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Contact resistance and wave damping limit thermal-wave impedance matching in layered Maxwell–Cattaneo–Vernotte conductors

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Abstract

At very short times heat can propagate as a damped wave with finite speed, as described by the Maxwell–Cattaneo–Vernotte (MCV) equation. We give a closed-form Laplace-domain solution for a finite layered slab under a surface temperature step, with a thermal contact (Kapitza) resistance $\mathbf{R}_c$ at the interface, in a 2×2 matrix form that applies to any number of layers. At short times the temperature front is reflected at an interface according to the thermal-wave impedance $\mathbf{Z} = \mathbf{e}/\sqrt{\tau}$, where $\mathbf{e}$ is the thermal effusivity and τ the relaxation time, so two materials can transmit the front without reflection even when their effusivities differ. The front carries only a fraction $\exp(-\Sigma)$ of the applied step, where Σ is half the transit time measured in relaxation times, so this matching matters only at times comparable with τ . Contact resistance enters the front coefficients only through the product $\mathbf{Z}\mathbf{R}_c$. For gold films, measured contact resistances give $\mathbf{Z}\mathbf{R}_c$ of about 10 to 200, so the contact controls the reflection. The MCV and Fourier models can still predict reflections of opposite sign: $-\mathbf{0.12}$ and $+\mathbf{0.46}$ for gold on sapphire. A finite-volume solver reproduces the predicted front amplitude to within 1%.

Keywords: non-Fourier heat conduction, Maxwell–Cattaneo–Vernotte equation, thermal waves, layered media, Kapitza resistance, Laplace transform

1 Introduction

Fourier's law assumes that the heat flux responds at once to a temperature gradient, which leads to a parabolic heat equation with infinite signal speed. For most engineering transients, which last seconds or longer, this does no harm. It fails in several settings that are now routine: second sound in dielectrics at low temperature^{1,2}, ultra-fast laser heating of metal films, where electrons and phonons stay out of equilibrium for a few picoseconds^{3,4}, and heat-pulse transport in superlattices and disordered crystals⁵⁻⁷. Whether the infinite speed is a genuine paradox or an artefact of applying a homogenised equation outside its range is still debated, and Auriault⁸ takes the second view. We adopt the Maxwell–Cattaneo–Vernotte (MCV) law here because it reproduces the finite propagation delay observed in these settings. The choice does not rest on any claim that Fourier's law is inconsistent.

The MCV law^{9,10} adds one relaxation time τ to the flux law, $\tau \partial_t q + q + k \partial_x T = 0$. It can be derived from the linearised phonon Boltzmann equation¹¹ or from a free-energy argument¹². Combined with the energy balance it gives a telegraph equation whose signals travel at $v = \sqrt{\alpha/\tau}$, with $\alpha = k/(\rho c)$ the thermal diffusivity. Reported relaxation times range from picoseconds in metals¹³ to values of the order of a second in heterogeneous media, where they remain disputed (see Discussion).

The homogeneous problem has been solved for many boundary conditions and sources^{1,14-19}. For layered media, Frankel, Vick and Özişik²⁰ gave the general one-dimensional formulation for composite media and showed that an interface between dissimilar materials reflects the wave with an amplitude set by the ratio of layer properties. Tang and Araki¹⁶ solved a two-layer slab by finite differences. Lor and Chu²¹ included interface thermal resistance, and Khadrawi et al.²² compared perfect and imperfect contact in composite slabs. Ramadan²³ discussed the form of the temperature-jump condition under non-Fourier conduction. In the dual-phase-lag model an interface produces a negative followed by a positive reflection when a pulse enters a region of larger τ_T ²⁴. Kheibari et al.²⁵ treated multilayer cylinders, including imperfect interfaces, by separation of variables, and Nosko²⁶ showed that for two semi-infinite bodies with an interfacial source the long-time temperature partition is fixed by the effusivity ratio. Outside layered slabs, fins have been treated with discontinuous Galerkin²⁷, lattice Boltzmann^{28,29} and Fourier-sine/Laplace transform³⁰ methods, graded cylinders with a state-space method³¹, and subsurface defects in graded materials through thermal-wave scattering³². Other work covers smoothed-particle hydrodynamics³³, an implicit scheme for the nonlinear MCV equation³⁴, a flux formulation³⁵, exact homogeneous solutions³⁶, dual-phase-lag solutions³⁷ and bio-heat problems such as laser heating of skin^{38,39}.

A second group of papers uses the language of impedance. Green⁴⁰ defined a thermal surface impedance for plane heat waves in layered materials in the diffusive setting, and Li and Zhang⁴¹ studied interface resistances in multilayer thermal-wave propagation. Shendeleva⁴² derived reflection and refraction at a plane interface in two dimensions. Camacho de la Rosa et al.^{43,44} obtained a thermal-wave impedance directly from the MCV solution and used the Fresnel-type coefficients to design Bragg mirrors from alternating layers with a large impedance contrast. Farhat et al.⁴⁵ applied scattering cancellation to MCV heat waves (see also ref.⁴⁶), Ordoñez-Miranda

and Alvarado-Gil⁴⁷ compared effective layered properties under the parabolic and hyperbolic models, and impedance-type boundary conditions have been used for the reflection of coupled thermoelastic waves with hyperbolic heat conduction⁴⁸. For layered diffusive problems, the transfer-matrix approach is described by Maillet et al.⁴⁹ as the thermal-quadrupole method. The impedance concept and the Fresnel form of the interface coefficients are therefore established, as is the effect of contact resistance on the reflected wave, and we claim none of them.

