

A Finite Analytic Method for Two-Dimensional Power-Law Fluid Flow in Heterogeneous Porous Media Using Unstructured Grids

Mengwei Qing

University of Science and Technology of China

Jun Hu

University of Science and Technology of China

Zhifeng Liu

lzf123@ustc.edu.cn

University of Science and Technology of China

Xiaohong Wang

University of Science and Technology of China

Research Article

Keywords: Finite analytical method, Flow in porous media, Unstructured grids, Heterogeneity, Power-law Fluid

Posted Date: October 16th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11177189/v1>

License:  This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

A Finite Analytic Method for Two-Dimensional Power-Law Fluid Flow in Heterogeneous Porous Media Using Unstructured Grids

Mengwei Qing¹, Jun Hu¹, Zhifeng Liu^{1*}, Xiaohong Wang¹

¹Department of Thermal Science and Energy Engineering, University of Science and Technology of China, Hefei 230026, P. R. China

Short Title for the running head: 2D FAM for power law fluid flow in heterogeneous media with unstructured grids

Email address of corresponding author: lzf123@ustc.edu.cn

Abstract: An efficient finite analytical method (FAM) is developed for two-dimensional steady single-phase power-law fluid flow in heterogeneous porous media. The method is founded on a local analytical solution near the singular point at a permeability discontinuity, where the nodal solution exhibits a power-law radial dependence r^α . The smallest eigenvalue α_1 and its corresponding eigenfunction are determined using the fourth-order Runge-Kutta method combined with Newton's iteration. Based on this nodal solution, an alternative triangle-based numerical scheme is constructed. Numerical examples demonstrate that the FAM achieves significant improvements in both accuracy and convergence over the traditional weighted harmonic mean (HM) method under mesh refinement. A key advantage of the proposed FAM is that its accuracy is largely insensitive to medium heterogeneity and the power-law index n . For practical applications, a mesh refinement parameter of $m=4$ is recommended, which yields a relative error of less than 8% even in the worst-case scenario. In comparison, the traditional HM method requires substantially higher refinement ratios to achieve comparable accuracy, particularly for strongly heterogeneous porous media.

Keywords: Finite analytical method; Flow in porous media; Unstructured grids; Heterogeneity; Power-law Fluid.

1. Introduction

Many critical energy and environmental applications demand discretization schemes that preserve local mass balance on irregularly shaped domains, particularly in the presence of discontinuous coefficients describing media properties. For instance, geometrically complex geological structures—such as faults, cross-stratified beddings, and large-scale inclined strata—can profoundly influence fluid flow through subsurface formations. Standard finite difference methods, typically formulated on rectangular grids, lack the geometric flexibility required to accurately model flow through or around such features. Consequently, unstructured mesh methods have been developed over the past several decades, substantially relaxing mesh constraints and enabling more realistic representations of complex domains.

Even with unstructured meshes, large discontinuities in medium properties pose a significant challenge. In groundwater modeling and reservoir simulation, permeability can vary over distances far shorter than the overall simulation domain, making it reasonable to treat the permeability field as discontinuous. Permeability contrasts spanning four to six orders of magnitude are sometimes encountered in petroleum reservoir simulation. Such extreme heterogeneity demands numerical schemes with a proper definition of internodal transmissibility across cell interfaces. On rectangular grids, assuming a linear pressure distribution within each cell, the internodal transmissibility reduces to the harmonic average form, derived from the continuity of potential and flux at cell edges. Most existing numerical methods preserve this principle when extended to unstructured grids.

Durlofsky [1] developed a triangle-based mixed finite element–finite volume technique to handle generally anisotropic, highly discontinuous permeability fields, in which basis functions are defined linearly over each element. Subsequently, Aavatsmark et al. [2,3] presented similar schemes on unstructured grids under the name of multi-point flux approximation (MPFA) methods. Edwards et al. [4–8] proposed locally conservative, flux-continuous formulations for unstructured grids composed of quadrilateral or triangular cells. Their approach, known as the

CVD-MPFA (control-volume distributed multi-point flux approximation) method, encompasses a family of schemes for general grids, including consistent nine-point stencils on structured quadrilateral meshes that accommodate all permeability tensor types—from diagonal isotropic to full tensors. MPFA schemes have been shown to outperform standard CVFE (control-volume finite element) methods for flow problems involving property discontinuities.

In general, these methods reduce to the standard five-point stencil for isotropic permeability fields on rectangular grids, where the internodal transmissibility takes the traditional harmonic mean form for discontinuous permeabilities. However, it is well established that the harmonic mean underestimates interface flux, particularly under strong heterogeneity, and the resulting numerical error grows unboundedly as the cell permeability ratio increases [9]. Cordazzo [10,11] simulated single-phase flow in a 4×4 cell domain and found that the harmonic average scheme requires refinement up to a 700×700 grid to obtain a reference solution. Romeu and Noetinger [12] analytically investigated the accuracy of numerical schemes for equivalent permeability and confirmed the existence of strong bias and very slow convergence in traditional methods. In other words, traditional schemes without sufficient refinement may introduce considerable errors, especially in strongly heterogeneous media.

For two-dimensional single-phase Darcy flow on rectangular grids, the pressure field exhibits power-law behavior, and its gradient diverges as a node joining regions of different permeability is approached [13,14]. This singular behavior causes traditional numerical schemes to converge very slowly under strong permeability heterogeneity. To improve computational efficiency, a finite analytic method (FAM) for single-phase Darcy flow has been proposed, constructed on the basis of the nodal analytic power-law solution. This method yields accurate solutions with only 2×2 or 3×3 subdivisions, and its convergence rate is independent of permeability heterogeneity. A two-dimensional FAM on unstructured grids has also been developed for Darcy flow [15], demonstrating high performance.

In this study, we extend the two-dimensional FAM on unstructured grids from Darcy flow to power-law fluid flow. As a representative class of non-Newtonian

fluids, power-law fluids are characterized by a relatively simple constitutive relation between stress and strain rate and are frequently encountered in practical applications. Understanding power-law fluid flow in porous media is relevant to various fields, including petroleum engineering, paper manufacturing, and composite manufacturing. In the one-dimensional case, the Blake–Kozeny equation—which describes a power-law relationship between seepage velocity and macroscopic pressure gradient—has been experimentally validated through observations of power-law fluid flow in packed tubes [16]. This one-dimensional relation is further corroborated by lattice Boltzmann simulations in two-dimensional porous media with a fixed macroscopic pressure gradient direction [17]. In higher dimensions ($d \geq 2$), an extended form of the Blake–Kozeny equation, known as the Darcy-scale filtration equation, is widely accepted for isotropic porous media [18,19].

In this study, we extend the 2D FAM on unstructured grids from Darcy flow to power-law fluid flow. As a representative class of non-Newtonian fluids, power-law fluids feature a relatively simple constitutive relation between stress and strain rate and arise frequently in practice. Knowledge of power-law fluid flow in porous media is relevant to many applications, including petroleum engineering, paper manufacturing, and composite manufacturing. In the one-dimensional case, the Blake – Kozeny equation—describing a power-law relation between the seepage velocity and the macroscopic pressure gradient—has been confirmed experimentally by observing power-law fluid flow through packed tubes [16]. This one-dimensional relation is further supported by lattice Boltzmann simulations in two-dimensional porous media with a fixed macroscopic pressure gradient direction [17]. In higher dimensions ($d \geq 2$), an extended form of the Blake – Kozeny equation (the Darcy-scale filtration equation) is widely accepted for isotropic porous media [18,19].

The remainder of this paper is organized as follows. Section 2 presents the governing equation for power-law fluid flow in porous media and its traditional discretization on unstructured grids. Section 3 derives the analytic solution of the governing equation in angular domains of arbitrary shape. Section 4 describes the construction of the FAM based on the analytic solution. Section 5 provides numerical

examples that illustrate the accuracy and efficiency of the proposed FAM. Section 6 discusses the key differences between the FAM construction for linear Darcy flow and that for nonlinear power-law fluid flow. Finally, Section 7 concludes the paper.

2. Governing Equation and Traditional Discretization on Unstructured Grids

We consider incompressible, steady-state power-law fluid flow in heterogeneous porous media. The governing equation is [20]:

$$\mathbf{v} = -\frac{k\nabla p}{\mu|\mathbf{v}|^{n-1}} \quad (2.1)$$

where $\mathbf{v}$ is the seepage velocity, p is the fluid pressure, n is the power-law fluid index, and μ is the reference viscosity. The permeability k is treated as a scalar in this study and is prescribed as a piecewise constant field, which is standard practice in numerical simulation. Equation (2.1) can then be rewritten as:

$$\mathbf{v} = -\left(\frac{k}{\mu}\right)^{1/n} |\nabla p|^{1/n-1} \nabla p \quad (2.2)$$

The unit of the permeability k is not essential in this study, so its unit is omitted through the following discussion.

