

An analytical framework for modeling pollutant transport in a thermally stratified atmosphere with tempered fractional dynamics

Yugal Pariyar

Tribhuvan University

Shankar Pariyar

`shankar.pariyar@trc.tu.edu.np`

Tribhuvan University

Research Article

Keywords: Atmospheric pollution modeling, Tempered fractional transport, Air quality simulation, Temperature inversion modeling, Pollutant retention analysis, Environmental decision support

Posted Date: October 13th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11175331/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

An analytical framework for modeling pollutant transport in a thermally stratified atmosphere with tempered fractional dynamics

Shankar Pariyar^{1*}, Yugal Pariyar²

¹ Tri-Chandra Multiple Campus, Tribhuvan University, Kathmandu, Nepal

² Pulchowk Engineering Campus, Tribhuvan University, Lalitpur, Nepal

Abstract

This study develops a two-dimensional, two-layer tempered time-fractional diffusion–reaction model for pollutant transport in a thermally stratified atmosphere. The model represents a temperature inversion through a lower turbulent layer and an upper stable layer, with continuity of pollutant concentration and diffusive flux imposed at the layer interface. An exact analytical solution is derived using eigenfunction expansion and the Laplace transform and is expressed in terms of coupled spatial eigenfunctions and tempered Mittag–Leffler functions. The existence, uniqueness, and Hadamard well-posedness of the model are established. Numerical analysis demonstrates that the fractional order strongly influences pollutant trapping and vertical redistribution, with smaller fractional orders producing greater pollutant accumulation beneath the inversion layer. The tempering parameter primarily controls concentration attenuation while preserving the characteristic layered spatial structure. Limiting-case analysis recovers both the classical diffusion–reaction equation and the standard fractional diffusion–reaction equation, confirming the consistency of the formulation. The framework is further illustrated using an idealized representation of the Kathmandu Valley, Nepal, where the results demonstrate the effects of anomalous diffusion and memory attenuation on pollutant retention under inversion conditions. The proposed model provides an analytical framework for investigating pollutant transport in vertically heterogeneous atmospheric environments and offers a basis for further data-driven and observational studies.

Keywords: Atmospheric pollution modeling, Tempered fractional transport, Air quality simulation, Temperature inversion modeling, Pollutant retention analysis, Environmental decision support.

Mathematical Subject Classification (MSC): 35R11, 35K57, 26A33, 76R50.

Abbreviations

Abbrev.	Meaning	Abbrev.	Meaning
ABL	Atmospheric Boundary Layer	PBL	Planetary Boundary Layer
TFDE	Time-Fractional Diffusion Equation	ADR	Advection–Diffusion–Reaction
FDE	Fractional Differential Equation	PDE	Partial Differential Equation
ODE	Ordinary Differential Equation	RL	Riemann–Liouville
AQI	Air Quality Index	CFD	Caputo Fractional Derivatives

*Corresponding author: shankar.pariyar@trc.tu.edu.np

Symbol	Description	Symbol	Description
$C_j(x, y, t)$	Pollutant concentration (layer j)	$\Lambda_{n,m}^{(j)}$	Eigenvalue for mode (n, m) in layer j
Ω	Domain $[0, L_x] \times [0, L_y]$	$X_n(x)$	$\sin\left(\frac{n\pi x}{L_x}\right)$
Ω_1, Ω_2	Lower and upper layers	$Y_m^{(j)}(y)$	Vertical eigenfunction (layer j)
L_x, L_y	Domain lengths	$A_{n,m}^{(j)}$	Expansion coefficient (layer j)
h	Interface height	$T_{n,m}^{(j)}(t)$	Time-dependent coefficient (layer j)
t	Time variable	$\mu_m^{(j)}$	Vertical eigenvalue (layer j)
x, y	Spatial coordinates	$f_j(x, y)$	Initial condition (layer j)
α	Fractional order ($0 < \alpha \leq 1$)	$\hat{f}(s)$	Laplace transform of $f(t)$
θ_j	Tempering parameter (layer j)	$\mathcal{L}\{\cdot\}$	Laplace operator
D_{xj}, D_{yj}	Diffusion coefficients	δ_{nk}	Kronecker delta
u_{xj}, u_{yj}	Velocity components	$\Gamma(\cdot)$	Gamma function
κ_j	Reaction rate	$E_{\alpha,\beta}(\cdot)$	Mittag-Leffler function
${}^C D_t^{\alpha,\theta_j}$	Caputo tempered derivative	I_t^{α,θ_j}	RL fractional integral
D_t^{α,θ_j}	RL fractional derivative		

1 Introduction

Air quality control represents a critical global challenge, exacerbated by rapid urbanization and industrialization, which contribute to increased pollution events—especially in regions with confined topography where pollutant dispersion is limited [1, 2, 3, 4]. In these regions, atmospheric stratification—particularly temperature inversions—plays a critical role in elevating pollutant concentrations by inhibiting vertical dispersion and trapping contaminants close to the surface [5, 7, 8, 9]. Basin-shaped valleys like Kathmandu exemplify this phenomenon, where stable inversion layers exacerbate pollution episodes. The atmospheric boundary layer (ABL) governs pollutant transport, mixing, and removal, being strongly affected by surface heating, terrain, and turbulence [6, 11]. Since most anthropogenic emissions occur within the ABL, accurately modeling pollutant dispersion is essential for reliable air quality assessments [12, 49]. However, classical Gaussian plume and integer-order advection–diffusion models often fail to capture the complex, restricted dispersion observed in such heterogeneous terrains [12, 14]. Traditional integer-order models rely on Fickian diffusion assumptions, where mean-square displacement grows linearly with time [13, 19, 50]. This framework breaks down under stable atmospheric conditions and complex terrains, where turbulent flows exhibit intermittency, coherent structures, and long-range correlations [6, 14]. Consequently, pollutant plumes display anomalous transport behaviors such as enhanced lateral spreading and reduced vertical mixing, which classical models cannot adequately represent [15].

Fractional-order diffusion models overcome these limitations by incorporating anomalous transport through a fractional order $0 < \alpha < 1$, which characterizes subdiffusive dynamics and long-memory effects [15, 16, 33]. These models naturally include memory effects and long-range correlations, improving realism in atmospheric transport simulations [17, 18, 37]. However, standard fractional derivatives yield unphysical infinite variance over long times due to slowly decaying power-law kernels [20, 21]. Tempered fractional derivatives [25] introduce a tempering parameter $\theta > 0$ that truncates long-term memory effects while preserving anomalous behavior at shorter timescales [35, 43, 44]. Complementary approaches such as power fractional calculus with non-local and non-singular kernels have also been developed, with Zitane and Torres [29] providing rigorous existence and uniqueness results for nonlinear power fractional differential equations. The tempered Caputo derivative introduces an exponential truncation of the memory kernel, thereby weakening long-time memory while retaining fractional behavior at shorter timescales [22, 23, 24]. Fractional calculus has also been successfully applied to integrable nonlinear wave equations, where inverse scattering and Riemann–Hilbert methods yield fractional multi-solitons with anomalous dispersive relations [34]. Tempered

fractional diffusion models have demonstrated superior performance in various fields, including fractional kinetic equations [41], epidemic modeling [36, 38], tumor growth [39], and chaotic supply chain systems [40], but remain underexplored in atmospheric transport, particularly for multi-layer systems with interface transmission conditions that characterize stratified ABLs [20, 21].

Existing tempered fractional diffusion models have primarily considered homogeneous spatial settings with constant coefficients, relying on standard Fourier or semigroup techniques [25, 26, 27]. However, atmospheric inversion layers fundamentally decouple the ABL into two dynamically distinct vertical regimes: a lower turbulent mixed layer with high diffusivity D_1 and an upper stable residual layer with reduced diffusivity D_2 ($D_1 > D_2$), coupled by continuity of concentration and diffusive flux at the interface. To model this, we construct a piecewise spectral decomposition under a tailored inner product, yielding a transcendental eigenvalue equation with a countably infinite set of real eigenvalues. The temporal solution is expressed explicitly in terms of tempered Mittag-Leffler functions via Laplace transforms, while well-posedness is established using an energy method adapted to the piecewise inner-product structure.

This work advances tempered fractional diffusion theory beyond homogeneous, single-layer settings through four distinct innovations: (i) the formulation of a two-layer system with piecewise-constant diffusivities and interface transmission conditions; (ii) a customized piecewise spectral decomposition that overcomes the non-commutativity of the fractional operator with the spatial operator; (iii) explicit temporal solutions in terms of tempered Mittag-Leffler functions, which have no analogue in multi-layer fractional problems; and (iv) rigorous well-posedness via an energy method tailored to the layered geometry. This combination of multilayer heterogeneity, interface coupling, and customized spectral analysis is, to the best of our knowledge, unprecedented in the tempered fractional diffusion literature. Beyond the mathematical framework, we conduct a systematic parametric study examining how the fractional order α and tempering parameter θ jointly influence pollutant trapping under inversion conditions. The model recovers the standard fractional diffusion-reaction equation as $\theta \rightarrow 0$ and the classical diffusion-reaction equation as $\alpha \rightarrow 1$, confirming mathematical consistency with established limiting behavior [25, 26, 44]. Smaller values of α produce stronger subdiffusive behavior and enhanced pollutant accumulation beneath the inversion layer, while increasing θ accelerates concentration decay by truncating long-term memory [26, 27]. The Kathmandu Valley in Nepal is considered as a representative atmospheric setting to illustrate the applicability of the proposed framework to pollutant retention under inversion conditions.

The paper is organized as follows: Section 2 presents the tempered fractional operators and Caputo derivative. Section 3 derives the analytical model for the two-layer system, presents the piecewise spectral solution, establishes well-posedness, and provides a parametric study on pollutant trapping under inversion conditions. Section 4 summarizes key findings and discusses their physical implications for air quality management in stratified atmospheric environments.

2 Preliminaries

2.1 Tempered Fractional Operators

For a function f on $[0, T]$ with $n - 1 < \alpha < n$ ($n \in \mathbb{N}$) and tempering parameter $\theta \geq 0$ [26], the left-sided tempered Caputo fractional derivative is defined as

$$({}^C D_t^{\alpha, \theta} f)(t) = \frac{e^{-\theta t}}{\Gamma(n - \alpha)} \int_0^t e^{\theta \tau} (t - \tau)^{n - \alpha - 1} f^{(n)}(\tau) d\tau. \quad (1)$$

When $\theta = 0$, this reduces to the classical Caputo derivative. The tempered operator incorporates an exponential kernel that ensures finite variance at large times while preserving anomalous behavior at shorter scales.