What is missing is a treatment that puts these pieces together for a finite slab. The finite slab with a Kapitza resistance has no closed Laplace-domain solution in the literature; existing treatments are numerical^{16,21}, restricted to semi-infinite bodies^{26,37}, or series solutions that need an eigenvalue search²⁵. We give that solution in 2×2 matrix form, which carries over unchanged to N layers with a resistance at each interface. The practical weight of the matching condition $Z_1 = Z_2$ has also not been assessed. The condition removes the leading discontinuity but says nothing about the wake behind it, and we show that the discontinuity carries only the fraction $\exp(-\Sigma)$ of the applied step, where Σ is half the transit time in units of τ . Matching is therefore a short-time statement whose window can be stated in advance. Published matching arguments also assume perfect contact, which is hard to justify for picosecond transport because Z is then very large. We show that a finite contact resistance R_c enters the coefficients only through the product ZR_c , and measured boundary conductances^{50–52} put ZR_c between about 10 and 200 for metal/dielectric films. In that range the contact sets the reflection and the choice of materials has little effect. The analysis also yields a prediction that can be tested. Because Z and e depend differently on τ , the two models can give reflections of opposite sign at the same interface; for gold on sapphire the MCV model gives $R \approx -0.12$ and the Fourier model $R_F \approx +0.46$. A change of sign is easier to detect than a change of amplitude, and it does not require an exactly matched pair.

2 Results

2.1 Closed-form solution for a bilayer with interfacial resistance

The slab has thickness $L = L_1 + L_2$ and two homogeneous layers (Fig. 1). Layer i occupies $x \in [x_{i-1}, x_i]$ with $x_0 = 0$, $x_1 = L_1$, $x_2 = L$, and has constant density ρ_i , specific heat c_i , conductivity k_i and relaxation time τ_i . We use $\alpha_i = k_i/(\rho_i c_i)$, $v_i = \sqrt{\alpha_i/\tau_i}$, the effusivity $e_i = \sqrt{k_i \rho_i c_i}$, and the thermal-wave impedance

$$Z_i = \rho_i c_i v_i = \frac{e_i}{\sqrt{\tau_i}} = \sqrt{\frac{k_i \rho_i c_i}{\tau_i}}, \quad [Z_i] = \text{W}/(\text{m}^2 \text{K}). \quad (1)$$

Z has the units of a conductance per unit area, while e is measured in $\text{W s}^{1/2}/(\text{m}^2 \text{K})$. If T is taken as the potential-like variable and the flux q as the flux-like one, a right-going wave has $q = ZT$, so Z is dimensionally a wave admittance and $1/Z$ is the wave resistance. We keep the established term “thermal-wave impedance”^{40,43} and fix the sign convention below. Because a thermal wave does not carry energy in the way an acoustic wave does⁵³, the acoustic analogy applies to the algebra of reflection and transmission only.

$$T(0, t) = T_0 + \Delta T H(t) \quad q(L, t) = 0$$

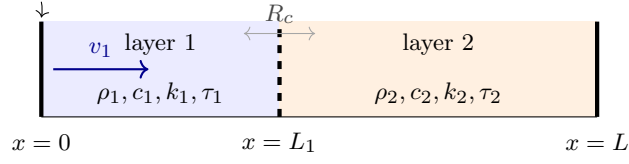


Fig. 1 Two-layer MCV slab. The front face at $x = 0$ receives a temperature step ΔT at $t = 0$, the back face at $x = L$ is insulated, and the interface at $x = L_1$ carries a Kapitza resistance R_c across which the heat flux is continuous and the temperature may jump. Perfect contact is the case $R_c = 0$.

The governing equations, boundary conditions and scaling are given in Methods. Writing $\theta_i = T_i - T_0$ and taking the Laplace transform, each layer satisfies $d^2\bar{\theta}_i/dx^2 = \mu_i^2\bar{\theta}_i$ with

$$\mu_i^2(s) = \frac{s(1 + \tau_i s)}{\alpha_i}, \quad \beta_i(s) := \frac{k_i \mu_i(s)}{1 + \tau_i s} = e_i \sqrt{\frac{s}{1 + \tau_i s}}, \quad (2)$$

where β_i is the Laplace-domain admittance and $\bar{q}_i = -(1 + \tau_i s)^{-1} k_i d\bar{\theta}_i/dx$. Most of what follows comes from the two limits of β_i . As $s \rightarrow \infty$ it tends to $e_i/\sqrt{\tau_i} = Z_i$, and as $s \rightarrow 0$ it tends to $e_i\sqrt{s}$, so the impedance controls short times and the effusivity controls long times.

Imposing the front-face step, the insulated back face, flux continuity and the temperature jump $\bar{\theta}_1 - \bar{\theta}_2 = R_c \bar{q}$ at $x = L_1$ (derivation in Methods), and writing

$$\Gamma(s) := \cosh(\mu_2 L_2) + R_c \beta_2 \sinh(\mu_2 L_2) \quad (3)$$

for the contact-resistance factor at the interface, gives

$$\bar{\theta}_1(x, s) = \frac{\Delta T}{s} \frac{\beta_1 \Gamma \cosh(\mu_1(L_1 - x)) + \beta_2 \sinh(\mu_2 L_2) \sinh(\mu_1(L_1 - x))}{D_c(s)}, \quad (4)$$

$$\bar{\theta}_2(x, s) = \frac{\Delta T}{s} \frac{\beta_1 \cosh(\mu_2(L - x))}{D_c(s)}, \quad (5)$$

$$\bar{q}_1(x, s) = \frac{\Delta T}{s} \frac{\beta_1 [\beta_1 \Gamma \sinh(\mu_1(L_1 - x)) + \beta_2 \sinh(\mu_2 L_2) \cosh(\mu_1(L_1 - x))]}{D_c(s)}, \quad (6)$$

$$\bar{q}_2(x, s) = \frac{\Delta T}{s} \frac{\beta_1 \beta_2 \sinh(\mu_2(L - x))}{D_c(s)}, \quad (7)$$

with the dispersion function

$$D_c(s) = \beta_1 \Gamma \cosh(\mu_1 L_1) + \beta_2 \sinh(\mu_1 L_1) \sinh(\mu_2 L_2). \quad (8)$$

Expanding Γ shows that R_c adds a single term to the perfect-contact dispersion function $D(s)$:

$$D_c(s) = \underbrace{\beta_1 \cosh(\mu_1 L_1) \cosh(\mu_2 L_2) + \beta_2 \sinh(\mu_1 L_1) \sinh(\mu_2 L_2)}_{D(s)} + R_c \beta_1 \beta_2 \cosh(\mu_1 L_1) \sinh(\mu_2 L_2). \quad (9)$$

With $R_c = 0$, $\Gamma = \cosh(\mu_2 L_2)$ and the perfect-contact solution is recovered; the limit $\tau_i \rightarrow 0$ gives the Carslaw–Jaeger composite slab⁵⁴. Evaluating equations (4) to (9) at complex s , we found the front-face value, the insulated back face, flux continuity and the temperature jump all satisfied to within 10^{-16} of $\Delta T/s$.

