \geq \underline{\psi}(z)$, the decreasing sequence satisfies $\psi_j(z) \geq \underline{\psi}(z)$ ($\forall j \geq 1$). Considering (3.18), we easily check the equi-continuity of $\{\psi_j\}_{j \geq 1}$, we apply the Arzera-Ascoli theorem to obtain a function $\tilde{\phi}(z)$ to which $\psi_j(\cdot)$ locally uniformly converges. As a result, $\mathcal{F}_\theta[\tilde{\phi}](z) = \tilde{\phi}(z) \geq \underline{\psi}(z)$. Furthermore, since $\{\psi_j\}_{j \geq 1}$ is a family of monotone increasing functions, $\tilde{\phi}(z)$ is also monotone increasing. Hence, $\tilde{\phi}(\cdot)$ is the desired fixed-point. $\square$

3.4 Completion of the proof of Theorem 1

Proof of Theorem 1 Let $\tilde{\phi}(z)$ be the fixed-point obtained in Theorem 6. We verify that $\tilde{\phi}(z)$ provides a solution to (1.3). It is notified that the fixed-point $\tilde{\phi}(z)$ is of class C^1 and $\tilde{\phi}'(z) \geq 0$. Let

$$\begin{aligned}
U(z) &= U_0 e^{-\beta \tilde{\phi}(z)}, \\
V(z) &= - \int_{-\infty}^z e^{-\frac{\sigma}{c}(z-\tau)} U'(\tau) d\tau = - \int_0^\infty e^{-\frac{\sigma}{c}\tau} U'(z-\tau) d\tau, \\
W(z) &= - \int_{-\infty}^z e^{-\frac{\delta}{c}(z-\tau)} [U'(\tau) + V'(\tau)] d\tau = - \int_0^\infty e^{-\frac{\delta}{c}s} [U'(z-s) + V'(z-s)] ds.
\end{aligned}$$

Then, since $V'(z) = -U'(z) - \frac{\sigma}{c}V(z)$,

$$W(z) = \frac{\sigma}{c} \int_0^\infty e^{-\frac{\delta}{c}s} V(z-s) ds.$$

We show that $(U(z), V(z), W(z))$ satisfies (1.3).

In view of (3.15),

$$\begin{aligned}
\tilde{\phi}(z) &= \frac{\sigma}{c^2} \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[U_0 - U_0 e^{-\beta \tilde{\phi}(\cdot-s-\tau)}](z) d\tau \right) ds \\
&= \frac{\sigma}{c^2} \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[U_0 - U(\cdot-s-\tau)](z) d\tau \right) ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{\phi}'(z) &= \frac{\sigma}{c^2} \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[-U'(\cdot-s-\tau)](z) d\tau \right) ds \\
&= \frac{\sigma}{c^2} \int_0^\infty e^{-\frac{\delta}{c}s} \mathcal{K}_\theta[V(\cdot-s)](z) ds \\
&= \frac{1}{c} \mathcal{K}_\theta[W](z).
\end{aligned}$$

This implies that the above defined $(U(z), V(z), W(z))$ provides a solution to (1.3) since

$$U'(z) = -\beta U(z) \tilde{\phi}'(z) = -\frac{\beta}{c} U(z) \mathcal{K}_\theta[W](z).$$

Meanwhile, it is easy to see from (1.6) and the definition of $c^*(\theta)$ that for $c > c^*(\theta)$, there exists ρ_0 satisfying $\ell(\rho_0, c, \theta) < 1$. Consequently, Theorem 6 leads us to the assertion of Theorem 1. $\square$

4 Proof of Theorems 2 and 3

4.1 Proof of Theorem 2

Proof of Theorem 2 We convert the system (1.9) with (1.4) into an integral equation. Consider the system (1.9). Put

$$\phi(z) = -\frac{1}{\beta} \log \left(\frac{U(z)}{U_0} \right), \quad \text{i.e.,} \quad U(z) = U_0 e^{-\beta \phi(z)}.$$

Then $\phi(-\infty) = 0, \phi(\infty) = \phi^*$ and

$$\phi_z = -\frac{1}{\beta} \frac{U_z}{U}, \quad \phi(z) = \frac{1}{c} \int_{-\infty}^z \mathcal{K}_\theta[W(\cdot)](s) ds.$$

Meanwhile, $c(U_z + V_z) = -\sigma V$ implies

$$V(z) = \int_{-\infty}^z e^{-\frac{\sigma}{c}(z-s)} [-U_z(s)] ds = \int_0^\infty e^{-\frac{\sigma}{c}\tau} [-U_z(z-\tau)] d\tau. \quad (4.1)$$

From the third equation of (1.9), we see

$$\mathcal{K}_\theta[W(\cdot)](z) = \frac{\sigma}{c} \int_0^\infty J\left(\frac{\eta}{c}\right) \mathcal{K}_\theta[V(\cdot - \eta)](z) d\eta,$$

thereby,

$$\begin{aligned} \phi(z) &= \frac{\sigma}{c^2} \int_{-\infty}^z \left\{ \int_0^\infty J\left(\frac{\eta}{c}\right) \mathcal{K}_\theta[V(\cdot - \eta)](s) d\eta \right\} ds \\ &= \frac{\sigma}{c^2} \int_0^\infty J\left(\frac{\eta}{c}\right) \left\{ \int_{-\infty}^z \mathcal{K}_\theta[V(\cdot - \eta)](s) ds \right\} d\eta. \end{aligned}$$

Using (4.1), we compute

$$\begin{aligned} \int_{-\infty}^z \mathcal{K}_\theta[V(\cdot - \eta)](s) ds &= \int_{-\infty}^z \left\{ \int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[-U_z(\cdot - \eta - \tau)](s) d\tau \right\} ds \\ &= \int_0^\infty e^{-\frac{\sigma}{c}\tau} \left\{ \int_{-\infty}^z \mathcal{K}_\theta[-U_z(\cdot - \eta - \tau)](s) ds \right\} d\tau \\ &= \int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[U_0 - U(\cdot - \eta - \tau)](z) d\tau, \end{aligned}$$

by which

$$\phi(z) = \frac{\sigma}{c^2} \int_0^\infty J\left(\frac{\eta}{c}\right) \left\{ \int_0^\infty e^{-\frac{\sigma}{c}\tau} \mathcal{K}_\theta[U_0 - U(\cdot - \eta - \tau)](z) d\tau \right\} d\eta. \quad (4.2)$$

Put

$$\tilde{H}^c(s, \tau) = \frac{\sigma}{c^2 J_\infty} J\left(\frac{s}{c}\right) e^{-\frac{\sigma}{c}\tau}, \quad \tilde{G}(w) := J_\infty U_0 (1 - e^{-\beta w}).$$

Then (4.2) is written as

$$\phi(z) = \tilde{\mathcal{F}}[\phi] := \int_0^\infty \int_0^\infty \tilde{H}^c(s, \tau) \mathcal{K}_\theta[\tilde{G}(\phi(\cdot - s - \tau))](z) d\tau ds. \quad (4.3)$$