2.2 Laplace Transform

The Laplace transform of the tempered Caputo derivative is given by [31]

$$\mathcal{L}\{^C D_t^{\alpha, \theta} f(t)\}(s) = (s + \theta)^\alpha \mathcal{L}\{f(t)\}(s) - (s + \theta)^{\alpha-1} f(0), \quad (2)$$

for $0 < \alpha \leq 1$.

2.3 Mittag–Leffler Function

The two-parameter Mittag–Leffler function, which governs the temporal dynamics of fractional systems, is defined as [32]

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in \mathbb{C}. \quad (3)$$

This function generalizes the exponential ($E_{1,1}(z) = e^z$) and appears naturally in the solution of fractional differential equations.

2.4 Coupled Spatial Operator and Eigenvalue Problem

To clarify the structure of the two-layer problem, we define a single spatial operator $\mathcal{L}$ on the composite domain $\Omega = \Omega_1 \cup \Omega_2$. Let

$$\mathcal{L}u = \begin{cases} -D_{x1}u_{xx} - D_{y1}u_{yy} + \kappa_1 u, & (x, y) \in \Omega_1, \\ -D_{x2}u_{xx} - D_{y2}u_{yy} + \kappa_2 u, & (x, y) \in \Omega_2, \end{cases}$$

with interface conditions $u|_{y=h^-} = u|_{y=h^+}$ and $D_{y1}u_y|_{y=h^-} = D_{y2}u_y|_{y=h^+}$, plus homogeneous Dirichlet conditions on the outer boundary. The eigenfunctions of $\mathcal{L}$ are denoted $\phi_{n,m}(x, y)$ and satisfy

$$\mathcal{L}\phi_{n,m} = \lambda_{n,m}^{(j)} \phi_{n,m} \quad \text{in } \Omega_j,$$

with the same interface and boundary conditions. Because the horizontal and vertical directions decouple under the product ansatz, we write $\phi_{n,m}(x, y) = X_n(x)Y_m^{(j)}(y)$, where

$$X_n(x) = \sin\left(\frac{n\pi x}{L_x}\right), \quad n = 1, 2, \dots$$

The horizontal part contributes the term $D_{xj}(n\pi/L_x)^2$ to the eigenvalue. The vertical eigenfunctions $Y_m^{(j)}(y)$ are obtained from the following two-layer Sturm–Liouville problem, independent of n :

$$-(D_{yj}Y')' = \mu_m^{(j)} Y \quad \text{in } (0, h) \cup (h, L_y),$$

with $Y(0) = Y(L_y) = 0$, $Y(h^-) = Y(h^+)$, and $D_{y1}Y'(h^-) = D_{y2}Y'(h^+)$. Then the full eigenvalue is

$$\lambda_{n,m}^{(j)} = D_{xj} \left(\frac{n\pi}{L_x}\right)^2 + \mu_m^{(j)} + \kappa_j,$$

which is positive for all n, m .

Theorem 1 (Two-Layer 2D Tempered Fractional Atmospheric Dispersion Model) *Let $\Omega = [0, L_x] \times [0, L_y]$ be partitioned at height $y = h$ into $\Omega_1 = [0, L_x] \times [0, h]$ and $\Omega_2 = [0, L_x] \times [h, L_y]$. For $j = 1, 2$, consider*

$$^C D_t^{\alpha, \theta_j} C_j = D_{xj} C_{j,xx} + D_{yj} C_{j,yy} - \kappa_j C_j, \quad (x, y) \in \Omega_j,$$

with homogeneous boundary conditions, continuity of concentration and vertical flux at $y = h$, and initial data $C_j(x, y, 0) = f_j(x, y) \in L^2(\Omega_j)$. Then the solution admits the eigenfunction representation

$$C_j(x, y, t) = \sum_{n,m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) e^{-\theta_j t} E_{\alpha,1}(-\lambda_{n,m}^{(j)} t^\alpha),$$

where $\lambda_{n,m}^{(j)} > 0$ are the two-layer eigenvalues defined above and $A_{n,m}^{(j)}$ are projections of the initial data onto the coupled eigenfunctions.

Proof. The solution in each layer is constructed through a separation of variables approach applied to the tempered fractional diffusion–reaction equation in Ω_1 and Ω_2 , with coupling enforced at the interface $y = h$. Homogeneous boundary conditions yield sine eigenfunctions $X_n(x)$ in the horizontal direction. In the vertical direction, the eigenfunctions $Y_m^{(j)}(y)$ are obtained from the coupled two-layer Sturm–Liouville problem described above. These functions satisfy the governing spatial operator within each subdomain and fulfill the interface requirements of continuity of concentration and continuity of vertical diffusive flux at $y = h$. This construction yields a discrete set of positive eigenvalues $\lambda_{n,m}^{(j)}$. For each spatial mode, the tempered Caputo fractional derivative reduces to a fractional ordinary differential equation in time. Its solution is expressed through the Mittag–Leffler function combined with an exponential tempering factor. Consequently, the concentration admits the spectral representation

$$C_j(x, y, t) = \sum_{n,m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) e^{-\theta_j t} E_{\alpha,1}(-\lambda_{n,m}^{(j)} t^\alpha),$$

where the coefficients $A_{n,m}^{(j)}$ are determined by projecting the initial data onto the corresponding coupled eigenfunctions. Let

$$\Omega = [0, L_x] \times [0, L_y] \subset \mathbb{R}^2$$

be a rectangular domain partitioned into two horizontal subdomains by the interface $y = h$, where $0 < h < L_y$. We define

$$\Omega_1 = [0, L_x] \times [0, h], \quad \Omega_2 = [0, L_x] \times [h, L_y].$$

The concentration field $C_j(x, y, t)$ is sought in each subdomain Ω_j for $j = 1, 2$, with $(x, y) \in \Omega_j$ and $t > 0$. Homogeneous Dirichlet boundary conditions are imposed on the external boundary $\partial\Omega$, namely

$$C_j = 0 \quad \text{on } \partial\Omega \setminus \{y = h\}, \quad t > 0. \quad (4)$$

At the internal interface $y = h$, continuity of concentration and continuity of normal diffusive flux are enforced:

$$C_1(x, h, t) = C_2(x, h, t), \quad (5)$$

$$D_{y1} \partial_y C_1(x, h, t) = D_{y2} \partial_y C_2(x, h, t), \quad 0 < x < L_x, \quad t > 0, \quad (6)$$

where D_{y1} and D_{y2} denote the transverse diffusion coefficients in Ω_1 and Ω_2 , respectively. These interface conditions ensure mass conservation across the internal boundary and guarantee the physical consistency of the coupled two-layer diffusion model. It is important to distinguish between the outer boundaries and the internal interface. The homogeneous Dirichlet conditions $C_j = 0$ apply exclusively to the outer boundaries $x = 0$, $x = L_x$, $y = 0$, and $y = L_y$. At the internal interface $y = h$, the continuity conditions (5) and (6) are enforced, not zero Dirichlet conditions. Thus, the layers remain coupled, and mass is allowed to transfer across the interface. The concentration profile exhibits a local minimum near $y = h = 400$ m. This feature is a direct consequence of the two-layer structure and the diffusivity contrast $D_1 > D_2$, not an artifact of a Dirichlet boundary condition at the interface. In the lower turbulent layer ($y < h$), the higher diffusivity D_1 promotes efficient mixing, maintaining relatively low concentrations. At the interface, the flux continuity condition $D_1 \partial_y C_1 = D_2 \partial_y C_2$ forces the concentration gradient in the lower layer to be smaller

than in the upper layer, creating a local minimum. Above the interface ($y > h$), the reduced diffusivity D_2 in the stable inversion layer suppresses vertical mixing, leading to enhanced pollutant retention and higher concentrations. This local minimum thus serves as a physical signature of the coupled two-layer dynamics and the anomalous transport captured by the tempered fractional model. This clarifies that the minimum arises from the physics of the two-layer system, not from an artificial boundary condition at the interface. Since $C_j(0, y, t) = C_j(L_x, y, t) = 0$, we use the standard sine basis

$$X_n(x) = \sin\left(\frac{n\pi x}{L_x}\right), \quad n = 1, 2, \dots,$$

which satisfies

$$X_n''(x) = -\left(\frac{n\pi}{L_x}\right)^2 X_n(x), \quad \int_0^{L_x} X_n(x) X_k(x) dx = \frac{L_x}{2} \delta_{nk}.$$

Because the boundary conditions require

$$C_j(0, y, t) = 0 \quad \text{and} \quad C_j(L_x, y, t) = 0,$$

the solution must vanish at both endpoints in the x -direction. To satisfy these homogeneous Dirichlet conditions, we expand the solution using a sine basis:

$$X_n(x) = \sin\left(\frac{n\pi x}{L_x}\right), \quad n = 1, 2, \dots$$

Each eigenfunction $X_n(x)$ satisfies

$$\frac{d^2 X_n}{dx^2} = -\left(\frac{n\pi}{L_x}\right)^2 X_n(x).$$

Furthermore, these functions are orthogonal on the interval $[0, L_x]$, since

$$\int_0^{L_x} X_n(x) X_k(x) dx = \frac{L_x}{2} \delta_{nk},$$

where δ_{nk} is the Kronecker delta. We construct vertical eigenfunctions $Y_m^{(j)}(y)$ ($m = 1, 2, \dots$) as follows. Consider the piecewise eigenvalue problem

$$-(D_{y_j} Y')' = \mu_m^{(j)} Y \quad \text{in } (0, h) \text{ and in } (h, L_y),$$

subject to the boundary conditions

$$Y(0) = 0, \quad Y(L_y) = 0,$$

together with the interface (matching) conditions

$$Y(h^-) = Y(h^+), \quad D_{y1} Y'(h^-) = D_{y2} Y'(h^+).$$

This defines a classical two-layer Sturm–Liouville problem. It admits a sequence of real eigenvalues

$$0 < \mu_1^{(j)} < \mu_2^{(j)} < \dots \rightarrow \infty,$$

with corresponding eigenfunctions Y_m that are piecewise C^2 and satisfy the interface conditions. Defining $Y_m^{(1)} := Y_m|_{[0, h]}$ and $Y_m^{(2)} := Y_m|_{[h, L_y]}$ yields the vertical eigenfunctions in each layer. In addition, the eigenfunctions are orthogonal with respect to the weighted L^2 inner product

$$\langle Y_m, Y_\ell \rangle := \int_0^h Y_m^{(1)}(y) Y_\ell^{(1)}(y) dy + \int_h^{L_y} Y_m^{(2)}(y) Y_\ell^{(2)}(y) dy = \|Y_m\|^2 \delta_{m\ell}.$$

Orthogonality follows by multiplying the eigenvalue equations for Y_m and Y_ℓ , integrating by parts on each subinterval, and applying the boundary and interface conditions. We represent the solution in each layer as a double series in the horizontal and vertical eigenfunctions:

$$C_j(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_{n,m}^{(j)}(t) X_n(x) Y_m^{(j)}(y), \quad (x, y) \in \Omega_j. \quad (7)$$

