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Complex Probabilistic Neutrosophic Sets for Multi-Criteria Decision-Making: New Aggregation Operators and Comparative Analysis

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Abstract. Although the complex neutrosophic (CN) set – a generalization of the neutrosophic (N) set – is a useful tool for expressing two-dimensional neutrosophic information, it overlooks the acceptance level of expert opinions, which emulates the uncertainty that permeates real-life situations. To address this deficiency, we propose a novel neutrosophic set called the complex probabilistic neutrosophic (CPN) set, where the uncertainty associated with real-life situations is captured through CN, while the probability degree plays an important role in display the level of the acceptance level of expert opinions. This makes the process of dealing with real-life situations embodied by data of uncertainty more flexible and reliable. We begin by defining three basic operations for CPN elements and developing a series of operational laws. Next, we introduce two aggregation operators (AOs), the CPN-weighted averaging operator (CPNWAO) and the CPN-weighted geometric operator (CPNWGO), which are particularly useful in multi-criteria decision-making (MCDM) problems, where multiple evaluation criteria must be synthesized into a single decision value. We also investigate several key properties of these operators. Building on this foundation, we outline a step-by-step MCDM framework within the CPN environment, utilizing the proposed AOs. To demonstrate the method's effectiveness and practicality, we present a case study on selecting the most suitable car. Finally, we compare our approach with existing methods and highlight the advantages of using the CPN framework in decision-making scenarios, especially when the data is uncertain and attributes are interdependent. The proposed method provides a robust and flexible solution for complex MCDM problems involving both probabilistic and neutrosophic information.

Keywords: Complex probabilistic neutrosophic set, aggregation operators, complex neutrosophic set, multi-criteria decision making, score function

Mathematics Subject Classification: 03E72, 90B50, 68T35

1. Introduction

Due to persistent progress in science and technology, which underpins our daily lives, there has been increased diversity in information technology, intelligence, and cognitive interconnectedness. As a result, users face significant difficulty in determining the optimal choice among the

available options. However, the offered alternatives may satisfy specific user preferences, while users may prefer other elements that are more consistent with their perspective. Therefore, to address this challenge in such practical problems, scientists around the globe have studied and analyzed these problems and developed a practical framework called multi-criteria decision-making (MCDM). This framework sorts and analyzes the available alternatives and then makes the optimal decision. These technologies have played and continue to play a pivotal role in addressing decision-making (DM) challenges, but these challenges involve increasing uncertainty day by day. Consequently, consideration is necessary to envision mathematical tools that work efficiently with MCDM techniques to address this uncertainty before completing a DM process. In implementing this requirement, Zadeh [1] introduced the concept of traditional fuzzy sets (FSs) in 1965 to address uncertainty via a truth-membership function (TMF), mapping from a reference set (domain) to the closed interval $[0, 1]$ (range). The mechanism of employing the FS in MCDM techniques is represented by the interaction process between the reference set that contains elements representing the available options and the TMF that represents the values confined within the closed interval $[0, 1]$, where these values represent the degree of preferences of the users for reference set elements. Later, scholars realized that the TMF in FS cannot cope with ambiguous information more broadly. To handle this point, Atanassov [2] extended the FSs idea to intuitionistic FSs (IFSs) when he added another membership degree in the mathematical structure of FS named falsity membership function (FMF) such that both TMF and FMF lie in a closed interval $[0,1]$. In 1998, from a philosophical perspective, Smarandache referred to the principle of neutrality in human cognition, explaining that the human minds exposure to any process of evaluating a situation matures in it three patterns of thinking (alignment with the situation, opposition to the situation, and neutrality toward the situation). The TMF and FMF in IFSs can deal with patterns of alignment and opposition, respectively, but there is no membership degree for patterns of neutrality. To address this shortcoming, Smarandache [3] introduced a novel concept called neutrosophic sets (NSs) by generalizing both FSs and IFSs by adding a third membership degree, called the indeterminacy membership function (IMF), to the mathematical structure of each FS and IFS. This extension (NS) offers a superior and comprehensive method for handling uncertainty, imprecision, incompleteness, and inconsistency that often arise in real-world problems. To make NS more applicable and adaptable to handling practical engineering issues, Wang et al. [4] introduced SVNNS when three NS functions (TMF, IMF, and FMF) have a single real value lying in the closed interval value $[0,1]$. Researchers have worked efficiently on the SVNNS environment to manage uncertain information. For instance, Shahzadi et al. [5] used SVNNS to address uncertainty in medical diagnostic applications. Pramanik et al. [6] investigated basic properties of the cross-entropy measure under an SVNNS environment. Luo et al. [7] demonstrated the

influence of SVNS on the selection of high-caliber suppliers that practice sustainable business practices. Yang et al. [8] discussed the relationship between two SVNSs and how to leverage this interaction to address real-life scenarios. Abdel-Aziz et al. [9] integrated a novel system-based text classification framework using neutrosophic logic. Naz et al. [10] proposed the operations of SVN-Graph (SVNG) and applied them to an airport travel-time application. Ahmed et al. [11] developed bipolar SVNSs (BSVNSs) and employed them to solve linear programming problems. Bataihah and Hazaymeh [12] introduced fixed-point theorems for SVNs and for situated SVN metric spaces. Mishra et al. [13] demonstrated a useful framework for the MCGDM approach to assess and prioritize energy storage technologies. To ensure transparency and avoid bias that could distort the DM process, it is recommended to involve a neutral user who can provide objective ratings. This enhances the legitimacy of the DM process and builds up its reliability and objectivity. To cover this update, Karaaslan [14] presented the idea of probabilistic SVNSs (PSVNSs) as an extension of probabilistic FSs (PFSs) [15] and probabilistic IFSs (PIFSs) [16] when they added probability degree with SVNS mathematical structures. In these mathematical structures, the degree of probability plays a dynamic role in stabilizing the DM process by assigning a probability (evaluation) in the interval $[0, 1]$ to each SVN number (SVNN). The point of difference from the previous concepts is the addition of a probability function whose value lies in the closed interval $[0,1]$, representing the opinion of the independent user who gives a final evaluation of the data construction process, on the basis of which the DM process is carried out. Aslam et al. [21] examined a critical assessment of some N-statistical methods. Saqlain et al. [22] proposed PN-ordered weighted averaging (NPOWA) by integrating the POWA operator with the N-environment. Zhao et al. [23] examined the application of PSVNS to MADM in a hypersoft environment. Gradually, it became clear that this idea has been discussed and applied across broad areas such as medical diagnosis, expert systems, and pattern recognition, with this research contributing to enriching the fuzzy environment and its extensions. Table 1 presents a comprehensive review of the literature on the concept of probability in fuzzy environments and the techniques used to address everyday-life problems.

However, researchers find from their studies that these contributions mentioned in the literature have a shortage in covering uncertainty in our daily lives, which can only handle one-dimensional information. As a result, to make the literature mentioned above, adapt to process two-dimensional information (periodicity) that exists widely in real life, Ramot et al. [24] proposed the concept of complex FSs (CFSs) to overcome these constraints. They used the strategy that switches the range of TMF in FS from $[0,1]$ to the complex planes unit circle. This model (CFS) received attention from several researchers for its ability to comprise simultaneously two-dimensional data information in one set, and therefore, they created

TABLE 1. A comprehensive review of current literature related to possibility settings.

Authors	Possibility Fuzzy Structures	Types of Applications and Tools Used
Shawkat and Salleh [15]	PFSs	Apply the similarity measure between two PFSs to diagnose if the patient is suffering from dengue fever
Rahman et al. [17]	PFSs	Building an algorithm based on the similarity between two PFSs in recruitment pattern recognition, which has the property of hypersoft settings.
Al-Sharqi et al. [18]	PIVFSs	Apply the similarity measure between two PIVFSs to diagnose if the patient is suffering from dengue fever
Bashir et al. [16]	PIFSs	Apply the similarity measure between two PIFSs to select the optimal school among three schools.
Karaaslan [14]	PNSs	Apply the algorithm that works based on AND-product operation and OR-product operation between two PNSs to select the optimal house among three houses.
Rahman et al. [19]	PNSs	Develop an algorithm that works based on PNS with hypersoft settings to deal with the cleveland data set for heart diseases.
Saqlain et al. [22]	PNSs	Proposed PN-Ordered Weighted Averaging (NPOWA) by integrating the POWA operator with the N-environment and employ these structures solving two decision-making methodologies.
Xue et al. [20]	PNCSs	Build an algorithm based on the SFV of PNCS to select the optimal school among four schools.

several contributions. For instance, distance measures based on the operation and equalities of CFSs were investigated by Dia et al. [25]. Employing this strategy on IFS, Alkouri and Salleh [26] created the complex IFS (CIFS) to deal with the areas of right and wrong (acceptance or rejection) in the nature of human thinking under the two-dimensional parodic environment. The differences and applications between the two approaches, namely CFS and CIFS, are discussed by several authors. For example, Garg and Rani [27] established the fundamental algebraic operations on complex interval-valued IFSs (CIVIFSs), and they satisfied corresponding AOs on this structure. Gulzar et al. [28] initiated CIF-subgroups (CIFSGs)

and discussed their various fundamental algebraic properties. Ma et al. [29] developed a new algorithm utilizing CFSs for signal applications. Bilal et al. [30] introduced particular novel operations on CIFs. Mahmood and Rehman [31] interpreted the bipolar phenomenon on CFSs and employed it in solving DM situations. As we mentioned overhead, both CFS and CIFs are weakening in managing both FS and IFS information in a periodic manner associated with real-life problems. Therefore, Ali and Smarandache [32] introduced the novelty of complex NS (CNS), where the CNS is portrayed by complex valued-TMF, complex valued-IMF, and complex valued-FMF, respectively. The CNSs successfully managed the NS information in a periodic environment. Based on this framework, researchers have made numerous contributions to improve DM mechanisms. Al-Quran and Alkhazaleh [33] checked the cartesian product property between two CNS, and they used it in creating a new algorithm for DM. Ali and Mahmood [34] initiated dice similarity measures for CNS and used these measures to examine the pattern recognition model for reliability and superiority. Alqahtani et al. [35] linked CNS with graphical properties. Ali et al. [36] redefined the traditional CNS in interval form. Khan et al. [37] introduced the CNS in matrix form and investigated some matrix properties in this environment. Karaaslan et al. [38] put up CNS under hesitant property, and they defined its Hamming and Hausdorff distance measures. As a result, several authors have constructed different AOs for FS and its extension, which have proven an effective technique in tackling MADM challenges. Hayat et al. [39] defined AOs on IF-soft set (IFSS) and defined the necessary properties of the preparatory work of DM in an algorithm. Alcantud [40] appointed a new weighted geometric mean AOs on IFNs and compared it with other AOs prompted by the geometric mean. Jana and Pal [41, 42] proposed SVN_{SWA}, SVN_{SWG}, and SVN_{DPA} to address DM information. Karaaslan et al. [43] initiated the design of Trapezoidal-NAOs (TNAOs) and illustrated it in some numerical examples. Riaz et al. [44] presented novel operational laws for SVN_Ss and compared them with current approaches to demonstrate their practicality. Ji et al. [45] utilized the proposed Frank operations to develop the SVN-Frank normalized prioritized Bonferroni mean (SVN_{FNPBM}) operator. Liu [46] utilized the Archimedean t-conorm (ATCN) and Archimedean t-norm (ATN) for SVN_Ns, employing these operations to introduce a novel approach to addressing MADM problems. In the CNS environment, Akram et al. [47] established the CIF-Hamacher weighted averaging (CIFHWA) operator and CIF-Hamacher ordered weighted averaging (CIFHOWA) operator and used these techniques in building a new algorithm to solve a MADM problem. Garg and Rani [48] used t-norm operations in designing weighted averaging AOs (WAAOs) and illustrated a numerical example to explain the running of the tools. Xu et al. [49] enhanced the SVCNEWA and SVCNEWG operators installed on SVCNS and looked at the Einstein product and Einstein sum. Liaqat et al. [50] developed dynamic IVCN Aczel-Alsina AOs to

deal with a multi-period DM scenario, where the user presents all information as IVCSVN over different time intervals. Naseem et al. [51] formulated Aczel-Alsina aggregation operational laws on CNN by employing the idea of Aczel-Alsina t-norm and Aczel-Alsina t-conorm and presented their major properties, namely idempotency, boundedness, and monotonicity. Garg et al. [52] proposed a DM approach based on generalised AOs with CNNs and described their algebraic and operational laws. Mahmood et al. [53] developed Dombi-normalised weighted Bonferroni mean (DNWBM) operators for multi-valued CN uncertain linguistic sets (MVC-NULSs). Building on what has been discussed in this section and following the motivation adopted by previous authors in developing fuzzy structures to serve the purpose of dealing with complex life problems, in this work we will work on organizing an innovative concept called complex probabilistic neutrosophic set (CPNS) that combines the property of randomness in probability theory and fuzziness in one framework. From an academic perspective on CPNS structures, then CPNS are considered more robust and common than existing FS, IFS, SVNS, CFS, CIFS, and CNS in treating complex and nuanced information. Therefore, this paper makes the following primary scientific contribution:

- (1) The interaction of both the probability degree and CN structures initiates a novel concept called CPNS to improve accuracy in the existing models mentioned in the literature to address the challenge of uncertainty or ambiguity in real-life scenarios.
- (2) On this structure, we defined the traditional algebraic operations, like the complement of a CPNS, union and intersection between CPNSs then examined their initial properties.
- (3) Build a score function (SF) value and an accuracy function (AF) based on CPNSs, and work on verifying their features.
- (4) We develop some expanded AOs based on complex probabilistic neutrosophic numbers (CPNNs) such as complex probabilistic neutrosophic weighted averaging operator (CPNWAO), complex probabilistic neutrosophic weighted geometric operator (CPNWGO), and then verify their properties.
- (5) A new multistep algorithm is established based on AOs and SFs, and we demonstrate its effectiveness in troubleshooting the optimal solution.
- (6) The advantages and strengths were discussed through a comparison analysis with some old modes.

Research gap and Motivation behind CPNS: The existing studies in the literature reveal that we are living in an environment that has become more informationalised and intelligent due to the tremendous development of science and technology. As a result, researchers in our world have not stopped developing existing mathematical tools to suit our real-life situations. The CNSs are designed by Ali and Smarandache [29] as a powerful and

effective mathematical tool for handling indeterminate and vague information that has two-dimensional data (periodicity). Additionally, despite the tremendous developments mentioned in literature, there is no existing MADM frameworks simultaneously heading fuzzy probability evaluations, CNS-structures and multidimensional uncertainty in practical decision contexts. Consequently, this lack motivates us to develop a new framework named CPNSs that combines randomness and fuzziness simultaneously, where the fuzzy probability function plays a dynamic role in evaluating the performance of the three membership functions in CNS. In fact, CPNS are more robust and comprehensive than current FS, IFS, NS, PNS, CFS, and CNS in capturing awkward and complex data. By bridging probabilistic fuzzy function with three CN-uncertainty, the CPNWA and CPNGA operators offer a novel and more significant aggregation techniques appropriate for complex DM environments where conflicting and incomplete information is predominant. Since the mentioned FS, IFS, NS, PNS, CFS, and CNS are all special subsets of CPNS, the features of CPNS are pointed out below:

- (1) The dynamic mechanism of interaction between NS and the degree of probability, under the umbrella of complex numbers, is the most important feature of our proposed model.
- (2) The suggested operator offers a comprehensive and flexible aggregation tool capable of simultaneously treating probabilistic, behavioural, and uncertain information by integrating the components of complex truth, complex indeterminacy, and complex falsehood.
- (3) If we ignore the imaginary part of each component in CPNS, then the CPNS would switch to PNS.
- (4) If we disregard the probability part of each component in CPNS, then the CPNS would switch to CNS.
- (5) If we disregard both parts, the imaginary and the probability of each component in CPNS, then our model CPNS would switch to NS.
- (6) If we ignore the IMF in both real and imaginary parts of CPNS, then CPNS would switch to PIFS.
- (7) If we ignore IMF and FMF in both real and imaginary parts of CPNS, then CPNS would switch to PFS.

The paper is displayed in seven sections as follows: First of all, we recall some fundamental representations relevant to our model, such as FS, NS, PFS, and CNS, and their properties in Section 2. In Section 3, we structured the model of the CPNS with its essential operations and results. Then, in Section 4, we build CPN-weighted averaging arithmetic operator (CPNWA) and CPN-weighted geometric operator (CPNWG) with some numerical examples to show their performance. Tailoring a methodology for dealing with MCDM issues using AOs and

SFs based on CPN information in Section 5. Finally, the comparison analysis and conclusion of this work are summarised in Sections 6 and 7, respectively.

2. Basic Concepts of CPNS

In this part, we analyse several previous foundational methods that are associated with our proposed methods.

Definition 2.1. [1] A mathematical structure of FS $\check{\mathfrak{F}}$ on $\hat{\mathbf{X}}$ given as following sort:

$$\check{\mathfrak{F}} = \left\{ \left(\left\langle \uparrow_{\check{\mathfrak{F}}}^{\mathcal{T}}(\hat{\tau}) \right\rangle \right); \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

Which contains $\uparrow_{\check{\mathfrak{F}}}^{\mathcal{T}}(\hat{\tau}) : \hat{\mathbf{X}} \rightarrow [0, 1]$ designate as a TMF of $\hat{\tau} \in \hat{\mathbf{X}}$.

