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Research Article

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Posted Date: October 13th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11165936/v1>

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Additional Declarations: No competing interests reported.

Exact pulse modulation and perturbation response in time-dependent two-mode KdV equations

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Abstract

We determine when localized pulses can change amplitude or speed in a time-dependent two-mode Korteweg–de Vries (KdV) family with differentiated nonlinear and dispersive fluxes. A one-sided tail condition forces constant spatial scale. For the quadratic localized profile, necessary and sufficient compatibility conditions require affine amplitude and give a reconstruction classification with exceptional opposite-sign parameter pairs admitting variable speed at constant amplitude. A classification of regular local point transformations shows that equivalence to an autonomous member of the same nondegenerate family requires coefficients already constant on the comparison interval. A zero-background Fourier calculation distinguishes pulse exceptions from repeated linear frequencies. Two exact benchmarks illustrate a pulse whose amplitude grows linearly as it slows down and a constant-amplitude pulse whose speed varies periodically, with both retaining a fixed spatial width. Assuming smooth evolution, an exact initial-response law shows that inward parameter detuning of the periodic-speed pulse initially raises its peak and reduces its half-height width without changing the centre acceleration. Refined one-period simulations distinguish changes in fitted pulse parameters from residual deformation.

Keywords: Two-mode KdV equation; time-dependent coefficients; pulse modulation; compatibility conditions; point equivalence; perturbation response.

MSC 2020: Primary 35Q53; Secondary 35C05, 35A30, 35B20.

1 Introduction

Two-mode KdV equations describe two propagation modes within a single scalar equation [1, 2]. The conventional model has two characteristic velocities and a second-order time derivative, so its initial-value problem specifies both the field and its rate of change. When coefficients vary with time, a basic question is whether a localized pulse can change amplitude or speed while retaining its spatial profile. Exact conditions for such motion identify which coefficient laws are compatible with modulation and provide reference solutions for studying departures from it.

Earlier work includes exact solitary waves [1], a Hamiltonian formulation [3], and multisoliton constructions using Bäcklund and bilinear methods [4, 5]. A solution ansatz with time-dependent coefficients was introduced for a constant-coefficient equation and produced non-traveling solitary-wave expressions [6]. Other studies describe Jacobi elliptic waves [2], binary-soliton solutions [7], and chirped waves with spectral analysis [8]. These results supply the autonomous and exact-solution background for the present problem.

Variable-coefficient two-mode KdV-type equations have also been considered [9, 10, 11]. The distinction here is the placement of the coefficient functions. We use a temporal extension of the

symmetric wave form obtained from the conventional constant-coefficient model by moving with its mean velocity. In [9, 10, 11], the coefficients multiply differentiated field expressions from the outside. Here the nonlinear and dispersive coefficients lie inside the differentiated fluxes, so their time derivatives enter the compatibility conditions. This choice specifies the mathematical family studied below without assuming a physical derivation for arbitrary coefficient laws. Recent work studied exact wave solutions of a potential KdV equation with time-dependent advection [12]. In that first-order model, a time-dependent spatial translation removes the variable coefficient. For the nondegenerate second-order two-mode family considered here, we show that point equivalence to an autonomous member of the same family requires the coefficients to be constant.

For this family, we seek necessary and sufficient conditions linking amplitude, phase motion, and coefficient variation in a fixed localized-profile class. The distinction between testing a prescribed equation and reconstructing its coefficients jointly with a pulse is essential. We also ask whether a point transformation can remove the resulting time dependence. Studies of equivalence transformations for variable-coefficient equations [13, 14] show why this question must be addressed for the whole equation rather than for an individual solution.

Section 2 establishes the scale restriction and compatibility conditions for a profile satisfying a quadratic differential equation. Section 3 solves these conditions, explains the exceptional parameter cases, and constructs two modulated pulses. Section 4 then classifies point transformations and tests autonomous reduction. The Fourier calculation in Section 5 distinguishes the pulse exceptions from repeated background frequencies. Section 6 uses the exact pulses to study departures from compatibility through an initial deformation law and one-period simulations. Section 7 discusses the scope and further development of the analysis.

2 Model and localized-wave compatibility

Let $I \subset \mathbb{R}$ be an open time interval and let $u(x, t)$ be a real-valued field on $\mathbb{R} \times I$. We consider

$$0 = u_{tt} - s(t)^2 u_{xx} + (\partial_t - a(t)s(t)\partial_x)(\alpha(t)uu_x) + (\partial_t - b(t)s(t)\partial_x)(\beta(t)u_{xxx}). \quad (2.1)$$

The coefficients s, α, β, a, b are real and smooth on I with $s > 0$ and $\alpha\beta \neq 0$. Here, s sets the speed of the linear wave part, α and β are the nonlinear and dispersive coefficients, and a, b weight the spatial derivatives of the corresponding fluxes. No restriction is imposed on a, b unless stated otherwise.