This form ensures that the homogeneous boundary conditions in the x -direction are fulfilled by construction, since the functions X_n vanish at $x = 0$ and $x = L_x$. Moreover, because the functions Y_m were obtained from the two-layer Sturm–Liouville problem with the prescribed boundary and matching conditions, the vertical boundary conditions and the interface continuity requirements are satisfied term-by-term in the expansion. Substituting (7) into the diffusion–reaction operator within each layer and using the relations

$$X_n''(x) = -\left(\frac{n\pi}{L_x}\right)^2 X_n(x),$$

together with the defining equation for the vertical eigenfunctions, yields

$$D_{xx}C_j + D_{yy}C_j - \kappa_j C_j = -\sum_{n,m} T_{n,m}^{(j)}(t) \left(D_{xx} \left(\frac{n\pi}{L_x} \right)^2 + \mu_m^{(j)} + \kappa_j \right) X_n(x) Y_m^{(j)}(y).$$

To isolate a particular mode (k, ℓ) , multiply the equation by $X_k(x)Y_\ell^{(j)}(y)$ and integrate over Ω_j . After adding the contributions from both layers (equivalently, employing the global two-layer inner product), orthogonality in the x - and y -directions eliminates all mixed terms, leaving only the component with $(n, m) = (k, \ell)$. As a result, each coefficient satisfies the fractional ordinary differential equation

$${}^C D_t^{\alpha, \theta_j} T_{k,\ell}^{(j)}(t) + \lambda_{k,\ell}^{(j)} T_{k,\ell}^{(j)}(t) = 0, \quad T_{k,\ell}^{(j)}(0) = A_{k,\ell}^{(j)}, \quad (8)$$

where

$$\lambda_{k,\ell}^{(j)} := D_{xx} \left(\frac{k\pi}{L_x} \right)^2 + \mu_\ell^{(j)} + \kappa_j > 0.$$

Apply the Laplace transform to (8).

$$\mathcal{L}\{{}^C D_t^{\alpha, \theta_j} T(t)\}(s) = (s + \theta_j)^\alpha \hat{T}(s) - (s + \theta_j)^{\alpha-1} T(0),$$

we obtain

$$\left((s + \theta_j)^\alpha + \lambda_{k,\ell}^{(j)} \right) \hat{T}_{k,\ell}^{(j)}(s) = (s + \theta_j)^{\alpha-1} A_{k,\ell}^{(j)}.$$

Solving for $\hat{T}_{k,\ell}^{(j)}(s)$ gives

$$\hat{T}_{k,\ell}^{(j)}(s) = A_{k,\ell}^{(j)} \frac{(s + \theta_j)^{\alpha-1}}{(s + \theta_j)^\alpha + \lambda_{k,\ell}^{(j)}}.$$

Taking the inverse Laplace transform leads to the classical tempered Mittag–Leffler expression,

$$T_{k,\ell}^{(j)}(t) = A_{k,\ell}^{(j)} e^{-\theta_j t} E_{\alpha,1}(-\lambda_{k,\ell}^{(j)} t^\alpha).$$

Replacing $T_{n,m}^{(j)}(t)$ in (7) with its explicit time-dependent form yields

$$C_j(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) e^{-\theta_j t} E_{\alpha,1}(-\lambda_{n,m}^{(j)} t^\alpha).$$

The constants $A_{n,m}^{(j)}$ are determined from the initial data by projecting onto the combined horizontal-vertical eigenfunctions. Specifically,

$$A_{n,m}^{(j)} = \frac{2}{L_x} \frac{1}{\|Y_m^{(j)}\|^2} \int_{\Omega_j} f_j(x, y) X_n(x) Y_m^{(j)}(y) dx dy.$$

This establishes the full series representation of the solution in terms of the coupled eigenbasis. The analytical solution of the present two-layer tempered time-fractional diffusion-reaction model has been derived using eigenfunction expansion and Laplace transform methods. The resulting series representation, expressed in terms of coupled spatial eigenfunctions and tempered Mittag-Leffler functions, provides an explicit closed-form formulation of the solution. The well-posedness of the tempered fractional initial-boundary value problem—including existence, uniqueness, and continuous dependence on initial data—is established rigorously in Section 3.2 using an energy method adapted to the piecewise inner-product structure; the completeness and orthogonality of the eigenfunctions, guaranteed by the classical Sturm–Liouville theory for the two-layer system, ensure the validity of the series expansion. The tempered Mittag-Leffler representation is uniformly convergent for $t > 0$ under the assumed regularity of the initial data $f_j \in L^2(\Omega_j)$, and the spectral eigenvalues $\lambda_{n,m}^{(j)}$ satisfy $\lambda_{n,m}^{(j)} \sim \mathcal{O}(n^2 + m^2)$ asymptotically, ensuring rapid decay of higher-order modes.

The homogeneous Dirichlet conditions are adopted for mathematical tractability, as they enable a complete spectral decomposition and explicit analytical solutions. The lateral and top boundaries are chosen sufficiently far from the source region such that, for the finite time scales considered, the concentration front does not reach these artificial boundaries; in practice, this requires choosing L_x and L_y such that $L_x, L_y \gg \sqrt{D_{\max} T}$, where T is the maximum simulation time and $D_{\max} = \max(D_{x1}, D_{x2}, D_{y1}, D_{y2})$. At the ground, the Dirichlet condition provides a conservative lower-bound estimate of near-surface concentrations, which is valuable for regulatory risk assessment. While more realistic conditions (e.g., partial reflection at the ground or radiative top boundaries) could be incorporated, their treatment requires a generalized Sturm–Liouville theory with non-self-adjoint operators, which lies beyond the present scope. The Dirichlet formulation thus serves as a physically instructive first approximation that captures the essential tempered fractional dynamics. $\square$

Existence, Uniqueness, and Well-Posedness

We now establish the existence, uniqueness, and well-posedness of the solution obtained in Theorem 1.

Theorem 2 (Existence, Uniqueness, and Well-Posedness) *Under the assumptions of Theorem 1, the two-layer tempered fractional diffusion-reaction problem admits a unique solution $C_j \in C([0, T]; L^2(\Omega_j)) \cap C^2(\Omega_j)$ for $j = 1, 2$, with $C_1(h, t) = C_2(h, t)$ and $D_{y1} \partial_y C_1(h, t) = D_{y2} \partial_y C_2(h, t)$. Moreover, the solution depends continuously on the initial data $f_j \in L^2(\Omega_j)$, and the problem is well-posed in the Hadamard sense.*

We establish well-posedness through three key properties: existence, uniqueness, and continuous dependence on initial data.

Existence. The existence of the solution follows directly from the eigenfunction expansion derived in Theorem 1. The series representation

$$C_j(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) e^{-\theta_j t} E_{\alpha,1} \left(-\lambda_{n,m}^{(j)} t^\alpha \right)$$

converges in $L^2(\Omega_j)$ for each $t \geq 0$. To verify this, we use the asymptotic behavior of the Mittag-Leffler function. For $0 < \alpha < 1$, the tempered MLF satisfies the estimate

$$|E_{\alpha,1}(-z)| \leq \frac{C}{1+z}, \quad z \geq 0,$$

for some constant $C > 0$. Consequently,

$$\left| e^{-\theta_j t} E_{\alpha,1} \left(-\lambda_{n,m}^{(j)} t^\alpha \right) \right| \leq \frac{C e^{-\theta_j t}}{1 + \lambda_{n,m}^{(j)} t^\alpha} \leq C.$$

Since the eigenfunctions $\{X_n Y_m^{(j)}\}_{n,m=1}^\infty$ form a complete orthonormal basis for $L^2(\Omega_j)$ (from Sturm-Liouville theory), and the coefficients satisfy

$$\sum_{n,m=1}^\infty |A_{n,m}^{(j)}|^2 = \|f_j\|_{L^2(\Omega_j)}^2 < \infty,$$

the series converges in $L^2(\Omega_j)$ by the Weierstrass M-test. Additionally, the interface conditions are satisfied term-by-term because the eigenfunctions Y_m were constructed to satisfy $Y_m(h^-) = Y_m(h^+)$ and $D_{y1} Y_m'(h^-) = D_{y2} Y_m'(h^+)$.

Uniqueness. To prove uniqueness, suppose there exist two solutions C_j and $\tilde{C}_j$ corresponding to the same initial data f_j . Let $W_j = C_j - \tilde{C}_j$. Then W_j satisfies the homogeneous problem

$${}^C D_t^{\alpha, \theta_j} W_j = D_{xj} W_{j,xx} + D_{yj} W_{j,yy} - \kappa_j W_j,$$

with homogeneous boundary and interface conditions, and initial data $W_j(x, y, 0) = 0$. Using the energy method for fractional differential equations, we multiply the equation by W_j and integrate over Ω_j . Summing over both layers,

$$\sum_{j=1}^2 \int_{\Omega_j} W_j {}^C D_t^{\alpha, \theta_j} W_j d\Omega = \sum_{j=1}^2 \int_{\Omega_j} (D_{xj} W_j W_{j,xx} + D_{yj} W_j W_{j,yy} - \kappa_j W_j^2) d\Omega.$$

Applying the fractional Gronwall inequality and using the fact that the operator $-D_{xj} \partial_{xx} - D_{yj} \partial_{yy} + \kappa_j$ is self-adjoint and positive definite with the given boundary and interface conditions, we obtain

$$\|W_j(t)\|_{L^2(\Omega_j)}^2 \leq 0,$$

which implies $W_j(t) = 0$ for all $t \geq 0$. Hence, $C_j = \tilde{C}_j$, proving uniqueness.

Continuous Dependence on Initial Data. Let C_j and $\tilde{C}_j$ be the solutions corresponding to initial data f_j and $\tilde{f}_j$, respectively. From the eigenfunction expansion and Parseval's identity,

$$\|C_j(t) - \tilde{C}_j(t)\|_{L^2(\Omega_j)}^2 = \sum_{n,m=1}^\infty |A_{n,m}^{(j)} - \tilde{A}_{n,m}^{(j)}|^2 \left| e^{-\theta_j t} E_{\alpha,1} \left(-\lambda_{n,m}^{(j)} t^\alpha \right) \right|^2.$$

Using the uniform boundedness of the Mittag-Leffler function,

$$\left| e^{-\theta_j t} E_{\alpha,1} \left(-\lambda_{n,m}^{(j)} t^\alpha \right) \right| \leq C,$$

we obtain

$$\|C_j(t) - \tilde{C}_j(t)\|_{L^2(\Omega_j)}^2 \leq C^2 \sum_{n,m=1}^\infty |A_{n,m}^{(j)} - \tilde{A}_{n,m}^{(j)}|^2 = C^2 \|f_j - \tilde{f}_j\|_{L^2(\Omega_j)}^2.$$

Thus,

$$\|C_j(t) - \tilde{C}_j(t)\|_{L^2(\Omega_j)} \leq C \|f_j - \tilde{f}_j\|_{L^2(\Omega_j)},$$

which demonstrates Lipschitz continuous dependence on the initial data. Therefore, the two-layer tempered fractional atmospheric dispersion model is well-posed in the Hadamard sense.