Definition 2.2. [1] Assume that $\check{\mathfrak{F}}^1 = \left\{ \left(\left\langle \uparrow_{\check{\mathfrak{F}}^1}^{\mathcal{T}}(\hat{\tau}) \right\rangle \right); \hat{\tau} \in \hat{\mathbf{X}} \right\}$ and $\check{\mathfrak{F}}^2 = \left\{ \left(\left\langle \uparrow_{\check{\mathfrak{F}}^2}^{\mathcal{T}}(\hat{\tau}) \right\rangle \right); \hat{\tau} \in \hat{\mathbf{X}} \right\}$ be two FSs on $\hat{\mathbf{X}}$. Then the next properties are fulfilled:

(1). $\check{\mathfrak{F}}^1$ is named **FS-empty set** if $\check{\mathfrak{F}}^1$ given as following

$$\check{\mathfrak{F}}^1 = \left\{ \hat{\tau}, \left(\langle 0 \rangle, \hat{\tau} \in \hat{X} \right) \right\}$$

.

(2). $\check{\mathfrak{F}}^1$ is called **FS-absolute set** if $\check{\mathfrak{F}}^1$ given as following

$$\check{\mathfrak{F}}^1 = \left\{ \hat{\tau}, \left(\langle 1 \rangle, \hat{\tau} \in \hat{X} \right) \right\}$$

.

(3). $\check{\mathfrak{F}}^2$ is called **containment** for $\check{\mathfrak{F}}^1$ if $\uparrow_{\check{\mathfrak{F}}^1}^{\mathcal{T}}(\hat{\tau}) \leq \uparrow_{\check{\mathfrak{F}}^2}^{\mathcal{T}}(\hat{\tau})$.

(4). $\check{\mathfrak{F}}^2$ is called **equality** for $\check{\mathfrak{F}}^1$ if $\uparrow_{\check{\mathfrak{F}}^1}^{\mathcal{T}}(\hat{\tau}) = \uparrow_{\check{\mathfrak{F}}^2}^{\mathcal{T}}(\hat{\tau})$.

(5). The **complement** of $\check{\mathfrak{F}}^1$ indicates as $\left(\check{\mathfrak{F}}^1 \right)^c$ and given as by following: $\uparrow_{\left(\check{\mathfrak{F}}^1 \right)^c}^{\mathcal{T}}(\hat{\tau}) = \uparrow_{\check{\mathfrak{F}}^1}^{\mathcal{T}}(\hat{\tau})$.

(6). The **union (intersection)** of $\check{\mathfrak{F}}^1$ and $\check{\mathfrak{F}}^2$ referred to as $\check{\mathfrak{F}}^1 \text{ (II) } (\text{II}) \check{\mathfrak{F}}^2$ and the TMD define as following:

$$\uparrow_{\check{\mathfrak{F}}^1 \text{ (II) } (\text{II}) \check{\mathfrak{F}}^2}^{\mathcal{T}}(\hat{\tau}) = \left[\uparrow_{\check{\mathfrak{F}}^1}^{\mathcal{T}}(\hat{\tau}) \max(\min) \uparrow_{\check{\mathfrak{F}}^2}^{\mathcal{T}}(\hat{\tau}) \right].$$

Definition 2.3. [15] A mathematical structure of PFS $\check{\mathcal{F}}_{\hat{\mathcal{P}}}$ on $\hat{\mathbf{X}}$ given as following sort:

$$\check{\mathcal{F}}_{\hat{\mathcal{P}}} = \left\{ \left(\left\langle \uparrow_{\check{\mathcal{F}}_{\hat{\mathcal{P}}}}^{\mathcal{T}}(\hat{\tau}) \right\rangle \middle| \hat{\mathcal{P}}(\hat{\tau}) \right); \hat{\tau} \in X \right\}$$

Which contains $\uparrow_{\check{\mathcal{F}}}^{\mathcal{T}}(\hat{\tau}) : \hat{\mathbf{X}} \rightarrow [0, 1]$ designate as a TMF of $\hat{\tau} \in \hat{\mathbf{X}}$ and $\hat{\mathcal{P}}(\hat{\tau}) \in [0, 1]$ designate to possopltly degree of $\uparrow_{\check{\mathcal{F}}}^{\mathcal{T}}(\hat{\tau})$.

Definition 2.4. [3] A mathematical structure of NS $\ddot{\mathcal{N}}$ on $\hat{\mathbf{X}}$ given as following sort:

$$\ddot{\mathcal{N}} = \left\{ \left(\left\langle \dagger_{\ddot{\mathcal{N}}}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{F}}(\hat{\tau}) \right\rangle \right); \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

Which contains $\dagger_{\ddot{\mathcal{N}}}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{F}}(\hat{\tau}) : \hat{\mathbf{X}} \rightarrow [0, 1]$ designate as a TMF, IMF and FMF of $\hat{\tau} \in \hat{\mathbf{X}}$.

Definition 2.5. [14] A mathematical structure of PNS $\ddot{\mathcal{N}}_{\widehat{\mathcal{P}}}$ on $\hat{\mathbf{X}}$ given as following sort:

$$\ddot{\mathcal{N}}_{\widehat{\mathcal{P}}} = \left\{ \left(\left\langle \dagger_{\ddot{\mathcal{N}}_{\widehat{\mathcal{P}}}}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}_{\widehat{\mathcal{P}}}}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}_{\widehat{\mathcal{P}}}}^{\mathcal{F}}(\hat{\tau}) \right\rangle \middle| \widehat{\mathcal{P}}(\hat{\tau}) \right); \hat{\tau} \in X \right\}$$

Which contains $\dagger_{\ddot{\mathcal{N}}}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{F}}(\hat{\tau}) : \hat{\mathbf{X}} \rightarrow [0, 1]$ designate as TMD, IMD and FMD of $\hat{\tau} \in \hat{\mathbf{X}}$ and $\widehat{\mathcal{P}}(\hat{\tau}) \in [0, 1]$ designate to possopltiy degree of $\dagger_{\ddot{\mathcal{N}}}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{N}}}^{\mathcal{F}}(\hat{\tau})$.

After that Ali and Smarandache [32] provided the following definitions for the CNS and its operations:

Definition 2.6. [32] A mathematical structure of CNS $\ddot{\mathcal{H}}$ on $\hat{\mathbf{X}}$ given as following sort:

$$\ddot{\mathcal{H}} = \left\{ \left(\left\langle \dagger_{\ddot{\mathcal{H}}}^{\mathcal{T}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{T}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{I}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{I}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{F}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{F}}(\hat{\tau}))} \right\rangle, \hat{\tau} \in \hat{\mathbf{X}} \right) \right\}$$

Which contains $\dagger_{\ddot{\mathcal{H}}}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}}^{\mathcal{F}}(\hat{\tau}) : \hat{\mathbf{X}} \rightarrow [0, 1]$ designate amplitude term real value and $\diamond^{\mathcal{T}}(\hat{\tau}), \diamond^{\mathcal{I}}(\hat{\tau}), \diamond^{\mathcal{F}}(\hat{\tau}) : \hat{\mathbf{X}} \rightarrow [0, 1]$ designate phase term real value.

Definition 2.7. [32] Let

$$\begin{aligned} \ddot{\mathcal{H}}^1 &= \left\{ \left(\left\langle \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{T}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{I}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{I}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{F}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{F}}(\hat{\tau}))} \right\rangle, \hat{\tau} \in \hat{X} \right) \right\} \text{ and} \\ \ddot{\mathcal{H}}^2 &= \left\{ \left(\left\langle \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{T}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{T}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{I}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{I}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{F}}(\hat{\tau}).e^{i2\pi(\diamond^{\mathcal{F}}(\hat{\tau}))} \right\rangle, \hat{\tau} \in \hat{X} \right) \right\} \text{ be two} \end{aligned}$$

CNSs on $\hat{\mathbf{X}}$. Then the following properties are satisfied:

(1). $\ddot{\mathcal{H}}^1$ is called **CN-empty set** if $\ddot{\mathcal{H}}^1$ given as following

$$\ddot{\mathcal{H}}^1 = \left\{ \hat{\tau} \left(\langle 0, 1, 1 \rangle, \hat{\tau} \in \hat{X} \right) \right\}$$

.

(2). $\ddot{\mathcal{H}}^1$ is called **CN-absolute set** if $\ddot{\mathcal{H}}^1$ given as following

$$\ddot{\mathcal{H}}^1 = \left\{ \hat{\tau} \left(\langle 1, 0, 0 \rangle, \hat{\tau} \in \hat{X} \right) \right\}$$

.

(3). $\ddot{\mathcal{H}}^2$ is called **containment** for $\ddot{\mathcal{H}}^1$ if

$$\dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}) \leq \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{I}}(\hat{\tau}) \geq \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{F}}(\hat{\tau}) \geq \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{F}}(\hat{\tau}) \text{ and in the same way for phase term.}$$

(4). $\ddot{\mathcal{H}}^2$ is called **equality** for $\ddot{\mathcal{H}}^1$ if

$$\dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}) = \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{T}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{I}}(\hat{\tau}) = \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{I}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{F}}(\hat{\tau}) = \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{F}}(\hat{\tau}) \text{ and in the same way for phase term.}$$

(5). The **complement** of $\ddot{\mathcal{H}}^1$ indicates as $(\ddot{\mathcal{H}}^1)^c$ and given as by following functions:

$$\dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{T}})^c} = \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{F}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{F}})},$$

,

$$\dagger_{(\ddot{\mathcal{H}}^1)^c}^{\mathcal{I}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{(\ddot{\mathcal{H}}^1)^c}^{\mathcal{I}})} = 1 - \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{I}}(\hat{\tau}) \cdot e^{i2\pi(1 - \diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{I}})},$$

,

$$\dagger_{(\ddot{\mathcal{H}}^1)^c}^{\mathcal{F}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{(\ddot{\mathcal{H}}^1)^c}^{\mathcal{F}})} = \dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{T}})}.$$

.

(6). The **union (intersection)** of $\ddot{\mathcal{H}}^1$ and $\ddot{\mathcal{H}}^2$ referred to as $\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2$ and the three memberships degrees (TMF, IMF and FMF) are define as following:

$$\dagger_{\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2}^{\mathcal{T}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2}^{\mathcal{T}})} = \left[\dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}) \max(\min) \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{T}}(\hat{\tau}) \right] \cdot e^{i2\pi[\diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{T}}(\hat{\tau}) \max(\min) \diamond_{\ddot{\mathcal{H}}^2}^{\mathcal{T}}(\hat{\tau})]},$$

$$\dagger_{\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2}^{\mathcal{I}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2}^{\mathcal{I}})} = \left[\dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{I}}(\hat{\tau}) \min(\max) \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{I}}(\hat{\tau}) \right] \cdot e^{i2\pi[\diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{I}}(\hat{\tau}) \min(\max) \diamond_{\ddot{\mathcal{H}}^2}^{\mathcal{I}}(\hat{\tau})]},$$

$$\dagger_{\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2}^{\mathcal{F}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1(\amalg(\amalg))\ddot{\mathcal{H}}^2}^{\mathcal{F}})} = \left[\dagger_{\ddot{\mathcal{H}}^1}^{\mathcal{F}}(\hat{\tau}) \min(\max) \dagger_{\ddot{\mathcal{H}}^2}^{\mathcal{F}}(\hat{\tau}) \right] \cdot e^{i2\pi[\diamond_{\ddot{\mathcal{H}}^1}^{\mathcal{F}}(\hat{\tau}) \min(\max) \diamond_{\ddot{\mathcal{H}}^2}^{\mathcal{F}}(\hat{\tau})]}.$$

3. The development of complex probabilistic neutrosophic set (CPNS)

As a development of techniques referred to in the literature in Section 1, in this part, we organise the major definition of a CPNS and its major algebraic laws. Next, based on this major definition, we construct the AFV and SFV, as well as several fundamental set-theoretical operations, including the complement of CPNS, the union, and the intersection between CPNSs, along with several associated properties.

Definition 3.1. An CPNS on $\hat{\mathbf{X}}$ given as following form:

$$\ddot{\mathcal{H}}_{\widehat{\mathcal{P}}} = \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{\mathcal{H}}}^{\mathcal{T}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\mathcal{T}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{I}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\mathcal{I}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{F}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\mathcal{F}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}(\hat{\tau}) \right), \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

Where

the collection $\left(\left\langle \dagger_{\ddot{\mathcal{H}}}^{\mathcal{T}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\mathcal{T}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{I}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\mathcal{I}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{F}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\mathcal{F}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}(\hat{\tau}) \right)$ is complex components that point out to complex neutrosophic information along probabilities to a set $\ddot{\mathcal{H}}_{\widehat{\mathcal{P}}}$ and $\dagger_{\ddot{\mathcal{H}}}^{\mathcal{T}}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{I}}, \dagger_{\ddot{\mathcal{H}}}^{\mathcal{F}} : \hat{\mathbf{X}} \rightarrow [0, 1]$ denote to a truth amplitude term, indeterminate amplitude term, false amplitude term respectively to the element $\hat{\tau}$ in $\hat{\mathbf{X}}$ with stander condition $\dagger_{\ddot{\mathcal{H}}}^{\mathcal{T}} + \dagger_{\ddot{\mathcal{H}}}^{\mathcal{I}} + \dagger_{\ddot{\mathcal{H}}}^{\mathcal{F}} \in [0, 3]$, whereas the domain for $\diamond_{\ddot{\mathcal{H}}}^{\mathcal{T}}, \diamond_{\ddot{\mathcal{H}}}^{\mathcal{I}}, \diamond_{\ddot{\mathcal{H}}}^{\mathcal{F}}$ from $\hat{\mathbf{X}}$ to $[0, 1]$ denote to a truth

phase term, indeterminate phase term, false phase term respectively to the element $\hat{\tau}$ in $\hat{\mathbf{X}}$. In addition for probabilities function $\mathcal{P}$ of CNS environment in $\ddot{\mathcal{H}}_{\hat{\mathcal{P}}}$ it belong to closed $[0, 1]$ such that $\sum_{j=1}^m \hat{\mathcal{P}}_j(\hat{\tau}) = 1$.

In the following, we supply a numerical example that illustrates the working dynamics of our proposed concept CPNS and how to employ the fuzzy probability function $\mathcal{P}$ in dealing with expert opinions, which avail the interests of the multi-criteria decision-making process.

Example 3.2. Consider a committees related to renewable energy issues intend to evaluates two renewable energy projects, P_1 and P_2 build on two traits: economic viability ($\hat{\tau}_1$) and environmental-impact($\hat{\tau}_2$). Here our model point out the interaction in the committee's opinions for both projects is represented as follows: $\ddot{\mathcal{H}}_{\hat{\mathcal{P}}_1}^1$ for the first project and $\ddot{\mathcal{H}}_{\hat{\mathcal{P}}_2}^2$ for the second project, as follows.

$$\begin{aligned} \ddot{\mathcal{H}}_{\hat{\mathcal{P}}_1}^1 &= \{(\hat{\tau}_1, \langle .3 \cdot e^{i2\pi(.7)}, .5 \cdot e^{i2\pi(.2)}, .9 \cdot e^{i2\pi(.4)} \rangle, .8), (\hat{\tau}_2, \langle .3 \cdot e^{i2\pi(.6)}, .7 \cdot e^{i2\pi(.7)}, .5 \cdot e^{i2\pi(.2)} \rangle, .5)\} \\ \ddot{\mathcal{H}}_{\hat{\mathcal{P}}_2}^2 &= \{(\hat{\tau}_2, \langle .5 \cdot e^{i2\pi(.8)}, .3 \cdot e^{i2\pi(.6)}, .4 \cdot e^{i2\pi(.4)} \rangle, .7), (\hat{\tau}_2, \langle .3 \cdot e^{i2\pi(.4)}, .1 \cdot e^{i2\pi(.1)}, .7 \cdot e^{i2\pi(.9)} \rangle, .2)\} \end{aligned}$$

In this case, the terms of the CPNS are distributed as follows:

- i. The term $\dagger^{\ddot{\mathcal{T}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{T}}}(\hat{\tau}))}$ refers to the degree of satisfaction with the project based on the two specified attributes and this value increases as it gets closer to 1.
- ii. The term $\dagger^{\ddot{\mathcal{I}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{I}}}(\hat{\tau}))}$ refers to refers to the degree of neutrality in providing an evaluation of the project based on the two specified attributes and this value increases as it gets closer to 1.
- iii. The term $\dagger^{\ddot{\mathcal{F}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{F}}}(\hat{\tau}))}$ refers to the degree of dissatisfaction with the project based on the two specified criteria, and this value increases as it gets closer to 1.
- iv. For the fuzzy probability function $\mathcal{P}$, its plays a dynamic role in clarifying the committee chair's opinion regarding this project evaluation according to the established criteria. This opinion is expressed as a numerical value within the closed interval $[0, 1]$.