The constant-coefficient equation is a moving-frame representation of the conventional two-mode KdV model [1, 2]. We use (2.1) to specify its temporal extension. In particular, the nonlinear and dispersive coefficients lie inside the temporal derivatives so the equation includes $\alpha'uu_x$ and $\beta'u_{xxx}$. The linear wave part contains no derivative of s .

Write $\mathcal{E}[u]$ for the differential expression in (2.1). We consider classical solutions on the whole line with all derivatives appearing in the equation continuous. The pulses decay to zero at spatial infinity and the classification is restricted to the specified profile classes. Additional regularity and decay assumptions are stated with each result.

We first allow an arbitrary fixed profile with a controlled decaying tail:

$$u = A(t)F(\zeta), \quad \zeta = K(t)x - \theta(t), \quad AK \neq 0, \quad (2.2)$$

where $A, K, \theta \in C^2(I)$ and $F \in C^4(\mathbb{R})$ is real.

Proposition 1. *Suppose F is nonzero for all sufficiently large ζ , and*

$$F \rightarrow 0, \quad F^{(j)}/F = O(1) \quad (1 \leq j \leq 4), \quad F''/F \rightarrow \lambda^2 > 0, \quad (2.3)$$

as $\zeta \rightarrow +\infty$. Then, every solution of (2.1) of the form (2.2) has constant K .

Proof. Since $K \neq 0$, put $h = K'/K$, $v = h\theta - \theta'$, and $q = \zeta_t = h\zeta + v$. A dot denotes time differentiation at fixed x . Thus, $\dot{q} = (h' + h^2)\zeta + hv + v'$. Substituting $u = A(t)F(\zeta)$ into $\mathcal{E}[u]$ and applying the product and chain rules gives

$$\begin{aligned}\mathcal{E}[u] = & A''F + (2A'q + A\dot{q})F' + A(q^2 - s^2K^2)F'' \\ & + (\alpha A^2K)'FF' + \alpha A^2K(q - asK)(FF')' \\ & + (\beta AK^3)'F''' + \beta AK^3(q - bsK)F'''.\end{aligned}\tag{2.4}$$

Fix $t \in I$. Then, $q = O(\zeta)$ and $\dot{q} = O(\zeta)$ as $\zeta \rightarrow +\infty$ whereas all coefficients evaluated at this time are fixed. Because $x \mapsto K(t)x - \theta(t)$ maps $\mathbb{R}$ onto $\mathbb{R}$, this profile-tail limit is available for either sign of K . Divide (2.4) by ζ^2F which is nonzero sufficiently far along the tail. The first two terms are respectively $O(\zeta^{-2})$ and $O(\zeta^{-1})$. The contribution from $-As^2K^2F''$ is $O(\zeta^{-2})$.

For the nonlinear terms, $F' = O(F)$ and $\frac{(FF')'}{F} = \left(\frac{F'}{F}\right)^2 F + F'' = O(F)$. Therefore, their divided contributions are $O(F/\zeta^2)$ and $O(F/\zeta)$. The last two dispersive terms give $O(\zeta^{-2})$ and $O(\zeta^{-1})$ by using the bounds for F'''/F and F''''/F . All these quantities tend to zero.

The only surviving term has limit

$$\lim_{\zeta \rightarrow +\infty} A \left(\frac{q}{\zeta}\right)^2 \frac{F''}{F} = Ah^2\lambda^2,$$

and the vanishing residual gives $Ah^2\lambda^2 = 0$. Since $A \neq 0$ and $\lambda^2 > 0$, we have $h = 0$ at every $t \in I$. Thus $K' = 0$ and K is constant on I . $\blacksquare$

The above criterion applies to $F = C \operatorname{sech}^\nu(\ell\zeta)$ for $C \neq 0$ and $\ell, \nu > 0$, with $\lambda^2 = (\nu\ell)^2$, and to smooth modifications away from the specified tail. It gives a necessary scale restriction for any solution in this class. Profiles with algebraic tails or unbounded derivative ratios require another argument as do profiles whose shape changes with time.