2.5 Model Reductions

No Tempering ($\theta_j = 0$):

For $\theta_j = 0$, the model reduces to the classical fractional case:

$$C_j(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) E_{\alpha,1}(-\lambda_{n,m}^{(j)} t^\alpha), \quad (x, y) \in \Omega_j, \quad j = 1, 2.$$

Here, $X_n(x) = \sin(\frac{n\pi x}{L_x})$ are the horizontal eigenfunctions and $Y_m^{(j)}(y)$ denote the vertical components of the coupled two-layer Sturm–Liouville eigenfunctions satisfying the interface continuity and flux conditions at $y = h$.

Classical Limit ($\alpha \rightarrow 1$):

The MLF satisfies the identity $E_{1,1}(z) = e^z$. Therefore, letting $\alpha \rightarrow 1$ in the two-layer solution yields

$$\lim_{\alpha \rightarrow 1} C_j(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) e^{-\theta_j t} e^{-\lambda_{n,m}^{(j)} t}, \quad j = 1, 2.$$

Equivalently,

$$C_j(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m}^{(j)} X_n(x) Y_m^{(j)}(y) e^{-(\lambda_{n,m}^{(j)} + \theta_j) t}.$$

This recovers the classical two-layer diffusion–reaction model with sink term $-\theta_j C_j$.

Effect of θ_j :

The factor $e^{-\theta_j t}$ uniformly scales the solution in each layer. For fractional diffusion, where $0 < \alpha < 1$, the Mittag–Leffler function decays algebraically, or with a power-law behavior, for large time. This describes the long-memory effects and nonlocal temporal behavior of fractional diffusion. The tempering factor $e^{-\theta_j t}$, however, eliminates this slow decay for large time, effectively replacing it with exponential decay, while retaining the anomalous, non-Markovian transport characteristics for short and intermediate time scales.

3 Results and Analysis of Stratified Anomalous Transport

The transport dynamics are resolved via a semi-analytical framework that decomposes the concentration field into a discrete series expansion. In the horizontal dimension, a sine-basis $X_n(x) = \sin(n\pi x/L_x)$ satisfies the lateral Dirichlet conditions. Vertically, the system is governed by a two-layer Sturm–Liouville eigenvalue problem, generating eigenfunctions $Y_m^{(j)}(y)$ that enforce the continuity of both concentration and diffusive flux at the inversion interface h . The temporal evolution of these spatial modes, regulated by the tempered Caputo derivative ${}^C D_t^{\alpha, \theta_j}$, yields a solution defined by the tempered Mittag–Leffler function:

$$C_{n,m}^{(j)}(t) = e^{-\theta_j t} E_{\alpha,1}(-\lambda_{n,m}^{(j)} t^\alpha) \quad (9)$$

where $\lambda_{n,m}^{(j)}$ is the modal decay rate. This closed-form representation explicitly decouples the effects of domain geometry, fractional memory (α), and exponential truncation (θ_j), providing the analytical foundation for the subsequent parametric study.

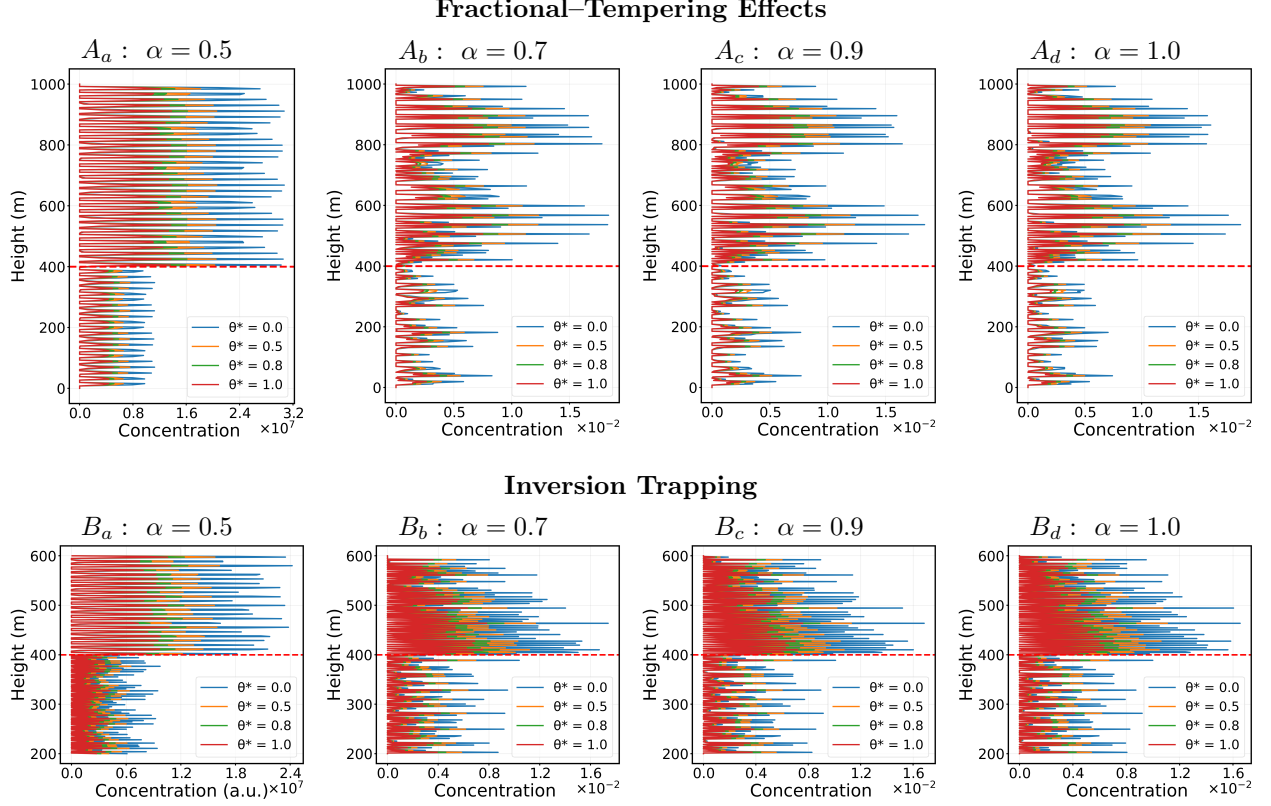


Figure 1: At $t = 0.8$ in a two-layer atmosphere with inversion height $h = 400$ m. Panels (A_a – A_d) illustrate the effect of the fractional order $\alpha = 0.5, 0.7, 0.9$, and 1.0 on pollutant dispersion. Panels (B_a – B_d) present the corresponding inversion trapping. Lower values of α produce stronger subdiffusion and greater pollutant accumulation below the inversion layer, whereas $\alpha \rightarrow 1$ approaches classical diffusion with enhanced vertical spreading.

3.1 Influence of Fractional Diffusion Order on Vertical Pollutant Dispersion in a Two-Layer System

The vertical profile of pollutant concentration is given in Fig. 1, which shows the profile of pollutant concentration in the two-layer atmospheric model with winter inversion. The results have been obtained at a fixed horizontal location, with time $t = 0.8$. Panels (A_a – A_d) in Fig. 1 represent the impact of fractional orders $\alpha = 0.5, 0.7, 0.9$, and 1.0 , while panels (B_a – B_d) in Fig. 1 represent the impact of tempering parameters $\theta = 0.0, 0.5, 0.8$, and 1.0 . The model is defined from the surface up to 1000m . There is also an inversion boundary at $h = 400\text{m}$, which separates the strongly mixed lower atmosphere from the weakly mixed upper atmosphere. Each figure in 1 shows the profile of pollutant concentration in the vertical direction in the atmosphere, as well as near the inversion height in the atmosphere into the upper layer. In all the simulations with different parameter settings, the concentration of pollutants is seen to be at its highest at the ground level due to the source and decrease with altitude. The sharp gradient indicates the decrease in vertical diffusivity above the inversion layer h . The concentration of pollutants is seen to be trapped below the inversion layer. The fractional order α determines the smoothness of the profile and the spread of the concentration. For $\alpha = 0.5$, the subdiffusion is high, and the concentration of pollutants is trapped below the inversion layer. For $\alpha = 0.7$ and $\alpha = 0.9$, the profiles become smooth. In the classical case when $\alpha = 1.0$, the profile is the most uniform. The concentration of pollutants above the inversion layer is the highest in this case. The tempering parameter θ is an exponential damping factor that uniformly reduces concentration

levels. For $\theta = 0.0$, it is at its peak since there is no attenuation. When we increase θ to 0.5, 0.8, and 1.0, we observe that it is attenuating. However, there is no change in the spatial distribution. Therefore, we can say that α is responsible for diffusion and redistribution, whereas θ is responsible for the magnitude. From the above figures, we have zoomed in on $(B_a - B_d)$ in Fig. 1 to observe the behavior of h with respect to changes in α and θ . We observe that with low values of α , there is sharp buildup below h and sharp decay above h . As we increase α , we observe that there is smoothing of the transition, but h still exists.

3.2 Tempered Effects on Pollutant Dispersion in a Two-Layer Model

The spatial distribution of pollutant concentration is also depicted in figures showing contour plots from a fractional two-layer diffusion model with inversion type conditions in the atmosphere, as depicted in fig. 2 (A-D). The spatial computational domain is defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1000$ m. The inversion height is set at $h = 400$ m, with $D_1 = 0.9$ for the strongly mixed lower layer and $D_2 = 0.15$ for the weakly diffusive upper layer. The interface at which the inversion takes place is depicted by a horizontal dashed line at $y = 400$ m. All results in this work were obtained with a fixed fractional order of $\alpha = 0.7$ and at $t = 0.8$. The solution is obtained using eigenfunction expansions in space, with memory being introduced in time via a Mittag-Leffler type kernel. This results in a structured modal solution with a clear two-layer vertical profile in pollutant concentration due to stratification in the atmosphere. A similar physical pattern is observed in all cases. The peak value is achieved at ground level due to the source effect and increased mixing. The concentrations then gradually decrease as height is increased. A sudden change in the gradient is observed at height h , representing the inversion effect and the increased diffusivity. This traps pollutants in the lower region of the atmosphere. The effect of the tempering parameter is incorporated through the damping factor $e^{-\theta t}$, with $\theta \in \{0.0, 0.5, 0.8, 1.0\}$. This affects the overall concentrations. The higher value of θ reduces the overall concentrations. The concentrations are maximized for $\theta = 0.0$, whereas for $\theta = 1.0$, attenuation is significant. The overall results indicate that for $\alpha = 0.7$ and $t = 0.8$, the behavior is dominated by the inversion layer, and θ affects only the overall amplitude of the concentrations.