Definition 3.3. (CPN-subset): Let $\ddot{\mathcal{H}}_{\hat{\mathcal{P}}_1}^1$ and $\ddot{\mathcal{H}}_{\hat{\mathcal{P}}_2}^2$ be two CPNSs (given bellow)

$$\begin{aligned} \ddot{\mathcal{H}}_{\hat{\mathcal{P}}}^1 &= \left\{ \hat{\tau}, \left(\left\langle \dagger^{\ddot{\mathcal{T}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{T}}}_{\ddot{\mathcal{H}}^1}(\hat{\tau}))}, \dagger^{\ddot{\mathcal{I}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{I}}}_{\ddot{\mathcal{H}}^1}(\hat{\tau}))}, \dagger^{\ddot{\mathcal{F}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{F}}}_{\ddot{\mathcal{H}}^1}(\hat{\tau}))} \right\rangle \middle| \hat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\} \\ \ddot{\mathcal{H}}_{\hat{\mathcal{P}}}^2 &= \left\{ \hat{\tau}, \left(\left\langle \dagger^{\ddot{\mathcal{T}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{T}}}_{\ddot{\mathcal{H}}^2}(\hat{\tau}))}, \dagger^{\ddot{\mathcal{I}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{I}}}_{\ddot{\mathcal{H}}^2}(\hat{\tau}))}, \dagger^{\ddot{\mathcal{F}}}(\hat{\tau}).e^{i2\pi(\diamond^{\ddot{\mathcal{F}}}_{\ddot{\mathcal{H}}^2}(\hat{\tau}))} \right\rangle \middle| \hat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\} \end{aligned}$$

defined on $\hat{\mathbf{X}}$. Then $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ is called to be complex probabilistic neutrosophic subset (CPN-subset) of $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and denotes $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \subseteq \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ if:

- (1) $\dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}) \leq \dagger_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}) \geq \dagger_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}), \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}) \geq \dagger_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau})$ for amplitudes degree terms.
- (2) $\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}) \leq \diamond_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}), \diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}) \geq \diamond_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}), \diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}) \geq \diamond_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau})$ for phases degree terms.
- and
- (3) $\mathcal{P}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \leq \mathcal{P}_{\ddot{\mathcal{H}}^2}^2(\hat{\tau})$ for possibility terms.

Definition 3.4. (CPN-equelset): Let $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ be two CPNSs (given bellow)

$$\ddot{\mathcal{H}}_{\mathcal{P}}^1 = \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

$$\ddot{\mathcal{H}}_{\mathcal{P}}^2 = \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\ddot{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

defined on $\hat{\mathbf{X}}$. Then $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ is said to be complex probabilistic neutrosophic equalset (CPN-equelset) of $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and denotes $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 = \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ if each term in $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ is equal in value to the corresponding term in $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$.

Definition 3.5. Let (CPN-emptyset): A CPNS $\ddot{\mathcal{H}}_{\mathcal{P}_1}$ (given bellow) defined on $\hat{\mathbf{X}}$

$$\ddot{\mathcal{H}}_{\mathcal{P}}^1 = \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\ddot{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

is said to be complex probabilistic empty neutrosophic set (CP-empty-NS) and denoted by $\ddot{\mathcal{H}}_{\phi_1}^{\phi}$ if $\ddot{\mathcal{H}}_{\phi}^{\phi} = \{\hat{\tau}, (\langle 0.0, 0.0, 0.0 \rangle | 0.0), \hat{\tau} \in \hat{\mathbf{X}}\}$.

Definition 3.6. (CPN-absoluteset): A CPNS $\ddot{\mathcal{H}}_{\mathcal{P}_1}$ (given bellow) defined on $\hat{\mathbf{X}}$

$$\ddot{\mathcal{H}}_{\mathcal{P}} = \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

is said to be complex probabilistic absolute neutrosophic set (CP-absolute-NS) and denoted by $\ddot{\mathcal{H}}_1^1$ if $\ddot{\mathcal{H}}_1^1 = \{\hat{\tau}, (\langle 1, 1, 1 \rangle | 1), \hat{\tau} \in \hat{\mathbf{X}}\}$.

Definition 3.7. (Complement of a CPN-set): The complement of a CPNS $\ddot{\mathcal{H}}_{\mathcal{P}}$ (given bellow) defined on $\hat{\mathbf{X}}$.

$$\ddot{\mathcal{H}}_{\mathcal{P}} = \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}))}, \dagger_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}).e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\ddot{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}}(\hat{\tau}) \right), \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

denotes to $(\ddot{\mathcal{H}}_{\mathcal{P}})^c$ where c is CNS-complement and for fuzzy possibility-complement it given as $(\widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1})^c = 1 - \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1$.

Definition 3.8. (CPN-union): Let $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ be two CPNSs (given bellow)

$$\ddot{\mathcal{H}}_{\mathcal{P}}^1 = \left\{ \hat{\tau}, \left(\left\langle \uparrow_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

$$\ddot{\mathcal{H}}_{\mathcal{P}}^2 = \left\{ \hat{\tau}, \left(\left\langle \uparrow_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

defined on $\hat{\mathbf{X}}$. Then the union operation between two CPN-sets $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ denotes by

$\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \amalg \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ is a CPNS-set $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3$ such that:

- (i) $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3 = \amalg \left\{ \ddot{\mathcal{H}}_{\mathcal{P}_1}^1, \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right\}$ where $\amalg$ is CN-union.
- (ii) $\widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^3}^3 = \max \left\{ \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1, \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2 \right\}$.

Definition 3.9. (CPN-intersection): Assume that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ be two CPNSs (given bellow)

$$\ddot{\mathcal{H}}_{\mathcal{P}}^1 = \left\{ \hat{\tau}, \left(\left\langle \uparrow_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

$$\ddot{\mathcal{H}}_{\mathcal{P}}^2 = \left\{ \hat{\tau}, \left(\left\langle \uparrow_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

defined on $\hat{\mathbf{X}}$. Then the intersection operation between two CPN-sets $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ denotes by $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \amalg \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ is a CPNS-set $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3$ such that:

- (i) $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3 = \amalg \left\{ \ddot{\mathcal{H}}_{\mathcal{P}_1}^1, \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right\}$ where $\amalg$ is CN-intersection.
- (ii) $\widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^3}^3 = \min \left\{ \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1, \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2 \right\}$.

Next, we will define the score function (SF), which plays a crucial role in converting a CPNS into a numerical value. The importance of this function lies in its ability to compare available alternatives to select the optimal option.

Definition 3.10. (Score function value (SFV) of a CPN-set): A CPNS $\ddot{\mathcal{H}}_{\mathcal{P}}$ (given bellow) defined on $\hat{\mathbf{X}}$.

$$\ddot{\mathcal{H}}_{\mathcal{P}} = \left\{ \hat{\tau}, \left(\left\langle \uparrow_{\ddot{\mathcal{H}}}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\hat{\tau}}(\hat{\tau}))}, \uparrow_{\ddot{\mathcal{H}}}^{\hat{\tau}}(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\hat{\tau}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}(\hat{\tau}) \right), \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

then the SFV of a CPN-set is given as following:

$$\widehat{\mathbb{S}}(\ddot{\mathcal{H}}_{\mathcal{P}}) = \frac{1}{6} \left(\left(\uparrow_{\ddot{\mathcal{H}}_j}^{\hat{\tau}} \cdot \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}_j} + \diamond_{\ddot{\mathcal{H}}_j}^{\hat{\tau}} \cdot \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}_j} \right) + \left(\uparrow_{\ddot{\mathcal{H}}_j}^{\hat{\tau}} \cdot \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}_j} + \diamond_{\ddot{\mathcal{H}}_j}^{\hat{\tau}} \cdot \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}_j} \right) + \left(\uparrow_{\ddot{\mathcal{H}}_j}^{\hat{\tau}} \cdot \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}_j} + \diamond_{\ddot{\mathcal{H}}_j}^{\hat{\tau}} \cdot \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}_j} \right) \right)$$

The SFV $\widehat{\mathbb{S}}(\ddot{\mathcal{H}}_{\mathcal{P}}) \in [0, 1]$ and its built here to compare between two CPN-sets $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ in $\hat{\mathbf{X}}$ for example if $\widehat{\mathbb{S}}(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1) \succ \widehat{\mathbb{S}}(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2)$ this means $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ is greater than $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and it symbolizes $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \succ \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$, if $\widehat{\mathbb{S}}(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1) \prec \widehat{\mathbb{S}}(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2)$ this means $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ is smaller than $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and it

symbolizes $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \prec \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and if $\widehat{\mathbb{S}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1\right) = \widehat{\mathbb{S}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2\right)$ this means $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$ is equal to $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and it symbolizes $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 = \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$.

Example 3.11. Assume that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 = \{(\hat{\tau}, \langle .3 \cdot \mathbf{e}^{i2\pi(.7)}, .5 \cdot \mathbf{e}^{i2\pi(.2)}, .9 \cdot \mathbf{e}^{i2\pi(.4)} \rangle, .8)\}$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2 = \{(\hat{\tau}, \langle .1 \cdot \mathbf{e}^{i2\pi(.9)}, .4 \cdot \mathbf{e}^{i2\pi(.8)}, .6 \cdot \mathbf{e}^{i2\pi(.2)} \rangle, .4)\}$ are two CPN-sets.

Then, according to definition 3.10 we get the following results:

$$\widehat{\mathbb{S}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1\right) = \frac{1}{6} * ((.3 * .8 + .7 * .8) + (.5 * .8 + .2 * .8) + (.9 * .8 + .4 * .8)) = .408$$

$$\widehat{\mathbb{S}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2\right) = \frac{1}{6} * ((.1 * .4 + .9 * .4) + (.4 * .4 + .8 * .4) + (.6 * .4 + .2 * .4)) = .204$$

We get $\widehat{\mathbb{S}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1\right) \succ \widehat{\mathbb{S}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2\right)$, therefor we conclude that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \succ \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$.

Definition 3.12. (Accuracy function value (AFV) of a CPN-set): A CPNS $\ddot{\mathcal{H}}_{\mathcal{P}}$ (given bellow) defined on $\hat{\mathbf{X}}$.

$$\ddot{\mathcal{H}}_{\mathcal{P}} = \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \dagger^{\hat{T}} \\ \ddot{\mathcal{H}}} \end{matrix} (\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\hat{T}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{I}} \\ \ddot{\mathcal{H}}} \end{matrix} (\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\hat{I}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{F}} \\ \ddot{\mathcal{H}}} \end{matrix} (\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{\mathcal{H}}}^{\hat{F}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}(\hat{\tau}) \right), \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

then the AFV of a CPN-set is given as following:

$$\hat{A}\left(\ddot{\mathcal{H}}_{\mathcal{P}}\right) = \frac{\begin{matrix} \dagger^{\hat{T}} \\ \ddot{\mathcal{H}}} \cdot \diamond_{\ddot{\mathcal{H}}}^{\hat{T}} - \begin{matrix} \dagger^{\hat{I}} \\ \ddot{\mathcal{H}}} \cdot \diamond_{\ddot{\mathcal{H}}}^{\hat{I}} \end{matrix} \left(1 - \left(\begin{matrix} \dagger^{\hat{T}} \\ \ddot{\mathcal{H}}} \cdot \diamond_{\ddot{\mathcal{H}}}^{\hat{T}}\right)\right) - \begin{matrix} \dagger^{\hat{F}} \\ \ddot{\mathcal{H}}} \cdot \diamond_{\ddot{\mathcal{H}}}^{\hat{F}} \end{matrix} \left(1 - \left(\begin{matrix} \dagger^{\hat{I}} \\ \ddot{\mathcal{H}}} \cdot \diamond_{\ddot{\mathcal{H}}}^{\hat{I}}\right)\right) \end{matrix}}{\widehat{\mathcal{P}}_{\ddot{\mathcal{H}}}}$$

Where $\hat{\mathbb{A}}\left(\ddot{\mathcal{H}}_{\mathcal{P}}\right) \in [-1, 1]$.

Example 3.13. Assume that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 = \{(\hat{\tau}, \langle .3 \cdot \mathbf{e}^{i2\pi(.7)}, .5 \cdot \mathbf{e}^{i2\pi(.2)}, .9 \cdot \mathbf{e}^{i2\pi(.4)} \rangle, .8)\}$ and $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2 = \{(\hat{\tau}, \langle .1 \cdot \mathbf{e}^{i2\pi(.9)}, .4 \cdot \mathbf{e}^{i2\pi(.8)}, .6 \cdot \mathbf{e}^{i2\pi(.2)} \rangle, .4)\}$ are two CPN-sets. Then, according to definition 3.12 we get the following results:

$$\hat{\mathbb{A}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1\right) = \frac{.3 * .7 - .5 * .2(1 - (.3 * .7)) - .9 * .4(1 - (.5 * .2))}{.8} = -.2817$$

$$\hat{\mathbb{A}}\left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2\right) = \frac{.1 * .9 - .4 * .8(1 - (.1 * .9)) - .6 * .2(1 - (.4 * .8))}{.4} = -.7071$$

Definition 3.14. Assume that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$, $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3$ be three CPN-sets (given bellow)

$$\ddot{\mathcal{H}}_{\mathcal{P}}^1 = \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \dagger^{\hat{T}} \\ \ddot{\mathcal{H}}^1 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{T}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{I}} \\ \ddot{\mathcal{H}}^1 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{I}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{F}} \\ \ddot{\mathcal{H}}^1 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^1}^{\hat{F}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^1}^1(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

$$\ddot{\mathcal{H}}_{\mathcal{P}}^2 = \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \dagger^{\hat{T}} \\ \ddot{\mathcal{H}}^2 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{T}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{I}} \\ \ddot{\mathcal{H}}^2 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{I}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{F}} \\ \ddot{\mathcal{H}}^2 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^2}^{\hat{F}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^2}^2(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

$$\ddot{\mathcal{H}}_{\mathcal{P}}^3 = \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \dagger^{\hat{T}} \\ \ddot{\mathcal{H}}^3 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^3}^{\hat{T}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{I}} \\ \ddot{\mathcal{H}}^3 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^3}^{\hat{I}}(\hat{\tau}))}, \begin{matrix} \dagger^{\hat{F}} \\ \ddot{\mathcal{H}}^3 \end{matrix} (\hat{\tau}) \cdot \mathbf{e}^{i2\pi(\diamond_{\ddot{\mathcal{H}}^3}^{\hat{F}}(\hat{\tau}))} \right\rangle \middle| \widehat{\mathcal{P}}_{\ddot{\mathcal{H}}^3}^3(\hat{\tau}) \right), \hat{\tau} \in \hat{X} \right\}$$

defined on $\hat{\mathbf{X}}$ and for $\psi \succ 0$. Then the following fundamental algebraic operations of CPN-sets are defined in the following way:

$$\begin{aligned}
\text{(i)} \quad \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 &= \left\{ \left(\left\langle \left(\begin{pmatrix} \dagger^{\ddot{T}} & \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} - \begin{pmatrix} \dagger^{\ddot{T}} & \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \right) \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{T}}_{\mathcal{H}^1} + \diamond^{\ddot{T}}_{\mathcal{H}^2} - \diamond^{\ddot{T}}_{\mathcal{H}^1} \cdot \diamond^{\ddot{T}}_{\mathcal{H}^2})}, \begin{pmatrix} \dagger^{\ddot{T}} & \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{T}}_{\mathcal{H}^1} \cdot \diamond^{\ddot{T}}_{\mathcal{H}^2})}, \right. \\ &\quad \left. \begin{pmatrix} \dagger^{\ddot{F}} & \dagger^{\ddot{F}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{F}}_{\mathcal{H}^1} \cdot \diamond^{\ddot{F}}_{\mathcal{H}^2})} \right\rangle \left| \frac{\widehat{P_{\mathcal{H}^1}^1} \cdot \widehat{P_{\mathcal{H}^2}^2}}{\widehat{P_{\mathcal{H}^1}^1 + P_{\mathcal{H}^2}^2}} \right| \right\} \\
\text{(ii)} \quad \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 &= \left\{ \left(\left\langle \begin{pmatrix} \dagger^{\ddot{T}} & \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{T}}_{\mathcal{H}^1} \cdot \diamond^{\ddot{T}}_{\mathcal{H}^2})}, \begin{pmatrix} \dagger^{\ddot{T}} & \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} - \begin{pmatrix} \dagger^{\ddot{T}} & \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{T}}_{\mathcal{H}^1} + \diamond^{\ddot{T}}_{\mathcal{H}^2} - \diamond^{\ddot{T}}_{\mathcal{H}^1} \cdot \diamond^{\ddot{T}}_{\mathcal{H}^2})}, \right. \\ &\quad \left. \begin{pmatrix} \dagger^{\ddot{F}} & \dagger^{\ddot{F}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} + \begin{pmatrix} \dagger^{\ddot{F}} & \dagger^{\ddot{F}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} - \begin{pmatrix} \dagger^{\ddot{F}} & \dagger^{\ddot{F}} \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{F}}_{\mathcal{H}^1} + \diamond^{\ddot{F}}_{\mathcal{H}^2} - \diamond^{\ddot{F}}_{\mathcal{H}^1} \cdot \diamond^{\ddot{F}}_{\mathcal{H}^2})} \right\rangle \left| \frac{\widehat{P_{\mathcal{H}^1}^1} \cdot \widehat{P_{\mathcal{H}^2}^2}}{\widehat{P_{\mathcal{H}^1}^1 + P_{\mathcal{H}^2}^2}} \right| \right\} \\
\text{(iii)} \quad \psi \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 &= \left\{ \left(\left\langle \left(1 - \left(1 - \begin{pmatrix} \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 \end{pmatrix} \right)^\psi \right) \cdot \mathfrak{e}^{i2\pi(1 - (1 - \diamond^{\ddot{T}}_{\mathcal{H}^1})^\psi)}, \begin{pmatrix} \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 \end{pmatrix}^\psi \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{T}}_{\mathcal{H}^1})^\psi}, \right. \right. \\ &\quad \left. \left. \begin{pmatrix} \dagger^{\ddot{F}} \\ \ddot{\mathcal{H}}^1 \end{pmatrix}^\psi \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{F}}_{\mathcal{H}^1})^\psi} \right\rangle \left| \frac{(\widehat{P_{\mathcal{H}^1}^1})^\psi}{\psi \widehat{P_{\mathcal{H}^1}^1}} \right| \right\} \\
\text{(vi)} \quad (\ddot{\mathcal{H}}_{\mathcal{P}_1}^1)^\psi &= \left\{ \left(\left\langle \begin{pmatrix} \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 \end{pmatrix}^\psi \cdot \mathfrak{e}^{i2\pi(\diamond^{\ddot{T}}_{\mathcal{H}^1})^\psi}, \left(1 - \left(1 - \begin{pmatrix} \dagger^{\ddot{T}} \\ \ddot{\mathcal{H}}^1 \end{pmatrix} \right)^\psi \right) \cdot \mathfrak{e}^{i2\pi(1 - (1 - \diamond^{\ddot{T}}_{\mathcal{H}^1})^\psi)}, \right. \right. \\ &\quad \left. \left. \left(1 - \left(1 - \begin{pmatrix} \dagger^{\ddot{F}} \\ \ddot{\mathcal{H}}^1 \end{pmatrix} \right)^\psi \right) \cdot \mathfrak{e}^{i2\pi(1 - (1 - \diamond^{\ddot{F}}_{\mathcal{H}^1})^\psi)} \right\rangle \left| \frac{(\widehat{P_{\mathcal{H}^1}^1})^\psi}{\psi \widehat{P_{\mathcal{H}^1}^1}} \right| \right\}
\end{aligned}$$