Plug $\phi = e^{\rho z}$ into the linearized problem around $\phi = 0$ for (4.3) and we obtain

$$\begin{aligned} e^{\rho z} &= \int_0^\infty \int_0^\infty \tilde{H}^c(s, \tau) \tilde{G}'(0) \mathcal{K}_\theta[e^{\rho(\cdot - s - \tau)}](z) d\tau ds \\ &= \beta J_\infty U_0 \left(\int_0^\infty \int_0^\infty \tilde{H}^c(s, \tau) e^{-\rho(s+\tau)} d\tau ds \right) \frac{1}{2} [\cosh(\rho n_1 r) + \cosh(\rho n_2 r)] e^{\rho z} \\ &= \frac{\beta \sigma U_0}{c^2} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} e^{-\rho\tau} d\tau \right) \left(\int_0^\infty J\left(\frac{s}{c}\right) e^{-\rho s} ds \right) \frac{1}{2} [\cosh(\rho n_1 r) + \cosh(\rho n_2 r)] e^{\rho z} \\ &= \frac{\beta \sigma U_0}{2(c\rho + \sigma)} \left(\frac{1}{c} \int_0^\infty J\left(\frac{s}{c}\right) e^{-\rho s} ds \right) [\cosh(\rho n_1 r) + \cosh(\rho n_2 r)] e^{\rho z} = \tilde{\ell}(\rho, c, \theta) e^{\rho z}. \end{aligned}$$

Once we obtain this equality, we can apply the same argument as in §3 and obtain the desired assertion of Theorem 2. Since the proof is very similar to that for Theorem 1, we omit it. $\square$

4.2 Proof of Theorem 3

Proof of Theorem 3 The key point for the proof is a derivation of a similar integral equation as in the previous sections. We only sketch it to avoid repeating similar computations.

Throughout this section, we assume (1.12). We put

$$(T(x, t_0), E(x, t_0), I(x, t_0)) = (T_0(x), E_0(x), I_0(x)).$$

From the equation $\frac{\partial}{\partial t} \log T(x, t) = -\beta \bar{I}(x, t)$, we define

$$u(x, t) := \int_{t_0}^t \bar{I}(x, s) ds, \quad \frac{T(x, t)}{T_0(x)} = e^{-\beta u(x, t)}. \quad (4.4)$$

In a similar way for deriving (3.9), we have

$$u(x, t) = a(t) \bar{I}_0(x) + \sigma \int_{\mathbb{R}} k(x - y) \left(\int_0^{t-t_0} e^{-\delta s} \left\{ \int_{t_0}^{t-s} E(y, \eta) d\eta \right\} ds \right) dy, \quad (4.5)$$

where $a(t)$ is defined in (3.10). Using

$$E(x, t) = e^{-\sigma(t-t_0)} E_0(x) - \int_0^{t-t_0} e^{-\sigma\tau} T_t(x, t - \tau) d\tau,$$

we have

$$\begin{aligned} \int_{t_0}^{t-s} E(y, \eta) d\eta &= \int_{t_0}^{t-s} e^{-\sigma(\eta-t_0)} E_0(y) d\eta - \int_{t_0}^{t-s} \left(\int_0^{\eta-t_0} e^{-\sigma\tau} T_t(y, \eta - \tau) d\tau \right) d\eta \\ &= \frac{1 - e^{-\sigma(t-s-t_0)}}{\sigma} E_0(y) - \int_0^{t-s-t_0} \left(\int_{t_0+\tau}^{t-s} e^{-\sigma\tau} T_t(y, \eta - \tau) d\eta \right) d\tau \\ &= \frac{1 - e^{-\sigma(t-s-t_0)}}{\sigma} E_0(y) - \int_0^{t-s-t_0} e^{-\sigma\tau} [T(y, t-s-\tau) - T_0(y)] d\tau. \end{aligned}$$

Substitute this into (4.5) and we have

$$\begin{aligned} u(x, t) &= a(t) \bar{I}_0(x) + b(t) \bar{E}_0(x) \\ &\quad - \sigma \int_{\mathbb{R}} k(x - y) \left\{ \int_0^{t-t_0} e^{-\delta s} \left(\int_0^{t-s-t_0} e^{-\sigma\tau} [T(y, t-s-\tau) - T_0(y)] d\tau \right) ds \right\} \\ &= a(t) \bar{I}_0(x) + b(t) \bar{E}_0(x) \\ &\quad + \sigma \int_{\mathbb{R}} k(x - y) T_0(y) \left\{ \int_0^{t-t_0} e^{-\delta s} \left(\int_0^{t-s-t_0} e^{-\sigma\tau} [1 - e^{-\beta u(y, t-s-\tau)}] d\tau \right) ds \right\} dy, \end{aligned} \quad (4.6)$$

where $b(t)$ is as in (3.12).

We drive an integral equation for $(U(z), V(z), W(z)) = (T(x + ct), E(x + ct), I(x + ct))$, $z = x + ct$, satisfying

$$\begin{cases} cU' = -\beta U(z) \bar{W}(z), \\ cV' = \beta U(z) \bar{W}(z) - \sigma V(z), \\ cW' = \sigma V(z) - \delta W(z), \end{cases} \quad z \in \mathbb{R}, \quad (4.7)$$

with (1.4), where $' = d/dz$ and $\bar{W}(z) = \int_{\mathbb{R}} k(z - y) W(y) dy$. Put

$$\phi(z) = -\frac{1}{\beta} \log \frac{U(z)}{U_0}, \quad \text{i.e.,} \quad \frac{U(z)}{U_0} = e^{-\beta \phi(z)}.$$

Then, ϕ satisfies (3.14). It is easy to check that as $t \rightarrow -\infty$, (4.6) becomes

$$u(x, t) = \sigma \int_{\mathbb{R}} k(x - y) U_0 \left\{ \int_0^\infty e^{-\delta s} \left(\int_0^\infty e^{-\sigma\tau} [1 - e^{-\beta u(y, t-s-\tau)}] d\tau \right) ds \right\} dy.$$

Assuming the translation invariance, $u(x + x', t) = u(x, t + x'/c)$, we have

$$\begin{aligned} u(x, t) &= \sigma U_0 \int_{\mathbb{R}} k(x - y) \left\{ \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} [1 - e^{-\beta u(y, t-s/c-\tau/c)}] d(\tau/c) \right) d(s/c) \right\} dy \\ &= \frac{\sigma}{c^2} U_0 \int_{\mathbb{R}} k(x - y) \left\{ \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} [1 - e^{-\beta u(y, t-s/c-\tau/c)}] d\tau \right) ds \right\} dy. \end{aligned}$$