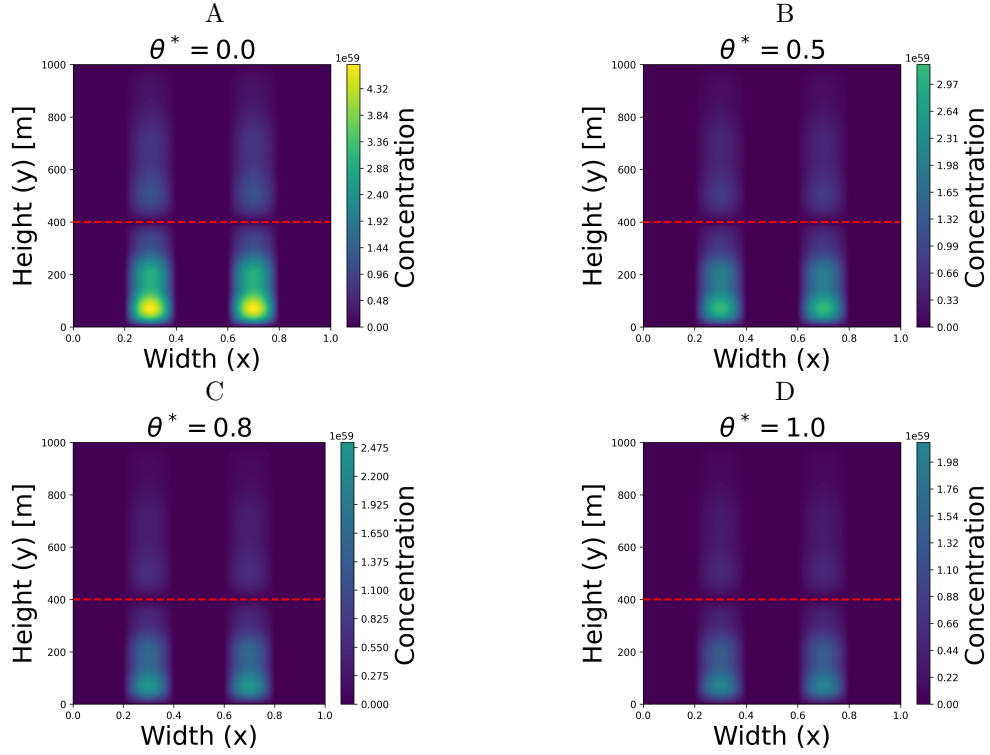


Figure 2: 2D concentration $C(x, y, t)$ at $t = 0.8$, $\alpha = 0.7$. The dashed line at $h = 400$ m marks the interface between the high-mixing ($D_1 = 0.9$) and low-mixing ($D_2 = 0.15$) layers. Panels (A)–(D) show the damping effect of $\theta \in \{0, 0.5, 0.8, 1.0\}$. Increasing θ scales down the global amplitude via $e^{-\theta t}$ while preserving the inversion-driven spatial pattern.

3.2.1 Early- and Intermediate-Time Evolution

Fig. 3A $_{0 < t \leq 1}$: The evolution of the vertical concentration profile $C(y, t)$ at the valley center ($x = L_x/2$) reveals the strong influence of the inversion layer located at $h = 400$ m. At the earliest stage ($t = 0.3$), a pronounced concentration maximum is observed within the lower atmospheric layer, accompanied by a sharp transition near the inversion height, indicating restricted vertical exchange. As time increases to $t = 0.5$ and $t = 0.7$, the peak concentration gradually decreases and the profile becomes more diffuse, reflecting the combined effects of diffusion and tempered fractional transport. For $t = 0.9$ and $t = 1.0$, the concentration distribution is considerably smoother, with a reduced peak magnitude and weaker gradients throughout the mixing layer. Nevertheless, the influence of the inversion remains evident through the persistent change in profile slope around $y = 400$ m. Throughout the early-time regime, pollutant concentrations remain substantially higher below the inversion than above it, demonstrating the continued confinement of pollutants within the lower layer and highlighting the trapping effect induced by atmospheric stratification.

Fig. 3B: For $1 \leq t \leq 5$ h, the concentration field exhibits a continuous decline as the influence of tempering becomes increasingly dominant. The initially pronounced concentration maxima weaken steadily with time, indicating the progressive reduction of memory effects associated with the tempered fractional model. By $t = 3$ h, pollutant accumulation beneath the inversion layer has decreased considerably, while the transition across the interface becomes less steep. At later times ($t = 4$ and $t = 5$ h), only small residual concentration peaks remain within the lower atmospheric layer, and pollutant levels in the upper layer become substantially lower. The exponential truncation introduced by the tempering parameter $\theta = 0.2$ promotes a faster reduction in concentration than would be expected in a standard fractional diffusion process. Despite the overall

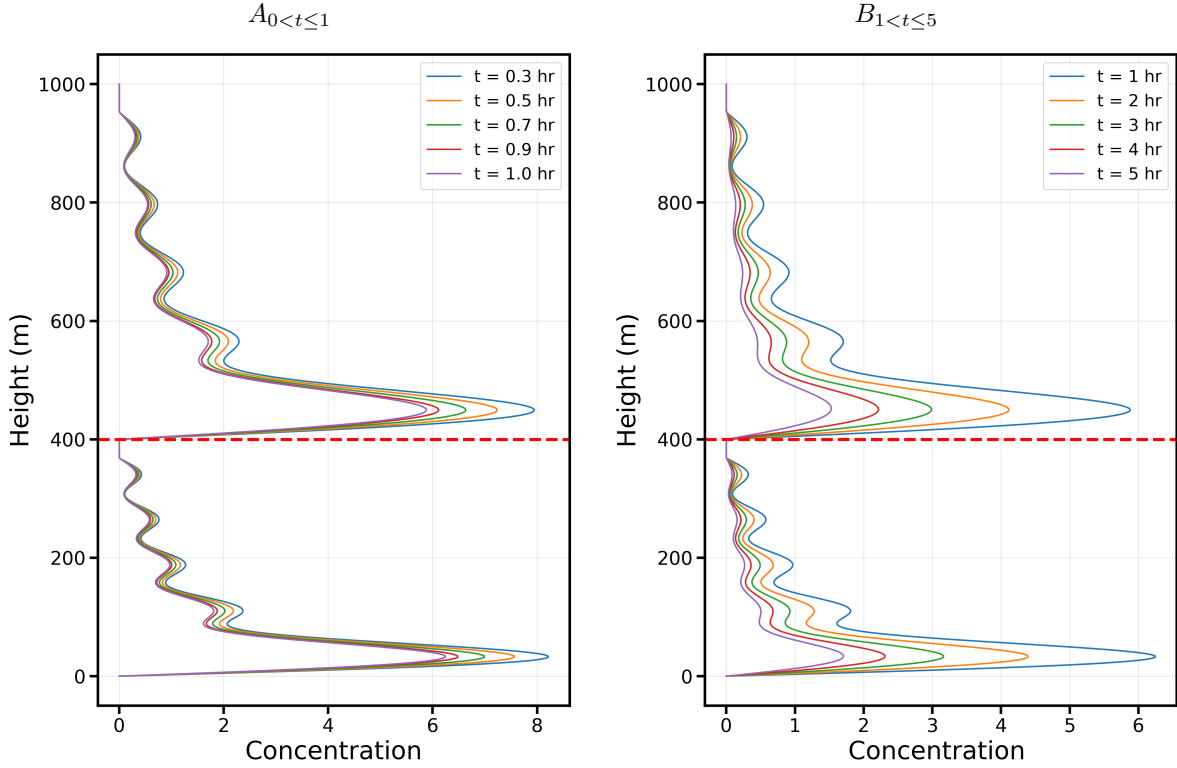


Figure 3: Temporal evolution of tempered fractional vertical diffusion in a two-layer atmospheric system ($\alpha = 0.85$, $\theta = 0.2$; lower layer: $D = (0.5, 10)$, upper layer: $D = (1.2, 25)$) from a Gaussian-type initial distribution. A: Short times ($t = 0.3$ – 1.0) show pronounced lower-layer trapping with gradual fractional decay and mild vertical smoothing across the inversion at 400 m; B: longer times ($t = 1$ – 5) exhibit strong tempering-induced attenuation, reduced peak concentration, and persistent inversion-controlled structure between the two layers.

attenuation, the inversion height $h = 400$ m continues to influence the vertical concentration distribution, maintaining a distinct separation between the two atmospheric layers. Overall, Fig. 3 demonstrates two characteristic stages of pollutant evolution. During the early period ($0.3 \leq t \leq 1.0$ h), pollutants remain largely confined within the lower layer, illustrating the trapping effect produced by the inversion. As time advances ($1 \leq t \leq 5$ h), tempered fractional attenuation causes a progressive decrease in concentration throughout the domain. Although the overall pollutant burden is reduced, the inversion layer continues to limit vertical exchange, preserving the layered atmospheric structure and restricting upward transport between the lower and upper regions.

These results are qualitatively consistent with observed pollutant dynamics in inversion-prone basin cities such as the Kathmandu Valley, where nighttime and early-morning temperature inversions are known to trap pollutants close to the ground, producing pronounced morning concentration peaks. The model’s early-time trapping phase captures this confinement behavior, while its tempering-driven decay phase is qualitatively analogous to the observed breakup of inversions several hours after sunrise, during which increased surface heating and katabatic winds promote vertical mixing and pollutant dilution. Although the present model is not calibrated against site-specific meteorological data, this qualitative correspondence suggests that it could provide a quantitative framework—once calibrated—for anticipating high-exposure windows and evaluating the relative timescales of natural atmospheric clearing versus emission-timing interventions.