Example 3.15. Assume that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 = \{(\hat{\tau}, \langle .3 \cdot \mathfrak{e}^{i2\pi(.7)}, .5 \cdot \mathfrak{e}^{i2\pi(.2)}, .9 \cdot \mathfrak{e}^{i2\pi(.4)} \rangle, .8)\}$, $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2 = \{(\hat{\tau}, \langle .1 \cdot \mathfrak{e}^{i2\pi(.9)}, .4 \cdot \mathfrak{e}^{i2\pi(.8)}, .6 \cdot \mathfrak{e}^{i2\pi(.2)} \rangle, .4)\}$ and $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3 = \{(\hat{\tau}, \langle .4 \cdot \mathfrak{e}^{i2\pi(.8)}, .7 \cdot \mathfrak{e}^{i2\pi(.2)}, .5 \cdot \mathfrak{e}^{i2\pi(.9)} \rangle, .6)\}$ are three CPN-sets and for $\psi=3$. Then according to definition 3.14 we get the following results:

$$\begin{aligned}
\text{(i)} \quad \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 &= \{(\langle (.3 + .1 - .3 \cdot .1) \cdot \mathfrak{e}^{i2\pi(.7+.9-.7 \cdot .9)}, (.5 \cdot .4) \cdot \mathfrak{e}^{i2\pi(.2 \cdot .8)}, \\ &\quad (.9 \cdot .6) \cdot \mathfrak{e}^{i2\pi(.4 \cdot .2)} \rangle \left| \frac{.8 \cdot .4}{.8+.4} \right|) \} \\ &= \{(\langle (.37) \cdot \mathfrak{e}^{i2\pi(.97)}, (.2) \cdot \mathfrak{e}^{i2\pi(.16)}, (.54) \cdot \mathfrak{e}^{i2\pi(.08)} \rangle \left| .27 \right|) \}
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 &= \{(\langle (.3 \cdot .1) \cdot \mathfrak{e}^{i2\pi(.7 \cdot .9)}, (.5 + .4 - .5 \cdot .4) \cdot \mathfrak{e}^{i2\pi(.2+.8-.2 \cdot .8)} \\ &\quad (.9 + .6 - .9 \cdot .6) \cdot \mathfrak{e}^{i2\pi(.4+.2-.4 \cdot .2)} \rangle \left| \frac{.8 \cdot .4}{.8+.4} \right|) \} \\ &= \{(\langle (.03) \cdot \mathfrak{e}^{i2\pi(.63)}, (.7) \cdot \mathfrak{e}^{i2\pi(.84)}, (.96) \cdot \mathfrak{e}^{i2\pi(.52)} \rangle \left| .27 \right|) \}
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad \psi \ddot{\mathcal{H}}_{\mathcal{P}_3}^3 &= \left\{ \left(\left\langle \left(1 - (1 - .4)^3 \right) \cdot \mathfrak{e}^{i2\pi(1 - (1 - .8)^3)}, (.7)^3 \cdot \mathfrak{e}^{i2\pi(.2)^3}, (.5)^3 \cdot \mathfrak{e}^{i2\pi(.9)^3} \right\rangle \left| \frac{(.6)^3}{3 \cdot .6} \right| \right) \right\} \\ &= \{(\langle (.784) \cdot \mathfrak{e}^{i2\pi(.992)}, (.343) \cdot \mathfrak{e}^{i2\pi(.008)}, (.125) \cdot \mathfrak{e}^{i2\pi(.729)} \rangle \left| .12 \right|) \}
\end{aligned}$$

$$\begin{aligned}
& \text{(vi)} \left(\ddot{\mathcal{H}}_{\mathcal{P}_3}^3 \right)^\psi = \\
& \left\{ \left(\left((.4)^3 \right) \cdot \mathfrak{e}^{i2\pi((.8)^3)}, \left(1 - (1 - .7)^3 \right) \cdot \mathfrak{e}^{i2\pi(1-(1-.2)^3)}, \left(1 - (1 - .5)^3 \right) \cdot \mathfrak{e}^{i2\pi(1-(1-.9)^3)} \right) \left| \frac{(.6)^3}{3*.6} \right. \right\} \\
& = \left\{ \left((.064) \cdot \mathfrak{e}^{i2\pi(.512)}, (.973) \cdot \mathfrak{e}^{i2\pi(.488)}, (.875) \cdot \mathfrak{e}^{i2\pi(.999)} \right) | .12 \right\}
\end{aligned}$$

Theorem 3.16. Assume that $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1$, $\ddot{\mathcal{H}}_{\mathcal{P}_2}^2$ and $\ddot{\mathcal{H}}_{\mathcal{P}_3}^3$ are three arbitrary CPN-sets on $\hat{\mathbf{X}}$ and for $\psi, \psi_1, \psi_2 \succ 0$ are two initially positive real numbers. Then the next points hold:

- (1) $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 = \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_1}^1$.
- (2) $\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right) \oplus \ddot{\mathcal{H}}_{\mathcal{P}_3}^3 = \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_3}^3 \right)$.
- (3) $\psi \left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right) = \psi \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \psi \ddot{\mathcal{H}}_{\mathcal{P}_2}^2$.
- (4) $\psi_1 \ddot{\mathcal{H}}_{\mathcal{P}} \oplus \psi_2 \ddot{\mathcal{H}}_{\mathcal{P}} = (\psi_1 + \psi_2) \ddot{\mathcal{H}}_{\mathcal{P}}$.
- (5) $\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 = \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_1}^1$.
- (6) $\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right) \otimes \ddot{\mathcal{H}}_{\mathcal{P}_3}^3 = \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \otimes \left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_3}^3 \right)$.
- (7) $\left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \otimes \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right)^\psi = \left(\ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \right)^\psi \otimes \left(\ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \right)^\psi$.
- (8) $\left(\ddot{\mathcal{H}}_{\mathcal{P}} \right)^{\psi_1} \otimes \left(\ddot{\mathcal{H}}_{\mathcal{P}} \right)^{\psi_2} = \left(\ddot{\mathcal{H}}_{\mathcal{P}} \right)^{\psi_1 + \psi_2}$.
- (9) $\left(\left(\ddot{\mathcal{H}}_{\mathcal{P}} \right)^c \right)^c = \ddot{\mathcal{H}}_{\mathcal{P}}$.

Proof. (1) Based on Definition 3.14(1) and use the commutative property with multiplying and adding, we have

$$\begin{aligned}
& \ddot{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 = \\
& \left\{ \left(\left(\begin{pmatrix} \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond_{\mathcal{H}^1}^1 + \diamond_{\mathcal{H}^2}^2 - \diamond_{\mathcal{H}^1}^2 \cdot \diamond_{\mathcal{H}^2}^1)}, \begin{pmatrix} \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \\ \ddot{\mathcal{H}}^1 & \ddot{\mathcal{H}}^2 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond_{\mathcal{H}^1}^1 \cdot \diamond_{\mathcal{H}^2}^1)} \right) \left| \frac{\widehat{P}_{\mathcal{H}^1}^1 \cdot \widehat{P}_{\mathcal{H}^2}^2}{\widehat{P}_{\mathcal{H}^1}^1 + \widehat{P}_{\mathcal{H}^2}^2} \right. \right\} \\
& = \left\{ \left(\left(\begin{pmatrix} \ddot{\mathcal{H}}^2 & \ddot{\mathcal{H}}^1 \\ \ddot{\mathcal{H}}^2 & \ddot{\mathcal{H}}^1 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond_{\mathcal{H}^2}^2 + \diamond_{\mathcal{H}^1}^1 - \diamond_{\mathcal{H}^2}^1 \cdot \diamond_{\mathcal{H}^1}^2)}, \begin{pmatrix} \ddot{\mathcal{H}}^2 & \ddot{\mathcal{H}}^1 \\ \ddot{\mathcal{H}}^2 & \ddot{\mathcal{H}}^1 \end{pmatrix} \cdot \mathfrak{e}^{i2\pi(\diamond_{\mathcal{H}^2}^2 \cdot \diamond_{\mathcal{H}^1}^2)} \right) \left| \frac{\widehat{P}_{\mathcal{H}^2}^2 \cdot \widehat{P}_{\mathcal{H}^1}^1}{\widehat{P}_{\mathcal{H}^2}^2 + \widehat{P}_{\mathcal{H}^1}^1} \right. \right\} \\
& = \ddot{\mathcal{H}}_{\mathcal{P}_2}^2 \oplus \ddot{\mathcal{H}}_{\mathcal{P}_1}^1. \quad \square
\end{aligned}$$

Proof. (2) Based on Definition 3.14(2) and taking advantage of the commutative property with multiplying and adding, we have

[illegible]

$$\begin{aligned}
& \psi \left(\widehat{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \widehat{\mathcal{H}}_{\mathcal{P}_2}^2 \right) \\
&= \psi \left(\left\{ \left\langle \left(\begin{smallmatrix} \dagger^{\dot{T}} & \dagger^{\dot{T}} & -\dagger^{\dot{T}} & \cdot \dagger^{\dot{T}} \\ \ddots \ddot{H}^1 & \ddots \ddot{H}^2 & \ddots \ddot{H}^1 & \ddots \ddot{H}^2 \end{smallmatrix} \right) \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{T}} + \diamond_{\ddot{H}^2}^{\dot{T}} - \diamond_{\ddot{H}^1}^{\dot{T}} \cdot \diamond_{\ddot{H}^2}^{\dot{T}})}, \left(\begin{smallmatrix} \dagger^{\dot{I}} & \cdot \dagger^{\dot{I}} \\ \ddots \ddot{H}^1 & \ddots \ddot{H}^2 \end{smallmatrix} \right) \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{I}} \cdot \diamond_{\ddot{H}^2}^{\dot{I}})}, \right. \right. \\
&\quad \left. \left(\begin{smallmatrix} \dagger^{\dot{F}} & \cdot \dagger^{\dot{F}} \\ \ddots \ddot{H}^1 & \ddots \ddot{H}^2 \end{smallmatrix} \right) \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{F}} \cdot \diamond_{\ddot{H}^2}^{\dot{F}})} \right\rangle \left| \frac{\widehat{\mathcal{P}}_{\ddot{H}^1}^2 \cdot \widehat{\mathcal{P}}_{\ddot{H}^2}^1}{\widehat{\mathcal{P}}_{\ddot{H}^2} + \widehat{\mathcal{P}}_{\ddot{H}^1}} \right\rangle \right\} \\
&= \left\{ \left(\left\langle \left(1 - \left(1 - \left(\begin{smallmatrix} \dagger^{\dot{T}} & \dagger^{\dot{T}} & -\dagger^{\dot{T}} & \cdot \dagger^{\dot{T}} \\ \ddots \ddot{H}^1 & \ddots \ddot{H}^2 & \ddots \ddot{H}^1 & \ddots \ddot{H}^2 \end{smallmatrix} \right) \right) \right)^{\psi} \right\rangle \cdot e^{i2\pi \left(1 - \left(1 - (\diamond_{\ddot{H}^1}^{\dot{T}} + \diamond_{\ddot{H}^2}^{\dot{T}} - \diamond_{\ddot{H}^1}^{\dot{T}} \cdot \diamond_{\ddot{H}^2}^{\dot{T}}) \right)^{\psi} \right)}, \right. \\
&\quad \left. \left(\begin{smallmatrix} \dagger^{\dot{I}} & \cdot \dagger^{\dot{I}} \\ \ddots \ddot{H}^1 & \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{I}} \cdot \diamond_{\ddot{H}^2}^{\dot{I}})^{\psi}}, \left(\begin{smallmatrix} \dagger^{\dot{F}} & \cdot \dagger^{\dot{F}} \\ \ddots \ddot{H}^1 & \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{F}} \cdot \diamond_{\ddot{H}^2}^{\dot{F}})^{\psi}} \right\rangle \left| \frac{(\widehat{\mathcal{P}}_{\ddot{H}^1}^2 \cdot \widehat{\mathcal{P}}_{\ddot{H}^2}^1)^{\psi}}{(\widehat{\mathcal{P}}_{\ddot{H}^2} + \widehat{\mathcal{P}}_{\ddot{H}^1})^{\psi}} \right\rangle \right\} \\
&= \left\{ \left(\left\langle \left(1 - \left(\left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right) \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right) \right) \right)^{\psi} \right\rangle \cdot e^{i2\pi \left(1 - \left((1 - \diamond_{\ddot{H}^1}^{\dot{T}}) (1 - \diamond_{\ddot{H}^2}^{\dot{T}}) \right)^{\psi} \right)}, \right. \\
&\quad \left. \left(\begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \left(\begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi \left((\diamond_{\ddot{H}^1}^{\dot{I}})^{\psi} (\diamond_{\ddot{H}^2}^{\dot{I}})^{\psi} \right)}, \left(\begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \left(\begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi \left((\diamond_{\ddot{H}^1}^{\dot{F}})^{\psi} (\diamond_{\ddot{H}^2}^{\dot{F}})^{\psi} \right)} \right\rangle \left| \frac{(\widehat{\mathcal{P}}_{\ddot{H}^1}^1)^{\psi} \cdot (\widehat{\mathcal{P}}_{\ddot{H}^2}^2)^{\psi}}{(\widehat{\mathcal{P}}_{\ddot{H}^1}^1)^{\psi} + (\widehat{\mathcal{P}}_{\ddot{H}^2}^2)^{\psi}} \right\rangle \right\} \\
&= \left\{ \left(\left\langle \left(\left(1 - \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} + 1 - \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} - \right. \right. \right. \\
&\quad \left. \left. \left(1 - \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \right) \left(1 - \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \right) \right) \right)^{\psi} \right\rangle \cdot e^{i2\pi \left(\frac{1 - (1 - \diamond_{\ddot{H}^1}^{\dot{T}})^{\psi} + 1 - (1 - \diamond_{\ddot{H}^2}^{\dot{T}})^{\psi} -}{(1 - (1 - \diamond_{\ddot{H}^1}^{\dot{T}})^{\psi}) (1 - (1 - \diamond_{\ddot{H}^2}^{\dot{T}})^{\psi})} \right)}, \right. \\
&\quad \left. \left(\begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \left(\begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi \left((\diamond_{\ddot{H}^1}^{\dot{I}})^{\psi} (\diamond_{\ddot{H}^2}^{\dot{I}})^{\psi} \right)}, \left(\begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \left(\begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi \left((\diamond_{\ddot{H}^1}^{\dot{F}})^{\psi} (\diamond_{\ddot{H}^2}^{\dot{F}})^{\psi} \right)} \right\rangle \left| \frac{(\widehat{\mathcal{P}}_{\ddot{H}^1}^1)^{\psi} \cdot (\widehat{\mathcal{P}}_{\ddot{H}^2}^2)^{\psi}}{(\widehat{\mathcal{P}}_{\ddot{H}^1}^1)^{\psi} + (\widehat{\mathcal{P}}_{\ddot{H}^2}^2)^{\psi}} \right\rangle \right\} \\
&= \left\{ \left(\left\langle \left(1 - \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \right) \right\rangle \cdot e^{i2\pi \left(1 - (1 - \diamond_{\ddot{H}^1}^{\dot{T}})^{\psi} \right)}, \left(\begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{I}})^{\psi}}, \left(\begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H}^1 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi(\diamond_{\ddot{H}^1}^{\dot{F}})^{\psi}} \right\rangle \left| \frac{(\widehat{\mathcal{P}}_{\ddot{H}^1}^1)^{\psi}}{\psi \widehat{\mathcal{P}}_{\ddot{H}^1}^1} \right\rangle \right\} \oplus \\
&\quad \left\{ \left(\left\langle \left(1 - \left(1 - \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \right) \right\rangle \cdot e^{i2\pi \left(1 - (1 - \diamond_{\ddot{H}^2}^{\dot{T}})^{\psi} \right)}, \left(\begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi(\diamond_{\ddot{H}^2}^{\dot{I}})^{\psi}}, \left(\begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H}^2 \end{smallmatrix} \right)^{\psi} \cdot e^{i2\pi(\diamond_{\ddot{H}^2}^{\dot{F}})^{\psi}} \right\rangle \left| \frac{(\widehat{\mathcal{P}}_{\ddot{H}^2}^2)^{\psi}}{\psi \widehat{\mathcal{P}}_{\ddot{H}^2}^2} \right\rangle \right\} \\
&= \psi \widehat{\mathcal{H}}_{\mathcal{P}_1}^1 \oplus \psi \widehat{\mathcal{H}}_{\mathcal{P}_2}^2. \quad \square
\end{aligned}$$