Thus, for

$$\phi(z) = -\frac{1}{\beta} \log \frac{U(z)}{U_0} = -\frac{1}{\beta} \log \frac{T(x+ct)}{T_0} = u(x+ct),$$

we have

$$\begin{aligned} \phi(z) &= \frac{\sigma U_0}{c^2} \int_{\mathbb{R}} k(z - (y+ct)) \left\{ \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} [1 - e^{-\beta u(y-s-\tau,t)}] d\tau \right) ds \right\} dy \\ &= \frac{\sigma U_0}{c^2} \int_{\mathbb{R}} k(z - (y+ct)) \left\{ \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} [1 - e^{-\beta \phi(y+ct-s-\tau)}] d\tau \right) ds \right\} dy \\ &= \frac{\sigma U_0}{c^2} \int_{\mathbb{R}} k(z - \eta) \left\{ \int_0^\infty e^{-\frac{\delta}{c}s} \left(\int_0^\infty e^{-\frac{\sigma}{c}\tau} [1 - e^{-\beta \phi(\eta-s-\tau)}] d\tau \right) ds \right\} d\eta \\ &= \int_{\mathbb{R}} \frac{U_0}{\delta} [1 - e^{-\beta \phi(\eta)}] \left\{ \frac{\delta \sigma}{c^2} \int_0^\infty \int_0^\infty e^{-\frac{\delta}{c}s} e^{-\frac{\sigma}{c}\tau} k(z - \eta - s - \tau) d\tau ds \right\} d\eta. \end{aligned} \quad (4.8)$$

Let $H^c(s, \tau)$ and $G(w)$ as in (3.16) and put

$$J^c(\xi) := \int_0^\infty \int_0^\infty H^c(s, \tau) k(\xi - s - \tau) ds d\tau.$$

Then we can write (4.8) as

$$\phi(z) = \hat{\mathcal{F}}[\phi](z) := \int_{-\infty}^\infty G(\phi(\eta)) J^c(z - \eta) d\eta. \quad (4.9)$$

Hence, the existence of a traveling wave solution to (4.7) is reduced to finding a fixed-point of $\hat{\mathcal{F}}$. It is notified that H^c and G satisfy (3.19). Moreover, (3.20) holds for G .

Linearized problem for (4.9) around $\phi = 0$ is given by

$$\psi(z) = G'(0) \int_{-\infty}^\infty \psi(\eta) J^c(z - \eta) d\eta. \quad (4.10)$$

Substituting $\psi = e^{\rho z}$ into (4.10) yields

$$e^{\rho z} = G'(0) \int_{-\infty}^\infty e^{\rho \eta} J^c(z - \eta) d\eta = G'(0) \int_{-\infty}^\infty e^{\rho(z-\xi)} J^c(\xi) d\xi, \quad (4.11)$$

thus,

$$\begin{aligned} 1 &= G'(0) \int_{-\infty}^\infty e^{-\rho \xi} J^c(\xi) d\xi \\ &= G'(0) \int_0^\infty \int_0^\infty H^c(s, \tau) ds d\tau \int_{-\infty}^\infty e^{-\rho \xi} k(\xi - s - \tau) d\xi \\ &= G'(0) \frac{\delta \sigma}{c^2} \left(\rho + \frac{\delta}{c} \right)^{-1} \left(\rho + \frac{\sigma}{c} \right)^{-1} \int_{-\infty}^\infty k(\eta) e^{-\rho \eta} d\eta \\ &= \frac{\beta \sigma U_0}{(c\rho + \delta)(c\rho + \sigma)} \int_{-\infty}^\infty k(\eta) e^{-\rho \eta} d\eta = \hat{\ell}(\rho, c) \end{aligned} \quad (4.12)$$

(recall (1.13)). It is easy to verify that given $\rho \in (0, \rho_*]$, there exists $c = \hat{c}(\rho)$ satisfying $\hat{\ell}(\rho, \hat{c}(\rho)) = 1$.

As a result, we obtain the following result:

Theorem 7 *Assume $\beta U_0/\delta > 1$. If there exists $\rho_0 \in (0, \rho_*]$ satisfying $\hat{\ell}(\rho_0, c) < 1$ for a positive c , then there exists a fixed-point $\hat{\phi}(z)$ of $\mathcal{F}$ defined in (4.9) and $\hat{\phi}(z)$ is a monotone increasing function satisfying $\hat{\phi}(-\infty) = 0, \hat{\phi}(\infty) = \phi^*$.*

Since the poof can be carried out in a similar way as done for Theorem 6, we omit it.

Finally, the proof of Theorem 3 is completed by applying Theorem 7 and modifying the argument in §3.4. We leave the details to the readers. $\square$

5 Numerical simulations

Numerical simulation of the one-space dimensional discrete system, namely, replacing $\mathcal{K}$ of (1.1) by $\mathcal{K}_1$ defined in (1.14), calculates the values in a finite number of nodes. As mentioned before, the discrete system can be regarded as a lattice dynamical system. Firstly, we introduce a simple spatial discretization on an interval $[x_{-m}, x_m]$. The interval is then divided into $2m+1$ equidistant subintervals with a grid size $r = (x_{i+1} - x_i)/(2m+1)$. The end points of each subinterval give nodes of the lattice. We assume that the infected cells are located in the middle of this interval, i.e., at $x = x_0$ and the infection spreads both left and right from x_0 . It is assumed that the target cells are located all over the domain initially and the corresponding initial conditions are given as:

$$\begin{aligned} T(x_i, 0) &= T_0, & \text{for } i &= -m, -m+1, \dots, m-1, m, \\ E(x_i, 0) &= 0, & \text{for } i &= -m, -m+1, \dots, m-1, m, \\ I(x_i, 0) &= \begin{cases} 0, & \text{for } i = -m, -m+1, \dots, -1, 1, \dots, m-1, m, \\ I_0, & \text{for } i = 0. \end{cases} \end{aligned} \quad (5.1)$$

In addition, the Neumann boundary condition is employed, namely, the spatially discrete derivatives at $x = x_{-m}$ and $x = x_m$ are 0. Denote $T_i(t) = T(x_i, t)$, $E_i(t) = E(x_i, t)$ and $I_i(t) = I(x_i, t)$. Then (1.1) can be written in the following form:

$$\begin{aligned} \frac{dT_i}{dt} &= -\frac{\beta}{2}T_i(t)[I_{i+1}(t) + I_{i-1}(t) - 2I_i(t)] - \beta T_i(t)I_i(t), \\ \frac{dE_i}{dt} &= \frac{\beta}{2}T_i(t)[I_{i+1}(t) + I_{i-1}(t) - 2I_i(t)] + \beta T_i(t)I_i(t) - \sigma E_i(t), \\ \frac{dI_i}{dt} &= \sigma E_i(t) - \delta I_i(t), \end{aligned} \quad (5.2)$$

for $i = -m+1, -m+2, \dots, m-2, m-1$.