For incompressible flow, the continuity equation reduces to:

$$\nabla \cdot \mathbf{v} = 0 \quad (2.3)$$

Substituting Eq. (2.1) into Eq. (2.3) and assuming constant viscosity μ yields:

$$\nabla \cdot \left(k^{1/n} |\nabla p|^{1/n-1} \nabla p \right) = 0 \quad (2.4)$$

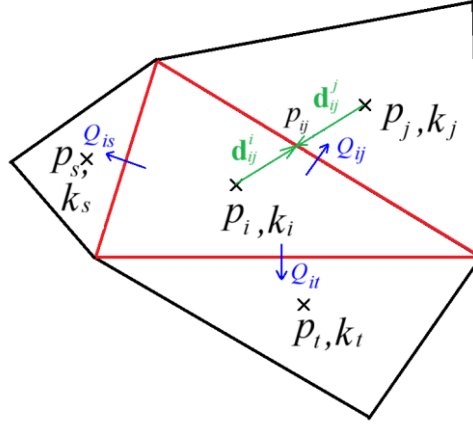


Figure 1. Schematic of the discretization of Eq. (2.4) on triangular meshes.

We briefly describe the traditional discretization of Eq. (2.4). As illustrated in Figure 1, the discrete continuity equation for control volume “ p_i ” on a triangular mesh reads:

$$Q_{ij} + Q_{is} + Q_{it} = 0 \quad (2.5)$$

where Q_{ij} , Q_{is} and Q_{it} denote the flow rates from the adjacent control volumes “ p_j ”, “ p_s ” and “ p_t ” into the control volume “ p_i ”, respectively. To compute the flow rate Q across an interface, the governing equation is projected onto the line connecting the two discrete pressure nodes. Taking Q_{ij} as an example, we project Eq. (2.1) onto the line segment connecting nodes “ p_i ” and “ p_j ”:

$$v_{ij} = -\frac{k^{1/n}}{\mu^{1/n}} \left| \frac{dp}{dl_{ij}} \right|^{1/n-1} \frac{dp}{dl_{ij}} \quad (2.6)$$

where l_{ij} denotes the line segment from node “ p_i ” to node “ p_j ”.

Since the permeabilities of control volumes “ p_i ” and “ p_j ”—denoted as k_i and k_j , respectively—differ, an auxiliary pressure variable p_{ij} is introduced at their shared interface. Enforcing continuity of the normal velocity across the interface yields:

$$\begin{cases} \mathbf{v}_{ij}^i = -\frac{k_i^{1/n}}{\mu^{1/n}} \left(\frac{p_i - p_{ij}}{d_{ij}^i} \right)^{1/n} \\ \mathbf{v}_{ij}^j = -\frac{k_j^{1/n}}{\mu^{1/n}} \left(\frac{p_{ij} - p_j}{d_{ij}^j} \right)^{1/n} \\ \mathbf{v}_{ij}^i = \mathbf{v}_{ij}^j \end{cases} \quad (2.7)$$

This condition yields the following expression for the interface pressure:

$$p_{ij} = \frac{k_i d_{ij}^j p_i + k_j d_{ij}^i p_j}{k_i d_{ij}^j + k_j d_{ij}^i} \quad (2.8)$$

The corresponding interface flow rate can thus be expressed as:

$$Q_{ij} = \mathbf{v}_{ij} S_{ij} \cos(\mathbf{v}_{ij}, \mathbf{n}_S) = -\frac{S_{ij}}{\mu^{1/n}} \frac{k_i^{1/n} k_j^{1/n} d_{ij}^{1/n}}{(k_i d_{ij}^j + k_j d_{ij}^i)^{1/n}} \left(\frac{p_i - p_j}{d_{ij}} \right)^{1/n} \cos(\mathbf{v}_{ij}, \mathbf{n}_{S_{ij}}) \quad (2.9)$$

where S_{ij} denotes the measure of the interface between control volumes “ p_i ” and “ p_j ”, d_{ij} is the distance between nodes p_i and p_j with $d_{ij} = d_{ij}^i + d_{ij}^j$, and $\mathbf{n}_{S_{ij}}$ represents the unit normal vector of the interface S_{ij} .

The average permeability k_{ij} between adjacent cells “ p_i ” and “ p_j ” is defined as:

$$k_{ij}|_{\text{HM}} = \frac{k_i k_j d_{ij}}{k_i d_{ij}^j + k_j d_{ij}^i} \quad (2.10)$$

The subscript “HM” denotes the weighted harmonic average. With this definition, the flow rate Q_{ij} can be rewritten as:

$$Q_{ij} = -\frac{k_{ij}^{1/n}|_{\text{HM}} S_{ij}}{\mu^{1/n}} \left(\frac{p_i - p_j}{d_{ij}} \right)^{1/n} \cos(\mathbf{v}_{ij}, \mathbf{n}_{S_{ij}}) \quad (2.11)$$

The remaining flow rates Q_{is} and Q_{it} are computed in an analogous manner.

The corresponding average permeabilities at their respective interfaces are given by:

$$k_{is}|_{\text{HM}} = \frac{k_i k_s d_{is}}{k_i d_{is}^i + k_s d_{is}^s}, \quad k_{it}|_{\text{HM}} = \frac{k_i k_t d_{it}}{k_i d_{it}^i + k_t d_{it}^t} \quad (2.12)$$

Collecting these terms, we obtain the nodal equation:

$$\begin{aligned}
& k_{ij}^{1/n} \Big|_{\text{HM}} S_{ij} \left(\frac{p_i - p_j}{d_{ij}} \right)^{1/n} \cos(\mathbf{v}_{ij}, \mathbf{n}_{S_{ij}}) + k_{is}^{1/n} \Big|_{\text{HM}} S_{is} \left(\frac{p_i - p_s}{d_{is}} \right)^{1/n} \cos(\mathbf{v}_{is}, \mathbf{n}_{S_{is}}) \\
& + k_{it}^{1/n} \Big|_{\text{HM}} S_{it} \left(\frac{p_i - p_t}{d_{it}} \right)^{1/n} \cos(\mathbf{v}_{it}, \mathbf{n}_{S_{it}}) = 0
\end{aligned} \tag{2.13}$$

In this formulation, the average permeability is derived by projecting the governing equation onto a one-dimensional direction, resulting in a form analogous to harmonic averaging. It is well established that harmonic averaging significantly underestimates flow conductivity in strongly heterogeneous media. The FAM algorithm developed in the following sections addresses this limitation by incorporating the local analytic solution near permeability discontinuities.

3. Local Analytic Solution Near a Singular Point

For Darcy flow (with the power-law fluid index $n=1$), the analytic solution can be obtained via separation of variables. However, the resulting pressure gradient diverges at singular points where permeabilities are discontinuous, which in turn leads to a divergent velocity. Motivated by this observation, we extend the separation-of-variables approach to power-law fluid flow near such singular points.

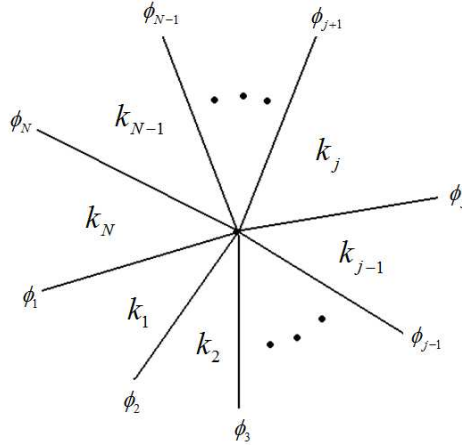


Figure 2. Schematic of the angular domain of arbitrary shape with different permeabilities.

As illustrated in Figure 2, the angular domain is partitioned into N subdomains. We can then rewrite problem (2.4) as:

$$\left\{ \begin{array}{l} \nabla \cdot (k_1 |\nabla p_1|^{1/n-1} \nabla p_1) = 0, \quad \theta \in [\phi_1, \phi_2] \\ \nabla \cdot (k_2 |\nabla p_2|^{1/n-1} \nabla p_2) = 0, \quad \theta \in [\phi_2, \phi_3] \\ \dots \\ \nabla \cdot (k_j |\nabla p_j|^{1/n-1} \nabla p_j) = 0, \quad \theta \in [\phi_j, \phi_{j+1}] \\ \dots \\ \nabla \cdot (k_N |\nabla p_N|^{1/n-1} \nabla p_N) = 0, \quad \theta \in [\phi_N, \phi_1 + 2\pi] \end{array} \right. \quad (3.1)$$

with the interface conditions of pressure continuity:

$$\left\{ \begin{array}{l} p_1|_{\theta=\phi_2} = p_2|_{\theta=\phi_2} \\ p_2|_{\theta=\phi_3} = p_3|_{\theta=\phi_3} \\ \dots \\ p_j|_{\theta=\phi_{j+1}} = p_{j+1}|_{\theta=\phi_{j+1}} \\ \dots \\ p_N|_{\theta=\phi_1+2\pi} = p_1|_{\theta=\phi_1} \end{array} \right. \quad (3.2)$$

and normal flux continuity:

$$\left\{ \begin{array}{l} k_1 |\nabla p_1|^{1/n-1} \frac{\partial p_1}{\partial \theta} \Big|_{\theta=\phi_2} = k_2 |\nabla p_2|^{1/n-1} \frac{\partial p_2}{\partial \theta} \Big|_{\theta=\phi_2} \\ k_2 |\nabla p_2|^{1/n-1} \frac{\partial p_2}{\partial \theta} \Big|_{\theta=\phi_3} = k_3 |\nabla p_3|^{1/n-1} \frac{\partial p_3}{\partial \theta} \Big|_{\theta=\phi_3} \\ \dots \\ k_j |\nabla p_j|^{1/n-1} \frac{\partial p_j}{\partial \theta} \Big|_{\theta=\phi_{j+1}} = k_{j+1} |\nabla p_{j+1}|^{1/n-1} \frac{\partial p_{j+1}}{\partial \theta} \Big|_{\theta=\phi_{j+1}} \\ \dots \\ k_N |\nabla p_N|^{1/n-1} \frac{\partial p_N}{\partial \theta} \Big|_{\theta=\phi_1+2\pi} = k_1 |\nabla p_1|^{1/n-1} \frac{\partial p_1}{\partial \theta} \Big|_{\theta=\phi_1} \end{array} \right. \quad (3.3)$$