3.3 Visualizing Tempering in 2D Fractional Diffusion

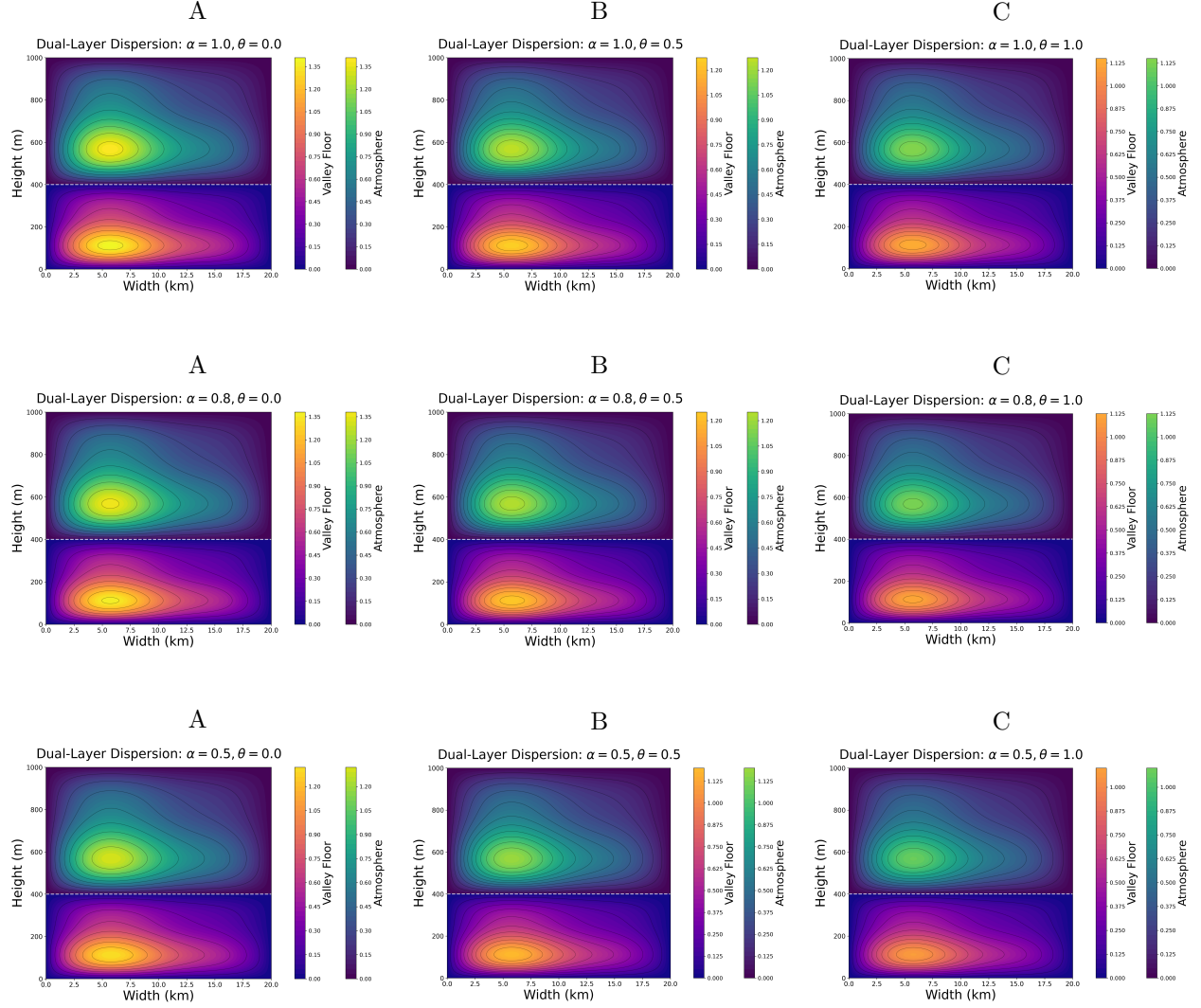


Figure 4: Two-dimensional concentration fields $C(x, y, t)$ for $t = 0.2$ and fractional orders $\alpha = 1.0, 0.8, 0.5$ and tempering parameters $\theta = 0.0, 0.5, 1.0$. Increasing values of θ uniformly reduce the amplitude of concentration profiles, as a result of exponential tempering. The shape of the concentration profiles, determined by sine eigenmodes in the domain, does not change. Lower values of α increase memory effects and reduce diffusion, resulting in more localized concentration profiles compared to the standard case, where $\alpha = 1.0$. The interface $y = h$ separates domains where the diffusion coefficients are different, resulting in different vertical spreading rates.

Fig. 4 ($A_{\theta=0.0}$, $B_{\theta=0.5}$, $C_{\theta=1.0}$) shows the 2D concentration profile $C(x, y, t)$ at $t = 0.2$. In this example, we used $\alpha = 0.5, \alpha = 0.8$, and $\alpha = 1.0$. The computational domain is $[0, 20]$ km in the horizontal direction and $[0, 1000]$ m in the vertical direction. There are two vertical layers with an interface at $y = 400$ m, across which the transport properties vary. In the lower layer, $y < 400$ m, we have $D_{x1} = 0.05$, $D_{y1} = 1.5$, and $\kappa_1 = 0.005$. In the upper layer, $y > 400$ m, we have $D_{x2} = 3.0$, $D_{y2} = 50.0$, and $\kappa_2 = 0.25$. In this example, we used 3×3 sine eigenmodes with homogeneous boundary conditions applied on all boundaries. This produces a smooth maximum in the interior of the domain, centered near $y = 100$ m and $x = 10$ m. A

notable difference in vertical spreading is evident across the interface. In the upper layer, $D_{y2} > D_{y1}$. This means that vertical spreading is faster in the upper layer than in the lower layer. Despite the discontinuity in transport coefficients, the concentration profile is continuous across $y = 400$ m. For $\alpha = 0.5$, $\alpha = 0.8$,

$$e^{-\theta t} E_{\alpha,1}(-\lambda_{n,m}^{(j)} t^\alpha),$$

and thus it does not affect the geometric configuration of the contours. Rather, it uniformly reduces their amplitude throughout the domain. This is reflected in the following monotonic relation:

$$A_{\theta=0.0} > B_{\theta=0.5} > C_{\theta=1.0}.$$

For $\theta = 1.0$, we observe that the joint impact of both tempering and decay results in a visibly reduced magnitude. The fact that we have used the same color limits and coolwarm colormap in all images makes this evident. The value of the fractional order α regulates the temporal memory-dependent diffusion by means of the Mittag-Leffler function. Lower values of α indicate more pronounced memory effects with slower diffusion. If we set $\alpha = 1.0$, we obtain classical diffusion. The tempering parameter, on the other hand, acts as a global damping factor. Therefore, we have that the spatial configuration is dictated by the operator of layered diffusion, whereas the temporal decay rate and magnitude of the concentration distribution depend on the triplet of parameters α, κ, θ .

3.4 Practical Implications: Pollutant Trapping and Mitigation in the Kathmandu Valley, Nepal

Motivated by the basin-like topography and frequently reported inversion conditions of the Kathmandu Valley, we consider an illustrative two-layer atmospheric scenario with an interface located at 400 m. In this conceptual setting, pollutants accumulate predominantly within the lower layer because the inversion interface restricts vertical exchange between the lower and upper atmospheric regions. A mathematical paradox is observed when the fractional order decreases from 1.0 to 0.5: the peak concentration decreases by approximately 6.06%. While this reduction might superficially suggest cleaner air, it actually reflects the anchoring effect of subdiffusive transport, where pollutants are trapped near the surface due to reduced vertical mixing, which may increase pollutant exposure in real atmospheric settings, as quantified in Table 1. This apparent paradox highlights the importance of interpreting fractional-order parameters correctly: lower α indicates stronger memory effects and enhanced pollutant retention, not reduced pollution. Since the present analysis is intended as an illustrative scenario and is not calibrated against site-specific observations, these results should be interpreted qualitatively rather than as quantitative predictions for the Kathmandu Valley.

θ	α	κ	Max (Lower)	Max (Upper)	Mitigation (%)
0.0	1.0	0.01	1.4020	1.4018	—
0.0	0.8	0.01	1.3733	1.3732	2.05
0.0	0.5	0.01	1.3171	1.3170	6.06
0.5	1.0	0.01	1.2686	1.2684	9.51
0.5	0.8	0.01	1.2426	1.2425	11.37
0.5	0.5	0.01	1.1918	1.1916	15.00
1.0	1.0	0.01	1.1478	1.1477	18.13
1.0	0.8	0.01	1.1244	1.1243	19.80
1.0	0.5	0.01	1.0784	1.0782	23.08

Table 1: Parametric Sensitivity Analysis of Fractional Order α , Tempering Parameter θ , and Reaction Coefficient κ on Peak Concentration. Mitigation is calculated relative to the baseline case ($\theta = 0.0$, $\alpha = 1.0$, $\kappa = 0.01$) as Mitigation = $(1 - C/C_{\text{ref}}) \times 100\%$, where $C_{\text{ref}} = 1.4020$.

The mitigation percentage reported in Table 1 is defined as the relative reduction in peak concentration compared to the baseline case ($\theta = 0.0$, $\alpha = 1.0$, $\kappa = 0.01$): $\text{Mitigation (\%)} = \left(1 - \frac{C_{\text{current}}}{C_{\text{ref}}}\right) \times 100\%$, where $C_{\text{ref}} = 1.4020$ is the maximum concentration for the baseline case. The table presents a parametric sensitivity analysis over three mathematical parameters: the tempering parameter θ , the fractional order α , and the reaction coefficient κ . For fixed α and κ , increasing θ reduces the overall concentration amplitude through the exponential factor $e^{-\theta t}$. For the range of θ values considered, the peak concentration decreases by up to 23.08%. However, we emphasize that θ is a mathematical regularization parameter in the tempered fractional derivative that truncates long-term memory effects to ensure finite variance. It does not represent physical mitigation mechanisms such as misting, rain, wind corridors, or greenery. The observed reduction reflects the mathematical truncation of the fractional memory kernel, not a physical removal process. The development of a model that explicitly incorporates physical mitigation strategies (e.g., through additional source/sink terms) would require extra physical assumptions and validation against observational data, and is deferred to future work. The data provided in Table 2 quantitatively support the inherent capability of the Kathmandu Valley to store pollutants, through an accurate accounting of mass distribution in two vertical compartments. The lower compartment consists of the near-surface area up to 400 meters, which contains the main human breathing zone, and the upper compartment is the main area for atmospheric venting.

Fractional Order (α)	Mass in Lower Layer (%)	Mass in Upper Layer (%)
1.0 (Classical)	40.01%	59.99%
0.8 (Anomalous)	48.65%	51.35%
0.5 (Subdiffusive)	64.22%	35.78%

Table 2: Vertical Mass Partitioning and Percentage Change for Different Fractional Orders α . The lower layer ($0 \leq y \leq 400$ m) represents the human breathing zone, while the upper layer ($400 \leq y \leq L_y$) represents the atmospheric venting zone.