Now we discretize a time interval $[t_0, t_n]$ in the same manner. The time interval is divided into n equidistant subintervals with a step size $\Delta t = (t_n - t_0)/n$. Applying the explicit Euler scheme to (5.2) gives

$$\begin{aligned} T_{i,j+1} &= T_{i,j} - \Delta t \left(\frac{\beta}{2}T_{i,j}[I_{i+1,j} + I_{i-1,j} - 2I_{i,j}] + \beta T_{i,j}I_{i,j} \right), \\ E_{i,j+1} &= E_{i,j} + \Delta t \left(\frac{\beta}{2}T_{i,j}[I_{i+1,j} + I_{i-1,j} - 2I_{i,j}] + \beta T_{i,j}I_{i,j} - \sigma E_{i,j} \right), \\ I_{i,j+1} &= I_{i,j} + \Delta t (\sigma E_{i,j} - \delta I_{i,j}), \end{aligned} \quad (5.3)$$

for $j = 0, 1, \dots, n-1$, where $T_{i,j}$, $E_{i,j}$ and $I_{i,j}$ are the values of $T_i(t)$, $E_i(t)$ and $I_i(t)$ at time $t = t_j$, respectively.

In the following simulation, the spatial domain $[-150, 150]$ and the time interval $[0, 25]$ are divided into 300 subintervals with a grid size $r = 1$ and 2500 subintervals with a step size $\Delta t = 0.01$, respectively. The initial condition and the parameters are fixed to $T_0 = 10$, $I_0 = 0.01$, $\beta = 2$, $\alpha = 1$, $\sigma = 4.17$ and $\delta = 7$, which satisfy $R_0 = \beta T_0 / \delta > 1$.

Figure 1 presents a bird's-eye view of I_{ij} over the domain $[-150, 150] \times [0, 25]$. It illustrates how the population of infected cells at (x, t) evolves: the number of infected cells initially increases at $x = 0$, followed by a rise in infection in neighboring cells. Thereafter, a gradual right/leftward shift of the infection waves is observed, while the wave shape remains nearly constant.

Figure 2 illustrates in more detail the temporal evolution of the number of infected cells at $t = 0$, the number of infected cells is very low, but it gradually increases, reaching a maximum around $t = 3.0$. Thereafter, the wave of infected cells splits in two, keeping the same shape and propagating at the same speed to the left and to the right.

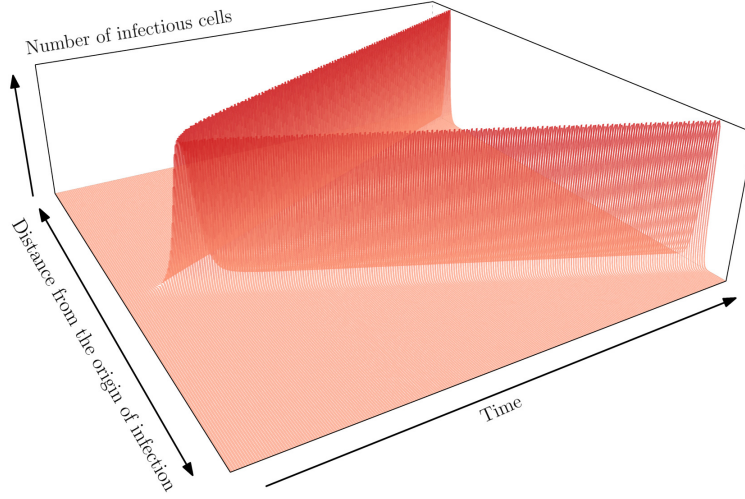


Fig. 1 Temporal changes of the population of infected cells simulated by (5.3).

Figure 3 illustrates the distribution of all four types of cells in a subdomain at $t = 12$. In the region where x , the distance from the origin of infection, exceeds 65, all cells remain uninfected because the wave of infection has not yet reached this area. In contrast, in the region where $x < 45$, the infection wave has passed leaving many cells dead, although some uninfected cells that escaped infection can still be observed. In the intervening regions, infection is ongoing and all four type of cells are present.

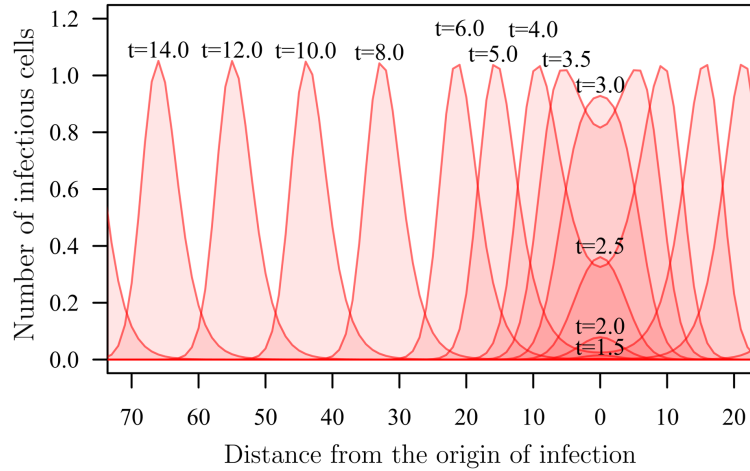


Fig. 2 Snapshots of infected cell distributions at the indicated times, shown in the same frame.

6 Proof of Theorem 4

6.1 Reduction to a 2-dimensional system

We reduce the system (1.15) to a 2-component system. We use the prime as the derivative with respect to z .

Adding the three equations yields

$$c(U + V + W)' + \delta W = 0. \quad (6.1)$$

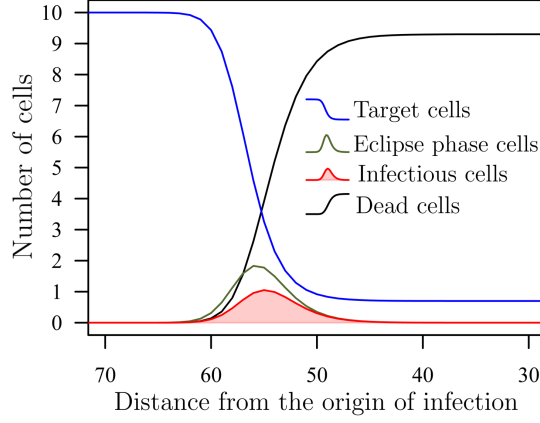


Fig. 3 Distribution of the four types of cells at $t = 12$.