2.2 Extension to N layers

Equations (4) to (8) are the two-layer case of a matrix recursion. With the state vector $\mathbf{u} = (\bar{\theta}, \bar{q})^\top$,

$$\begin{aligned} \mathbf{u}(x_i^-) &= \mathbf{M}_i \mathbf{u}(x_{i-1}^+), & \mathbf{M}_i &= \begin{pmatrix} \cosh(\mu_i L_i) & -\beta_i^{-1} \sinh(\mu_i L_i) \\ -\beta_i \sinh(\mu_i L_i) & \cosh(\mu_i L_i) \end{pmatrix}, \\ \mathbf{u}(x_i^+) &= \mathbf{J}_i \mathbf{u}(x_i^-), & \mathbf{J}_i &= \begin{pmatrix} 1 & -R_{c,i} \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (10)$$

where $\mathbf{M}_i$ is the hyperbolic form of the thermal quadrupole⁴⁹, $\det \mathbf{M}_i = 1$, and $\mathbf{J}_i$ carries the state across a resistive interface. Chaining gives $\mathbf{u}(L) = \mathbf{G}(s) \mathbf{u}(0)$ with $\mathbf{G} = \mathbf{M}_N \mathbf{J}_{N-1} \mathbf{M}_{N-1} \cdots \mathbf{J}_1 \mathbf{M}_1$. With $\bar{\theta}(0) = \Delta T/s$ and $\bar{q}(L) = 0$ the second row yields the front-face flux $\bar{q}(0, s) = -(G_{21}/G_{22}) \Delta T/s$, and every field follows by forward multiplication. For $N = 2$ one finds $\beta_1 G_{22} = D_c(s)$. We confirmed the matrix form against the closed form in 60-digit arithmetic, with agreement to 10^{-23} . In double precision the plain matrix product is unstable at the large $|s|$ reached on the inversion contour (Methods), and the scaled closed forms are used there.

2.3 Wavefront reflection and impedance matching

For $s \rightarrow \infty$, $\mu_i \rightarrow s/v_i + 1/(2v_i \tau_i)$ and $\beta_i \rightarrow Z_i$. Keeping the dominant exponential in equation (5) at $R_c = 0$ and inverting gives the leading wavefront in layer 2,

$$T_2(x, t) - T_0 \sim T_{12} \Delta T e^{-\Sigma(x)} H(t - \tau_w(x)), \quad (11)$$

$$\tau_w = \frac{L_1}{v_1} + \frac{x - L_1}{v_2}, \quad \Sigma = \sigma_1 L_1 + \sigma_2 (x - L_1), \quad (12)$$

with $\sigma_i = 1/(2v_i \tau_i)$. The front arrives at the ballistic time τ_w and is attenuated by the optical depth Σ along its path.

Sign conventions for these coefficients differ between papers, so we state ours. A right-going wave in layer i has $q = Z_i T$ and a left-going wave has $q = -Z_i T$.

For incident, reflected and transmitted temperature amplitudes T_I , T_R , T_T we define $R = T_R/T_I$ and $T_{12} = T_T/T_I$. Continuity of T and q at a bonded face then gives

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T_{12} = \frac{2Z_1}{Z_1 + Z_2} = 1 + R. \quad (13)$$

This is the convention of the photothermal literature^{41,42}, and it agrees with the acoustic pressure convention once $1/Z$ is taken as the wave resistance. The heat-flux reflection coefficient is $-R$, and readers comparing with sources that quote $(Z_2 - Z_1)/(Z_1 + Z_2)$ should note the change of sign. A second layer of higher impedance, $Z_2 > Z_1$, gives $R < 0$. Values $T_{12} > 1$ occur when $Z_2 < Z_1$; this is allowed because T is not an energy density.

The temperature reflection coefficient of the leading wavefront therefore vanishes if and only if

$$Z_1 = Z_2 \iff \frac{k_1 \rho_1 c_1}{\tau_1} = \frac{k_2 \rho_2 c_2}{\tau_2} \iff \frac{e_1}{\sqrt{\tau_1}} = \frac{e_2}{\sqrt{\tau_2}}, \quad (14)$$

and the jump then crosses with $T_{12} = 1$, attenuated only by $e^{-\sigma_i L_i}$. Equation (14) is the thermal counterpart of terminating a transmission line in its characteristic impedance, and it does not depend on frequency or layer thickness. The thermal-wave Bragg mirrors of refs^{43,44} work differently: they rely on interference between reflections from several interfaces, so their response depends on both layer thickness and frequency.

The contrast with Fourier theory comes from the identity $Z_2/Z_1 = (e_2/e_1)/\sqrt{\tau_2/\tau_1}$ derived in Methods. In the diffusive limit the interface coefficient is $R_F = (e_1 - e_2)/(e_1 + e_2)$, and it vanishes when the two effusivities match. Equation (14) can hold at $e_1 \neq e_2$ if the relaxation times compensate, and Fourier theory then still predicts a reflection. In principle, then, τ can be used to adjust the short-time reflection without changing the long-time response. R and R_F are different quantities. R is the reflection of a propagating discontinuity and is governed by $Z = e/\sqrt{\tau}$, which diverges as $\tau \rightarrow 0$; R_F is the long-time effusivity partition of the contact temperature. The two agree only when τ_1 and τ_2 tend to zero together.