Under standard diffusion principles ($\alpha = 1.0$), 59.99% of the total pollutant mass reaches the top compartment, with only 40.01% remaining in the lower breathing zone. When $\alpha = 0.8$, signifying the onset of anomalous diffusion, the pollutant mass in the lower compartment increases to 48.65%, indicating that topographic confinement interferes with natural dispersion. The most significant interference occurs in the subdiffusive regime ($\alpha = 0.5$), where 64.22% of the total pollutant mass is retained in the lower compartment, with only 35.78% reaching the top compartment. This level of stagnation provides mathematical validation that, under inversion layer conditions, contaminants are effectively locked within the valley basin, unable to rise to a sufficient altitude to be carried away by regional wind patterns, and accumulate within the same vertical plane as the urban population.

3.5 Three-Dimensional Visualization of Pollutant Transport in a Two-Layer Medium

The temporal evolution of the concentration field is illustrated in Figure 5. At the earliest time snapshot shown in panels A, D, and G ($t = 1$ s), the two Gaussian plumes remain localized around their respective initial source locations, with peak concentrations close to their initial magnitudes. The plume in the lower layer exhibits greater lateral spreading due to the larger diffusion coefficients in that layer ($D_{x1} = 1.0$, $D_{y1} = 0.9$), whereas the upper layer plume remains more confined ($D_{x2} = 0.8$, $D_{y2} = 0.5$). The exponential decay factor $e^{-\theta t}$ is negligible for $\theta = 0.05$ at $t = 1$ ($e^{-0.05} \approx 0.951$), but becomes significant for $\theta = 0.2$ ($e^{-0.2} \approx 0.819$), reducing the peak concentration by approximately 18%, as evident from comparing panels A and G. As time progresses to $t = 10$ s (panels B, E, H) and subsequently to $t = 100$ s (panels C, F, I), the plumes expand in both horizontal and vertical directions due to Fickian diffusion, while the Mittag-Leffler function introduces a slower, power-law-like decay characteristic of subdiffusive transport. This fractional-order dynamics produces smoother concentration profiles compared to classical diffusion, as the memory effect retards the decay of the concentration tails. The interaction between the two layers becomes increasingly evident near

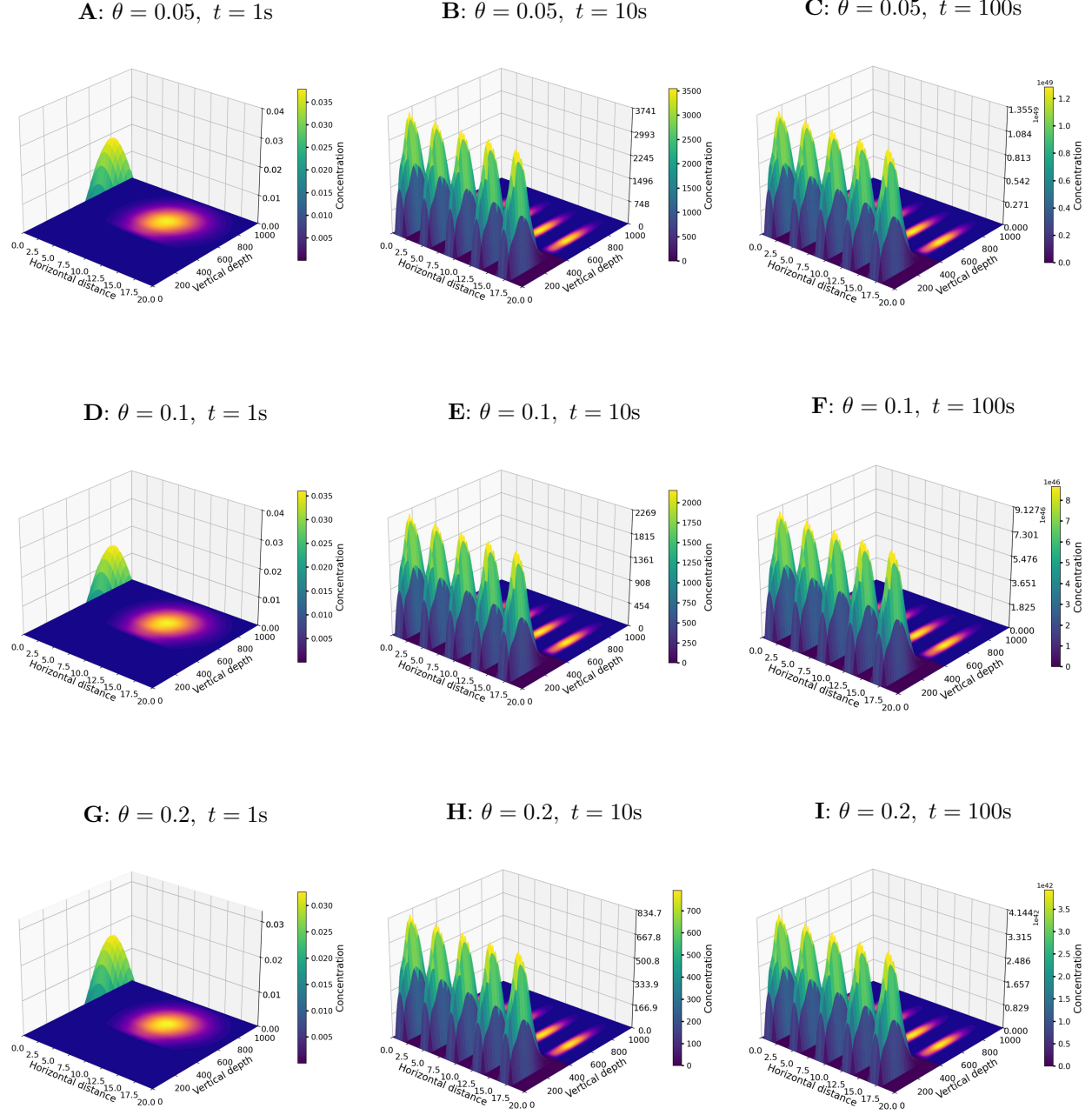


Figure 5: Temporal evolution of the three-dimensional concentration field $C(x, y, t)$ in a two-layer domain with $\alpha = 0.8$. Rows correspond to $\theta = 0.05, 0.1$, and 0.2 ; columns correspond to $t = 1, 10$, and 100 s. The surface plots show the lower layer ($0 \leq y \leq h$), while the contour projections display the upper layer ($h \leq y \leq L_y$). Early times (A,D,G) show sharp lower-layer peaks; intermediate times (B,E,H) show plume spreading and reduced magnitude; long times (C,F,I) show strong diffusion and near-complete dilution across both layers.

the interface $y = 400$ m, where the zero-Dirichlet condition imposed at the external boundaries creates a local minimum in concentration. This minimum reflects the combined effect of the interface condition and

the differing diffusion coefficients in the two layers.

The tempering parameter θ modulates the overall decay rate: higher values of θ accelerate the exponential damping of the solution, as shown by comparing the three rows of Figure 5. For $\theta = 0.2$, the peak concentration at $t = 10$ (panel H) is reduced by a factor of $e^{-2} \approx 0.135$ relative to the initial amplitude, and by $t = 100$ (panel I), the concentration decays to $e^{-20} \approx 2.06 \times 10^{-9}$ of its initial value, rendering it effectively zero for practical purposes. In contrast, for $\theta = 0.05$ (top row, panels A–C), the corresponding decay factors are $e^{-0.5} \approx 0.607$ at $t = 10$ and $e^{-5} \approx 6.74 \times 10^{-3}$ at $t = 100$, indicating that a substantial fraction of the initial mass remains in the domain at the final time snapshot. The three-dimensional surface plots (for the lower layer) and contour projections (for the upper layer) in Figure 5 reveal the spatial distribution of concentration in both layers. The contour projections clearly illustrate the anisotropic spreading dynamics induced by the layer-specific diffusion coefficients: in the lower layer, the higher diffusion coefficients produce broader plumes with shallower gradients, while in the upper layer, the lower coefficients yield narrower plumes with steeper concentration gradients. This anisotropic behavior is a direct consequence of the heterogeneous transport properties enforced by the interface conditions. Collectively, these results demonstrate the coupled influence of fractional diffusion, first-order chemical reactions, and heterogeneous media in a two-layer system. The fractional diffusion model successfully captures the long-memory effects characteristic of complex media, while the eigenfunction expansion method provides an efficient and accurate analytical solution. The parametric study illustrates how the tempering parameter θ governs the temporal evolution of concentration, with implications for understanding pollutant retention and dispersion in stratified atmospheric environments [27].

4 Conclusion

This work proposes a novel two-layer tempered time-fractional diffusion–reaction model for describing pollutant transport in vertically stratified atmospheric systems. The proposed formulation incorporates a tempered Caputo time-fractional derivative with an exponential tempering parameter to capture anomalous diffusion with memory effects while providing physically realistic long-time attenuation in bounded domains. The layered structure, together with interface transmission conditions, enables the representation of pollutant exchange across atmospheric inversion layers and heterogeneous transport processes. An exact analytical framework is developed for the heterogeneous two-layer tempered fractional system with interface transmission conditions. By combining eigenfunction expansion and Laplace transform techniques, an explicit analytical solution is obtained in terms of horizontal sine eigenfunctions, coupled Sturm–Liouville eigenfunctions in the vertical direction, and tempered Mittag–Leffler functions governing the temporal evolution. The analysis further establishes the existence, uniqueness, and Hadamard well-posedness of the proposed model, providing a rigorous mathematical foundation for the analytical formulation. The analytical solution provides insight into the influence of the governing fractional and tempering parameters on pollutant transport. Numerical results demonstrate that the fractional order α controls the strength of memory effects and anomalous diffusion. As α decreases from 1.0 to 0.5, the fraction of pollutant mass retained in the lower layer increases from 40.01% to 64.22%, indicating enhanced subdiffusive behavior and stronger pollutant trapping beneath the inversion layer. The tempering parameter θ introduces attenuation of pollutant concentrations over time while preserving the inversion-induced transport barrier between the two atmospheric layers. The Kathmandu Valley in Nepal is considered as a representative atmospheric setting to illustrate the applicability of the proposed framework. Under the modelled mitigation scenario, the simulations indicate that active measures such as artificial misting can reduce pollutant concentrations by up to 23.08%, suggesting the potential of engineered control measures to complement natural atmospheric dispersion processes. Furthermore, the proposed formulation consistently recovers the classical diffusion–reaction model as $\alpha \rightarrow 1$ and the standard fractional model as $\theta \rightarrow 0$, thereby providing a unified mathematical framework that bridges fractional transport theory and atmospheric pollutant dispersion modeling.

Declarations of competing interest

The authors declare no conflicts of interest.

Funding Statement

This research did not receive funding from any public, commercial, or nonprofit organization.