Plug this into the U equation and we have

$$c(U'/U) + \alpha W'' - (c\beta/\delta)(U + V + W)' = 0, \quad (6.2)$$

hence, integrating this yields

$$c \log U + \alpha W' - (c\beta/\delta)(U + V + W) = -cA_1, \quad A_1 := (\beta/\delta)U_0 - \log U_0. \quad (6.3)$$

Define

$$g(U) := \frac{\beta}{\delta}U - \log U - A_1 = \frac{\beta}{\delta}U - \log U - \left(\frac{\beta}{\delta}U_0 - \log U_0\right) \quad (U > 0), \quad (6.4)$$

so that $g(U_0) = 0$ holds. Since $g(U)$ has the minimum at δ/β ,

$$g(\delta/\beta) = 1 - \log(\delta/\beta) - A_1 < 0,$$

by the condition $R_0 > 1$, i.e., $U_0 > \delta/\beta$, there are two positive zeros of $g(U)$. One of them is U_0 and the other one is U_* . Namely, we have

$$g(U_0) = g(U_*) = 0, \quad 0 < U_* < \delta/\beta < U_0, \quad g'(U_*) < 0 < g'(U_0). \quad (6.5)$$

Using $g(U)$, we write (6.3) as

$$\frac{\alpha}{c}W' - g(U) - \frac{\beta}{\delta}(W + V) = 0. \quad (6.6)$$

We rewrite V -equation of (1.15) as

$$V = (\delta/\sigma)W + (c/\sigma)W', \quad (6.7)$$

by which we eliminate V in (6.6) as

$$\left(\frac{\alpha}{c} - \frac{c\beta}{\delta\sigma}\right)W' - g(U) - \beta\left(\frac{1}{\delta} + \frac{1}{\sigma}\right)W = 0. \quad (6.8)$$

Hence,

$$W' = -\frac{c\delta\sigma}{c^2\beta - \alpha\delta\sigma} \left[g(U) + \frac{\beta(\delta + \sigma)}{\delta\sigma} W \right]. \quad (6.9)$$

On the other hand, differentiating W -equation of (1.15) yields

$$cW'' - \sigma V' + \delta W' = 0, \quad (6.10)$$

by which we eliminate W'' in U -equation of (1.15) as

$$cU' + \alpha U(-(\delta/c)W' + (\sigma/c)V') + \beta UW = 0. \quad (6.11)$$

From (6.1), $V' = -(\delta/c)W - U' - W'$ follows. Using this, we eliminate V' in (6.11) as

$$\begin{aligned} & cU' + \alpha U \left[-\frac{\delta}{c}W' + \left(\frac{\sigma}{c}\right) \left\{ -\left(\frac{\delta}{c}\right)W - U' - W' \right\} \right] + \beta UW \\ &= cU' - \frac{\alpha\delta}{c}UW' - \frac{\alpha\delta\sigma}{c^2}UW - \frac{\alpha\sigma}{c}UU' - \frac{\alpha\sigma}{c}UW' + \beta UW = 0, \end{aligned}$$

from which we have

$$\left(c - \frac{\alpha\sigma}{c}U \right) U' - \frac{\alpha}{c}(\delta + \sigma)UW' + \left(\beta - \frac{\alpha\delta\sigma}{c^2} \right) UW = 0. \quad (6.12)$$

As a consequence, the following equation is obtained:

$$\begin{aligned} U' &= \frac{c}{c^2 - \alpha\sigma U} \left\{ \frac{\alpha}{c}(\delta + \sigma)UW' - \left(\beta - \frac{\alpha\delta\sigma}{c^2} \right) UW \right\} \\ &= \frac{\alpha(\delta + \sigma)U}{c^2 - \alpha\sigma U} \left\{ W' - \left(\frac{c^2\beta - \alpha\delta\sigma}{c\alpha(\delta + \sigma)} \right) W \right\} \\ &= -\frac{\alpha(\delta + \sigma)U}{c^2 - \alpha\sigma U} \cdot \frac{c\delta\sigma}{c^2\beta - \alpha\delta\sigma} \left[g(U) + \frac{\beta(\delta + \sigma)}{\delta\sigma} W + \frac{(c^2\beta - \alpha\delta\sigma)^2}{c^2\alpha\delta\sigma(\delta + \sigma)} W \right], \end{aligned}$$

where we used (6.9).

Assuming (1.16) (hence, $c^2\beta - \alpha\delta\sigma > 0$), and putting

$$B(c) := \frac{\delta\sigma}{c^2\beta - \alpha\delta\sigma},$$

we have the system of first order ordinary differential equations for (U, W)

$$\begin{aligned} U' &= -\frac{cB(c)\alpha(\delta + \sigma)U}{c^2 - \alpha\sigma U} \left[g(U) + \frac{\beta(\delta + \sigma)}{\delta\sigma} W + \frac{(c^2\beta - \alpha\delta\sigma)^2}{c^2\alpha\delta\sigma(\delta + \sigma)} W \right], \\ W' &= -cB(c) \left[g(U) + \frac{\beta(\delta + \sigma)}{\delta\sigma} W \right], \end{aligned} \quad (6.13)$$

and search for a heteroclinic orbit satisfying

$$\lim_{z \rightarrow -\infty} (U(z), W(z)) = (U_0, 0), \quad \lim_{z \rightarrow \infty} (U(z), W(z)) = (U_*, 0), \quad (6.14)$$

for $c > 0$.

Before examining (6.13), we verify that (1.15) can be derived from (6.13). First notice that (6.12) and (6.8) follows from (6.13). By (6.7), we have W -equation. Using (6.7) in (6.8), we obtain (6.6). On the other hand, (6.3) follows from (6.6), eventually, (6.2) follows. Hence, we derived (6.2) and (6.12) from (6.13). We next derive the remaining two equations of (1.15). Write (6.12) as

$$c \frac{U'}{U} - \frac{\alpha\sigma}{c} U' - \frac{\alpha}{c} (\delta + \sigma) W' + \left(\beta - \frac{\alpha\delta\sigma}{c^2} \right) W = 0. \quad (6.15)$$

Using (6.2), we eliminate U'/U as

$$-\alpha W'' + \frac{c\beta}{\delta} (U + W + V)' - \frac{\alpha\sigma}{c} U' - \frac{\alpha}{c} (\delta + \sigma) W' + \left(\beta - \frac{\alpha\delta\sigma}{c^2} \right) W = 0,$$

while (6.10) follows from W -equation. We apply it to the above equation to obtain

$$\begin{aligned} & -\frac{\alpha\sigma}{c} V' + \frac{\alpha\delta}{c} W' + \frac{c\beta}{\delta} (U + W + V)' - \frac{\alpha\sigma}{c} U' - \frac{\alpha}{c} (\delta + \sigma) W' + \left(\beta - \frac{\alpha\delta\sigma}{c^2} \right) W \\ & = \frac{c^2\beta - \alpha\delta\sigma}{c\delta} (U + W + V)' + \frac{c^2\beta - \alpha\delta\sigma}{c^2} W = 0. \end{aligned}$$

This implies (6.1). Hence, we easily obtain U -equation and V -equation by (6.1) and (6.2).