With the jump condition in place of temperature continuity the coefficients become

$$R(R_c) = \frac{Z_1 - Z_2 + Z_1 Z_2 R_c}{Z_1 + Z_2 + Z_1 Z_2 R_c}, \quad T_{12}(R_c) = \frac{2Z_1}{Z_1 + Z_2 + Z_1 Z_2 R_c}, \quad (15)$$

now related by $1 + R = T_{12}(1 + R_c Z_2)$, where the extra factor is the temperature drop across the contact. The resistance appears only in the product $Z_1 Z_2 R_c$, so its size is judged against the wave resistance: the relevant dimensionless group is $Z R_c$, with Z of the order of Z_1 and Z_2 . A mismatched pair becomes reflection-free when $R_c = 1/Z_1 - 1/Z_2$ (possible only for $Z_2 > Z_1$), which makes the contact a resistive matching element that is independent of frequency at leading order. Conversely, a

Table 1 Test cases and interface coefficients. Dimensionless layer-2 properties in the scaling of Methods (layer 1 has all properties equal to 1), and the resulting wavefront (MCV) and diffusive (Fourier) coefficients in the convention of [equation \(13\)](#), $T_{12} = 1 + R$.

Case	layer-2 properties				ratios		MCV		Fourier
	k	τ	v	e	Z_2/Z_1	e_2/e_1	R	T_{12}	R_F
Uniform	1.0	1.0	1.0	1.000	1.000	1.000	+0.000	1.000	+0.000
B: Z -mismatch	4.0	1.0	2.0	2.000	2.000	2.000	-0.333	0.667	-0.333
A: Z -matched, e -mismatch	0.5	0.5	1.0	0.707	1.000	0.707	+0.000	1.000	+0.172

matched pair acquires a reflection from any finite resistance,

$$R(R_c)|_{Z_1=Z_2=Z} = \frac{ZR_c}{2 + ZR_c} \xrightarrow{ZR_c \ll 1} \frac{1}{2}ZR_c, \quad (16)$$

rising to $R \rightarrow +1$ as $R_c \rightarrow \infty$; an insulating interface acts as a free end and the face temperature doubles. [Equation \(14\)](#) is therefore the $ZR_c \ll 1$ limit of [equation \(15\)](#).

2.4 Ballistic fraction

[Equation \(11\)](#) describes only the jump. Behind the front the MCV equation carries an exponentially damped, Bessel-modulated wake, which is well known from the semi-infinite problem^{1,14}. The wake is contained in [equation \(5\)](#) and is reproduced by both solvers, but it is not part of [equation \(11\)](#) and does not obey [equation \(13\)](#). To measure how much of the response the jump carries, we define the ballistic fraction

$$\Phi(x) := T_{12} e^{-\Sigma(x)}, \quad \Sigma(x) = \frac{1}{2} \sum_i \frac{t_i(x)}{\tau_i}, \quad (17)$$

where t_i is the transit time in layer i ; the second form follows from $\sigma_i L_i = L_i/(2v_i\tau_i)$. The optical depth is thus half the transit time in units of the relaxation time. At a matched interface ($T_{12} = 1$) the discontinuity carries $\Phi = 0.61$ of the applied step after a transit of one relaxation time, 0.37 after two, 0.135 after four and 0.007 after ten. Impedance matching removes the reflection of the jump, but it cannot make the interface transparent to the wake: that would need $\beta_1(s) = \beta_2(s)$ for all s , which means $e_1 = e_2$ and $\tau_1 = \tau_2$. An experiment built on [equation \(14\)](#) therefore has to probe times of order τ .

2.5 Independent tuning of the short- and long-time responses

Three dimensionless cases are used, with $L_1 = L_2 = 1$, $\Delta T = 1$, $T_0 = 0$, $\rho_i c_i = 1$ and $R_c = 0$; layer 1 is the same in all of them and the layer-2 properties are listed in [Table 1](#). In every MCV case the temperature stays exactly zero until the front arrives at $\hat{t}_w = \hat{x}/v$ and then jumps, whereas the Fourier limit gives a non-zero temperature at every probe immediately after $\hat{t} = 0$.

Case B has $Z_2 = 2Z_1$, so [equation \(13\)](#) gives $R = -1/3$ and $T_{12} = 2/3$. Spatial snapshots (Supplementary Fig. S1) show the collision and its split into a transmitted

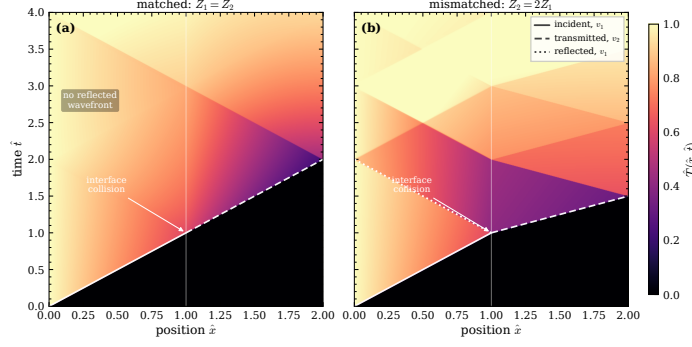


Fig. 2 Spacetime temperature field for the matched and mismatched bilayers. $\hat{T}(\hat{x}, \hat{t})$ from the characteristic finite-volume solver for the matched case A ($Z_1 = Z_2$, left) and the mismatched case B ($Z_2 = 2Z_1$, right). White lines are the analytic characteristics. In the matched case no reflected characteristic leaves the interface at $\hat{x} = 1$; in the mismatched case the incident front splits at $(\hat{x}, \hat{t}) = (1, 1)$ into transmitted and reflected fronts.