Proof. The proof of 4-8 is straightforward and with similar athletic steps in proof 1-3. $\square$

Proof. (9): Take advantage of the Definition 3.7 such that for every

$$\widehat{\mathcal{H}}_{\widehat{\mathcal{P}}} = \left\{ \hat{\tau}, \left(\left\langle \begin{smallmatrix} \dagger^{\dot{T}} \\ \ddots \ddot{H} \end{smallmatrix} (\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{H}}^{\dot{T}}(\hat{\tau}))}, \begin{smallmatrix} \dagger^{\dot{I}} \\ \ddots \ddot{H} \end{smallmatrix} (\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{H}}^{\dot{I}}(\hat{\tau}))}, \begin{smallmatrix} \dagger^{\dot{F}} \\ \ddots \ddot{H} \end{smallmatrix} (\hat{\tau}) \cdot e^{i2\pi(\diamond_{\ddot{H}}^{\dot{F}}(\hat{\tau}))} \right\rangle \left| \widehat{\mathcal{P}}_{\ddot{H}}(\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{\mathbf{X}} \right\}$$

then,

$$\begin{aligned} \left(\ddot{H}_{\widehat{P}}^1\right)^c &= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \uparrow^{\dot{T}} \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{(\ddot{H}^1)^c}^{\dot{T}} (\hat{\tau}) \right)}, \begin{matrix} \uparrow^i \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{(\ddot{H}^1)^c}^i (\hat{\tau}) \right)} \right\rangle \left| \widehat{P}_{(\ddot{H}^1)^c}^1 (\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{X} \right\} \\ &= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \uparrow^{\dot{T}} \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{\ddot{H}^1}^{\dot{T}} (\hat{\tau}) \right)}, 1 - \begin{matrix} \uparrow^i \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(1 - \diamond_{\ddot{H}^1}^i (\hat{\tau}) \right)} \right\rangle \left| 1 - \widehat{P}_{\ddot{H}^1}^1 (\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{X} \right\} \end{aligned}$$

Thus,

$$\begin{aligned} \left(\left(\ddot{H}_{\widehat{P}}^1\right)^c\right)^c &= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \uparrow^{\dot{T}} \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{(\ddot{H}^1)^c}^{\dot{T}} (\hat{\tau}) \right)}, 1 - \begin{matrix} \uparrow^i \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(1 - \diamond_{(\ddot{H}^1)^c}^i (\hat{\tau}) \right)} \right\rangle \left| 1 - \widehat{P}_{(\ddot{H}^1)^c}^1 (\hat{\tau}) \right\rangle \right\} \\ &= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \uparrow^{\dot{T}} \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{\ddot{H}^1}^{\dot{T}} (\hat{\tau}) \right)}, 1 - \left(1 - \begin{matrix} \uparrow^i \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(1 - \diamond_{\ddot{H}^1}^i (\hat{\tau}) \right)} \right) \right\rangle \left| 1 - \left(1 - \widehat{P}_{\ddot{H}^1}^1 (\hat{\tau}) \right) \right\rangle, \hat{\tau} \in \hat{X} \right\} \\ &= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \uparrow^{\dot{T}} \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{\ddot{H}^1}^{\dot{T}} (\hat{\tau}) \right)}, \begin{matrix} \uparrow^i \\ \ddot{H}^1 \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{\ddot{H}^1}^i (\hat{\tau}) \right)} \right\rangle \left| \widehat{P}_{\ddot{H}^1}^1 (\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{X} \right\} \\ &= \ddot{H}_{\widehat{P}}^1. \end{aligned}$$

□

4. The development of AOs of CPNEs and their properties

In this part, we introduce CPNWAOs and CPNWGOs, which are advantageous in MCDM. Based on these aggregation operators, we examine some of their notable features and provide numerical examples.

Definition 4.1. Assume that $\ddot{H}_{\widehat{P}}^j =$

$\left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \uparrow^{\dot{T}} \\ \ddot{H}^j \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{\ddot{H}^j}^{\dot{T}} (\hat{\tau}) \right)}, \begin{matrix} \uparrow^i \\ \ddot{H}^j \end{matrix} (\hat{\tau}) . e^{i2\pi \left(\diamond_{\ddot{H}^j}^i (\hat{\tau}) \right)} \right\rangle \left| \widehat{P}_{\ddot{H}^j}^j (\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{X} \right\}$
 $(j = 1, 2, 3, \dots, m)$ is a collection of CPNEs, and $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \dots, \bar{\omega}_m)$ is a collection of weight vector of $\ddot{H}_{\widehat{P}}^j$ ($j = 1, 2, 3, \dots, m$) with $\bar{\omega}_j \in [0, 1]$ ($j = 1, 2, 3, \dots, m$) and $\sum_{j=1}^m \bar{\omega}_j = 1$. Then the CPN-weighted averaging arithmetic operator (CPNWAAO) and CPN-weighted arithmetic geometric operator (CPNWAGO) are given as following:

$$CPNWAAO \left(\ddot{H}_{\widehat{P}}^1 (\hat{\tau}), \ddot{H}_{\widehat{P}}^2 (\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^m (\hat{\tau}) \right) = \bigoplus_{j=1}^m \bar{\omega}_j \ddot{H}_{\widehat{P}}^j (\hat{\tau}),$$

$$\mathcal{CPNWAGO} \left(\ddot{H}_{\mathcal{P}}^1(\hat{\tau}), \ddot{H}_{\mathcal{P}}^2(\hat{\tau}), \dots, \ddot{H}_{\mathcal{P}}^m(\hat{\tau}) \right) = \bigotimes_{j=1}^m \left(\ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \right)^{\bar{\omega}_j}.$$

Example 4.2. Assume that $\ddot{H}_{\mathcal{P}_1}^1$, $\ddot{H}_{\mathcal{P}_2}^2$ and $\ddot{H}_{\mathcal{P}_3}^3$ are three CPN-sets on $\hat{\mathbf{X}}$ given as follows:

$$\begin{aligned} \ddot{H}_{\mathcal{P}_1}^1 &= \{(\hat{\tau}, \langle 0.3 \cdot e^{i2\pi(0.7)}, 0.5 \cdot e^{i2\pi(0.2)}, 0.9 \cdot e^{i2\pi(0.4)} \rangle, 0.8)\}, \\ \ddot{H}_{\mathcal{P}_2}^2 &= \{(\hat{\tau}, \langle 0.1 \cdot e^{i2\pi(0.9)}, 0.4 \cdot e^{i2\pi(0.8)}, 0.6 \cdot e^{i2\pi(0.2)} \rangle, 0.4)\} \text{ and} \\ \ddot{H}_{\mathcal{P}_3}^3 &= \{(\hat{\tau}, \langle 0.4 \cdot e^{i2\pi(0.8)}, 0.7 \cdot e^{i2\pi(0.2)}, 0.5 \cdot e^{i2\pi(0.9)} \rangle, 0.6)\} \text{ and } \omega = (0.7, 0.5, 0.3). \end{aligned}$$

Then based on definition 3.1,

$$\begin{aligned} \mathcal{CPNWAAO} \left(\ddot{H}_{\mathcal{P}}^1(\hat{\tau}), \ddot{H}_{\mathcal{P}}^2(\hat{\tau}), \ddot{H}_{\mathcal{P}}^3(\hat{\tau}) \right) &= 0.7 \cdot \ddot{H}_{\mathcal{P}}^1(\hat{\tau}) \oplus 0.5 \cdot \ddot{H}_{\mathcal{P}}^2(\hat{\tau}) \oplus 0.3 \cdot \ddot{H}_{\mathcal{P}}^3(\hat{\tau}) \\ &= \left\{ \left(\left(\left(1 - (1 - 0.3)^{0.7} * (1 - 0.1)^{0.5} * (1 - 0.4)^{0.3} \right) \cdot e^{i2\pi(1 - (1 - 0.7)^{0.7} * (1 - 0.9)^{0.5} * (1 - 0.8)^{0.3})}, \right. \right. \\ &\quad \left((0.5)^{0.7} \cdot (0.4)^{0.5} \cdot (0.7)^{0.3} \right) \cdot e^{i2\pi((0.2)^{0.7} \cdot (0.8)^{0.5} \cdot (0.2)^{0.3})}, \\ &\quad \left. \left((0.9)^{0.7} \cdot (0.6)^{0.5} \cdot (0.5)^{0.3} \right) \cdot e^{i2\pi((0.4)^{0.7} \cdot (0.2)^{0.5} \cdot (0.9)^{0.3})} \right) \left| \frac{(0.8)^{0.7} \cdot (0.4)^{0.5} \cdot (0.6)^{0.3}}{(0.8)^{0.7} + (0.4)^{0.5} + (0.6)^{0.3}} \right\} \\ &= \{((0.3660) \cdot e^{i2\pi(0.9162)}, (0.3498) \cdot e^{i2\pi(0.2281)}, (0.5844) \cdot e^{i2\pi(0.2281)}) | 0.1979\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{CPNWAGO} \left(\ddot{H}_{\mathcal{P}}^1(\hat{\tau}), \ddot{H}_{\mathcal{P}}^2(\hat{\tau}), \ddot{H}_{\mathcal{P}}^3(\hat{\tau}) \right) &= (\ddot{H}_{\mathcal{P}}^1(\hat{\tau}))^{0.7} \oplus (\ddot{H}_{\mathcal{P}}^2(\hat{\tau}))^{0.5} \oplus (\ddot{H}_{\mathcal{P}}^3(\hat{\tau}))^{0.3} \\ &= \left\{ \left(\left((0.3)^{0.7} \cdot (0.1)^{0.5} \cdot (0.4)^{0.3} \right) \cdot e^{i2\pi((0.7)^{0.7} \cdot (0.9)^{0.5} \cdot (0.8)^{0.3})}, \right. \right. \\ &\quad \left(1 - \left((1 - 0.5)^{0.7} \cdot (1 - 0.4)^{0.5} \cdot (1 - 0.7)^{0.3} \right) \right) \cdot e^{i2\pi(1 - ((1 - 0.2)^{0.7} \cdot (1 - 0.8)^{0.5} \cdot (1 - 0.2)^{0.3})}), \\ &\quad \left(1 - \left((1 - 0.9)^{0.7} \cdot (1 - 0.6)^{0.5} \cdot (1 - 0.5)^{0.3} \right) \right) \cdot e^{i2\pi(1 - ((1 - 0.4)^{0.7} \cdot (1 - 0.2)^{0.5} \cdot (1 - 0.9)^{0.3})}) \left. \right) \left| \frac{(0.8)^{0.7} \cdot (0.4)^{0.5} \cdot (0.6)^{0.3}}{(0.8)^{0.7} + (0.4)^{0.5} + (0.6)^{0.3}} \right\} \\ &= \{((0.1034) \cdot e^{i2\pi(0.6912)}, (0.6677) \cdot e^{i2\pi(0.6423)}, (0.8975) \cdot e^{i2\pi(0.6865)}) | 0.1979\} \end{aligned}$$

Theorem 4.3. Let $\ddot{H}_{\mathcal{P}}^j =$

$$\left\{ \hat{\tau}, \left(\left(\left(\ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \right) \cdot e^{i2\pi(\diamond_{\mathcal{H}}^j(\hat{\tau}))}, \ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\mathcal{H}}^j(\hat{\tau}))}, \ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \cdot e^{i2\pi(\diamond_{\mathcal{H}}^j(\hat{\tau}))} \right) \right) \left| \widehat{P}_{\mathcal{H}^j}^j(\hat{\tau}) \right|, \hat{\tau} \in \hat{X} \right\}$$

($j = 1, 2, 3, \dots, m$) be a collection of CPNEs, then aggregated value of them utilizing the CPNWA-operator and CPNWAG-operator is also a CPNEs, and

$$\begin{aligned} \mathcal{CPNWAAO} \left(\ddot{H}_{\mathcal{P}}^1(\hat{\tau}), \ddot{H}_{\mathcal{P}}^2(\hat{\tau}), \dots, \ddot{H}_{\mathcal{P}}^m(\hat{\tau}) \right) &= \\ &\left\{ \left(1 - \prod_{j=1}^m \left(1 - \ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(1 - \prod_{j=1}^m \left(1 - \diamond_{\mathcal{H}}^j(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \left(\prod_{j=1}^m \left(\ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(\diamond_{\mathcal{H}}^j(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \right. \\ &\quad \left. \left(\prod_{j=1}^m \left(\ddot{H}_{\mathcal{P}}^j(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(\diamond_{\mathcal{H}}^j(\hat{\tau}) \right)^{\bar{\omega}_j} \right)} \left| \frac{\prod_{j=1}^m (P^j(\hat{\tau}))^{\bar{\omega}_j}}{\sum_{j=1}^m \left(\prod_{j=1}^m (P^j(\hat{\tau}))^{\bar{\omega}_j} \right)} \right\}, \\ \mathcal{CPNWAGO} \left(\ddot{H}_{\mathcal{P}}^1(\hat{\tau}), \ddot{H}_{\mathcal{P}}^2(\hat{\tau}), \dots, \ddot{H}_{\mathcal{P}}^m(\hat{\tau}) \right) &= \end{aligned}$$

$$\left\{ \left(\prod_{j=1}^m \left(\dagger_{\ddot{H}}^{T^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(\diamond_{\ddot{H}}^{T^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \left(\prod_{j=1}^m \left(1 - \dagger_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(1 - \diamond_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \right. \\ \left. \left(\prod_{j=1}^m \left(1 - \dagger_{\ddot{H}}^{F^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(1 - \diamond_{\ddot{H}}^{F^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)} \right\} \left| \frac{\prod_{j=1}^m (P^j(\hat{\tau}))^{\bar{\omega}_j}}{\sum_{j=1}^m \left(\prod_{j=1}^m (P^j(\hat{\tau}))^{\bar{\omega}_j} \right)} \right\}$$