6.2 Phase plane analysis for heteroclinic orbits

Making the change of variables $s = cB(c)z$ in (6.13) yields

$$\begin{aligned} \frac{dU}{ds} &= -\frac{\alpha(\delta + \sigma)U}{c^2 - \alpha\sigma U} \left[g(U) + \frac{\beta(\delta + \sigma)}{\delta\sigma} W + \frac{(c^2\beta - \alpha\delta\sigma)^2}{c^2\alpha\delta\sigma(\delta + \sigma)} W \right], \\ \frac{dW}{ds} &= -\left[g(U) + \frac{\beta(\delta + \sigma)}{\delta\sigma} W \right], \\ \lim_{s \rightarrow -\infty} (U(s), W(s)) &= (U_0, 0), \quad \lim_{s \rightarrow \infty} (U(s), W(s)) = (U_*, 0). \end{aligned} \quad (6.16)$$

Furthermore, putting

$$\begin{aligned} p_c(U) &:= \frac{\alpha(\delta + \sigma)U}{c^2 - \alpha\sigma U} > 0 \quad (0 < U \leq U_0), \\ q &:= \frac{\beta(\delta + \sigma)}{\delta\sigma}, \quad r_c := \frac{(c^2\beta - \alpha\delta\sigma)^2}{c^2\alpha\delta\sigma(\delta + \sigma)} = \frac{\beta^2(c^2 - \alpha\sigma(\delta/\beta))^2}{c^2\alpha\delta\sigma(\delta + \sigma)}, \end{aligned} \quad (6.17)$$

we write the system as

$$\begin{aligned} \frac{dU}{ds} &= -p_c(U)[g(U) + (q + r_c)W], \\ \frac{dW}{ds} &= -[g(U) + qW]. \end{aligned} \quad (6.18)$$

We look for a heteroclinic orbit of (6.18), that is, (6.16). Recall the equilibria

$$(U, W) = (U_*, 0), \quad (U_0, 0)$$

satisfy

$$g(U_*) = 0, \quad g'(U_*) < 0, \quad g(U_0) = 0, \quad g'(U_0) > 0,$$

respectively. Our main result is as follows:

Theorem 8 For an arbitrarily given $U_0 > \delta/\beta$, (6.16) has a positive solution for any $c > 0$ satisfying (1.16).

Proof First check the nullclines of (6.16) given by

$$\mathcal{N}_U : (q + r_c)W + g(U) = 0 \quad (U_* \leq U \leq U_0), \quad \mathcal{N}_W : qW + g(U) = 0 \quad (U_* \leq U \leq U_0).$$

On $\mathcal{N}_U$, we see $U' = 0$ and $W' > 0$ except for $W = 0$. On the other hand, on $\mathcal{N}_W$, we have $W' = 0$ and $U' < 0$ except for $W = 0$. Both curves are convex, any orbit starting at a point in the region

$$\left\{ (U, W) : U_* < U < U_0, \quad \frac{-g(U)}{q + r_c} < W < \frac{-g(U)}{q} \right\},$$

eventually leaves through the part of $\mathcal{N}_W$

$$C_W := \{(U, W) : qW + g(U) = 0 \quad U_* < U < \delta/\beta\} \quad (6.19)$$

(see Figure 4).

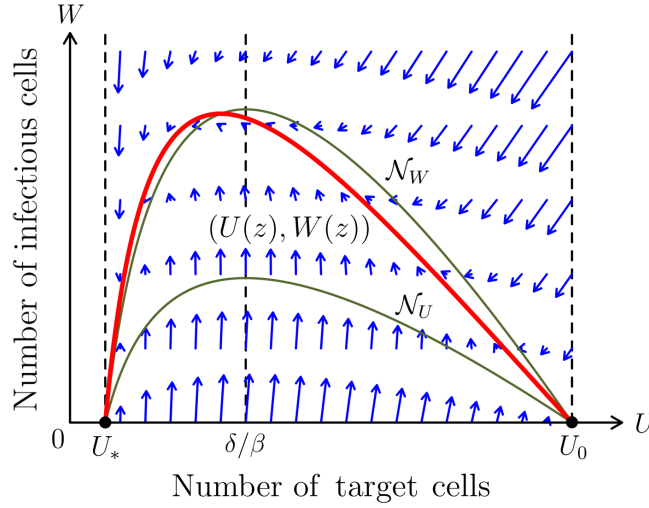


Fig. 4 Nullclines of $\mathcal{N}_U$ and $\mathcal{N}_W$ in (U, W) -plane.

Under this consideration, in order to prove the existence of heteroclinic orbit connecting $(U_0, 0)$ and $(U_*, 0)$, we will check that in a neighborhood of $(U_0, 0)$, the unstable manifold of $(U_0, 0)$ lies in the region sandwiched by $\mathcal{N}_U$ and $\mathcal{N}_W$, and then the orbit leaving C_W converges to $(U_*, 0)$ as $t \rightarrow \infty$.

In the first step, we examine the behavior of orbits in a neighborhood of $(\tilde{U}, 0) = (U_*, 0)$ and $(U_0, 0)$. To this end, we consider the linearized system for $(\tilde{U}, 0) = (U_*, 0)$ or $(U_0, 0)$,

$$\begin{aligned} \frac{du}{ds} &= -p_c(\tilde{U})g'(\tilde{U})u - p_c(\tilde{U})(q + r_c)w, \\ \frac{dw}{ds} &= -g'(\tilde{U})u - qw. \end{aligned}$$

The eigenvalues of the Jacobian matrix

$$J(\tilde{U}, 0) := \begin{pmatrix} -p_c(\tilde{U})g'(\tilde{U}) & -p_c(\tilde{U})(q + r_c) \\ -g'(\tilde{U}) & -q \end{pmatrix},$$

are given by solutions to

$$\lambda^2 + (q + g'(\tilde{U})p_c(\tilde{U}))\lambda - r_cg'(\tilde{U})p_c(\tilde{U}) = 0. \quad (6.20)$$

Since $r_cg'(U_0)p_c(U_0) > 0$, the equilibrium $(U_0, 0)$ is a saddle point.

On the other hand, by $r_cg'(U_*)p_c(U_*) < 0$, $(U_*, 0)$ is a stable node if

$$(q + g'(U_*)p_c(U_*))^2 + 4r_cg'(U_*)p_c(U_*) > 0 \quad \text{and} \quad q + g'(U_*)p_c(U_*) > 0 \quad (6.21)$$

hold. We can prove the next lemma:

Lemma 9 *If (1.16) holds, then (6.21) holds.*

The proof of this lemma will be given later.