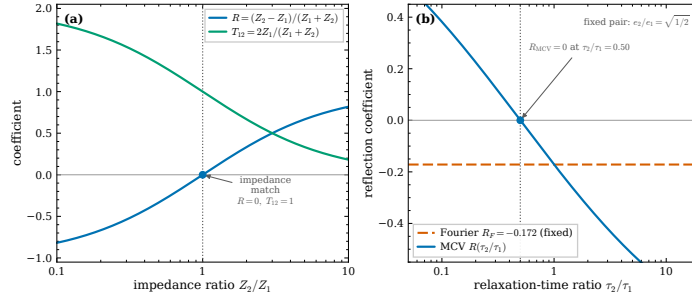


Fig. 3 Fresnel-form coefficients and their dependence on the relaxation-time ratio. (a) Wavefront reflection R and transmission T_{12} against the impedance ratio Z_2/Z_1 from equation (13). The dot marks the matched point $Z_2/Z_1 = 1$, where $R = 0$ and $T_{12} = 1$. (b) MCV reflection R (solid) and Fourier reflection R_F (dashed) against τ_2/τ_1 at a fixed effusivity ratio $e_2/e_1 = \sqrt{1/2}$. R crosses zero at $\tau_2/\tau_1 = 1/2$, whereas $R_F = +0.172$ does not depend on τ .

front, which moves into layer 2 at $v_2 = 2$, and a reflected front returning into layer 1. The reflected front lowers the temperature behind it, as $R < 0$ requires, and each front is a sharp monotone step at the predicted position. Figure 2 compares the mismatched and matched cases in the $(\hat{x}, \hat{t})$ plane. In the mismatched panel the incident characteristic splits at $(\hat{x}, \hat{t}) = (L_1, 1)$ and seeds the reverberation pattern; in the matched panel no reflected characteristic leaves the interface. The absence of reflection applies to the discontinuity only, and the wake behind it is still partly reflected.

The matched- Z pair has $R = 0$ while Fourier theory gives $R_F = +0.172$, because the effusivities differ by $\sqrt{2}$ (Table 1). Figure 3a plots R and T_{12} against Z_2/Z_1 . Figure 3b fixes $e_2/e_1 = \sqrt{1/2}$ and sweeps τ_2/τ_1 : the MCV reflection crosses zero at $\tau_2/\tau_1 = 1/2$, the value that enforces $Z_1 = Z_2$, while R_F stays at $+0.172$. Figure 4 contours $|R|$ over the plane $(\tau_2/\tau_1, e_2/e_1)$; the MCV locus is the diagonal $e_2/e_1 = \sqrt{\tau_2/\tau_1}$, the Fourier locus is the line $e_2/e_1 = 1$, and they meet only at $(1, 1)$.

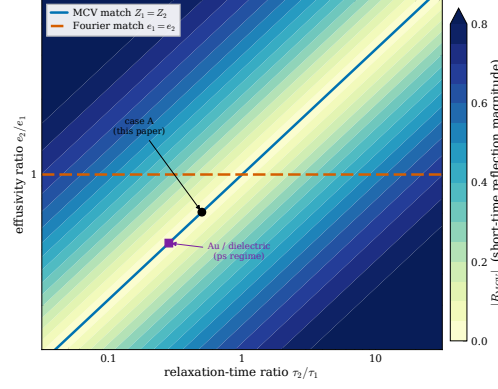


Fig. 4 Wavefront reflection magnitude over the material design plane. Contours of $|R|$ against the relaxation-time ratio τ_2/τ_1 and the effusivity ratio e_2/e_1 . The solid line is the MCV matching locus $e_2/e_1 = \sqrt{\tau_2/\tau_1}$, where $R = 0$; the dashed line is the Fourier matching locus $e_2/e_1 = 1$, where $R_F = 0$. The two meet only at $(1, 1)$. The circle marks case A and the square the gold/aluminium nitride (Au/AlN) pair discussed under real material pairs.

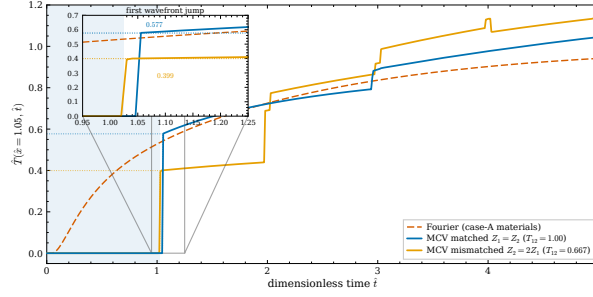


Fig. 5 Probe response just inside layer 2. $\hat{T}(\hat{x} = 1.05, \hat{t})$ for the matched MCV case A (blue), the mismatched MCV case B (orange) and the Fourier solution for the case-A materials (dashed). The inset resolves the first wavefront arrival; dotted horizontal lines mark the closed-form amplitudes $T_{12}e^{-\Sigma} = 0.577$ (matched) and 0.399 (mismatched) from [equation \(11\)](#).

Figure 5 shows the probe response at $\hat{x} = 1.05$. For the matched pair the front arrives at $\hat{t}_w = 1.05$ with amplitude $T_{12}e^{-\Sigma} = 0.577$ and nothing returns into layer 1. For case B the front arrives earlier, since v_2 is larger, and carries 0.399 . The Fourier curve has neither the delay nor the jump.

2.6 Validation of the jump amplitude

Table 2a compares $T_{12}e^{-\Sigma}$ from [equation \(11\)](#) with the jump taken from the finite-volume field as the steepest spatial drop in $\hat{T}(\hat{x})$ when the front reaches the probe. In the continuum the discontinuous part is exactly $T_{12}\Delta T e^{-\Sigma}$, because the wake grows from zero only for $\hat{t} > \hat{\tau}_w$. A total-variation-diminishing scheme spreads the front over a few cells and the estimator reads $\hat{T}$ a fixed cell offset behind it, so a residual is expected and should scale with the mesh. Table 2b confirms this: the observed order is $p = 1.05, 1.01$ and 1.08 at the three probes, and Richardson extrapolation of the

Table 2 Validation of the leading-jump amplitude. (a) Analytic amplitude $T_{12}e^{-\Sigma}$ from equation (11) against the finite-volume jump $\Delta\hat{T}_{\text{FVM}}$ at $n_x = 6000$ cells. (b) Mesh refinement for case A, where the residual is largest, with the observed convergence order p and the Richardson-extrapolated residual from the two finest meshes.