Proof. We will work to prove this theorem by using the following mathematical induction method. When $m=2$, according to Definition 3.14, we have

$$\begin{aligned} & CPNWAAO \left(\ddot{H}_{\bar{P}}^1(\hat{\tau}), \ddot{H}_{\bar{P}}^2(\hat{\tau}) \right) \\ &= \bar{\omega}_1 \ddot{H}_{\bar{P}}^1(\hat{\tau}) \oplus \bar{\omega}_2 \ddot{H}_{\bar{P}}^2(\hat{\tau}) \\ &= \left\{ \left(1 - \left(1 - \dagger_{\ddot{H}}^{T^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) + \left(1 - \left(1 - \dagger_{\ddot{H}}^{T^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) - \left(1 - \left(1 - \dagger_{\ddot{H}}^{T^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) * \right. \\ & \quad \left. \left(1 - \left(1 - \dagger_{\ddot{H}}^{T^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) \right\} \cdot \\ & e^{i2\pi \left(\left(1 - \left(1 - \diamond_{\ddot{H}}^{T^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) + \left(1 - \left(1 - \diamond_{\ddot{H}}^{T^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) - \left(1 - \left(1 - \diamond_{\ddot{H}}^{T^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) * \left(1 - \left(1 - \diamond_{\ddot{H}}^{T^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) \right)}, \\ & \left(\left(\left(\dagger_{\ddot{H}}^{I^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) * \left(\left(\dagger_{\ddot{H}}^{I^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) \right) \cdot \\ & e^{i2\pi \left(\left(\left(\diamond_{\ddot{H}}^{I^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) * \left(\left(\diamond_{\ddot{H}}^{I^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) \right)}, \left(\left(\left(\dagger_{\ddot{H}}^{F^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) * \left(\left(\dagger_{\ddot{H}}^{F^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) \right) \cdot \\ & e^{i2\pi \left(\left(\left(\diamond_{\ddot{H}}^{F^1}(\hat{\tau}) \right)^{\bar{\omega}_1} \right) * \left(\left(\diamond_{\ddot{H}}^{F^2}(\hat{\tau}) \right)^{\bar{\omega}_2} \right) \right)}, \left\{ \frac{\prod_{j=1}^2 (P^j(\hat{\tau}))^{\bar{\omega}_j}}{\sum_{j=1}^2 (P^j(\hat{\tau}))^{\bar{\omega}_1} * \sum_{j=1}^2 (P^j(\hat{\tau}))^{\bar{\omega}_2}} \right\} \\ &= \left\{ \left(1 - \prod_{j=1}^2 \left(1 - \dagger_{\ddot{H}}^{T^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(1 - \prod_{j=1}^2 \left(1 - \diamond_{\ddot{H}}^{T^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \left(\prod_{j=1}^2 \left(\dagger_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^2 \left(\diamond_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \right. \\ & \quad \left. \left(\prod_{j=1}^2 \left(\dagger_{\ddot{H}}^{F^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^2 \left(\diamond_{\ddot{H}}^{F^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)} \right\} \left| \frac{\prod_{j=1}^2 (P^j(\hat{\tau}))^{\bar{\omega}_j}}{\sum_{j=1}^2 \left(\prod_{j=1}^2 (P^j(\hat{\tau}))^{\bar{\omega}_j} \right)} \right\} \end{aligned}$$

After we show it is true in $m=2$, we assume that it is true for $m = k$, then when $m = k+1$.

$$\begin{aligned} & CPNWAAO \left(\ddot{H}_{\bar{P}}^1(\hat{\tau}), \ddot{H}_{\bar{P}}^2(\hat{\tau}), \dots, \ddot{H}_{\bar{P}}^{k+1}(\hat{\tau}) \right) \\ &= \left(\bigoplus_{j=1}^k \bar{\omega}_j \ddot{H}_{\bar{P}}^j(\hat{\tau}) \right) \oplus \left(\bar{\omega}_{k+1} \ddot{H}_{\bar{P}}^{k+1}(\hat{\tau}) \right) \\ &= \left\{ \left(1 - \prod_{j=1}^k \left(1 - \dagger_{\ddot{H}}^{T^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(1 - \prod_{j=1}^k \left(1 - \diamond_{\ddot{H}}^{T^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \left(\prod_{j=1}^k \left(\dagger_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^k \left(\diamond_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \right. \\ & \quad \left. \left(\prod_{j=1}^k \left(\dagger_{\ddot{H}}^{F^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^k \left(\diamond_{\ddot{H}}^{F^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)} \right\} \left| \frac{\prod_{j=1}^k (P^j(\hat{\tau}))^{\bar{\omega}_j}}{\sum_{j=1}^k \left(\prod_{j=1}^k (P^j(\hat{\tau}))^{\bar{\omega}_j} \right)} \right\} \oplus \\ & \quad \left\{ \left(1 - \left(1 - \dagger_{\ddot{H}}^{T^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \cdot e^{i2\pi \left(1 - \left(1 - \diamond_{\ddot{H}}^{T^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)}, \left(\left(\dagger_{\ddot{H}}^{I^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \cdot e^{i2\pi \left(\left(\diamond_{\ddot{H}}^{I^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)}, \right. \\ & \quad \left. \left(\left(\dagger_{\ddot{H}}^{F^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \cdot e^{i2\pi \left(\left(\diamond_{\ddot{H}}^{F^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)} \right\} \end{aligned}$$

$$\begin{aligned}
& e^{i2\pi \left(\left(\diamond_{\ddot{H}}^{j^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)}, \\
& \left(\left(\left(\dagger_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \cdot e^{i2\pi \left(\left(\diamond_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)} \right) \left| \frac{\prod_{j=1}^{k+1} (P^{k+1}(\hat{\tau}))^{\bar{\omega}_{k+1}}}{\sum_{j=1} \left((P^{k+1}(\hat{\tau}))^{\bar{\omega}_{k+1}} \right)} \right\} \\
& = \left\{ \left(1 - \left(1 - \dagger_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \cdot e^{i2\pi \left(1 - \left(1 - \diamond_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)}, \left(\left(\dagger_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \right\} \\
& e^{i2\pi \left(\left(\diamond_{\ddot{H}}^{j^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)}, \\
& \left(\left(\left(\dagger_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right) \cdot e^{i2\pi \left(\left(\diamond_{\ddot{H}}^{\dot{F}^{k+1}}(\hat{\tau}) \right)^{\bar{\omega}_{k+1}} \right)} \right) \left| \frac{\prod_{j=1}^{k+1} (P^{k+1}(\hat{\tau}))^{\bar{\omega}_{k+1}}}{\sum_{j=1} \left((P^{k+1}(\hat{\tau}))^{\bar{\omega}_{k+1}} \right)} \right\} \\
& = \left\{ \left(\dagger_{\ddot{H}}^{\dot{F}^1}(\hat{\tau}) \right) \cdot e^{i2\pi \left(\diamond_{\ddot{H}}^{\dot{F}^1}(\hat{\tau}) \right)}, \left(\dagger_{\ddot{H}}^{\dot{F}^j}(\hat{\tau}) \right) \cdot e^{i2\pi \left(\diamond_{\ddot{H}}^{\dot{F}^j}(\hat{\tau}) \right)}, \right. \\
& \left. \left(\dagger_{\ddot{H}}^{\dot{F}^1}(\hat{\tau}) \right) \cdot e^{i2\pi \left(\diamond_{\ddot{H}}^{\dot{F}^1}(\hat{\tau}) \right)} \mid P^1(\hat{\tau}) \right\} \\
& = \ddot{H}_P^1(\hat{\tau}).
\end{aligned}$$

□

Theorem 4.4. (Idempotency): Let $\ddot{H}_P^j$
 $= \left\{ \hat{\tau}, \left(\left\langle \dagger_{\ddot{H}^j}^{\dot{F}}(\hat{\tau}) \cdot e^{i2\pi \left(\diamond_{\ddot{H}^j}^{\dot{F}}(\hat{\tau}) \right)}, \dagger_{\ddot{H}^1}^{\dot{F}}(\hat{\tau}) \cdot e^{i2\pi \left(\diamond_{\ddot{H}^1}^{\dot{F}}(\hat{\tau}) \right)}, \dagger_{\ddot{H}^j}^{\dot{F}}(\hat{\tau}) \cdot e^{i2\pi \left(\diamond_{\ddot{H}^j}^{\dot{F}}(\hat{\tau}) \right)} \right\rangle \mid \widehat{P}_{\ddot{H}^j}^j(\hat{\tau}) \right\}, \hat{\tau} \in \hat{X}$
 $(j = 1, 2, 3, \dots, m)$ all are equal for, i.e., $\ddot{H}_P^j = \ddot{H}_P$ for all j , then
 $CPNWAAO \left(\ddot{H}_P^1(\hat{\tau}), \ddot{H}_P^2(\hat{\tau}), \dots, \ddot{H}_P^m(\hat{\tau}) \right) = CPNWAAO \left(\ddot{H}_P(\hat{\tau}), \ddot{H}_P(\hat{\tau}), \dots, \ddot{H}_P(\hat{\tau}) \right) = \ddot{H}_P(\hat{\tau}),$
 $CPNWGAO \left(\ddot{H}_P^1(\hat{\tau}), \ddot{H}_P^2(\hat{\tau}), \dots, \ddot{H}_P^m(\hat{\tau}) \right) = CPNWAAO \left(\ddot{H}_P(\hat{\tau}), \ddot{H}_P(\hat{\tau}), \dots, \ddot{H}_P(\hat{\tau}) \right) = \ddot{H}_P(\hat{\tau}).$

Proof. Based on the Definition 4.1 and Theorem 3.16, we can straightforwardly obtain:

$$\begin{aligned}
& CPNWAAO \left(\ddot{H}_P^1(\hat{\tau}), \ddot{H}_P^2(\hat{\tau}), \dots, \ddot{H}_P^m(\hat{\tau}) \right) = CPNWAAO \left(\ddot{H}_P(\hat{\tau}), \ddot{H}_P(\hat{\tau}), \dots, \ddot{H}_P(\hat{\tau}) \right) \\
& = \bigoplus_{j=1}^m \bar{\omega}_j \ddot{H}_P(\hat{\tau}) \\
& = \left(\sum_{j=1}^m \bar{\omega}_j \right) \ddot{H}_P(\hat{\tau}) \\
& = 1 * \ddot{H}_P(\hat{\tau}) = \ddot{H}_P(\hat{\tau})
\end{aligned}$$

Again, taking benefit from Definition 4.1 and Theorem 3.16, we have

$$\begin{aligned}
& CPNWGAO \left(\ddot{H}_P^1(\hat{\tau}), \ddot{H}_P^2(\hat{\tau}), \dots, \ddot{H}_P^m(\hat{\tau}) \right) = CPNWGAO \left(\ddot{H}_P(\hat{\tau}), \ddot{H}_P(\hat{\tau}), \dots, \ddot{H}_P(\hat{\tau}) \right) \\
& = \bigotimes_{j=1}^m \left(\ddot{H}_P(\hat{\tau}) \right)^{\bar{\omega}_j} \\
& = \left(\ddot{H}_P(\hat{\tau}) \right)^{\sum_{j=1}^m \bar{\omega}_j}
\end{aligned}$$

$$= \left(\ddot{H}_{\widehat{P}}(\hat{\tau}) \right)^1 = \ddot{H}_{\widehat{P}}(\hat{\tau})$$

□

Theorem 4.5. (Commutativity): If $\ddot{H}_{\widehat{P}}^j(\hat{\tau})$

$$= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \dagger^{\dot{T}^j}(\hat{\tau}) \\ \ddot{H} \end{matrix} . e^{i2\pi(\diamond^{\dot{T}^j}(\hat{\tau}))}, \begin{matrix} \dagger^{ij}(\hat{\tau}) \\ \ddot{H} \end{matrix} . e^{i2\pi(\diamond^{ij}(\hat{\tau}))}, \begin{matrix} \dagger^{\dot{F}^j}(\hat{\tau}) \\ \ddot{H} \end{matrix} . e^{i2\pi(\diamond^{\dot{F}^j}(\hat{\tau}))} \right) \middle| \widehat{P}^j(\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{X} \right\}$$

$(j = 1, 2, 3, \dots, m)$ and $\ddot{H}_{\widehat{P}}^{j''}(\hat{\tau})$

$$= \left\{ \hat{\tau}, \left(\left\langle \begin{matrix} \dagger^{\dot{T}^{j''}}(\hat{\tau}) \\ \ddot{H} \end{matrix} . e^{i2\pi(\diamond^{\dot{T}^{j''}}(\hat{\tau}))}, \begin{matrix} \dagger^{ij''}(\hat{\tau}) \\ \ddot{H} \end{matrix} . e^{i2\pi(\diamond^{ij''}(\hat{\tau}))}, \begin{matrix} \dagger^{\dot{F}^{j''}}(\hat{\tau}) \\ \ddot{H} \end{matrix} . e^{i2\pi(\diamond^{\dot{F}^{j''}}(\hat{\tau}))} \right) \middle| \widehat{P}^{j''}(\hat{\tau}) \right\rangle, \hat{\tau} \in \hat{X} \right\}$$

$(j'' = 1'', 2'', 3'', \dots, m'')$ are two CPNEs and $\ddot{H}_{\widehat{P}}^{j''}(\hat{\tau})$ is an arbitrary permutation of $\ddot{H}_{\widehat{P}}^j(\hat{\tau})$ $(j = 1, 2, 3, \dots, m)$, then

$$CPNWAAO\left(\ddot{H}_{\widehat{P}}^1(\hat{\tau}), \ddot{H}_{\widehat{P}}^2(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^m(\hat{\tau})\right) = CPNWAAO\left(\ddot{H}_{\widehat{P}}^{1''}(\hat{\tau}), \ddot{H}_{\widehat{P}}^{2''}(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^{m''}(\hat{\tau})\right),$$

$$CPNWGAO\left(\ddot{H}_{\widehat{P}}^1(\hat{\tau}), \ddot{H}_{\widehat{P}}^2(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^m(\hat{\tau})\right) = CPNWGAO\left(\ddot{H}_{\widehat{P}}^{1''}(\hat{\tau}), \ddot{H}_{\widehat{P}}^{2''}(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^{m''}(\hat{\tau})\right).$$

Theorem 4.6. (Monotonicity): If $\ddot{H}_{\widehat{P}}^{1j}(\hat{\tau}) \leq \ddot{H}_{\widehat{P}}^{2j}(\hat{\tau})$ $(j = 1, 2, 3, \dots, m)$ then

$$CPNWAAO\left(\ddot{H}_{\widehat{P}}^{11}(\hat{\tau}), \ddot{H}_{\widehat{P}}^{12}(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^{1m}(\hat{\tau})\right) \leq CPNWAAO\left(\ddot{H}_{\widehat{P}}^{21}(\hat{\tau}), \ddot{H}_{\widehat{P}}^{22}(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^{2m}(\hat{\tau})\right),$$

$$CPNWGAO\left(\ddot{H}_{\widehat{P}}^{11}(\hat{\tau}), \ddot{H}_{\widehat{P}}^{12}(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^{1m}(\hat{\tau})\right) \leq CPNWGAO\left(\ddot{H}_{\widehat{P}}^{21}(\hat{\tau}), \ddot{H}_{\widehat{P}}^{22}(\hat{\tau}), \dots, \ddot{H}_{\widehat{P}}^{2m}(\hat{\tau})\right).$$

Proof.

Considering the assumption $\ddot{H}_{\widehat{P}}^{1j}(\hat{\tau}) \leq \ddot{H}_{\widehat{P}}^{2j}(\hat{\tau})$ $(j = 1, 2, 3, \dots, m)$ we have for each $1 \leq j \leq m$

$$\frac{1}{6} \left(\left(\sum_{j=1}^m \begin{matrix} \dagger^{\dot{T}} \\ \ddot{H}_j \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_j} + \sum_{j=1}^m \begin{matrix} \diamond^{\dot{T}} \\ \ddot{H}_j \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_j} \right) + \left(\sum_{j=1}^m \begin{matrix} \dagger^{\dot{T}} \\ \ddot{H}_j \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_j} + \sum_{j=1}^m \begin{matrix} \diamond^{\dot{T}} \\ \ddot{H}_j \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_j} \right) + \left(\sum_{j=1}^m \begin{matrix} \dagger^{\dot{F}} \\ \ddot{H}_j \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_j} + \sum_{j=1}^m \begin{matrix} \diamond^{\dot{F}} \\ \ddot{H}_j \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_j} \right) \right)$$

then

$$\frac{1}{6} \left(\sum_t \left(\prod_{j=1}^m \left(\begin{matrix} \dagger^{\dot{T}} \\ \ddot{H}_{t1j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t1j}} \right)^{\bar{\omega}_j} + \prod_{j=1}^m \left(\begin{matrix} \diamond^{\dot{T}} \\ \ddot{H}_{t1j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t1j}} \right)^{\bar{\omega}_j} + \prod_{j=1}^m \left(\begin{matrix} \dagger^{\dot{F}} \\ \ddot{H}_{t1j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t1j}} \right)^{\bar{\omega}_j} + \prod_{j=1}^m \left(\begin{matrix} \diamond^{\dot{F}} \\ \ddot{H}_{t1j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t1j}} \right)^{\bar{\omega}_j} \right) \right)$$

$$\leq \frac{1}{6} \left(\sum_t \left(\prod_{j=1}^m \left(\begin{matrix} \dagger^{\dot{T}} \\ \ddot{H}_{t2j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t2j}} \right)^{\bar{\omega}_j} + \prod_{j=1}^m \left(\begin{matrix} \diamond^{\dot{T}} \\ \ddot{H}_{t2j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t2j}} \right)^{\bar{\omega}_j} + \prod_{j=1}^m \left(\begin{matrix} \dagger^{\dot{F}} \\ \ddot{H}_{t2j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t2j}} \right)^{\bar{\omega}_j} + \prod_{j=1}^m \left(\begin{matrix} \diamond^{\dot{F}} \\ \ddot{H}_{t2j} \end{matrix} . \widehat{\mathcal{P}}_{\ddot{H}_{t2j}} \right)^{\bar{\omega}_j} \right) \right)$$

which point out that

$$\begin{aligned} & CPNWGAO \left(\ddot{H}_{\hat{P}}^{11}(\hat{\tau}), \ddot{H}_{\hat{P}}^{12}(\hat{\tau}), \dots, \ddot{H}_{\hat{P}}^{1m}(\hat{\tau}) \right) \\ & \leq CPNWGAO \left(\ddot{H}_{\hat{P}}^{21''}(\hat{\tau}), \ddot{H}_{\hat{P}}^{22''}(\hat{\tau}), \dots, \ddot{H}_{\hat{P}}^{2m''}(\hat{\tau}) \right). \end{aligned}$$

The rest of the inequalities can be proven using the same steps. $\square$

5. Suggest MCDM methodology for CPNWAO and CPNWGO (based on aggregation operators)

This section put forth a novel MCDM approach in order to handle the ambiguity within a CPN setting. Then, a scenario embodying one of the problems of daily life is introduced to expound the put-forward decision process. To formulate an MCDM problem with CPNS information, assume there are n alternatives designated by $\mathcal{A}_i (i = 1, 2, 3, \dots, n)$ that are supported by m attributes denoted as $\mathcal{C}_j (j = 1, 2, 3, \dots, m)$ along with the corresponding attribute weight vector given as $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \dots, \bar{\omega}_m)$ where its satisfies the condition $\sum_{j=1}^m \varpi_j = 1$.