For the parameter regime allowing (6.21), we search a heteroclinic orbit by the phase plane analysis. We examine a neighborhood of $(U, W) = (U_0, 0)$. The eigenvalues $\lambda = \mu_1, \mu_2$, satisfying $\mu_2 < 0 < \mu_1$, of (6.20) with $\tilde{U} = U_0$ are given by

$$\begin{aligned}\mu_1 &= \frac{1}{2} \left\{ -(q + g'(U_0)p_c(U_0)) + \sqrt{(q + g'(U_0)p_c(U_0))^2 + 4r_c g'(U_0)p_c(U_0)} \right\}, \\ \mu_2 &= \frac{1}{2} \left\{ -(q + g'(U_0)p_c(U_0)) - \sqrt{(q + g'(U_0)p_c(U_0))^2 + 4r_c g'(U_0)p_c(U_0)} \right\}.\end{aligned}$$

A corresponding eigenvector of $J(U_0, 0)$ to μ_1 is $(-1, g'(U_0)/(q + \mu_1))^T$. Recall the nullclines of (6.16) $\mathcal{N}_U$ and $\mathcal{N}_W$ whose tangential directions at $(U, W) = (U_0, 0)$ are

$$(-1, g'(U_0)/q)^T, \quad (-1, g'(U_0)/(q + r_c))^T,$$

respectively. Notice that

$$\begin{aligned}0 &= \mu_1^2 + (q + g'(U_0)p_c(U_0))\mu_1 - r_c g'(U_0)p_c(U_0) \\ &< r_c^2 + (q + g'(U_0)p_c(U_0))r_c - r_c g'(U_0)p_c(U_0) = r_c^2 + q r_c\end{aligned}$$

implies $\mu_1 < r_c$. Hence, by

$$\frac{g'(U_0)}{q + r_c} < \frac{g'(U_0)}{q + \mu_1} < \frac{g'(U_0)}{q},$$

in a neighborhood of $(U_0, 0)$, the unstable manifold emanating from $(U_0, 0)$ stays in a region sandwiched by the two nullclines.

We next examine a neighborhood of $(U, W) = (U_*, 0)$. In a similar way to the above, the eigenvalues $\lambda_2 \leq \lambda_1 < 0$ of (6.20) with $\tilde{U} = U_*$ are given by

$$\begin{aligned}\lambda_1 &= \frac{1}{2} \left\{ -(q + g'(U_*)p_c(U_*)) + \sqrt{(q + g'(U_*)p_c(U_*))^2 + 4r_c g'(U_*)p_c(U_*)} \right\}, \\ \lambda_2 &= \frac{1}{2} \left\{ -(q + g'(U_*)p_c(U_*)) - \sqrt{(q + g'(U_*)p_c(U_*))^2 + 4r_c g'(U_*)p_c(U_*)} \right\},\end{aligned}$$

and the corresponding eigenvectors are

$$\left(1, -\frac{g'(U_*)}{q + \lambda_1}\right)^T, \quad \left(1, -\frac{g'(U_*)}{q + \lambda_2}\right)^T,$$

respectively. We note

$$\begin{aligned}\lambda_1 + q &\geq \lambda_2 + q = \frac{1}{2} \left\{ q - g'(U_*)p_c(U_*) - \sqrt{(q + g'(U_*)p_c(U_*))^2 + 4r_c g'(U_*)p_c(U_*)} \right\}, \\ &= \frac{-2(q + r_c)g'(U_*)p_c(U_*)}{q - g'(U_*)p_c(U_*) + \sqrt{(q + g'(U_*)p_c(U_*))^2 + 4r_c g'(U_*)p_c(U_*)}} > 0.\end{aligned}$$

On the other hand, at $U = U_*$, the nullcline $\mathcal{N}_W$, i.e., $w = -g(u)/q$ has the tangential slope $-g'(U_*)/q$. Hence, since

$$\frac{-g'(U_*)}{q} \leq \frac{-g'(U_*)}{q + \lambda_1} \leq \frac{-g'(U_*)}{q + \lambda_2},$$

orbits approaching $(U_*, 0)$ along the eigenvector for λ_1 lie beyond the nullcline $\mathcal{N}_W$ in a neighborhood of $(U_*, 0)$. Note that in the region $qW + g(U) > 0$ ($U_* < U < \delta/\beta$), orbits move along the direction with $U' < 0$ and $W' < 0$.

We define the curve

$$\mathcal{C}_* : (q + \lambda_1)W + g(U) = 0 \quad (U_* \leq U \leq \delta/\beta).$$

In view of $g''(U_*) > 0$, we see $g(U) - g'(U_*)(U - U_*) > 0$ ($U > U_*$). Thus,

$$W = \frac{-g'(U_*)}{q + \lambda_1}(U - U_*) \geq \frac{-g(U)}{q + \lambda_1} \geq \frac{-g(U)}{q} \quad (U_* \leq U \leq \delta/\beta), \quad (6.22)$$

and the equal sign holds if and only if $U = U_*$. As seen below, we prove that

$$\Sigma_0 := \{(U, W) : U_* < U < \delta/\beta, -g(U)/q < W < -g(U)/(q + \lambda_1)\}, \quad (6.23)$$

is invariant, namely, if the orbit reaches in the region Σ_0 , then it cannot go out of this region. As a result, the orbit converges to the equilibrium point $(U_*, 0)$ as $t \rightarrow \infty$, which implies the existence of the desired heteroclinic orbit. We note that by (6.22) C_* lie in the region sandwiched by the curve $\mathcal{N}_W$ and the line $W = -(g'(U_*)/(q + \lambda_1))(U - U_*)$ in $\{(U, W) : U_* < U < \delta/\beta, W > 0\}$. This implies that when the orbit converges to $(U_*, 0)$, it is impossible that it converges along the direction of the eigenvector $(1, -(g'(U_*)/(q + \lambda_1)))$.

We verify the invariance of Σ_0 . Put

$$\mathbf{a} = \left(-1, \frac{g'(U)}{q + \lambda_1}\right)^T, \quad \mathbf{b} = (U', W')^T = \left(-p_c(U)[g(U) + (q + r_c)W], -(g(U) + qW)\right)^T.$$

Then, it suffices to show $\mathbf{a} \times \mathbf{b} > 0$ at the curve C_* . Compute it as

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= g(U) + qW + \frac{g'(U)}{q + \lambda_1} p_c(U)[g(U) + (q + r_c)W] \\ &= (g(U) + qW) \left(1 + \frac{g'(U)p_c(U)}{q + \lambda_1}\right) + \frac{g'(U)p_c(U)r_c W}{q + \lambda_1} \\ &= \frac{1}{q + \lambda_1} \left(-\lambda_1 W[q + \lambda_1 + g'(U)p_c(U)] + g'(U)p_c(U)r_c W\right) \\ &= \frac{W}{q + \lambda_1} \left(-\lambda_1^2 - q\lambda_1 - g'(U)p_c(U)\lambda_1 + g'(U)p_c(U)r_c\right). \end{aligned}$$

In view of (6.20),

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \frac{W}{q + \lambda_1} \left(g'(U_*)p_c(U_*)\lambda_1 - g'(U_*)p_c(U_*)r_c - g'(U)p_c(U)\lambda_1 + g'(U)p_c(U)r_c\right) \\ &= \frac{W}{q + \lambda_1} (r_c - \lambda_1)[g'(U)p_c(U) - g'(U_*)p_c(U_*)]. \end{aligned}$$

We examine the sign $g'(U)p_c(U) - g'(U_*)p_c(U_*)$. Recall (6.17). We have

$$g'(U)p_c(U) = \left(\frac{\beta}{\delta} - \frac{1}{U}\right) \frac{\alpha(\delta + \sigma)U}{c^2 - \alpha\sigma U} = \frac{\alpha(\delta + \sigma)\beta}{\delta} \cdot \frac{U - \delta/\beta}{c^2 - \alpha\sigma U}.$$