(a)					
Case	$\hat{x}$	τ_w	$T_{12}e^{-\Sigma}$	$\Delta\hat{T}_{\text{FVM}}$	rel. err.
A: matched	1.10	1.100	0.5488	0.5436	0.95%
A: matched	1.30	1.300	0.4493	0.4443	1.11%
A: matched	1.50	1.500	0.3679	0.3642	1.00%
B: mismatched	1.10	1.050	0.3944	0.3909	0.88%
B: mismatched	1.30	1.150	0.3751	0.3721	0.80%
B: mismatched	1.50	1.250	0.3568	0.3542	0.72%
(b) relative error of $\Delta\hat{T}_{\text{FVM}}$ (%), case A					
probe $\hat{x}$	$n_x = 1500$	$n_x = 3000$	$n_x = 6000$	$n_x = 12000$	p / Richardson
1.10	3.79	1.82	0.95	0.43	1.05 / -0.09%
1.30	4.16	2.16	1.11	0.51	1.01 / -0.09%
1.50	4.32	2.18	1.00	0.46	1.08 / -0.08%

two finest meshes leaves residuals of -0.09% , -0.09% and -0.08% , i.e. zero within the extrapolation error. The 1% agreement at $n_x = 6000$ is therefore discretisation error, and $T_{12}e^{-\Sigma}$ is the continuum amplitude. The Talbot inversion of the closed form and the finite-volume reference also agree at every arrival, every reverberation and through the diffusive tail at four probes spanning both layers (Supplementary Fig. S2); the remaining differences are the $\mathcal{O}(M^{-1})$ Gibbs ringing of the spectral inversion near each discontinuity and the slight smearing of the front by the limiter, which have opposite sign at the front, so the pair brackets the exact solution.

2.7 Real material pairs

To see whether real materials can satisfy the matching condition, we consider metal films on dielectric substrates, where τ is a few picoseconds and pump-probe thermoreflectance can resolve it⁵⁵. For gold we take $\tau = 8.5$ ps from lagging-model fits¹³. No independent measurement of a Cattaneo relaxation time exists for the dielectrics, so we estimate it from kinetic theory. For a phonon gas $\alpha = \frac{1}{3}v_p^2\tau$, hence

$$\tau = \frac{3\alpha}{v_p^2} = \frac{3k}{\rho c v_p^2}, \quad (18)$$

with v_p the mean phonon group velocity. The same identification gives $v = v_p/\sqrt{3}$, so it is consistent with equation (1).

No common substrate matches gold exactly (Table 3). The closest is AlN, with $Z_2/Z_1 = 0.91$ and $|R| = 0.05$; reaching $R = 0$ would need a tuned alloy or a superlattice with an adjustable effective τ . For most pairs the two models give clearly different coefficients. For Au/diamond and Au/*a*-SiO₂ the Fourier value is about twice the MCV value, and for Au/sapphire the two have opposite signs, $R = -0.118$ against $R_F = +0.464$. A difference in sign does not depend on absolute calibration: just behind

Table 3 Impedance and effusivity of gold and five substrates. Dielectric relaxation times from [equation \(18\)](#); gold from lagging-model fits¹³. R and R_F are for layer 1 = Au, layer 2 = substrate, perfect contact. Property values are representative room-temperature literature values.

Material	k W/(m K)	ρc MJ/(m ³ K)	v_p m/s	τ ps	Z 10 ⁹ W/(m ² K)	e 10 ⁴ W s ^{1/2} /(m ² K)	R	R_F
Au	318	2.49	n/a	8.5	9.65	2.81	n/a	n/a
Diamond	2000	1.79	13000	19.8	13.4	5.98	-0.164	-0.360
AlN	285	2.41	6300	8.9	8.78	2.62	+0.048	+0.035
Si	148	1.63	6400	6.7	6.02	1.55	+0.231	+0.289
Sapphire	35	3.03	7000	0.71	12.2	1.03	-0.118	+0.464
α -SiO ₂	1.4	1.63	4100	0.15	3.85	0.151	+0.429	+0.898

the reflected front, the temperature falls below the incident level in the hyperbolic model and rises above it in the diffusive one.

The predictions are also sensitive to τ , and this can be turned to advantage. Because $Z \propto \tau^{-1/2}$, a factor of two in τ changes Z by $\sqrt{2}$. For Au/sapphire, R crosses zero at $\tau_{\text{sap}} = 1.14$ ps, compared with the kinetic estimate of 0.71 ps, and reaches -0.28 if τ_{sap} is halved. For Au/AlN, R moves from +0.22 to -0.13 as τ_{Au} goes from 4.25 to 17 ps. Conversely, a measurement of the leading jump at a well-characterised interface would fix τ to within a factor of two, which steady measurements cannot do.

2.8 The interface dominates in thin films

[Table 3](#) assumes $ZR_c \rightarrow 0$, and this assumption does not hold for thin films because Z is so large. With $Z_{\text{Au}} \approx 9.7 \times 10^9$ W/(m² K) and measured boundary conductances $G = 1/R_c \approx 5 \times 10^7$ W/(m² K) to 1×10^9 W/(m² K)^{50–52,56},

$$Z_{\text{Au}}R_c \approx 10 \text{ to } 200, \quad (19)$$

where the lower end corresponds to the best epitaxial contact reported⁵² and the upper end to an ordinary evaporated film. By [equation \(15\)](#) the reflection is then set almost entirely by R_c . At $ZR_c = 10$ a perfectly matched pair already reflects $R = 0.83$, and the impedance contrast is a small correction. Observing [equation \(14\)](#) in a thin-film stack would require $ZR_c \lesssim 0.1$, that is, $G \gtrsim 1 \times 10^{11}$ W/(m² K), two orders of magnitude above the best measured metal/dielectric values. We do not expect the matched condition to be reachable in this regime with present interfaces. The sign test of the previous section is still possible, because [equation \(15\)](#) shifts R by a known amount once G has been measured on the same sample, as is routine in thermoreflectance⁵⁵.