In this context, we will build the following mathematical multi-step algorithm that uses CPNWAO and CPNWGO to find out the best alternative presented in the next example.

Step 1. Create the CPN decision matrix, which contains two categories of criteria: benefit attributes and cost attributes. Typically, we should categorise all attributes into one of these categories before making decisions, where we should work on changing all attributes to match the type of available categories. Therefore, we used the following formula to convert the whole to the benefit category that is better.

$$\hat{\Pi} = \begin{cases} \langle \ddot{H}_{\hat{P}} \rangle & \text{for benefit criteria} \\ \langle (\ddot{H}_{\hat{P}})^c \rangle & \text{for cost criteria} \end{cases}$$

Then, we gain the normalized CPN decision matrix $Q = (\hat{\Pi}_{ij})_{n \times m}$.

Step 2. Using the AO of CPN elements to determine the total CPN information of each alternative $\mathcal{A}_i$. Here, we utilized the advantage of the CPNWA operator as follows formula.

$$\begin{aligned} \hat{\Pi}_i &= CPNWA \left(\hat{\Pi}_{i1}, \hat{\Pi}_{i2}, \dots, \hat{\Pi}_{im} \right) = \bigoplus_{j=1}^m w_j \hat{\Pi}_{ij} \\ &= \left\{ \left(1 - \prod_{j=1}^m \left(1 - \dagger_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(1 - \prod_{j=1}^m \left(1 - \diamond_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)}, \left(\prod_{j=1}^m \left(\dagger_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(\diamond_{\ddot{H}}^{I^j}(\hat{\tau}) \right)^{\bar{\omega}_j} \right)} \right\}, \end{aligned}$$

$$\left(\prod_{j=1}^m \left(\begin{smallmatrix} \dot{F}^j \\ \ddot{H} \end{smallmatrix} (\hat{\tau}) \right)^{\bar{\omega}_j} \right) \cdot e^{i2\pi \left(\prod_{j=1}^m \left(\diamond_{\dot{H}}^{\dot{F}^j} (\hat{\tau}) \right)^{\bar{\omega}_j} \right)} \left| \frac{\prod_{j=1}^m (P^j(\hat{\tau}))^{\bar{\omega}_j}}{\sum_{j=1}^m \left(\prod_{j=1}^m (P^j(\hat{\tau})) \right)^{\bar{\omega}_j}} \right|,$$

Surely, in this step, we can also apply the CPNWGA formula that we have proposed in this work to determine the total CPN information of each alternative $\mathcal{A}_i$.

Step 3. Compute the SFV of each alternative $\mathcal{A}_i$ by taking advantage of the following formula.

$$\widehat{\mathbb{S}}(\hat{\Pi}_j) = \frac{1}{6} \left(\left(\begin{smallmatrix} \dot{\mathcal{P}} \\ \ddot{\Pi}_j \end{smallmatrix} \cdot \widehat{\mathcal{P}}_{\hat{\Pi}_j} + \diamond_{\dot{\Pi}_j}^{\dot{\mathcal{P}}} \cdot \widehat{\mathcal{P}}_{\hat{\Pi}_j} \right) + \left(\begin{smallmatrix} \dot{\mathcal{P}} \\ \ddot{\Pi}_j \end{smallmatrix} \cdot \widehat{\mathcal{P}}_{\hat{\Pi}_j} + \diamond_{\dot{\Pi}_j}^{\dot{\mathcal{P}}} \cdot \widehat{\mathcal{P}}_{\hat{\Pi}_j} \right) + \left(\begin{smallmatrix} \dot{\mathcal{P}} \\ \ddot{\Pi}_j \end{smallmatrix} \cdot \widehat{\mathcal{P}}_{\hat{\Pi}_j} + \diamond_{\dot{\Pi}_j}^{\dot{\mathcal{P}}} \cdot \widehat{\mathcal{P}}_{\hat{\Pi}_j} \right) \right).$$

Step 4. Based on the SFV output, arrange the candidate alternatives in descending order and then select the optimal alternative (highest value).

Step 5. End Algorithm.

For more details about this algorithm and to logically explain how it works, we present Figure (1) a visual representation of the methodologies sequential process to the mechanism for implementing this algorithm.

Now, we organize a workable MCDM example consistent with what we discussed at the beginning of this section to test the power of the aforementioned algorithm. This MCDM example talks about how customer X chooses the best car among five cars.

Example 5.1. At a modern Motor Show, a customer X expressed his intention to purchase a new car. At first glance, five cars $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ satisfied the customer. Also, it turns out that the customer pays attention to three important criteria for the perfection of the car selection process. These criteria are mathematically organized as follows: $\mathcal{C}_j (j = 1, 2, 3)$ where $\mathcal{C}_1$: Overall Cost, $\mathcal{C}_2$: Quality and $\mathcal{C}_3$: Exterior appearance of the car, all these criteria related to weight vector $\bar{\omega} = (0.4, 0.2, 0.4)$. Now, the customer X has been invited an expert to make an evaluation of each car corresponding to each attribute according to his/her experience. Then, we take out the following CPN information matrix presented in Table 2.

Here we notice that $\mathcal{C}_1$ is a cost attribute, while $\mathcal{C}_2$ and $\mathcal{C}_3$ are benefit features (attributes); therefore, we should first convert $\mathcal{C}_2$ to the benefit features by the above formula. Hence, Table 3 presents the CPN decision matrix with benefit features.

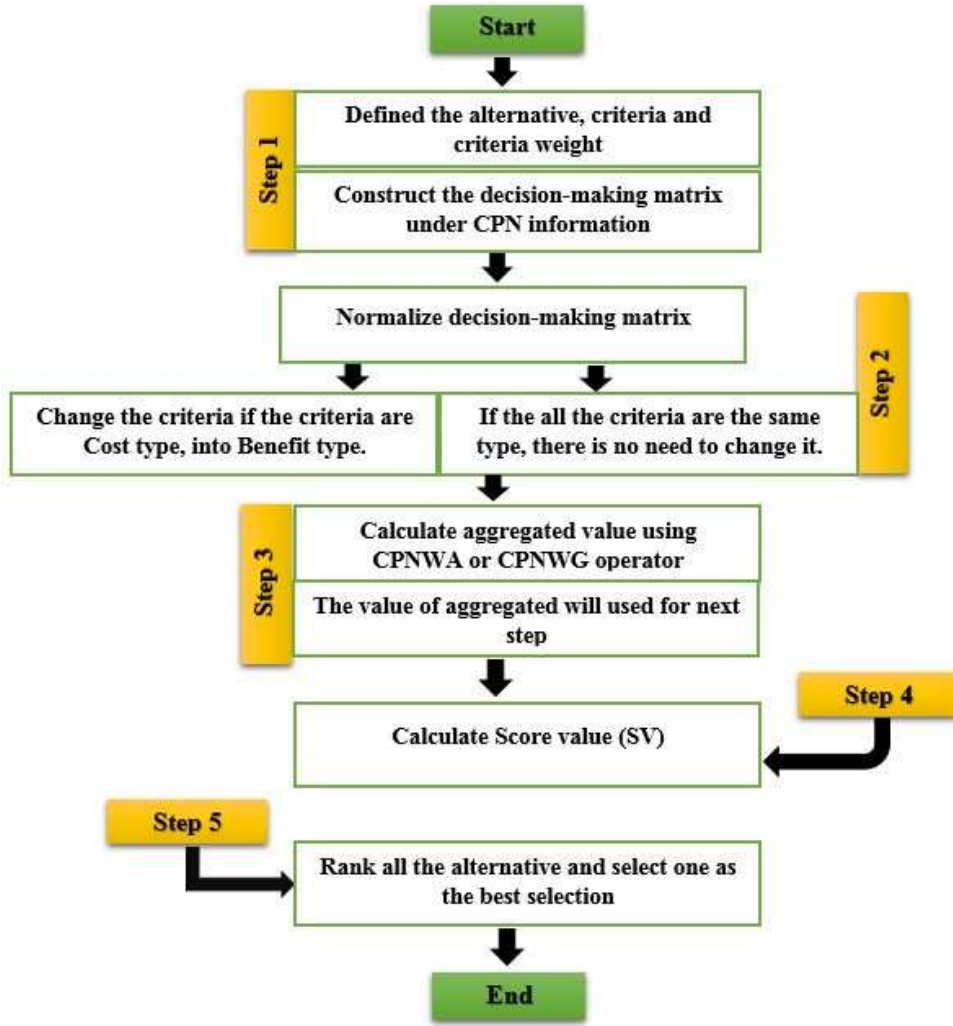


FIGURE 1. Steps to implement the proposed algorithm logically

Next, we apply Step 2, where we will adopt CPNWA operator to evaluation of each alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ as follows:

$$\begin{aligned} \hat{\Pi}_1 = & \left\{ \left(1 - (.4)^4 (.4)^4 (.6)^4 \right) \cdot e^{i2\pi(1 - (.2)^4 (.4)^4 (.1)^4)}, \left((.1)^2 (.8)^2 (.3)^2 \right) \cdot e^{i2\pi((.2)^2 (.4)^2 (.3)^2)}, \right. \\ & \left. \left((.7)^4 (.7)^4 (.7)^4 \right) \cdot e^{i2\pi((.7)^4 (.9)^4 (.2)^4)} \mid \frac{(.8)^4 (.5)^2 (.3)^4}{(.8)^4 (.5)^4 (.3)^4 + (.8)^2 (.5)^2 (.3)^2 + (.8)^4 (.5)^4 (.3)^4} \right\} \\ & = \left\{ .6083 \cdot e^{i2\pi(.8550)}, .4742 \cdot e^{i2\pi(.4204)}, .6518 \cdot e^{i2\pi(.4366)} \mid 0.3257 \right\}, \end{aligned}$$

Then, using the same steps, we obtain the rest of the values as follows:

$$\begin{aligned} \hat{\Pi}_2 &= \left\{ .4238 \cdot e^{i2\pi(.7186)}, .7413 \cdot e^{i2\pi(.7548)}, .2249 \cdot e^{i2\pi(.3245)} \mid .3701 \right\}, \\ \hat{\Pi}_3 &= \left\{ .5894 \cdot e^{i2\pi(.6340)}, .4373 \cdot e^{i2\pi(.4258)}, .2249 \cdot e^{i2\pi(.4282)} \mid .3548 \right\}, \\ \hat{\Pi}_4 &= \left\{ .8550 \cdot e^{i2\pi(.7298)}, .6258 \cdot e^{i2\pi(.7413)}, .5697 \cdot e^{i2\pi(.4282)} \mid 0.2463 \right\}, \end{aligned}$$

TABLE 2. The initial CPN decision matrix.

$X/Criteria$	C_1	C_2	C_3
$\mathcal{A}_1$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.7)} \\ .9 \cdot e^{i2\pi(.8)}, .2 \\ .6 \cdot e^{i2\pi(.8)} \end{pmatrix}$	$\begin{pmatrix} .6 \cdot e^{i2\pi(.6)} \\ .8 \cdot e^{i2\pi(.4)}, .5 \\ .7 \cdot e^{i2\pi(.9)} \end{pmatrix}$	$\begin{pmatrix} .4 \cdot e^{i2\pi(.9)} \\ .3 \cdot e^{i2\pi(.3)}, .3 \\ .7 \cdot e^{i2\pi(.2)} \end{pmatrix}$
$\mathcal{A}_2$	$\begin{pmatrix} .2 \cdot e^{i2\pi(.5)} \\ .6 \cdot e^{i2\pi(.3)}, .8 \\ .4 \cdot e^{i2\pi(.7)} \end{pmatrix}$	$\begin{pmatrix} .4 \cdot e^{i2\pi(.3)} \\ .8 \cdot e^{i2\pi(.5)}, .6 \\ .2 \cdot e^{i2\pi(.3)} \end{pmatrix}$	$\begin{pmatrix} .3 \cdot e^{i2\pi(.8)} \\ .7 \cdot e^{i2\pi(.7)}, .4 \\ .6 \cdot e^{i2\pi(.4)} \end{pmatrix}$
$\mathcal{A}_3$	$\begin{pmatrix} .2 \cdot e^{i2\pi(.8)} \\ .9 \cdot e^{i2\pi(.9)}, .3 \\ .7 \cdot e^{i2\pi(.7)} \end{pmatrix}$	$\begin{pmatrix} .1 \cdot e^{i2\pi(.1)} \\ .4 \cdot e^{i2\pi(.7)}, .5 \\ .2 \cdot e^{i2\pi(.3)} \end{pmatrix}$	$\begin{pmatrix} .6 \cdot e^{i2\pi(.7)} \\ .4 \cdot e^{i2\pi(.2)}, 0.7 \\ .6 \cdot e^{i2\pi(.5)} \end{pmatrix}$
$\mathcal{A}_4$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.8)} \\ .8 \cdot e^{i2\pi(.6)}, .3 \\ .9 \cdot e^{i2\pi(.8)} \end{pmatrix}$	$\begin{pmatrix} .9 \cdot e^{i2\pi(.9)} \\ .8 \cdot e^{i2\pi(.7)}, .1 \\ .5 \cdot e^{i2\pi(.3)} \end{pmatrix}$	$\begin{pmatrix} .2 \cdot e^{i2\pi(.4)} \\ .6 \cdot e^{i2\pi(.8)}, .9 \\ .7 \cdot e^{i2\pi(.5)} \end{pmatrix}$
$\mathcal{A}_5$	$\begin{pmatrix} .9 \cdot e^{i2\pi(.9)} \\ .6 \cdot e^{i2\pi(.5)}, 0.4 \\ .9 \cdot e^{i2\pi(.8)} \end{pmatrix}$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.8)} \\ .8 \cdot e^{i2\pi(.7)}, .7 \\ .8 \cdot e^{i2\pi(.9)} \end{pmatrix}$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.8)} \\ .8 \cdot e^{i2\pi(.9)}, .4 \\ .5 \cdot e^{i2\pi(.7)} \end{pmatrix}$

TABLE 3. Normalized CPN decision matrix.