We assume a solution of the separated-variable type in polar coordinates, whose radial dependence follows a power law r^α , with α being the eigenvalue. This leads to:

$$p_j = p_0 + A^n r^\alpha f_j(\theta; \alpha), \quad i=1,2,\dots,N \quad (3.4)$$

where the subscript i denotes the quadrant index, p_0 is the pressure at the origin, A is a constant determined by the outer boundary conditions, and $f_j(\theta; \alpha)$ represents the the part of the eigenfunction in the i th angular domain. Substituting Eq. (3.4) into Eq. (2.2) yields:

$$\mathbf{v}_j = v_{r,j} \mathbf{e}_r + v_{\theta,j} \mathbf{e}_\theta = -Ar^{\frac{\alpha-1}{n}} \frac{k_j^{1/n}}{\mu^{1/n}} \left[\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \left(\alpha f_j \mathbf{e}_r + \frac{df_j}{d\theta} \mathbf{e}_\theta \right) \quad (3.5)$$

In polar coordinates, the continuity equation $\nabla \cdot \mathbf{v}_j = 0$ takes the following form:

$$\frac{\partial v_{r,j}}{\partial r} + \frac{1}{r} \frac{\partial v_{\theta,j}}{\partial \theta} = 0 \quad (3.6)$$

Substituting Eq. (3.5) into Eq. (3.6) yields an ordinary differential equation (ODE) in θ for f_j :

$$\frac{\alpha-1+n}{n} \alpha f_j \left(\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right)^{\frac{1}{2n} - \frac{1}{2}} + \frac{d \left[\left(\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right)^{\frac{1}{2n} - \frac{1}{2}} \frac{df_j}{d\theta} \right]}{d\theta} = 0 \quad (3.7)$$

subject to the interface conditions:

$$\begin{cases} f_1|_{\theta=\phi_2} = f_2|_{\theta=\phi_2} \\ f_2|_{\theta=\phi_3} = f_3|_{\theta=\phi_3} \\ \dots \\ f_j|_{\theta=\phi_{j+1}} = f_{j+1}|_{\theta=\phi_{j+1}} \\ \dots \\ f_N|_{\theta=\phi_1+2\pi} = f_1|_{\theta=\phi_1} \end{cases} \quad (3.8)$$

$$\left\{ \begin{array}{l}
k_1^{1/n} \left[\alpha^2 f_1^2 + \left(\frac{df_1}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_1}{d\theta} \bigg|_{\theta=\phi_2} = k_2^{1/n} \left[\alpha^2 f_2^2 + \left(\frac{df_2}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_2}{d\theta} \bigg|_{\theta=\phi_2} \\
k_2^{1/n} \left[\alpha^2 f_2^2 + \left(\frac{df_2}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_2}{d\theta} \bigg|_{\theta=\phi_3} = k_3^{1/n} \left[\alpha^2 f_3^2 + \left(\frac{df_3}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_3}{d\theta} \bigg|_{\theta=\phi_3} \\
\ldots \\
k_j^{1/n} \left[\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_j}{d\theta} \bigg|_{\theta=\phi_{j+1}} = k_{j+1}^{1/n} \left[\alpha^2 f_{j+1}^2 + \left(\frac{df_{j+1}}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_{j+1}}{d\theta} \bigg|_{\theta=\phi_{j+1}} \\
\ldots \\
k_N^{1/n} \left[\alpha^2 f_N^2 + \left(\frac{df_N}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_N}{d\theta} \bigg|_{\theta=\phi_1+2\pi} = k_1^{1/n} \left[\alpha^2 f_1^2 + \left(\frac{df_1}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_1}{d\theta} \bigg|_{\theta=\phi_1}
\end{array} \right. \quad (3.9)$$

Equations (3.7)–(3.9) constitute an eigenvalue problem that determines a sequence of eigenvalues α (i.e., $\alpha_1, \alpha_2, \alpha_3, \dots$) along with the corresponding eigenfunctions $f_j(\theta; \alpha)$. In this study, the fourth-order Runge–Kutta method combined with Newton's iteration is employed to solve Eq. (3.7) subject to the conditions (3.8) and (3.9).

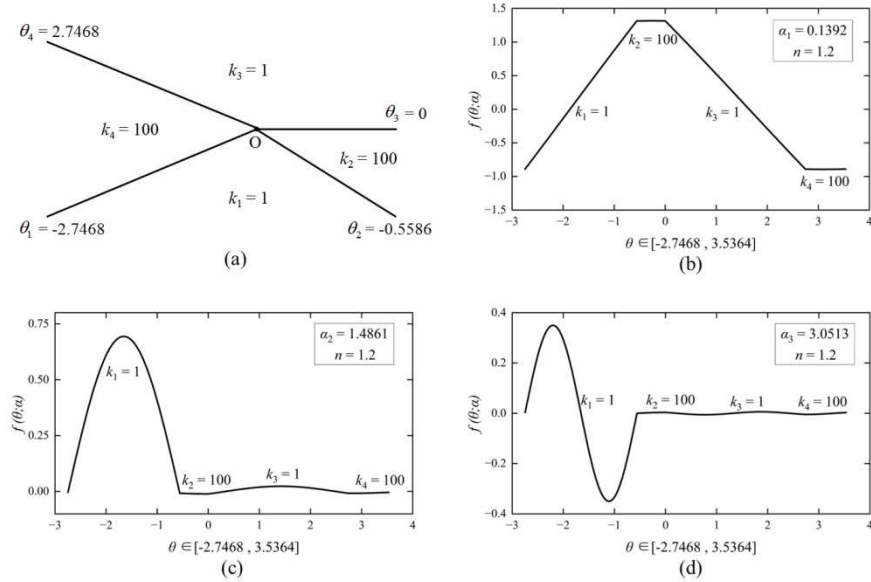


Figure 3. The calculated functions $f(\theta)$ for the first three eigenvalues (α_1, α_2 and α_3) with $n = 1.2$, $k_1 = k_3 = 1$ and $k_2 = k_4 = 100$.

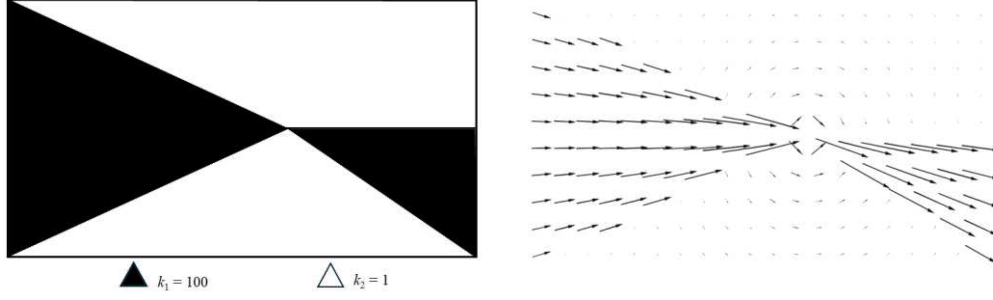


Figure 4. The velocity distribution in the vicinity of the singular point p_0 under the eigenvalue $\alpha_1 = 0.1392$ with $n = 1.2$, $k_1 = k_3 = 1$ and $k_2 = k_4 = 100$.

A specific example is presented here. Figure 3 shows the eigenfunction $f(\theta)$ with $n = 1.2$, $k_1 = k_3 = 1$ and $k_2 = k_4 = 100$. The first three eigenvalues are obtained as $\alpha_1 = 0.1392$, $\alpha_2 = 1.4861$ and $\alpha_3 = 3.0513$, with the corresponding eigenfunctions $f(\theta; \alpha_i)$ plotted in Figure 3. The velocity field computed using the first eigenvalue $\alpha_1 = 0.1392$ is displayed in Figure 4. Since $\alpha_1 < 1$ and $|v| \propto |\nabla P|^{1/n} \propto r^{(\alpha-1)/n}$, the velocity diverges at the singular point “O”. It is observed that the flow is channeled through a very narrow neck-like region as it passes from one high-permeability domain to another. It is worth noting that the original governing equation (2.4) is nonlinear except for $n=1$; therefore, solutions corresponding to different eigenvalues cannot be linearly superimposed. Thus, only one eigenfunction with its eigenvalue $\alpha \in (0,1)$ is relevant for constructing the FAM.