Differentiate this with respect to U and obtain

$$\frac{d}{dU} g'(U)p_c(U) = \frac{\alpha(\delta + \sigma)\beta}{\delta} \cdot \frac{c^2 - \alpha\sigma\delta/\beta}{(c^2 - \alpha\sigma U)^2} = \frac{\alpha(\delta + \sigma)}{\delta} \cdot \frac{c^2\beta - \alpha\sigma\delta}{(c^2 - \alpha\sigma U)^2} > 0$$

for $U \in [U_*, \delta/\beta]$ by (1.16). This implies

$$g'(U)p_c(U) > g'(U_*)p_c(U_*) \quad (U_* < U < \delta/\beta),$$

as a result $\mathbf{a} \times \mathbf{b} > 0$ ($U_* < U < \delta/\beta$). This complete the proof of thmosition 8. $\square$

Proof of Lemma 9 We prove that (1.16) is a sufficient condition (6.21). We first check the second condition of (6.21).

$$\begin{aligned} q + g'(U_*)p_c(U_*) &= \frac{\beta(\delta + \sigma)}{\delta\sigma} + \frac{\alpha(\delta + \sigma)\beta}{\delta} \cdot \frac{U_* - \delta/\beta}{c^2 - \alpha\sigma U_*} \\ &= \frac{\beta(\delta + \sigma)}{\delta\sigma} \left(1 + \frac{\alpha\sigma(U_* - \delta/\beta)}{c^2 - \alpha\sigma U_*}\right) = \frac{\beta(\delta + \sigma)}{\delta\sigma} \cdot \frac{c^2 - \alpha\sigma\delta/\beta}{c^2 - \alpha\sigma U_*} > 0, \end{aligned}$$

by (1.16). We next check the first condition.

$$\begin{aligned} &(q + g'(U_*)p_c(U_*))^2 + 4r_c g'(U_*)p(U_*) \\ &= \left(\frac{\beta(\delta + \sigma)}{\delta\sigma} \cdot \frac{c^2 - \alpha\sigma\delta/\beta}{c^2 - \alpha\sigma U_*}\right)^2 + 4\frac{\beta^2(c^2 - \alpha\sigma(\delta/\beta))^2}{c^2\alpha\delta\sigma(\delta + \sigma)} \cdot \frac{\alpha(\delta + \sigma)\beta}{\delta} \cdot \frac{U_* - \delta/\beta}{c^2 - \alpha\sigma U_*} \\ &= \frac{\beta^2(c^2 - \alpha\sigma\delta/\beta)^2}{\delta^2\sigma(c^2 - \alpha\sigma U_*)^2} \left(\frac{(\delta + \sigma)^2}{\sigma} - 4\frac{\beta(\delta/\beta - U_*)(c^2 - \alpha\sigma\delta/\beta)}{c^2}\right) \\ &= \frac{\beta^2(c^2 - \alpha\sigma\delta/\beta)^2}{c^2\delta^2\sigma^2(c^2 - \alpha\sigma U_*)^2} \left\{((\delta - \sigma)^2 + 4\beta\sigma U_*)c^2 + 4\alpha\delta\sigma^2\left(\frac{\delta}{\beta} - U_*\right)\right\} > 0. \end{aligned}$$

Hence, we have checked (6.21) and obtained the desired assertion. $\square$

We provide the profile of the solution to (6.13) with (6.14), obtained above, in Figure 5.

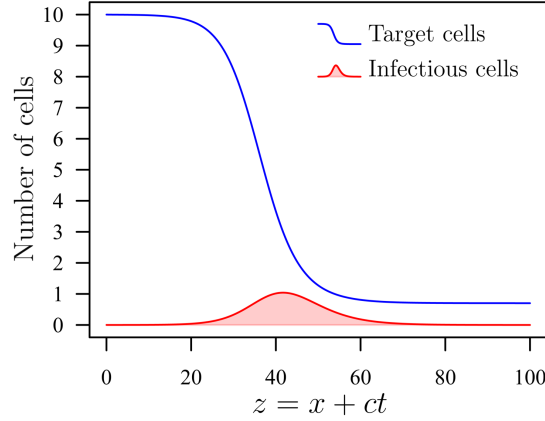


Fig. 5 Profile of the traveling wave solution $(U(z), W(z))$. The horizontal line indicates the z -axis. U and W express the concentration of target cells and infectious cells respectively.

6.3 Completion of the proof of Theorem 4

Proof of Theorem 4 In view of Theorem 8, together with Lemma 9, if $U_0 > \delta/\beta$, then for any $c > 0$ satisfying (1.16), there exists a positive solution $(U_c(z), W_c(z))$ to (6.13) with (6.14). Hence, $(U_c(z), V_c(z), W_c(z))$ is the desired solution to (1.15) with (1.4), where

$$V_c(z) := \frac{1}{\sigma} \left(c \frac{dW}{dz} + \delta W(z) \right).$$

This complete the proof of Theorem 4. □

Remark 5 As for the operator $\mathcal{K}_1$, we have a similar result to Theorem 1 for the existence of traveling waves. Although we don't describe the details, it is easy to see that the corresponding condition to $\ell(\rho, c, \theta) = 1$ becomes

$$\frac{\beta\sigma U_0}{(c\rho + \delta)(c\rho + \sigma)} \cosh(\rho r) = 1.$$

This implies

$$\frac{c^2\rho^2 + (\delta + \sigma)c\rho + \delta\sigma}{\beta\sigma U_0} = \cosh(\rho r). \quad (6.24)$$

We take an approximation for $\cosh(\rho r) \approx 1 + \rho^2 r^2/2$ so as

$$c^2\rho^2 + (\delta + \sigma)c\rho + \delta\sigma \approx (1 + \frac{1}{2}\rho^2 r^2)\beta\sigma U_0.$$

Then the approximate equation reads

$$\left(c^2 - \frac{\beta\sigma r^2 U_0}{2} \right) \rho^2 + c(\delta + \sigma)\rho - \sigma(\beta U_0 - \delta) = 0. \quad (6.25)$$

If $c^2 > \beta\sigma r^2 U_0/2$, then (6.25) has a unique positive solution ρ since $\beta U_0 > \delta$. This condition coincides with (1.16) because of $\alpha = \beta r^2/2$. We remark that even though $c^2 < \beta\sigma r^2 U_0/2$, the equation of (6.25) may allow two positive solutions ρ for arbitrarily small $c > 0$. However, given $r > 0$, the original condition (6.24) does not allow a positive solution ρ for sufficiently small $c > 0$. Hence, it is natural to exclude this case in the current approximate problem. On the other hand, when $c^2 = \beta\sigma r^2 U_0/2$, it is difficult to obtain the desired solution (see §6.1).

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