$X/Criteria$	C_1	C_2	C_3
$\mathcal{A}_1$	$\begin{pmatrix} .6 \cdot e^{i2\pi(.8)} \\ .1 \cdot e^{i2\pi(.2)}, .8 \\ .7 \cdot e^{i2\pi(.7)} \end{pmatrix}$	$\begin{pmatrix} .6 \cdot e^{i2\pi(.6)} \\ .8 \cdot e^{i2\pi(.4)}, .5 \\ .7 \cdot e^{i2\pi(.9)} \end{pmatrix}$	$\begin{pmatrix} .4 \cdot e^{i2\pi(.9)} \\ .3 \cdot e^{i2\pi(.3)}, .3 \\ .7 \cdot e^{i2\pi(.2)} \end{pmatrix}$
$\mathcal{A}_2$	$\begin{pmatrix} .4 \cdot e^{i2\pi(.7)} \\ .4 \cdot e^{i2\pi(.7)}, .2 \\ .2 \cdot e^{i2\pi(.5)} \end{pmatrix}$	$\begin{pmatrix} .4 \cdot e^{i2\pi(.3)} \\ .8 \cdot e^{i2\pi(.5)}, .6 \\ .2 \cdot e^{i2\pi(.3)} \end{pmatrix}$	$\begin{pmatrix} .3 \cdot e^{i2\pi(.8)} \\ .7 \cdot e^{i2\pi(.7)}, .4 \\ .6 \cdot e^{i2\pi(.4)} \end{pmatrix}$
$\mathcal{A}_3$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.7)} \\ .1 \cdot e^{i2\pi(.1)}, .7 \\ .2 \cdot e^{i2\pi(.8)} \end{pmatrix}$	$\begin{pmatrix} .1 \cdot e^{i2\pi(.1)} \\ .4 \cdot e^{i2\pi(.7)}, .5 \\ .2 \cdot e^{i2\pi(.3)} \end{pmatrix}$	$\begin{pmatrix} .6 \cdot e^{i2\pi(.7)} \\ .4 \cdot e^{i2\pi(.2)}, 0.7 \\ .6 \cdot e^{i2\pi(.5)} \end{pmatrix}$
$\mathcal{A}_4$	$\begin{pmatrix} .9 \cdot e^{i2\pi(.8)} \\ .2 \cdot e^{i2\pi(.4)}, .3 \\ .7 \cdot e^{i2\pi(.8)} \end{pmatrix}$	$\begin{pmatrix} .9 \cdot e^{i2\pi(.9)} \\ .8 \cdot e^{i2\pi(.7)}, .1 \\ .5 \cdot e^{i2\pi(.3)} \end{pmatrix}$	$\begin{pmatrix} .2 \cdot e^{i2\pi(.4)} \\ .6 \cdot e^{i2\pi(.8)}, .9 \\ .7 \cdot e^{i2\pi(.5)} \end{pmatrix}$
$\mathcal{A}_5$	$\begin{pmatrix} .9 \cdot e^{i2\pi(.8)} \\ .4 \cdot e^{i2\pi(.5)}, 0.4 \\ .9 \cdot e^{i2\pi(.9)} \end{pmatrix}$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.8)} \\ .8 \cdot e^{i2\pi(.7)}, .7 \\ .8 \cdot e^{i2\pi(.9)} \end{pmatrix}$	$\begin{pmatrix} .7 \cdot e^{i2\pi(.8)} \\ .8 \cdot e^{i2\pi(.9)}, .4 \\ .5 \cdot e^{i2\pi(.7)} \end{pmatrix}$

$$\hat{\Pi}_5 = \left\{ .8480 \cdot e^{i2\pi(.8550)}, .7614 \cdot e^{i2\pi(.7937)}, .6645 \cdot e^{i2\pi(.7969)} \mid .2863 \right\}.$$

Next, in accordance with Step 3, we calculate the SV of alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$. For example,

$$\widehat{S}(\hat{\Pi}_1) = \frac{1}{6} ((.6083 \times .3257 + .8550 \times .3257) + (.4742 \times .3257 + .4204 \times .3257) + (.6518 \times .3257 + .4366 \times .3257)) = .1870.$$

Then, using the same steps, we obtain the rest of the values as follows:

$$\widehat{S}(\hat{\Pi}_2) = .1965, \widehat{S}(\hat{\Pi}_3) = .1619, \widehat{S}(\hat{\Pi}_4) = .1620, \widehat{S}(\hat{\Pi}_5) = .2251$$

Finely, from Step 3, it gives the ranking of SFV as follows: $\widehat{S}(\hat{\Pi}_5) \succ \widehat{S}(\hat{\Pi}_2) \succ \widehat{S}(\hat{\Pi}_1) \succ \widehat{S}(\hat{\Pi}_4) \succ \widehat{S}(\hat{\Pi}_3)$. Therefore, the ranking of all alternatives will be as follows:

$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$. Therefore, we conclude that car $\mathcal{A}_5$ is the best choice, according to the experts opinion.

Furthermore, we can use other operators introduced in this research work to evaluate CPN information. Now, we will resort to CPNWG to conduct the optimal selection from the given alternatives.

$$\begin{aligned} \hat{\Pi}_1 &= \left\{ \left((.6)^4 (.6)^4 (.4)^4 \right) \cdot e^{i2\pi((.8)^4 (.6)^4 (.9)^4)}, \left(1 - (.9)^2 (.2)^2 (.7)^2 \right) \cdot e^{i2\pi(1 - (.8)^2 (.6)^2 (.7)^2)}, \right. \\ &\quad \left. \left(1 - (.3)^4 (.3)^4 (.3)^4 \right) \cdot e^{i2\pi(1 - (.3)^4 (.1)^4 (.8)^4)} \mid \frac{(.8)^4 (.5)^2 (.3)^4}{(.8)^4 (.5)^4 (.3)^4 + (.8)^2 (.5)^2 (.3)^2 + (.8)^4 (.5)^4 (.3)^4} \right\} \\ &= \left\{ .4606 \cdot e^{i2\pi(.7148)}, .3392 \cdot e^{i2\pi(.1959)}, .7641 \cdot e^{i2\pi(.7751)} \mid .2046 \right\}, \end{aligned}$$

Then, using the same steps, we obtain the rest of the values as follows:

$$\begin{aligned} \hat{\Pi}_2 &= \left\{ .3343 \cdot e^{i2\pi(.4899)}, .7355 \cdot e^{i2\pi(.4644)}, .4832 \cdot e^{i2\pi(.4644)} \mid .2887 \right\}, \\ \hat{\Pi}_3 &= \left\{ .2813 \cdot e^{i2\pi(.2992)}, .2018 \cdot e^{i2\pi(.2640)}, .5798 \cdot e^{i2\pi(.3451)} \mid .3454 \right\}, \\ \hat{\Pi}_4 &= \left\{ .4828 \cdot e^{i2\pi(.6077)}, .4230 \cdot e^{i2\pi(.4857)}, .7108 \cdot e^{i2\pi(.6549)} \mid .3904 \right\}, \\ \hat{\Pi}_5 &= \left\{ .7207 \cdot e^{i2\pi(.7650)}, .7751 \cdot e^{i2\pi(.5683)}, .8415 \cdot e^{i2\pi(.9020)} \mid .3025 \right\}, \end{aligned}$$

In the same manner, we calculate the SV of each alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$. For example,

$$\widehat{S}(\hat{\Pi}_1) = \frac{1}{6} ((.4606 \times .2046 + .7148 \times .2046) + (.3392 \times .2046 + .1959 \times .2046) + (.7641 \times .2046 + .7751 \times .2046)) = .1107.$$

Then, using the same steps, we obtain the rest of the values as follows:

$$\widehat{S}(\hat{\Pi}_2) = 0.1429, \widehat{S}(\hat{\Pi}_3) = 0.1134, \widehat{S}(\hat{\Pi}_4) = 0.1793, \widehat{S}(\hat{\Pi}_5) = 0.2304$$

Finely, from Step 3, it gives the ranking of SFV as follows: $\widehat{S}(\hat{\Pi}_5) \succ \widehat{S}(\hat{\Pi}_4) \succ \widehat{S}(\hat{\Pi}_3) \succ \widehat{S}(\hat{\Pi}_2) \succ \widehat{S}(\hat{\Pi}_1)$. Therefore, the ranking of all alternatives will be as follows: $\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$. Therefore, we conclude



FIGURE 2. Statistical comparison between SV come by CPNWA & CPNWGA

that car $\mathcal{A}_5$ is the best choice, according to the experts opinion.

After running both operators CPNWA and CPNWGA on the set of CPNEs, Table 4 and Figure 2 indicate that the values generated by CPNWA are always significantly larger than those computed by CPNWGA. Furthermore, the difference between the ranking results is slight, as $\mathcal{A}_5$ is the best choice for both operators, while $\mathcal{A}_2$ is the worst choice of the five options.

TABLE 4. Comparison between the SV come by CPNWA & CPNWGA and the ranking of five alternatives. .

Operators	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	Ranking
CPNWA	.1870	.1965	.1619	.1620	.2251	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
CPNWGA	.1107	.1429	.1134	.1793	.2304	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$.

5.1. Exploring the sensitivity of weights

In this subsection, we examine the influence of weight vector changes of CPNWA and BCPNWGA operators on the sensitivity of the final decision. The SF scores for different values of w_i are presented in Table 5 and illustrated in Figure 3.

Table 4 shows that the ranking results for the alternatives remain constant within the established range even when the weight values change. Upon reviewing Table 4, it becomes clear that alternative $\mathcal{A}_5$ is the preferred alternative, while alternative $\mathcal{A}_3$ is the least preferred, and

TABLE 5. Sensitivity analysis results for the CPNWA and CPNWG operators

Operators	(w_1, w_2, w_3)	$\widehat{\mathbb{S}}(\hat{\Pi}_1)$	$\widehat{\mathbb{S}}(\hat{\Pi}_2)$	$\widehat{\mathbb{S}}(\hat{\Pi}_3)$	$\widehat{\mathbb{S}}(\hat{\Pi}_4)$	$\widehat{\mathbb{S}}(\hat{\Pi}_5)$	Ranking
<i>CPNWA</i>	(0.2, 0.5, 0.3)	.1443	.1483	.1006	.1288	.1910	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
	(0.3, 0.3, 0.4)	.1584	.1700	.1298	.2219	.2361	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
<i>CPNWG</i>	(0.2, 0.5, 0.3)	.2207	.2163	.1514	.1622	.2717	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
	(0.3, 0.3, 0.4)	.1903	.1891	.1463	.2219	.2565	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3$

the remaining alternatives range in preference value between A_5 and A_3 within the permissible range as illustrated in Figure 3. Thus, a high level of homogeneity is shown in the optimal selection from among all weights associated with the corresponding criteria for the available alternatives, and this represents conclusive evidence of the success and reliability of the method proposed in this work.

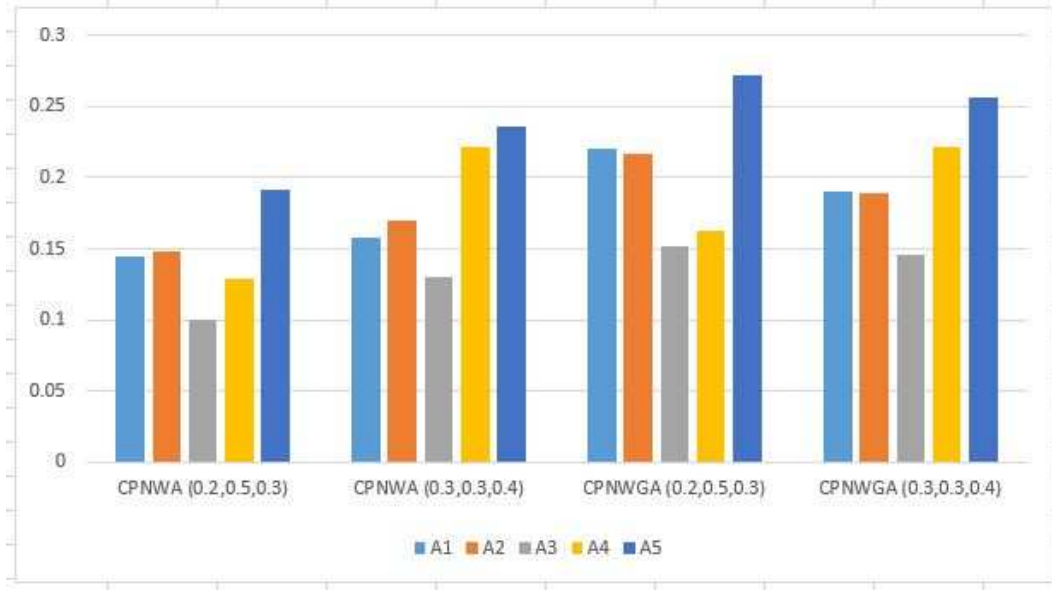


FIGURE 3. Accuracy of CPNWA and CPNWG operators with different weight values

6. Discussion and comparison

This section highlights the effectiveness of our suggested method in comparison to previous mathematical methods. To achieve this, we present a comparative analysis between our interpreted methods and the current studies [14, 32] mentioned in Section 1. Firstly, in reference [14], Karaaslan invented the effect of the degree of probability on one-dimensional information in NS structures, but these tools appear unable to handle periodic data with a time dimension. Secondly, in reference [32], Ali and Smarandache ignore the impact of the degree of probability on two-dimensional information in CNS structures. Based on the aforementioned discussion, it is clear that the methods in [32] are unable to handle MCDM present in CPNS environments. This reveals that our proposed method is superior to the current methods. CPNS amplifies the current methods: when the phase term of three membership functions (TMF, IMF, FMF) equals zero, it converts into PNS method, and if the degree of probability part equals zero, it transforms into CNSs method. In addition to the above points, in Table 3, we present a standard comparison between our proposed concept and some notions linked to the root genetic structure of our proposed concept, where the symbol $\checkmark$ indicates the success of the corresponding methods with respect to this property compared to our proposed concept, while the symbol $\times$ indicates failure. Based on Table 6, we conclude that the concepts with which our proposed concept was compared are unable to help the decision maker (user) in dealing with information from the CPN environment. Therefore, it can be judged that the

TABLE 6. Comparative examination of existing models using key criteria.

Methods	Domain	Co-domain	Presence of three membership degrees	Periodic na- ture	Probability de- gree
PFS [15]	$\hat{\mathbf{X}}$	Real unit interval	$\times$	$\times$	$\checkmark$
PIFS [15]	$\hat{\mathbf{X}}$	Real unit interval	$\times$	$\times$	$\checkmark$
NS [3]	$\hat{\mathbf{X}}$	Real unit interval	$\checkmark$	$\times$	$\times$
PNS [14]	$\hat{\mathbf{X}}$	Real unit interval	$\checkmark$	$\times$	$\checkmark$
CNS [32]	$\hat{\mathbf{X}}$	Complex unit inter- val	$\checkmark$	$\checkmark$	$\times$
The proposed method (CPNS)	$\hat{\mathbf{X}}$	Complex unit inter- val	$\checkmark$	$\checkmark$	$\checkmark$

method presented in this work addresses a gap in previous studies that were employed in the decision-making process.

6.1. *Limitations and future works*

Our proposed method exhibits some practical limitations, as outlined below. These practical limitations can be studied by researchers and used to enhance the practical utility and scalability of CPNS in the near future.

- (1) This study focuses primarily on the theoretical development and simulation stages of CPNS in the context of individual decision-making. It does not yet address group decision-making, dynamic data environments, or empirical verification using real-world, non-virtual datasets in specific fields such as healthcare, finance, supply chains, engineering, or environmental science.
 - (2) The mathematical structure of our proposed concept is built on single values, and this imposes constraints on data construction. Therefore, this structure can be developed by utilizing interval values, which provide greater flexibility for the user.
 - (3) Our proposed concept lacks mechanisms for structured parameterisation (soft set) [54], where this mechanism plays a vital role in interpreting the parametric properties of the components of the universal set.
 - (4) The number of experts reviewing the initial alternative values, relative to the parameter values, can be increased from one expert to a group of experts by integrating CPNS with the soft expert set [55].
 - (5) To ensure a clearer future roadmap, we recommend leverage the rich environment of currently available aggregation operators by applying them to our proposed concept. An example of such a Aggregation Operators is Aczel-Alsina aggregation operators, Dombi aggregation operators, Archimedean aggregation operators and so on.
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7. Conclusions

The randomness and ambiguity simultaneously inherent in everyday life situations cannot be ignored because they have a dynamic impact on the DM mechanism (select the optimal option). To meet this, we invented a new mathematical tool, which we point to as CPNS where the evaluation of any action using CNS structures has a probability score ranging from 0 to 1. Next, we established the traditional set theory operations, such as union and intersection between CPNSs, the complement of CPNS, and several properties of these operations, as well as discussed them with some numerical examples. Further, due to the high efficiency of AOs to cope with uncertain data intertwined with everyday life scenarios, we developed novel mathematical formulas for both CPNWA and CPNGA operators and study their results with some numerical examples. These operators have been successfully applied to deal with MCDM problems when a customer call an optimal selection based on several criteria. The suggested operator offers a comprehensive and flexible aggregation tool capable of simultaneously treating probabilistic, behavioural, and uncertain information by integrating the components of complex truth, complex indeterminacy, and complex falsehood. We explained the comparative analysis of the proposed method with current methods to highlight the advantages, reliability, and superiority of our method compared to existing methods. Finally, for future studies, there is a lot of scope for developing these mathematical tools to be more effective in capturing randomness and ambiguity data.

Abbreviations

FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
NS	Neutrosophic Set
SVNS	Single Value Neutrosophic Set
PFS	Possibility Fuzzy set
PNS	Possibility Neutrosophic Set
CNS	Complex Neutrosophic Set
PIFS	Possibility Intuitionistic Set
AF	Accuracy Function
SF	Score Function
CPNWAOs	Complex Possibility Neutrosophic Weighted Averaging Operators
CPNWGOs	Complex Possibility Neutrosophic Weighted Geometric Operators
MCDM	Multi-Criteria Decision Making
DM	Decision-Making
TMF	Truth-Membership Function
IMF	Indeterminacy Membership Function
FMF	Falsity-Membership Function
$\hat{\mathbf{X}}$	Universal Set
$\overset{\cdot}{\underset{\cdot}{\mathfrak{H}}} \overset{\cdot}{\mathfrak{T}}(\hat{\tau}) . e^{i2\pi(\diamond \overset{\cdot}{\mathfrak{T}}(\hat{\tau}))}$	Complex Truth-Membership Function
$\overset{\cdot}{\underset{\cdot}{\mathfrak{H}}} \overset{\cdot}{\mathfrak{I}}(\hat{\tau}) . e^{i2\pi(\diamond \overset{\cdot}{\mathfrak{I}}(\hat{\tau}))}$	Complex Indeterminacy-Function
$\overset{\cdot}{\underset{\cdot}{\mathfrak{H}}} \overset{\cdot}{\mathfrak{F}}(\hat{\tau}) . e^{i2\pi(\diamond \overset{\cdot}{\mathfrak{F}}(\hat{\tau}))}$	Complex Falsity-Membership Function
$\widetilde{\mathcal{P}}(\hat{\tau})$	Fuzzy Possibility-Membership Function

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Data availability

All data generated or analysed during this study are included in this published article.

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Conflicts of interest

Authors declare no conflicts of interest.

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