4. The FAM for Power-Law Fluid Flow Based on triangular mesh

The FAM was originally introduced by Chen [21, 22], combining analytic and numerical approaches: the governing algebraic equations are derived by integrating the local nodal analytic solution. The primary advantage of the FAM lies in its high computational accuracy; however, its main limitation is that a nodal analytic solution must be available in advance.

Fortunately, the nodal analytic solution for power-law fluid flow has been derived and is expressed by relation. Based on this solution, a two-dimensional FAM is constructed to solve Eq. (2.4) on unstructured grids. The method is described here for triangular grids but can be readily extended to other unstructured grid types, such

as polygonal meshes.

As shown in Figure 5, the triangular cell Δ_{ABC} is partitioned into three influence regions, each associated with one of its vertices “A”, “B”, and “C”. For instance, the fluid flow within the shaded region is governed by the nodal solution at vertex “A”. Accordingly, each cell edge is divided into two segments, which are assigned to the influence regions of the two endpoints of that edge. The discretized form of Eq. (2.4) for the control volume (outlined in red) is given by:

$$Q_{A,1} + Q_{A,2} + Q_{B,1} + Q_{B,2} + Q_{C,1} + Q_{C,2} = 0 \quad (4.1)$$

where the subscripts “A”, “B”, and “C” denote the dominant singular points. The discrete pressure points are defined at cell centers, making this a cell-centered scheme.

We illustrate the core idea of the FAM by taking the calculation of $Q_{A,1}$ as an example. Restricting the eigenvalue α to the interval $(0,1)$ yields a solution in which the velocity diverges at the singular point, which is the most physically significant among all admissible eigenvalues. We denote the smallest eigenvalue by α_1 ($0 < \alpha_1 < 1$) and employ its corresponding eigenfunction to construct the FAM. In numerical schemes, the key issue lies in accurately computing the flow rate across the interface between two adjacent cells. The conventional approach, based on relation (2.10), tends to consistently underestimate the flow rate, as noted earlier. In the FAM, the interface flux is instead evaluated using the local analytical solution (3.4) with $\alpha = \alpha_1$. It should be noted that the eigenfunction $f_i(\theta)$ cannot be expressed in closed form and is only available numerically. Nevertheless, since it is the solution of an one-dimensional ODE, it can be regarded as numerically exact when computed with a sufficiently fine discretization.

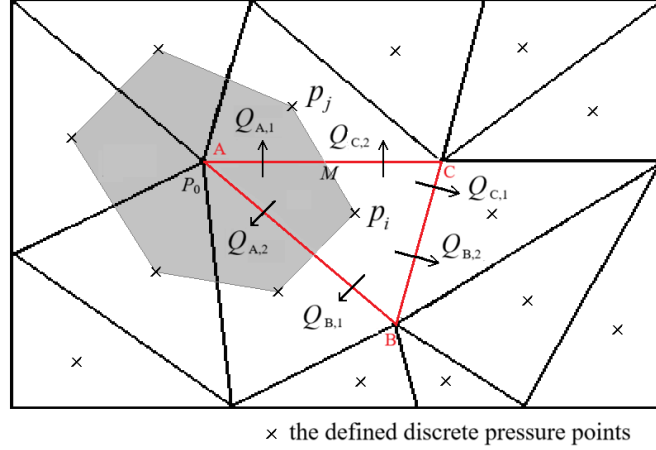


Figure 5. Schematic of the discretization of Eq. (2.4) based on triangular meshes; the shaded region represents the influence area of grid node p_0 .

To compute $Q_{A,1}$, we first rewrite the exact solution at the singular point as follows:

$$\begin{cases} p_i = p_0 + A^n r_i^\alpha f_i(\theta_i; \alpha) \\ p_j = p_0 + A^n r_j^\alpha f_j(\theta_j; \alpha) \end{cases} \quad (4.2)$$

where (r_i, θ_i) and (r_j, θ_j) denote the polar coordinates of the discrete pressures p_i and p_j , respectively. Consequently, we obtain:

$$A^n = \frac{p_i - p_j}{r_i^\alpha f_i(\theta_i; \alpha) - r_j^\alpha f_j(\theta_j; \alpha)} \quad (4.3)$$

Combining relations (4.3) and (3.5), the corresponding flux $Q_{A,1}$ across the interface $\overline{P_0 M}$ is given by:

$$\begin{aligned} Q_{(a),1} &= \int_0^{|P_0 M|} v_\theta dr = -A \frac{k_j^{1/n}}{\mu^{1/n}} \left\{ \left[\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_j}{d\theta} \right\}_{\theta=\phi_j} \int_0^{|P_0 M|} r^{\frac{\alpha-1}{n}} dr \\ &= -A \frac{n}{\alpha + n - 1} |P_0 M|^{\frac{\alpha+n-1}{n}} \frac{k_j^{1/n}}{\mu^{1/n}} \left\{ \left[\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_j}{d\theta} \right\}_{\theta=\phi_j} \\ &= - \left[\frac{p_i - p_j}{r_i^\alpha f_i(\theta_i; \alpha) - r_j^\alpha f_j(\theta_j; \alpha)} \right]^{1/n} \frac{n}{\alpha + n - 1} |P_0 M|^{\frac{\alpha+n-1}{n}} \frac{k_j^{1/n}}{\mu^{1/n}} \left\{ \left[\alpha^2 f_j^2 + \left(\frac{df_j}{d\theta} \right)^2 \right]^{\frac{1}{2n} - \frac{1}{2}} \frac{df_j}{d\theta} \right\}_{\theta=\phi_j} \end{aligned} \quad (4.4)$$

The remaining five fluxes across the interfaces of the control volume Δ_{ABC} , namely $Q_{A,2}$, $Q_{B,1}$, $Q_{B,2}$, $Q_{C,1}$ and $Q_{C,2}$, can be evaluated in a similar manner. Substituting these flux expressions into Eq. (4.1) yields the nodal equation for the

discrete pressure variable p_i . Upon incorporating the boundary conditions, the resulting system of equations for the discrete pressures $\{p_i, p_j, \dots\}$ can be solved, thereby yielding the numerical solution to Eq. (2.4).

5 Numerical examples

In the preceding section, the FAM scheme was developed for solving Eq. (2.4), which governs power-law fluid flow in heterogeneous fields, on triangular cells. The eigenfunction corresponding to the smallest eigenvalue α_1 ($0 < \alpha_1 < 1$) is employed to compute the flux across each cell interface. In this article, numerical simulations are performed on triangular-based grids, with discrete pressure points located at the centroids of the triangular elements, although other choices of collocation points are also permissible. To achieve higher accuracy, the original grids are refined into $m \times m$ congruent triangular subgrids, as illustrated in Figure 6.

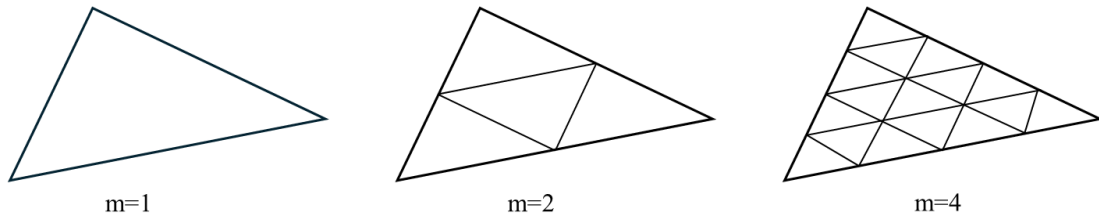


Figure 6. Schematic illustration of mesh refinement for triangular grids.

5.1 Numerical Test I: 2D Checkerboard Problem

To evaluate the numerical performance of the proposed FAM in strongly heterogeneous permeability fields, a classic two-dimensional checkerboard benchmark problem is adopted in this study. This test case is widely recognized as a rigorous validation tool for porous media flow simulators, owing to the availability of an analytical solution for the equivalent permeability ($\bar{k} = \sqrt{k_1 k_2}$) in the Darcy flow limit ($n=1$). As illustrated in Figure 7, the computational domain features a checkerboard pattern composed of alternating triangular elements with distinct permeability values: $k_1 = 1$ (dark gray regions) and k_2 (white regions), which span multiple orders of magnitude to produce varying permeability contrasts. Dirichlet boundary conditions of $P_{in} = 1$ and $P_{out} = 0$ are prescribed on the left and right

boundaries, respectively, while the remaining two sides are set as impermeable.