3 Discussion

The condition $ZR_c \ll 1$ is easier to meet when Z is small, that is, when τ is large, and heterogeneous media at room temperature have been proposed for this purpose. The experimental support for large τ in such media is contested. The relaxation times of 10 s to 30 s often quoted for processed meat come from Mitra et al.⁵⁷, and those

experiments were not reproduced: Graßmann and Peters⁵⁸, Herwig and Beckett^{59,60} and Scott et al.⁶¹ found no hyperbolic signature in comparable materials. Roetzel et al.⁶² traced the disagreement to measuring the thermophysical properties separately from τ . Fitting all parameters from one experiment, they did find non-Fourier behaviour, but with $\tau \approx 1.8$ s for meat and 2.3 s for sand. Antaki⁶³ obtained $\tau \approx 2$ s from a dual-phase-lag reinterpretation, and Maillet⁶⁴ judged much of the experimental basis unreliable. With $\tau \approx 2$ s, $k \approx 0.5$ W/(m K) and $\rho c \approx 3.8$ MJ/(m³ K) we obtain $v \approx 0.26$ mm/s and $Z \approx 9.7 \times 10^2$ W/(m² K), seven orders of magnitude below the thin-film value. Contiguous tissue layers have no Kapitza resistance, so $ZR_c \rightarrow 0$ and equation (14) applies directly. Here the difficulty is the reverse. A front crossing 1 mm takes about 4 s and retains $\Phi \approx 0.37$, so a window exists, but the experiment is slow and tissue properties scatter widely. We give these numbers for completeness and do not propose tissue as a test system.

Two further limitations apply to the picosecond estimates. At a few picoseconds in a metal the electron and phonon subsystems are not in equilibrium, and a two-temperature model^{3,4} is closer to the physics than a single τ ; the Guyer–Krumhansl equation is another alternative with room-temperature support^{7,11}. Pulsed heating also makes k , ρc and τ temperature-dependent, which leads to the nonlinear regime treated by Kovács and Rogolino³⁴. The present results are the linear, single- τ baseline against which such refinements can be measured.

In summary, we have derived a closed-form Laplace-domain solution for transient conduction in a finite MCV slab with a Kapitza resistance at the interface, in a 2×2 matrix form that extends to N layers. It reduces to the perfect-contact solution at $R_c = 0$ and to the layered Fourier solution as $\tau_i \rightarrow 0$, and it agrees with an independent transfer-matrix evaluation in extended precision. At short times the leading discontinuity follows Fresnel relations in $Z_i = \rho_i c_i v_i = e_i / \sqrt{\tau_i}$, so a pair with $Z_1 = Z_2$ but $e_1 \neq e_2$ transmits the front without reflection while Fourier theory predicts a reflection. A characteristic finite-volume solver reproduces the front amplitude $T_{12}e^{-\Sigma}$ to within 1% at $n_x = 6000$, and the residual extrapolates to below 0.1%. The practical reach of impedance matching is limited in two ways. The discontinuity carries only $\Phi = T_{12}e^{-\Sigma}$ of the applied step, so matching matters only for times of order τ . For gold films, measured boundary conductances give ZR_c of about 10 to 200, so the contact resistance controls the reflection, and bringing ZR_c below 0.1 would need conductances two orders of magnitude above present values. The two models can nevertheless be told apart. For gold on sapphire they predict reflections of opposite sign, and because $Z \propto \tau^{-1/2}$, a measured jump amplitude at an interface of known conductance would fix τ to within a factor of two. We think this inverse use of the reflection formulae is the most direct route to an experimental test.

4 Methods

4.1 Governing equations, boundary conditions and scaling

Energy balance and the MCV constitutive law in layer i are

$$\rho_i c_i \partial_t T_i + \partial_x q_i = 0, \quad \tau_i \partial_t q_i + q_i + k_i \partial_x T_i = 0, \quad (20)$$

and eliminating q_i gives the telegraph equation $\tau_i \partial_t^2 T_i + \partial_t T_i = \alpha_i \partial_x^2 T_i$. The limit $\tau_i \rightarrow 0$ is singular: the time order drops from two to one and $v_i \rightarrow \infty$. The medium starts quiescent, $T_i(x, 0) = T_0$, $q_i(x, 0) = 0$, and the boundary conditions are $T_1(0, t) = T_0 + \Delta T H(t)$ and $q_2(L, t) = 0$. The insulated back face keeps the reverberations inside the slab; other boundary conditions follow by the same algebra. At $x = L_1$ the flux is continuous and the temperature may jump across a Kapitza resistance $R_c = 1/G$,

$$q_1(L_1, t) = q_2(L_1, t) \equiv q_{\text{int}}, \quad T_1(L_1, t) - T_2(L_1, t) = R_c q_{\text{int}}. \quad (21)$$

Lengths are scaled by L_1 , times by L_1/v_1 and temperature by the step: $\hat{x} = x/L_1$, $\hat{t} = tv_1/L_1$, $\hat{T} = (T - T_0)/\Delta T$. Five dimensionless groups govern the problem: L_2/L_1 , v_2/v_1 , $Z_1 R_c$, and any two of Z_2/Z_1 , e_2/e_1 and τ_2/τ_1 , which are related by $Z_2/Z_1 = (e_2/e_1)/\sqrt{\tau_2/\tau_1}$. It is this identity that lets Z and e be set independently through τ . Hats are dropped in the analysis and restored on figure axes, where $L_1 = L_2 = 1$, $\Delta T = 1$, $T_0 = 0$ and $\rho_i c_i = 1$.

4.2 Derivation of the closed forms

The general solution of the transformed problem is

$$\bar{\theta}_1 = A_1 \cosh(\mu_1(L_1 - x)) + B_1 \sinh(\mu_1(L_1 - x)), \quad \bar{\theta}_2 = C \cosh(\mu_2(L - x)),$$

where the layer-2 form makes $\bar{q}_2(L, s) = 0$ automatic. The front-face condition gives $A_1 \cosh(\mu_1 L_1) + B_1 \sinh(\mu_1 L_1) = \Delta T/s$. At $x = L_1$ one has $\bar{\theta}_1 = A_1$, $\bar{q}_1 = \beta_1 B_1$, $\bar{\theta}_2 = C \cosh(\mu_2 L_2)$ and $\bar{q}_2 = \beta_2 C \sinh(\mu_2 L_2)$. Flux continuity gives $\beta_1 B_1 = \beta_2 C \sinh(\mu_2 L_2)$ for any R_c , and the jump condition [equation \(21\)](#) gives $A_1 = C \Gamma$ with Γ as in [equation \(3\)](#). Substituting both into the front-face condition and multiplying by β_1 gives $C = (\Delta T/s) \beta_1 / D_c(s)$ with D_c as in [equation \(8\)](#); A_1 and B_1 follow, and reassembling gives [equations \(4\) to \(7\)](#).