To obtain a complete compatibility test, we now choose a profile satisfying a quadratic second-order equation. For fixed $k \neq 0$, $p > 0$, and $r \neq 0$, let

$$u = A(t)F(\zeta), \quad \zeta = kx - \theta(t), \quad \theta' = c, \quad F(\zeta) = -\frac{3p}{2r} \operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\zeta\right),\tag{2.5}$$

where $A \in C^2(I)$ is nonzero and $c \in C^1(I)$. The translation speed in the chosen coordinate is c/k while the spatial width is fixed. The profile satisfies

$$F'' = pF + rF^2, \quad (F')^2 = pF^2 + \frac{2r}{3}F^3,\tag{2.6}$$

$$F''' = (p + 2rF)F', \quad (FF')' = 2pF^2 + \frac{5r}{3}F^3,\tag{2.7}$$

$$F'''' = p^2F + 5prF^2 + \frac{10r^2}{3}F^3.\tag{2.8}$$

These identities will reduce the pulse residual to a polynomial in F and an odd part proportional to F' .

Up to translation, this profile exhausts the nonzero smooth solutions of $F'' = pF + rF^2$ with $F, F' \rightarrow 0$ at both infinities. Multiplication by F' gives the first integral in (2.6) with zero constant by decay. A finite zero of F would also have $F' = 0$ which gives the trivial solution by uniqueness for the locally Lipschitz system $(F, F')' = (F', pF + rF^2)$. Thus a nontrivial profile has fixed sign.

Since $|F|$ decays at both ends, it attains a positive maximum at a finite point. There, $F' = 0$ and the first integral gives $F = -3p/(2r)$. The translated sech-squared profile has the same value and derivative at this point, so uniqueness identifies the two profiles. Periodic and nonzero-background solutions require a different first-integral setting.

The profile in (2.5) satisfies Proposition 1 with $\lambda^2 = p$. Thus, its constant spatial scale is forced by the PDE within the modulated-profile class.

Direct substitution, including $u_{tt} = A''F - (2A'c + Ac')F' + Ac^2F''$, gives

$$\begin{aligned} 0 &= A''F - (2A'c + Ac')F' + A(c^2 - k^2s^2)F'' \\ &\quad + k(\alpha A^2)'FF' - k\alpha A^2(c + kas)(FF')' \\ &\quad + k^3(\beta A)'F''' - k^3\beta A(c + kbs)F'''. \end{aligned} \quad (2.9)$$

Proposition 2. *The function (2.5) solves (2.1) if and only if*

$$A'' = 0, \quad (2.10a)$$

$$2A'c + Ac' = pk^3(\beta A)', \quad (2.10b)$$

$$(\alpha A^2)' + 2rk^2(\beta A)' = 0, \quad (2.10c)$$

$$c^2 - k^2s^2 = pk^3\beta(c + kbs), \quad (2.10d)$$

$$\alpha A(c + kas) + 2rk^2\beta(c + kbs) = 0. \quad (2.10e)$$

Proof. Assume first that (2.5) solves (2.1) and fix $t \in I$. Since F is even, the odd and even parts of (2.9) must vanish separately.

The odd part is

$$[-2A'c - Ac' + pk^3(\beta A)']F' + [k(\alpha A^2)' + 2rk^3(\beta A)']FF'.$$

On the half-line $\zeta > 0$, $F' \neq 0$ and F ranges over a nontrivial interval. Thus, dividing by F' shows that both coefficients vanish. Since $k \neq 0$, this gives (2.10b) and (2.10c).

Set $D = c^2 - k^2s^2$, $U = c + kas$, and $V = c + kbs$. The even part is $E_1F + E_2F^2 + E_3F^3$, where

$$\begin{aligned} E_1 &= A'' + ApD - k^3\beta Ap^2V, \\ E_2 &= ArD - 2k\alpha A^2pU - 5k^3\beta AprV, \\ E_3 &= -\frac{5rkA}{3}(\alpha AU + 2rk^2\beta V). \end{aligned}$$

On the same half-line $\zeta > 0$, F is nonzero. Dividing by F gives

$$E_1 + E_2F + E_3F^2 = 0.$$

Because F ranges over a nontrivial interval, this polynomial identity requires $E_1 = E_2 = E_3 = 0$. Since $rkA \neq 0$, the equality $E_3 = 0$ gives (2.10e). Using this relation reduces E_2 to $Ar(D - pk^3\beta V)$. Thus, $E_2 = 0$ gives (2.10d). Substituting this condition into $E_1 = 0$ then yields $A'' = 0$, which is (2.10a). This proves necessity.