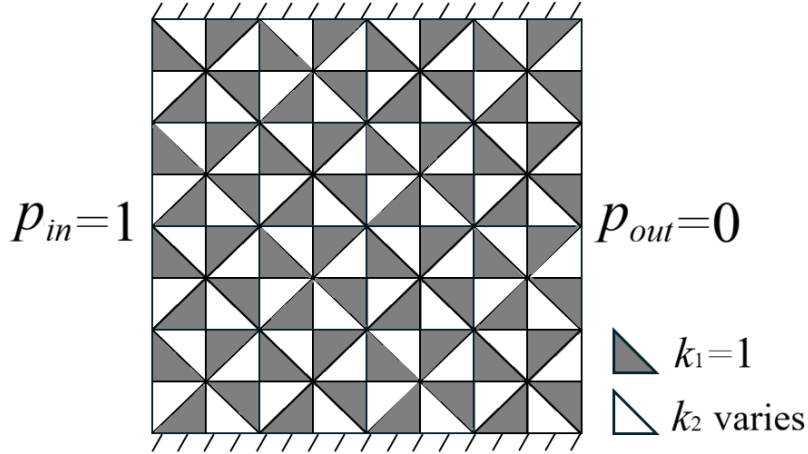


Figure 7. Schematic illustration of the two-dimensional checkerboard benchmark problem.

The equivalent permeability in the x -direction, denoted as $\bar{k}_x$, is given by:

$$\bar{k}_x = \frac{\mu L_x}{\Delta p_x} \left(\frac{Q_x}{L_y} \right)^n \quad (5.1)$$

where Q_x denotes the total flow rate in the x -direction, and L_x and L_y represent the length and width of the two-dimensional domain, respectively.

To validate the proposed method against theoretical results, we first set the power-law index to $n=1$ and fix the refinement parameter at $m=4$, a relatively coarse resolution. Figure 8 compares the equivalent permeability $\bar{k}_x$ obtained from the FAM, the traditional harmonic mean (HM) algorithm, and the analytical solution, across a range of permeability contrasts k_2/k_1 at $n=1.0$. The FAM predictions exhibit excellent agreement with the analytical solution over the entire range of k_2/k_1 . In sharp contrast, the HM algorithm suffers from severe systematic underestimation of the equivalent permeability. This behavior stems from the inherent limitation of the harmonic mean approach: the inter-nodal transmissibility, calculated as $2k_1k_2/(k_1 + k_2)$, is always smaller than the theoretical equivalent permeability $\sqrt{k_1k_2}$. As the permeability contrast increases, this discrepancy grows rapidly, ultimately rendering the HM method incapable of providing accurate solutions in strongly heterogeneous media under coarse grids.

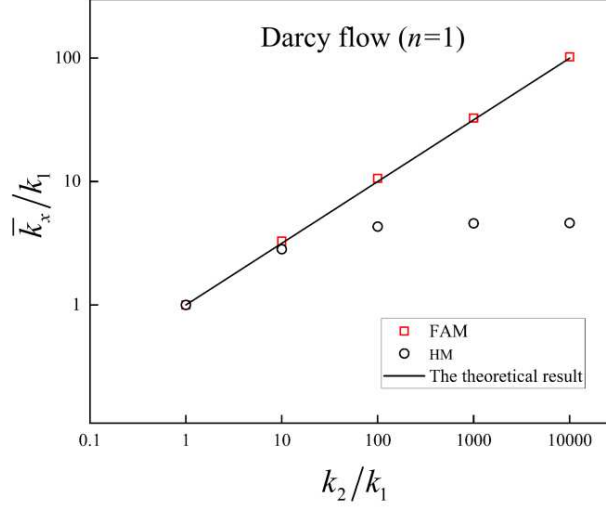


Figure 8. Comparisons of the equivalent permeability $\bar{k}_x$ between the theoretical result and the calculations of different numerical schemes (FAM and HM) for different permeability contrasts k_2/k_1 at $n=1.0$.

We next examine the convergence behavior of the FAM and the traditional HM method toward the reference solution as the grid refinement parameter m increases. In this test, we fix $k_1 = 1$ and $k_2 = 100$, while the power-law fluid index n varies from 0.8 to 1.5. Figure 9 presents the comparison results, from which several observations can be drawn. First, the accuracy of the FAM is nearly independent of the power-law fluid index n . In contrast, the HM method exhibits a clear dependence on n : the smaller the value of n , the poorer the accuracy. This trend is physically intuitive, since as $n \rightarrow \infty$, the velocity field becomes uniform and the numerical error tends to vanish. Second, the FAM achieves significantly higher accuracy than the HM method, and its results converge rapidly to the reference value as m increases. Although the relative errors of the FAM may exceed 15% on the coarsest grid $m=1$ for relatively large n (e.g., $n=1.2$ or 1.5), they drop sharply to 7% and 4% at $m=2$ and $m=4$, respectively, regardless of the value of n . In stark contrast, the HM method suffers from severe inaccuracies under coarse discretization: the relative errors reach as high as -90%, -73%, -57%, and -40% at $m=1$ for $n=0.8$, 1.0, 1.2 and 1.5, respectively. Furthermore, its convergence rate with respect to m is considerably slower than that of the FAM.

In summary, the proposed FAM demonstrates significantly higher accuracy than the traditional HM method. More importantly, its accuracy remains consistently high

and is largely independent of both the heterogeneity strength—characterized by the permeability ratio k_2/k_1 —and the power-law fluid index n . The relative errors remain below 7% and 4% for the relatively coarse grids $m=2$ and $m=4$, respectively, indicating rapid convergence to the reference solution as the refinement parameter m increases. In contrast, the accuracy of the HM method is highly sensitive to both the heterogeneity strength and the power-law fluid index n . For strongly heterogeneous media, the HM method consistently and severely underestimates the equivalent permeability, yielding errors that are not controllable. For instance, even with highly refined grids $m=256$, the relative error still reaches -58% for $n=0.8$ (see Figure 9).

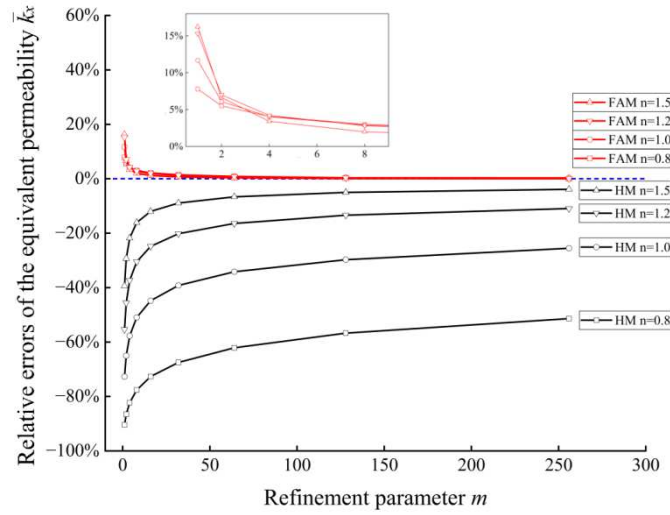


Figure 9. Relative errors of the equivalent permeability $\bar{k}_x$ as a function of the subdivision parameters m under various power-law fluid indices n for the checkerboard case ($k_1 = 1$, $k_2 = 100$).

5.2 Numerical Test II: Regular Hexagonal Model

The simulation setup is illustrated in Figure 10a: the domain is a regular hexagon partitioned into six original equilateral triangles, with permeability values specified as shown. Refined grids corresponding to $m=2$ and $m=3$ are displayed in Figure 10b and 10c, respectively. The upper and lower boundaries are set as impermeable, whereas the left and right boundaries are subject to first-type (Dirichlet) conditions. Upon solving Eq. (2.4), the total flux ($Q = Q_{in} = Q_{out}$) is evaluated. Figure 11 reports the relative error of Q , defined as $\varepsilon(m) = \frac{Q(m) - Q_{ref}}{Q_{ref}} \times 100\%$, as a function of m for both the FAM and the HM method. Here, Q_{ref} is the converged reference flux,

obtained with a sufficiently large m . In principle, $\lim_{m \rightarrow \infty} \varepsilon(m) = 0$ for any consistent scheme, and the convergence rate determines the efficiency of the method. In this work, the FAM result at $m=512$ is taken as the reference, since it is indistinguishable from the result at $m=256$, confirming that the solution has reached convergence. The HM result, derived from the relation in Eq. (2.11), is also included for comparison.

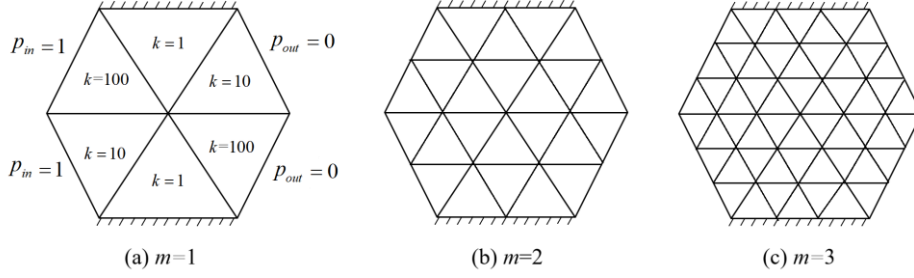


Figure 10. Schematic illustration of the regular hexagonal domain problem.