The transient poles satisfy $D_c(s_n) = 0$, that is,

$$\tanh(\mu_1 L_1) \tanh(\mu_2 L_2) + R_c \beta_2 \tanh(\mu_2 L_2) = -\beta_1 / \beta_2.$$

For a uniform medium with $R_c = 0$ the high-frequency poles approach $\text{Re}(s) = -1/(2\tau)$, where μ_i is purely imaginary, and the relation reduces to $\tan(|\mu|L_1) = \cot(|\mu|L_2)$, recovering the Dirichlet–Neumann quantisation $\mu_n L = i\pi(n - \frac{1}{2})$. This reduction requires $\beta_1 = \beta_2$ and fails for $\tau_1 \neq \tau_2$, where β_1/β_2 is complex and the poles leave the vertical line. No closed-form pole set then exists, and we invert numerically.

4.3 Numerical inversion

We use the fixed-Talbot algorithm^{65,66}, which deforms the Bromwich contour onto a cotangent path along which the integrand decays quickly. With M nodes,

$$f(t) \approx \frac{r}{M} \left[\frac{1}{2} F(r) e^{rt} + \sum_{k=1}^{M-1} \text{Re}(F(s_k)(1 + i\sigma_k) e^{s_k t}) \right], \quad (22)$$

599 with node parameters

600

$$601 \quad r = \frac{2M}{5t}, \quad \theta_k = \frac{k\pi}{M}, \quad (23)$$

$$602 \quad s_k = r\theta_k(\cot \theta_k + i), \quad \sigma_k = \theta_k + (\theta_k \cot \theta_k - 1) \cot \theta_k. \quad (24)$$

603

604
605 Smooth transforms reach twelve digits at $M = 24$, and our implementation reproduces
606 five transforms with known inverses to $\sim 10^{-10}$ (Supplementary Table S1). Our signals
607 contain a step at every wavefront arrival, so convergence there degrades to $\mathcal{O}(M^{-1})$
608 and Gibbs oscillations of about one percent remain at $M = 32$. These oscillations are
609 an artefact of the inversion. Wherever the front itself has to be resolved we use the
610 finite-volume solver below, and the two results bracket the exact answer.

611 On the Talbot contour $\text{Re}(\mu_i L_i)$ can be of order M , and $\cosh(\mu_i L_i)$ overflows a
612 double for $\text{Re}(\mu_i L_i) \gtrsim 700$. We factor e^{-z} out of each hyperbolic function, $\tilde{C}(z) =$
613 $e^{-z} \cosh z = \frac{1}{2}(1 + e^{-2z})$ and $\tilde{S}(z) = e^{-z} \sinh z = \frac{1}{2}(1 - e^{-2z})$, with shifted analogues
614 for the x -dependent arguments; the common factor $e^{-(\mu_1 L_1 + \mu_2 L_2)}$ cancels between
615 numerator and denominator. Accuracy fails well before overflow: the raw matrix chain
616 of [equation \(10\)](#) loses all significant digits in double precision by $\text{Re}(\mu_i L_i) \approx 8$, because
617 the growing mode is formed and cancelled numerically. Repeating the evaluation at
618 60 digits confirms the scaled closed forms to 10^{-23} . For more than two or three layers
619 a scattering-matrix recursion is recommended in place of the plain product.

620

621 4.4 Characteristic finite-volume solver

622

623 System [equation \(20\)](#) is linear and hyperbolic with Riemann invariants $w_i^\pm = \frac{1}{2}(T \pm$
624 $q/Z_i)$ advecting at $\pm v_i$. The energy is advanced by a Godunov update whose face flux
625 is the exact solution of the local Riemann problem. Across a face separating (T_L, q_L)
626 and (T_R, q_R) with impedances Z_L, Z_R ,

627

$$628 \quad q^* = \frac{Z_L Z_R (T_L - T_R) + Z_R q_L + Z_L q_R}{Z_L + Z_R}, \quad T^* = T_L + \frac{q_L - q^*}{Z_L}. \quad (25)$$

629

630
631 Minmod reconstruction of $w^\pm$ keeps the scheme total-variation-diminishing, so fronts
632 advect without the $2\Delta x$ oscillations of a staggered discretisation. The stiff source $-q/\tau$
633 is integrated exactly, $q^{n+1} = q^n e^{-\Delta t/\tau}$, and the two updates are combined by Godunov
634 splitting. The resulting scheme is first-order accurate in time and keeps fronts sharp;
635 the Courant number is 0.4.

636 Feeding a unit right-going invariant $(T_L, q_L) = (1, Z_1)$ into a quiescent layer 2,
637 [equation \(25\)](#) returns $T^* = T_{12}$ and $T^* - T_L = R$, in agreement with [equation \(13\)](#). This
638 agreement is built in, since the Riemann flux uses the same Z , so it only shows that the
639 code is self-consistent. The independent tests are the comparison of the closed form
640 with the full time evolution and the mesh convergence of the jump amplitude ([Table 2](#)).
641 The spectral implementation also satisfies interface continuity to $\sim 10^{-11}$ at complex s .
642 First-order accuracy is a limitation of the present scheme. A second-order version, with
643 Strang splitting and a monotone upstream-centred (MUSCL) reconstruction, should
644 reduce the residuals of [Table 2](#) by about two orders of magnitude and is worth using

when the jump amplitude is needed to better than 0.1%. Both solvers are implemented in a single Python module.

4.5 Use of generative AI

During the preparation of this work the authors used ChatGPT to improve the language and readability of the manuscript, and GitHub Copilot to assist in writing the plotting code for the figures. No AI tool was used to derive the analytical results, to generate, alter or interpret the numerical data, or to select or verify the references. The authors reviewed and edited all AI-assisted text and code and take full responsibility for the content of the article.

Data availability. The Python module implementing both solvers, together with the scripts that generate every figure and table, is available from the corresponding author on request.

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