Conversely, assume that (2.10) holds. Equations (2.10b) and (2.10c) make the odd coefficients vanish. The remaining three conditions give $E_1 = E_2 = E_3 = 0$. Hence, the entire residual vanishes for every x, t and then (2.5) solves (2.1). $\blacksquare$

The affine-amplitude condition also has a profile-independent explanation. Write the equation as $u_{tt} + \partial_x \mathcal{P} = 0$, where

$$\mathcal{P} = -s^2u_x + \partial_t(\alpha u^2/2) - as\alpha uu_x + \partial_t(\beta u_{xx}) - bs\beta u_{xxx}.$$

For example, assume $u \in C^2(I; L^1(\mathbb{R}))$ and that $\mathcal{P}$ has zero limits at both spatial infinities at every time. Integrating over $[-R, R]$ and letting $R \rightarrow \infty$ yields

$$M'' = 0, \quad M(t) = \int_{\mathbb{R}} u(x, t) dx. \quad (2.11)$$

Thus, M is affine with slope fixed by the initial integral of u_t . For (2.5), $M = A|k|^{-1} \int_{\mathbb{R}} F(\zeta) d\zeta$ and $\int_{\mathbb{R}} F = -6\sqrt{p}/r \neq 0$, so (2.11) requires $A'' = 0$ for either sign of k . Constant mass would further force constant amplitude but requires the additional condition $M' = 0$.

Corollary 3. *If α, β are nonzero constants, then every pulse (2.5) satisfying (2.1) has constant amplitude and speed, even when a, b, s vary.*

Proof. By (2.10a), $A'' = 0$, so $A(t) = A_0 + A_1 t$ for constants A_0, A_1 . Equation (2.10c) then becomes $A'(\alpha A + rk^2\beta) = 0$. If $A_1 \neq 0$, the second factor forces A constant which is a contradiction. Thus, $A' = 0$ and (2.10b) gives $c' = 0$. ■

Remark 1. *The zero-background assumption matters in comparing this obstruction with [6]. In their notation, equations (17)–(21) give $U = A_0(T) - 12G^2 \tanh^2(GX + H(T))$ with constant $G \neq 0$ and quadratic H . Its spatial background is $A_0(T) - 12G^2$. A zero background forces $A_0 = 12G^2$ and direct substitution into their constant-coefficient PDE then forces $H'' = 0$. Therefore, their time-dependent-background ansatz lies outside the localized class (2.5).*

3 Parameter classification and exact modulated waves

3.1 Reconstruction and exceptional pairs

Proposition 2 tests a pulse in a prescribed equation. We now solve its conditions to reconstruct coefficient laws jointly with a pulse. Set $B = \beta A$ and $J = \alpha A^2 + 2rk^2B$. The distinction between equal and unequal modal parameters leads to the following classification.

Theorem 4. *Fix $k \neq 0$, $p > 0$, and $r \neq 0$. Every solution of (2.1) of the form (2.5) has affine nonzero A and constant J .*

On any connected open subinterval where either $a = b$ or $a \neq b$ holds throughout, every admissible reconstruction belongs to one of the following cases:

$$\begin{cases} \text{(3.3) and (3.4),} & a = b, \quad J = 0, \\ \text{(3.5),} & a = b = \varepsilon, \quad \varepsilon \in \{-1, 1\} \text{ constant,} \quad J \neq 0, \\ \text{(3.6) and (3.7),} & a \neq b, \quad J \neq 0. \end{cases}$$

Conversely, each construction gives an exact pulse whenever the displayed relations admit smooth functions satisfying $s > 0$ and $\alpha\beta \neq 0$ throughout the subinterval. In particular, the first case requires a smooth positive root of (3.4) on the entire subinterval.

Proof. By (2.10a), A is affine. Since $B = \beta A$ and $J = \alpha A^2 + 2rk^2B$, equation (2.10c) gives $J' = 0$. Multiplying (2.10b) by A yields

$$(A^2c)' = pk^3AB'. \quad (3.1)$$

Multiplying (2.10e) by A , substituting $\alpha A^2 = J - 2rk^2B$, and collecting terms yields

$$J(c + kas) + 2rk^3sB(b - a) = 0. \quad (3.2)$$

Because $A \neq 0$, these steps are reversible. Thus, the five compatibility conditions are equivalent to $A'' = 0$, $J' = 0$, (3.1), (3.2), and (2.10d). Also, $sB \neq 0$, so (3.2) shows that $J = 0$ requires $a = b$ throughout the pulse interval.

Case 1: $a = b$ and $J = 0$. Write $a = b = \gamma(t)$. Choose an affine nonzero A , a smooth nonzero B , and a constant C_0 . The identity $J = 0$ determines α while integration of (3.1) determines c :

$$\beta = \frac{B}{A}, \quad \alpha = -\frac{2rk^2B}{A^2}, \quad c = \frac{C_0 + pk^3 \int_{t_0}^t A(\eta)B'(\eta) d\eta}{A^2}. \quad (3.3)$$

Here, t_0 is a fixed time in the subinterval. Equation (3.2) is automatically satisfied. The remaining condition (2.10d) becomes

$$k^2 s^2 + pk^4 \gamma \beta s + pk^3 \beta c - c^2 = 0. \quad (3.4)$$

Therefore, a smooth positive root s completes the reconstruction and satisfies all five compatibility conditions. The root must remain smooth and positive throughout the subinterval.