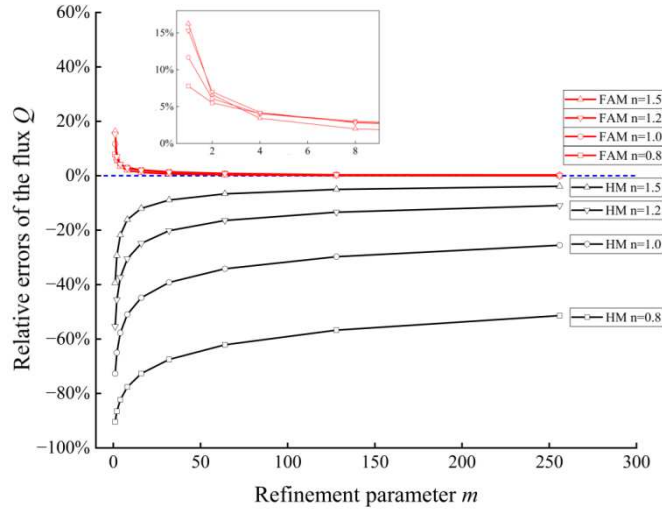


Figure 11. Relative errors of Q as a function of the subdivision parameters m under various power-law flow indices n for the regular hexagon problem.

Figure 11 presents the relative errors of the flux Q obtained with various subdivision parameters m across a range of power-law fluid indices n . This numerical experiment further corroborates the superior performance of the proposed FAM. Consistent with the previous examples, the FAM yields highly accurate results even with modest refinement: for $m=4$, the relative errors remain below 10% for all tested values of n . Moreover, the FAM solutions exhibit rapid convergence as m increases. In contrast, the conventional HM scheme requires considerably finer discretization to

achieve comparable accuracy, which limits its practicality in real-world engineering applications.

5.3 Numerical Example III: Irregular Triangular Mesh with Strong Permeability Heterogeneity

To further assess the capability of the proposed FAM in handling complex, unstructured geometries and strongly heterogeneous permeability fields, a numerical experiment is carried out on a domain subdivided into irregular polygonal subdomains. As depicted in Figure 12(a), the rectangular computational domain is partitioned into multiple irregular polygonal zones, each assigned a distinct permeability value spanning several orders of magnitude (from $k=1$ to $k=1500$), resulting in sharp and discontinuous contrasts in material properties. Dirichlet boundary conditions are imposed, with $P_{in} = 1$ applied on the left boundary and $P_{out} = 0$ on the right boundary, while the remaining two boundaries are set as impermeable (no-flux) conditions.

Discretization of the computational domain is accomplished via an unstructured triangular control-volume mesh, as depicted in Figure 12(b). To rigorously assess the numerical convergence, a systematic mesh-refinement study is conducted, wherein the refinement parameter m is progressively elevated from 1 to 128. This systematic increase in mesh resolution enables a thorough and comparative evaluation of the convergence characteristics of the proposed FAM against the conventional HM method.

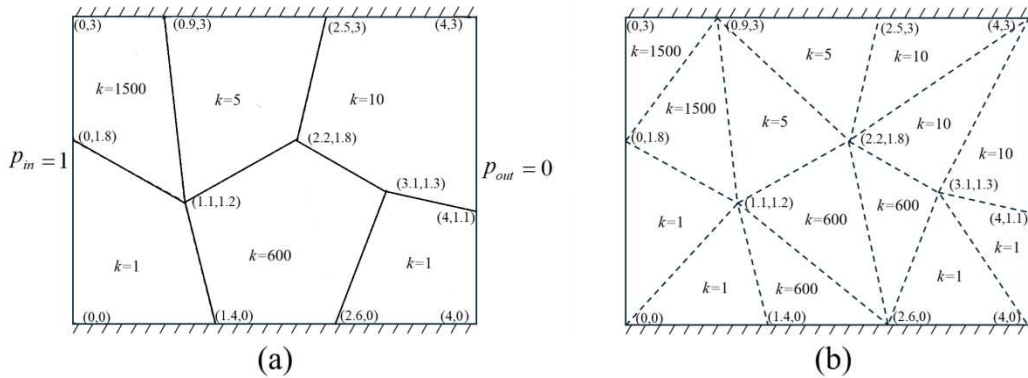


Figure 12. Schematic illustration of the irregular triangular mesh problem.

Figure 13 presents the relative errors of the x -direction equivalent permeability $\bar{k}_x$ as a function of the refinement parameter m , for four power-law indices ($n=0.8, 1.0, 1.2$, and 1.5). A key observation is that the errors of the proposed FAM achieves rapid and uniform convergence toward zero as m increases, with the error curves for different n values nearly coinciding (see red lines in Figure 13). This indicates that the convergence behavior of the FAM is highly robust and essentially insensitive to the power-law indices. Specifically, at the coarsest grid ($m=1$), the relative errors are approximately 25% for all tested indices n . These errors, however, decrease sharply with refinement—falling to about 7% at $m=4$ —and remain consistently low thereafter. In contrast, the conventional HM method exhibits substantial negative errors that persist even under mesh refinement. The convergence of these errors with increasing m is markedly slow, and the errors increase significantly as n decreases. These numerical results underscore a fundamental limitation of the HM scheme, whose systematic bias becomes increasingly pronounced under stronger permeability contrasts.

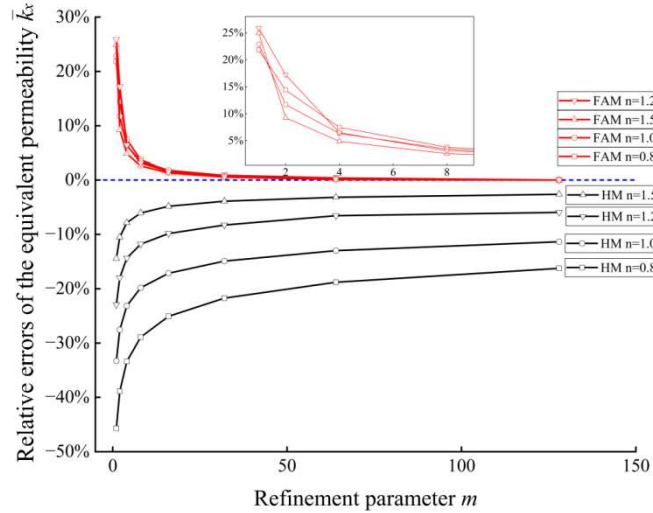


Figure 13. Relative errors of the equivalent permeability $\bar{k}_x$ as a function of the subdivision parameters m under various power-law indices n for the irregular triangular mesh problem.

5.4 Numerical Example IV: Log-normal permeability field

In subsurface media, where strong heterogeneity is ubiquitous, the permeability field is typically modeled as following a log-normal distribution. The corresponding probability density function can be written as:

$$f(\ln k) = \frac{1}{\sqrt{2\pi}\sigma_{\ln k}} \exp\left(-\frac{(\ln k - \bar{\mu})^2}{2\sigma_{\ln k}^2}\right) \quad (5.1)$$

To realistically emulate subsurface reservoir conditions, the permeability field is parameterized with a mean of $\bar{\mu} = 100$ and a logarithmic standard deviation of $\sigma_{\ln k} = 2.0$, thereby capturing spatial variability across several orders of magnitude. As illustrated in Figure 9, the resulting 2D log-normal permeability field exhibits values spanning from 2.1×10^{-1} to 2.7×10^3 . The abrupt variations in the color scale vividly demonstrate the stark contrast between high-permeability flow channels and low-permeability barriers, confirming that the stochastic field accurately reflects the strong heterogeneity inherent to subsurface formations.

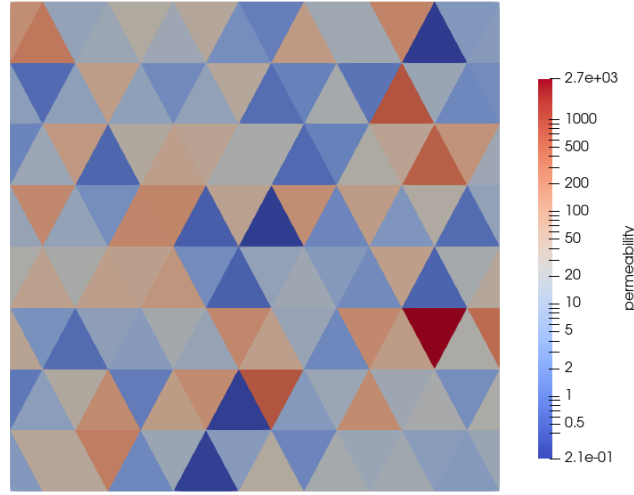


Figure 14. Schematic illustration of a two-dimensional log-normal permeability field.