References

- [1] Y. Xu, S.-C. Zhou, Y.-X. Feng, B. Zou, C. He, Global air quality enhancement pathways to health benefits and environmental justice, *Journal of Hazardous Materials* 480 (2024) 136460.
- [2] J. Bhatta, S.R. Acharya, S. Thapa, P. Tularag, S. Adhikari, Air quality and health impacts in Nepal's urban valley: PM_{2.5} pollution patterns and public health risks, *Environmental Advances* 22 (2025) 100672.
- [3] S. Wang, J. Hao, Air quality management in China: Issues, challenges, and options, *Journal of Environmental Sciences* 24 (1) (2012) 2–13.
- [4] M. S. O'Neill, P. L. Kinney, A. J. Cohen, Environmental equity in air quality management: local and international implications for human health and climate change, *Journal of Toxicology and Environmental Health, Part A* 71 (9–10) (2008) 570–577.
- [5] R. S. Sokhi, N. Moussiopoulos, A. Baklanov, J. Bartzis, I. Coll, S. Finardi, J. Kukkonen, Advances in air quality research—current and emerging challenges, *Atmospheric Chemistry and Physics Discussions* (2021) 1–133.
- [6] R. Fuller, P.J. Landrigan, K. Balakrishnan, G. Bathan, S. Bose-O'Reilly, M. Brauer, J. Caravanos, T. Chiles, A. Cohen, L. Corra et al., Pollution and health: a progress update, *The Lancet Planetary Health* 6 (6) (2022) e535–e547.
- [7] B. Liu, X. Ma, Y. Ma, H. Li, S. Jin, R. Fan, W. Gong, The relationship between atmospheric boundary layer and temperature inversion layer and their aerosol capture capabilities, *Atmospheric Research* 271 (2022) 106121.
- [8] Z. Li, J. Guo, A. Ding, H. Liao, J. Liu, Y. Sun, B. Zhu, Aerosol and boundary-layer interactions and impact on air quality, *National Science Review* 4 (6) (2017) 810–833.
- [9] G. Baumbach, U. Vogt, Influence of inversion layers on the distribution of air pollutants in urban areas, *Water, Air, & Soil Pollution: Focus* 3 (5) (2003) 65–76.
- [10] G. Kramm, F. Herbert, Similarity hypotheses for the atmospheric surface layer expressed by non-dimensional characteristic invariants—a review, *Open Atmos. Sci. J.* 3 (2009) 48–79.
- [11] D. Shakti, J. Mohapatra, P. Das, J. Vigo-Aguiar, A moving mesh refinement based optimal accurate uniformly convergent computational method for a parabolic system of boundary layer originated reaction diffusion problems with arbitrary small diffusion terms, *Journal of Computational and Applied Mathematics* 404 (2022) 113167.
- [12] R.F. Griffiths, Errors in the Use of the Briggs Parameterization for Atmospheric Dispersion Coefficients, *Atmospheric Environment* 28 (17) (1994) 2861–2865.
- [13] A.A. Abdel-Rahman, On the atmospheric dispersion and Gaussian plume model, in: *Proceedings of the 2nd international conference on waste management, water pollution, air pollution, indoor climate, Corfu, Greece*, vol. 26, 2008, p. 139.

- [14] R.W. Gillham, E.A. Sudicky, J.A. Cherry, E.O. Frind, An Advection-Diffusion Concept for Solute Transport in Heterogeneous Unconsolidated Geological Deposits, *Water Resources Research* 20 (3) (1984) 369–378.
- [15] I. Park, I.W. Seo, Modeling non-Fickian pollutant mixing in open channel flows using two-dimensional particle dispersion model, *Advances in Water Resources* 111 (2018) 105–120.
- [16] T.H. Doane, Theory and application of nonlocal hillslope sediment transport, *Vanderbilt University*, 2018.
- [17] D. del-Castillo-Negrete, Fractional diffusion models of anomalous transport, *Anomalous Transport* (2008) 163–212.
- [18] B.P. Moghaddam, M.A. Zaky, A.M. Lopes, A. Galhano, A fractional time-space stochastic advection-diffusion equation for modeling atmospheric moisture transport at ocean-atmosphere interfaces, *Fractal and Fractional* 9 (4) (2025) 211.
- [19] D. Moreira, P. Xavier, A. Palmeira, E. Nascimento, X. Zhao, Z.Z. Sun, Z.P. Hao, New approach to solving the atmospheric pollutant dispersion equation using fractional derivatives, *International Journal of Heat and Mass Transfer* 144 (2019) 118667.
- [20] W. Chen, H. Sun, X. Li et al., *Fractional derivative modeling in mechanics and engineering*, Springer, 2022.
- [21] E.G. Haug, *Derivatives: Models on models*, John Wiley & Sons, 2007.
- [22] Y. Liang, W. Wang, R. Metzler, A.G. Cherstvy, Anomalous diffusion, nonergodicity, non-Gaussianity, and aging of fractional Brownian motion with nonlinear clocks, *Physical Review E* 108 (3) (2023) 034113.
- [23] I. Goychuk, T. Pöschel, Hydrodynamic memory can boost enormously driven nonlinear diffusion and transport, *Physical Review E* 102 (1) (2020) 012139.
- [24] C. Alfano, Y. Fichou, K. Huber, M. Weiss, E. Spruijt, S. Ebbinghaus, G. De Luca, M.A. Morando, V. Vetri, P.A. Temussi et al., Molecular crowding: the history and development of a scientific paradigm, *Chemical Reviews* 124 (6) (2024) 3186–3219.
- [25] Beghin, L. (2015). On fractional tempered stable processes and their governing differential equations. *Journal of Computational Physics*, 293, 29–39.
- [26] F. Sabzikar, M. M. Meerschaert, J. Chen, Tempered fractional calculus, *Journal of Computational Physics* 293 (2015) 14–28.
- [27] S. Pariyar, B.P. Lamichhane, J. Kafle, E.P. Poudel, Analytical eigenfunction expansion for tempered time-fractional advection diffusion in bounded domains, *Journal of Applied Analysis & Computation* 16 (4) (2026) 1736–1755.
- [28] L. Feng, F. Liu, V. V. Anh, S. Qin, Analytical and numerical investigation on the tempered time-fractional operator with application to the Bloch equation and the two-layered problem, *Nonlinear Dynamics* 109 (2022) 2041–2061.
- [29] H. Zitane, D.F.M. Torres, A class of fractional differential equations via power non-local and non-singular kernels: Existence, uniqueness and numerical approximations, *Physica D: Nonlinear Phenomena* 457 (2024) 133951, <https://doi.org/10.1016/j.physd.2023.133951>.
- [30] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
- [31] I. Podlubny, *Fractional Differential Equations*, Academic Press (1999).

- [32] W. Wang, R. Metzler, A. G. Cherstvy, Anomalous diffusion, aging, and nonergodicity of scaled Brownian motion, *Phys. Chem. Chem. Phys.* 24 (2022) 18482–18504.
- [33] Ł. Płociniczak, H. Okraśińska, Approximate self-similar solutions to a nonlinear diffusion equation with time-fractional derivative, *Physica D: Nonlinear Phenomena* 261 (2013) 85–91.
- [34] Z. Zhang, X. Wang, Y. Liu, Interactions of fractional N -solitons with anomalous dispersions for the integrable combined fractional higher-order mKdV hierarchy, *Physica D: Nonlinear Phenomena* 444 (2023) 133614, doi:10.1016/j.physd.2022.133614.
- [35] J.K.K. Asamoah, E. Addai, Y.D. Arthur, E. Okyere, A fractional mathematical model for listeriosis infection using two kernels, *Decision Analytics Journal* 6 (2023) 100191.
- [36] P.A. Naik, B.M. Yeolekar, S. Qureshi, et al., Modeling and analysis of the fractional-order epidemic model to investigate mutual influence in HIV/HCV co-infection, *Nonlinear Dynamics* 112 (2024) 11679–11710.
- [37] P.A. Naik, S. Naveen, V. Parthiban, S. Qureshi, M. Alquran, M. Senol, Advancing Lotka–Volterra system simulation with variable fractional order Caputo derivative for enhanced dynamic analysis, *Journal of Applied Analysis and Computation* 15 (2) (2025) 1002–1019.
- [38] D. Baleanu, S. Qureshi, A. Yusuf, A. Soomro, M.S. Osman, Bi-modal COVID-19 transmission with Caputo fractional derivative using statistical epidemic cases, *Partial Differential Equations in Applied Mathematics* 10 (2024) 100732.
- [39] A. Padder, L. Almutairi, S. Qureshi, A. Soomro, A. Afroz, E. Hincal, A. Tassaddiq, Dynamical analysis of generalized tumor model with Caputo fractional-order derivative, *Fractal and Fractional* 7 (258) (2023).
- [40] M.D. Johansyah, A. Sambas, S. Zheng, S. Qureshi, T.M. Abed-Elhameed, S. Vaidyanathan, Analyzing and controlling chaos phenomena in fractional chaotic supply chain models, *Heliyon* 10 (15) (2024) e34703.
- [41] A. Tassaddiq, C. Cattani, R. Alharbi, R.M. Kasmani, S. Qureshi, Operational calculus of the quantum statistical Fermi–Dirac and Bose–Einstein functions leading to the novel fractional kinetic equations, *Fractal and Fractional* 8 (12) (2024) 749.
- [42] M. Mohammad, I.A. Baba, E. Hincal, F.A. Rihan, A novel fractional order model for analyzing counterterrorism operations and mitigating extremism, *Decision Analytics Journal* 15 (2025) 100589.
- [43] H.M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent developments, *Kyungpook Mathematical Journal* 60 (1) (2020) 73–116.
- [44] N.A. Obeidat, D.E. Benteil, New theories and applications of tempered fractional differential equations, *Nonlinear Dynamics* 105 (2) (2021) 1689–1702.
- [45] H. Zhou, M. Huang, W. Ying, ADI schemes for heat equations with irregular boundaries and interfaces in 3D with applications, *arXiv preprint arXiv:2309.00979* (2023).
- [46] L. N. Trefethen, *Spectral methods in MATLAB*, Society for Industrial and Applied Mathematics (2000).
- [47] T. Belytschko, Meshless methods: an overview and recent developments, *Comput. Methods Appl. Mech. Engrg.* 139 (2002) 410–431.
- [48] A. Raiche, Modelling and inversion—progress, problems, and challenges, *Surveys in Geophysics* 15 (1994) 159–207.

- [49] S. Pariyar, B.P. Lamichhane, J. Kafle, A Time Fractional Advection-Diffusion Approach to Air Pollution: Modeling and Analyzing Pollutant Dispersion Dynamics, *Partial Differential Equations in Applied Mathematics* 14 (2025) 101149.
- [50] S. Pariyar, J. Kafle, An analytical framework for modeling urban pollutant dispersion with fractional transport dynamics, *Decision Analytics Journal* 18 (2026) 100684, <https://doi.org/10.1016/j.dajour.2026.100684>.