Case 2: $a = b$ and $J \neq 0$. Again write $a = b = \gamma(t)$. Equation (3.2) now gives $c = -k\gamma s$. Hence, $c + kbs = 0$ and (2.10d) reduces to $k^2 s^2 (\gamma^2 - 1) = 0$. Since $ks \neq 0$, we obtain $\gamma^2 = 1$. Continuity on the connected subinterval then gives $\gamma = \varepsilon$ for a constant $\varepsilon \in \{-1, 1\}$. Substitution into (3.1) gives

$$a = b = \varepsilon, \quad c = -\varepsilon ks, \quad (A^2 s)' = -\varepsilon pk^2 AB', \quad (3.5)$$

with $\beta = B/A$ and $\alpha = (J - 2rk^2 B)/A^2$. Conversely, these relations imply (3.1) and (3.2). Both sides of (2.10d) vanish. Combining this with the affineness of A and the constancy of J yields all five compatibility conditions.

Case 3: $a \neq b$. Equation (3.2) forces $J \neq 0$. Set $z = c/(ks)$. Then (3.2) and (2.10d) become

$$J(z + a) + 2rk^2 B(b - a) = 0, \quad s(z^2 - 1) = \frac{pk^2 B}{A}(z + b).$$

We first check which values of z are excluded. If $z = \pm 1$, the second identity gives $z + b = 0$. Since $\alpha A \neq 0$, equation (2.10e) then gives $z + a = 0$, which contradicts $a \neq b$. Similarly, $z + b = 0$ would force $z^2 = 1$. Finally, setting $z + a = 0$ in the first identity gives $B(b - a) = 0$, which is impossible. Thus, $z \neq \pm 1, -a, -b$. Now, we may solve for B , s , and c :

$$B = -\frac{J(z + a)}{2rk^2(b - a)}, \quad s = \frac{pk^2 B}{A} \frac{z + b}{z^2 - 1}, \quad c = \frac{pk^3}{A} \mathcal{H}, \quad \text{where} \quad \mathcal{H} = B \frac{z(z + b)}{z^2 - 1}. \quad (3.6)$$

Since $A^2 c = pk^3 A \mathcal{H}$, equation (3.1) reduces to

$$(A \mathcal{H})' = AB'. \quad (3.7)$$

All derivatives are total time derivatives including the contributions from $a(t)$ and $b(t)$.

Conversely, take affine nonzero A , constant $J \neq 0$, and smooth functions a, b, z with $a \neq b$ and $z \neq \pm 1, -a, -b$. Define $B, s, c, \mathcal{H}$ by (3.6), and set $\beta = B/A$ and $\alpha = (J - 2rk^2 B)/A^2$. These definitions satisfy (3.2) and (2.10d). If (3.7) holds, then (3.1) also holds. These identities, together with $A'' = 0$ and $J' = 0$, recover all five compatibility conditions. Proposition 2 therefore gives an exact pulse whenever $s > 0$ and $\alpha\beta \neq 0$ throughout the subinterval. ■

Suppose $a(t_0) = b(t_0)$ at some time t_0 and $J \neq 0$. Equations (3.2) and (2.10d) then imply $a(t_0) = b(t_0) = \pm 1$ and $c(t_0) = -ka(t_0)s(t_0)$. Smooth continuation through t_0 requires the original compatibility conditions (2.10) to remain satisfied, so reconstruction formulas from different branches must extend consistently through this time.

For constant $a \neq b$, the differential condition simplifies further. Write $d = -J/[2rk^2(b - a)]$. Then

$$\mathcal{H} - B = d \frac{(z + a)(bz + 1)}{z^2 - 1}, \quad \mathcal{H}_z - d = -d \frac{P(z)}{(z^2 - 1)^2},$$

where $P(z) = (1 + ab)(z^2 + 1) + 2(a + b)z$. Expanding (3.7) yields $A' \mathcal{H} + A(\mathcal{H}_z - d)z' = 0$. Since $d \neq 0$ and $z^2 \neq 1$, then multiplying by $(z^2 - 1)^2/d$ gives

$$AP(z)z' = A'z(z + a)(z + b)(z^2 - 1). \quad (3.8)$$

The reconstructed coefficients are

$$\beta = \frac{d(z + a)}{A}, \quad \alpha = -\frac{2rk^2 d(z + b)}{A^2}, \quad s = \frac{pk^2 d(z + a)(z + b)}{A(z^2 - 1)}, \quad c = ksz. \quad (3.9)$$

On a subinterval where $z(z+a)(z+b)(z^2-1) \neq 0$, we divide (3.8) by $Az(z+a)(z+b)(z^2-1)$ and integrate from t_0 to t . The chain rule gives

$$\int_{z_0}^{z(t)} \frac{P(w)}{w(w+a)(w+b)(w^2-1)} dw = \log \left| \frac{A(t)}{A(t_0)} \right|, \quad (3.10)$$

where $z_0 = z(t_0)$. The undivided equation (3.8) remains valid at singular levels.