Against this highly heterogeneous background, Figure 15 compares the relative error evolution of the FAM and the traditional HM method under four power-law indices ($n=0.8, 1.0, 1.2$ and 1.5). As shown in Figure 15, the FAM demonstrates significant advantages in both accuracy and convergence. Though at the coarse grid level ($m=1$), the relative errors (about 33%, 26%, 23% and 14% for $n=0.8, 1.0, 1.2$ and 1.5 , respectively) are rather large for the FAM, they are still much smaller than the errors of the HM (about -60%, -47%, 38% and 28% for $n=0.8, 1.0, 1.2$ and 1.5 , respectively). As the mesh refinement parameter m increases, the errors of the FAM decays rapidly: when $m=4$, the relative errors for all the tested indices n fall within 11%; when $m=8$, the relative errors fall within 7%; when $m=64$, the relative errors

nearly approach zero. In contrast, the convergence speed of the HM method is very slow as m increases. Even when $m=64$, the relative errors of the HM method are still about -16%, -9%, -6% and -3% for $n=0.8$, 1.0, 1.2 and 1.5, respectively.

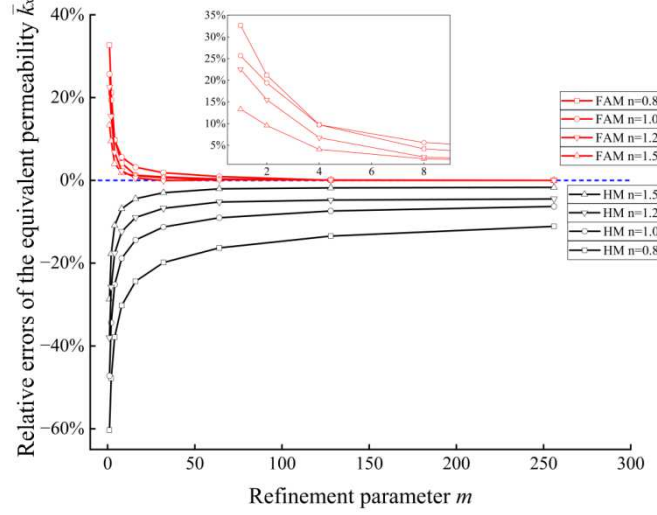


Figure 15. Relative errors of the equivalent permeability $\bar{k}_x$ as a function of the subdivision parameter m under different power-law flow indices n in the log-normal permeability field.

6 Discussion

In the construction of the FAM, the analytical solution associated with the smallest eigenvalue α_1 ($0 < \alpha_1 < 1$) is adopted, as it represents the most physically significant mode among all eigenfunctions. In some configurations—typically angular domains exhibiting structural symmetry—the smallest eigenvalue α_1 may correspond to multiple independent analytical solutions. For the linear case ($n=1$), all of these solutions can be incorporated into the FAM framework [15], because any linear combination thereof remains a valid solution of the original equation. In contrast, the power-law flow problem addressed herein is nonlinear, and thus such superposition is no longer admissible. Consequently, only one analytical solution is arbitrarily selected from the set of candidates to construct the FAM. It is worth noting that the choice among these candidate solutions may introduce a certain degree of variability in the resulting numerical accuracy.

We consider the problem depicted in Figure 16. In the angular domain surrounding the central singularity, with a power-law flow index of $n=1.2$, two eigenvalues are identified within the range $0 < \alpha < 1$: namely, $\alpha_1 = 0.1354$ and

$\alpha_2 = 0.1387$. Owing to the symmetric distribution of permeability, each eigenvalue gives rise to three independent eigenfunctions. The eigenfunctions corresponding to $\alpha_1 = 0.1354$ are illustrated in Figure 17, and those corresponding to $\alpha_2 = 0.1387$ are displayed in Figure 18.

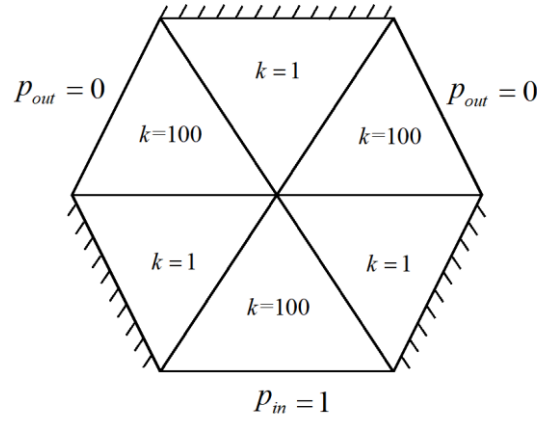


Figure 16. Schematic of a regular hexagon with a symmetric permeability field.

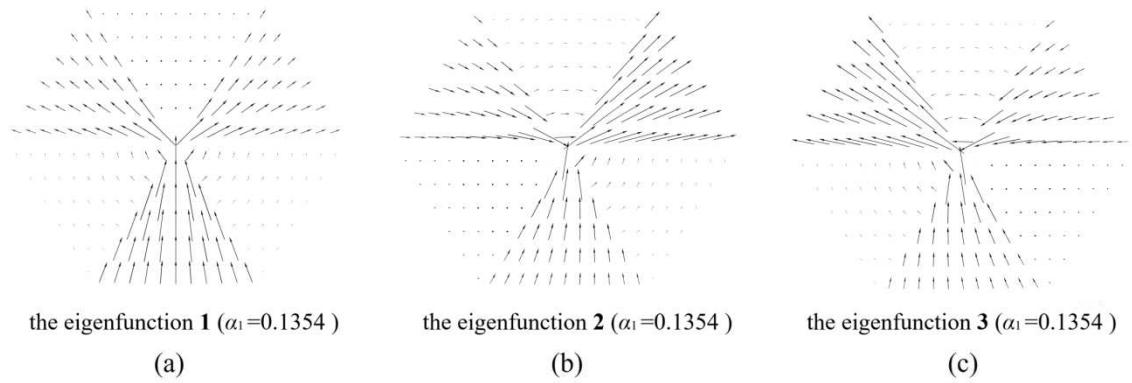


Figure 17. The velocity distributions of the three eigenfunctions corresponding to the eigenvalue $\alpha_1 = 0.1354$.

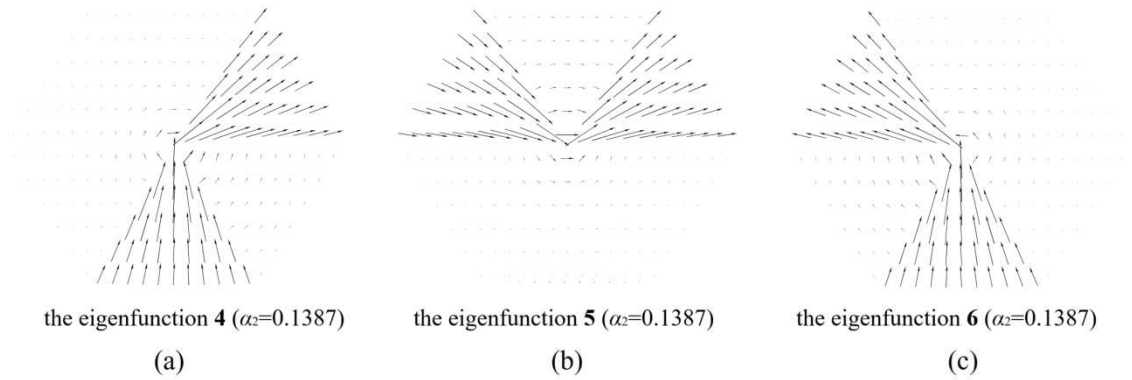


Figure 18. The velocity distributions of the three eigenfunctions corresponding to the eigenvalue $\alpha_2 = 0.1387$.

The original FAM for the linear problem ($n=1$) is constructed by linearly superposing all eigenfunctions whose eigenvalues are less than 1 [15]. This approach relies on the principle of linear superposition, which is valid only for linear governing equations. For the power-law case with ($n=1.2$), the problem becomes nonlinear, and such linear combinations are no longer admissible—they would not satisfy the governing equation. As a result, one eigenfunction (shown in Figures 17 and 18) has to be chosen individually to construct the FAM, following the same procedural steps as before. The relative errors of the calculated equivalent permeability $\bar{k}_x$ as a function of the subdivision parameter m using the FAM constructed of different eigenfunctions (the eigenfunctions 1, 2, 3, 4, 5 and 6) are shown in Figure 19.

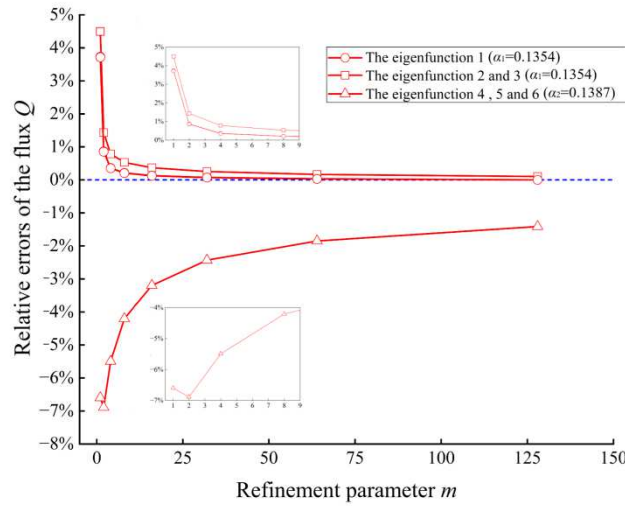


Figure 19. Relative errors of the equivalent permeability $\bar{k}_x$ as a function of the subdivision parameter m using the FAM constructed of different eigenfunctions (the eigen functions 1, 2, 3, 4, 5 and 6 in Figure 17 and 18) under the power-law flow index $n=1.2$ for the regular hexagon problem with the symmetrical permeability field.