Proposition 5. *For constant $a \neq b$ and constant amplitude, every admissible pulse has constant z unless $(a, b) = (1, -1)$ or $(-1, 1)$. For these exceptional pairs, arbitrary smooth z is allowed subject to (3.9) and nondegeneracy. When the affine amplitude is nonconstant, any constant admissible z must be zero. In this nonconstant-amplitude case, the exceptional pairs require $z = 0$ throughout the interval.*

Proof. If A is constant, equation (3.8) reduces to $P(z)z' = 0$.

First suppose that P is not the zero polynomial. If $z'(t_0) \neq 0$ at some time t_0 , continuity of z' gives a small interval around t_0 on which z' remains nonzero and has the same sign. On this interval, $P(z(t)) = 0$ while z keeps increasing or decreasing. Thus, P would have infinitely many distinct roots which is impossible for a nonzero polynomial. Therefore, $z' = 0$ throughout the interval and z is constant.

The polynomial P is identically zero precisely when $1 + ab = 0$ and $a + b = 0$. The second equality gives $b = -a$ and the first then gives $a^2 = 1$. Hence, the only exceptional pairs are $(a, b) = (1, -1)$ and $(-1, 1)$.

For either exceptional pair, $B = d(z + a)$ and $\mathcal{H} = dz$ where $d \neq 0$ is constant. Hence, equation (3.7) becomes

$$(Adz)' = Adz'.$$

Expanding the derivative and cancelling the common term Adz' gives $A'dz = 0$. When A is constant, this identity imposes no further condition on z . Therefore, any smooth z is allowed provided that (3.9) satisfies the stated positivity and nonvanishing conditions.

Now, suppose that the affine amplitude A is nonconstant. Its derivative A' is then a nonzero constant. If z is constant, equation (3.8) gives

$$z(z+a)(z+b)(z^2-1) = 0.$$

The restrictions $z \neq -a, -b, \pm 1$ leave only $z = 0$. For either exceptional pair, the identity $A'dz = 0$ already forces $z(t) = 0$ at every time because $A'd \neq 0$. This last conclusion does not require assuming that z is constant. ■

Table 1 summarizes the reconstruction conditions, including the restrictions on their use for prescribed coefficients.

Table 1: Compatibility within the quadratic localized-profile class. In every row, A is affine and nonzero, J is constant, and the reconstructed coefficients must satisfy $s > 0$ and $\alpha\beta \neq 0$. Off-diagonal rows also require $z \neq \pm 1, -a, -b$.

Parameter case	Necessary and sufficient reconstruction or further restriction
$a = b, J = 0$	Use (3.3) and a smooth positive root of (3.4) for s .
$a = b, J \neq 0$	Necessarily $a = b = \varepsilon \in \{-1, 1\}$; use (3.5) and $\alpha = (J - 2rk^2B)/A^2$.
$a \neq b$, possibly varying	Use (3.6)–(3.7) with $J \neq 0$. These determine coefficients jointly with the pulse.
Constant unequal a, b , outside the two exceptional pairs	Use (3.8)–(3.9). Constant A forces constant z and constant reconstructed coefficients.
$(a, b) = (1, -1)$ or $(-1, 1)$	For constant A , smooth z is free subject to (3.9) and the stated restrictions. For nonconstant A , necessarily $z = 0$.

The constant-speed traveling-wave constructions in [5, 7] provide the autonomous background. Proposition 5 identifies the additional differential compatibility needed for a varying phase and jointly reconstructed coefficients. Wazwaz’s displayed multisoliton restriction is diagonal whereas the variable-speed reconstruction above occurs at the opposite-sign pairs.

3.2 Modal interpretation and characteristic factors

To interpret the exceptional pairs, define the nonlinear modal coefficients by $\alpha_1 = \alpha(1 + a)/2$ and $\alpha_2 = \alpha(1 - a)/2$, and the dispersive ones by $\beta_1 = \beta(1 + b)/2$ and $\beta_2 = \beta(1 - b)/2$. The two nonlinear coefficients have the same strict sign when $|a| < 1$ and the analogous dispersive condition is $|b| < 1$. Allowing zero contributions gives the closed square $[-1, 1]^2$. These sign restrictions are optional as (2.1) allows all real a, b .

At $(a, b) = (1, -1)$, $\alpha_2 = \beta_1 = 0$ and at $(-1, 1)$, $\alpha_1 = \beta_2 = 0$. Therefore, the exceptional pairs represent complementary modal contributions to one field. The identity

$$P(z) = \frac{(1+a)(1+b)}{2}(z+1)^2 + \frac{(1-a)(1-b)}{2}(z-1)^2 \quad (3.11)$$

shows that $P(z) > 0$ for every real z in the strict interior of the square. On its boundary, the nonnegative weights give $P(z) > 0$ at every admissible off-diagonal level $z \neq \pm 1$ except at the two opposite-sign corners, where both weights vanish.

More explicitly, for $(a, b) = (\varepsilon, -\varepsilon)$, $\varepsilon = \pm 1$, and constant A , (3.9) becomes

$$s = \frac{pk^2d}{A}, \quad \beta = \frac{d(z + \varepsilon)}{A}, \quad \alpha = -\frac{2rk^2d(z - \varepsilon)}{A^2}, \quad c = ksz. \quad (3.12)$$

Here, $d/A > 0$ and $z \neq \pm 1$ with any additional sign restrictions on α, β retained. The characteristic speed s is constant although the pulse speed $c/k = sz$ may vary. This freedom belongs to the reconstruction: in a prescribed PDE, β fixes $z = A\beta/d - \varepsilon$. Nonconstant A instead forces $z = 0$ by Proposition 5.

The distinction from a common characteristic factor can be made at equation level. Define $D_\varepsilon = \partial_t + \varepsilon s \partial_x$, $N = \alpha u u_x$, and $Q = \beta u_{xx}$. For $\varepsilon \in \{-1, 1\}$, direct expansion gives, for any a, b ,

$$\mathcal{E}[u] = D_{-\varepsilon}(D_\varepsilon u + N + Q) + (\varepsilon - a)sN_x + (\varepsilon - b)sQ_x - \varepsilon s' u_x. \quad (3.13)$$

Since $Q_x = \beta u_{xxx}$, its coefficient first forces $b = \varepsilon$ if the remainder is to vanish for arbitrary derivatives. Next, $N_x = \alpha(u_x^2 + uu_{xx})$ and $\alpha \neq 0$ force $a = \varepsilon$. The remaining coefficient of u_x

then forces $s' = 0$. Conversely, these three conditions annul the remainder. This characterizes the displayed common-factor form. At the opposite-sign corners the exact identity is instead

$$\mathcal{E}[u] = D_{-\varepsilon}(D_{\varepsilon}u + N) + D_{\varepsilon}Q - \varepsilon s' u_x, \quad (a, b) = (\varepsilon, -\varepsilon). \quad (3.14)$$

Thus, the nonlinear and dispersive fluxes act through opposite characteristic derivatives. In (3.12), $s' = 0$ removes the correction but does not give the common factor in (3.13). Together with (3.11), this distinguishes the opposite-sign compatibility exception from the same-sign common-factor cases.

3.3 Two exact modulated pulses

The following examples illustrate changes in pulse speed and amplitude while preserving spatial width. Their explicit formulas also provide reference solutions for numerical computations. In both examples, we take $k = 1$, $p = 4$, and $r = -6$, so the profile is $F(\zeta) = \text{sech}^2 \zeta$.

Speed modulation with constant amplitude: Choose the exceptional pair $(a, b) = (1, -1)$ with $A = 1$ and $d = 1/4$. For any smooth speed function $c(t) > 1$ on an interval containing zero, the reconstruction formulas give

$$s = 1, \quad \alpha = 3(c - 1), \quad \beta = (c + 1)/4, \quad u = \text{sech}^2 \left(x - \int_0^t c(\eta) d\eta \right). \quad (3.15)$$

The pulse has constant peak height and width. Its centre is $X(t) = \int_0^t c(\eta) d\eta$, so its speed is $X'(t) = c(t)$. Choosing this speed also determines the coefficient functions α and β .

To verify the solution directly, substitution into (2.1) gives

$$c' \left[-F' + 3FF' + \frac{1}{4}F'''' \right] + (c^2 - 1) \left[F'' - 3(FF')' - \frac{1}{4}F'''' \right] = 0. \quad (3.16)$$

The first bracket vanishes because $F''' = 4F' - 12FF'$. Differentiating this identity shows that the second bracket also vanishes.