First, the three FAM implementations employing the eigenfunctions 1, 2 and 3 associated with the smallest eigenvalue ($\alpha_1 = 0.1354$) consistently outperform those using $\alpha_2 = 0.1387$, which is expected given that $\alpha_1 < \alpha_2$. In general, selecting the eigenfunction associated with the smallest eigenvalue yields higher precision. Second, among the three implementations with α_1 , the one based on eigenfunction 1 performs slightly better than those based on eigenfunctions 2 or 3. This can be attributed to the fact that, under the prescribed boundary conditions, the flow pattern near the singularity corresponds to eigenfunction 1. In more complex scenarios, however, it is

often impractical to identify in advance the eigenfunctions that match the local flow behavior, and obtaining all eigenfunctions for the smallest eigenvalue may be challenging. Nevertheless, the numerical results indicate that any of the eigenfunctions considered still provides satisfactory accuracy.

7 Conclusion

This work proposes an efficient finite analytical method (FAM) for steady single-phase power-law fluid flow in heterogeneous porous media. The principal novelty is the use of a local analytical solution near the singular point at a permeability discontinuity, where the solution takes the form of a power-law radial dependence r^α . The smallest eigenvalue α_1 and its corresponding eigenfunction are computed using the fourth-order Runge-Kutta method with Newton's iteration. Subsequently, an alternative triangle-based numerical scheme is constructed based on this analytical foundation.

In certain cases, particularly for angular domains with symmetrical structures, the smallest eigenvalue α_1 may correspond to multiple eigenfunctions. For linear problems, all such eigenfunctions can be incorporated into the FAM, since any linear combination of these solutions still satisfies the governing equation. However, the power-law flow considered in this work is nonlinear, and therefore these eigenfunctions cannot be linearly superposed. Instead, only one eigenfunction can be selected to construct the FAM. Although the choice of eigenfunction introduces a slight variation in accuracy, the resulting differences remain marginal.

Numerical experiments demonstrate that the proposed FAM achieves significant improvements in both accuracy and convergence for simulating power-law fluid flow. Across multiple benchmark tests — including the regular hexagon model, checkerboard problem, irregular triangular mesh, and log-normal permeability field — the FAM converges considerably faster than the traditional weighted harmonic mean (HM) method under mesh refinement. Notably, for power-law indices ($n=0.8, 1.0, 1.2$ and 1.5), the error curves of the FAM nearly coincide, indicating that the method is largely insensitive to variations in n and thus exhibits broad applicability. Even under

extreme heterogeneity, such as a permeability contrast of 10^4 in the checkerboard problem, the FAM accurately recovers the theoretical equivalent permeability. In contrast, the traditional HM method suffers from severe systematic underestimation, with errors further amplified as n decreases. A key advantage of the proposed FAM is that its accuracy does not depend on the degree of medium heterogeneity or the power-law index n . For practical applications, a mesh refinement parameter of $m=4$ is recommended, as it yields a relative error of less than 8% even in the worst-case scenario. In comparison, the traditional scheme requires substantially higher refinement ratios to achieve comparable accuracy, particularly for strongly heterogeneous porous media.

Funding Statement

This work was supported by the National Science and Technology Major Project of China under Grant No. 2025ZD1404403. The funder had no role in the design of the study, data collection, analysis, interpretation of data, or writing of the manuscript.

Conflict of Interest Statement

The authors declare that there are no conflicts of interest regarding the publication of this manuscript. No financial, commercial, professional, or personal relationships have influenced the research design, data collection, analysis, interpretation of results, or manuscript writing and submission.

Data Availability Statement

The data that support the findings of this study are openly available. All raw and processed data generated and analyzed during the current study are available upon reasonable request from the corresponding author.

References

- [1] Durlafsky LJ. A triangle based mixed finite element – finite volume technique for modeling two phase flow through porous media. *Journal of Computational Physics* 1993; 105: 252-266.

- [2] Aavatsmark I, Barkve T, Bøe Ø, Mannseth T. Discretization on unstructured grids for inhomogeneous, anisotropic media. Part I: derivation of the methods. SIAM J. Sci. Comput. 1998; 19: 1700-1716.
- [3] Aavatsmark I, Barkve T, Bøe Ø, Mannseth T. Discretization on unstructured grids for inhomogeneous, anisotropic media. Part II: discussion and numerical results. SIAM J. Sci. Comput. 1998, 19: 1717-1736.
- [4] Edwards MG. Control-Volume distributed sub-cell flux schemes for unstructured and flow based grids. SPE Reservoir Simulation Symposium, paper SPE-79710-MS, Houston, Texas, 3-5 February, 2003.
- [5] Edwards MG. Unstructured, control-volume distributed, full-tensor finite-volume schemes with flow based grids. Computational Geosciences. 2002, 6: 433-452.
- [6] Edwards MG, Zheng HW. Quasi-positive families of continuous Darcy-flux finite volume schemes on structured and unstructured grids. Journal of Computational and Applied Mathematics. 2010, 234: 2152-2161.
- [7] Friis HA, Edwards MG. A family of MPFA finite volume schemes with full pressure support for the general tensor pressure equation on cell-centred triangular grids. Journal of Computational Physics. 2011, 230: 205-231.
- [8] Paul M, Edwards MG. q-families of CVD(MPFA) schemes on general elements: numerical convergence and the maximum principle. Archives of Computational Methods in Engineering. 2010, 17(2): 137-189.
- [9] Yeo IW, Zimmerman RW. Accuracy of the renormalization method for computing effective conductivities of heterogeneous media. Transport in Porous Media. 2001, 45: 129-138.
- [10] Cordazzo J, Hurtado FSV, Maliska CR, Da Silva AFC. Numerical techniques for solving partial differential equations in heterogeneous media. XXIV Iberian Latin-American Congress on Computational Methods in Engineering. 2003.
- [11] Cordazzo J, Maliska CR, Romeu RK. A considerations about the internodal permeability evaluation in reservoir simulation. The 2nd Brazilian Congress on R&D in Petroleum and Gas, Rio de Janeiro. 15-18 June, 2003.
- [12] Romeu RK, Noetinger B. Calculation of internodal transmissibilities in finite

difference models of flow in heterogeneous porous media. *Water Resource Research*. 1995, 31(4): 943-959.

[13]Liu ZF, Wang XH. Finite analytic numerical method for two-dimensional fluid flow in heterogeneous porous media. *Journal of Computational Physics*. 2013, 235: 286-301.

[14]Qu ZX, Liu ZF, Wang XH, Zhao P. Finite analytic numerical method for solving two-dimensional quasi-Laplace equation. *Numerical Methods for Partial Differential Equations*. 2014, 30(6):1755-69.

[15]Wang GW, Liu ZF, Wang XH. Finite analytic method for 2D steady fluid flows in heterogeneous porous media with unstructured grids. *International Journal for Numerical Methods in Engineering*. 2018, 113(4): 742-766.

[16]Christopher RH, Middleman S. Power-law flow through a packed tube. *Industrial & Engineering Chemistry Fundamentals*. 1965, 4(4): 422-465.

[17]Sullivan SP, Gladden LF, Johns ML. Simulation of power-law fluid flow through porous media using lattice Boltzmann techniques. *Journal of Non-Newtonian Fluid Mechanics*. 2006, 133(2): 91-98.

[18]Fadili A, Tardy PMJ, Pearson, JRA. A 3D filtration law for power-law fluid in heterogeneous porous media. *Journal of Non-Newtonian Fluid Mechanics*. 2002, 106(2): 121-146.

[19]Federico VD, Pinelli M, Ugarelli R. Estimates of effective permeability for non-Newtonian fluid flow in randomly heterogeneous porous media. *Stochastic Environmental Research and Risk Assessment*. 2010, 24(7): 1067-1076.

[20]Wang XH, Jia JT, Liu ZF. Derivation of the Darcy-scale filtration equation for power-law fluids with the volume averaging method. *Journal of Porous Media*. 2014, 17(8): 741-750.

[21]Chen CJ, Chen HC. Finite analytic numerical method for unsteady two dimensional Navier-Stokes equations. *Journal of Computational Physics*. 1984, 53(2): 209-226.

[22]Chen CJ, Naseri-Neshat H, Ho KS. Finite analytic method numerical solution of heat transfer in two-dimensional cavity flow. *Numerical Heat Transfer*. 1981,

4(2): 179-197.