For the periodic choice $c(t) = 2 + \frac{1}{2} \sin t$, the exact pulse is

$$u = \text{sech}^2 \left(x - 2t - \frac{1}{2}(1 - \cos t) \right), \quad (3.17)$$

with positive periodic coefficients $\alpha = 3 + \frac{3}{2} \sin t$ and $\beta = \frac{3}{4} + \frac{1}{8} \sin t$. Its speed varies between 1.5 and 2.5 while its peak height and width remain unchanged.

Amplitude modulation with changing speed: Now use the diagonal branch $a = b = 0$ with $J = 0$. Choose $A = 1 + t$, $B = 1$, and $C_0 = 8$ in (3.3). The resulting coefficient and speed laws are

$$A = 1 + t, \quad \alpha = 12/A^2, \quad \beta = 1/A, \quad c = 8/A^2, \quad s^2 = 32(1 - t)/(1 + t)^4. \quad (3.18)$$

For $0 \leq t < 1$, take the positive square root for s . Integrating $\theta' = c$ with $\theta(0) = 0$ gives $\theta(t) = 8t/(1 + t)$. Thus, the exact pulse is

$$u = (1 + t) \text{sech}^2 \left(x - \frac{8t}{1 + t} \right).$$

Its peak height increases as $1 + t$ while its centre moves with decreasing speed $8/(1 + t)^2$. The spatial width remains constant.

Here $\alpha A^2 = 12$, $\beta A = 1$, $2A'c + Ac' = 0$, and $A(c^2 - s^2) = 4c$. Consequently, the residual reduces to $c[4F'' - 12(FF')' - F'''] = 0$ which follows from the differentiated profile identity above. The restriction $t < 1$ ensures $s > 0$ as required by the model.

The amplitude and speed modulation in these examples are illustrated in Figures 2 and 3, respectively. Initial fields for both examples are given in Section 6, where the periodic-speed pulse serves as the reference solution for the response calculations.

4 Point transformations and autonomous reducibility

The examples obtain modulation by varying the coefficients. We now ask whether this time dependence can be removed by a point transformation of the whole equation. We compare nondegenerate members of (2.1) with target coefficients $\tilde{\alpha}(\tau)$, $\tilde{\beta}(\tau)$, $\tilde{s}(\tau)$, $\tilde{a}(\tau)$ and $\tilde{b}(\tau)$. A regular local point transformation has nonzero Jacobian in (t, x, u) and is prolonged on an open graph chart where the transformed independent variables are valid coordinates. Equivalence means that the transformation identifies the zero sets of the equations as relations among the field and its derivatives. The next three lemmas restrict the independent coordinates, the field transformation, and the time dependence before the full coefficient action is stated.

Lemma 6. *Every regular local point transformation between nondegenerate members of (2.1) has independent coordinates $\tau = T(t)$ and $\xi = K(t)x + \chi(t)$ with $T'K \neq 0$.*

Proof. Consider a general smooth point transformation

$$\tau = T(t, x, u), \quad \xi = X(t, x, u), \quad v = U(t, x, u). \quad (4.1)$$

We first determine T and then X .

The fourth-order terms of the source equation are $\beta u_{xxxt} - bs\beta u_{xxxx}$. Their principal polynomial is

$$\beta \lambda_x^3 (\lambda_t - bs\lambda_x), \quad (4.2)$$

where λ_t, λ_x are formal variables corresponding to the time and spatial derivatives. It has a unique factor of multiplicity three and a distinct simple factor, including when $b = 0$.

On a regular graph chart, the chain rule transforms the highest-order derivatives by an invertible linear change of derivative directions and a nonzero common factor. Therefore, the multiplicities of the two linear factors are preserved. For the total spatial derivative, the chain rule gives

$$D_x = (D_x T) \tilde{D}_\tau + (D_x X) \tilde{D}_\xi.$$

Since the triple factor of the target equation is purely spatial, $D_x T = 0$ or $T_x + T_u u_x = 0$. The source equation can be solved for u_{tt} and places no restriction on the value of u_x at a point. Thus, the last identity must hold as u_x varies. This gives $T_x = T_u = 0$, so $T = T(t)$. Write $q = T' \neq 0$.

We now determine the dependence of X on x and u . Set $L = D_x X = X_x + X_u u_x$. The Jacobian of the independent coordinates is qL , so $L \neq 0$. Differentiating $v = U(t, x, u(t, x))$ with respect to x gives

$$v_\xi = \frac{U_x + U_u u_x}{L}.$$

Consequently,

$$U_u - v_\xi X_u = \frac{X_x U_u - X_u U_x}{L} \neq 0,$$

because the point Jacobian is $q(X_x U_u - X_u U_x)$. We may therefore define $\Lambda = (U_u - v_\xi X_u)^{-1}$.

Differentiating the field transformation with respect to t and x gives

$$u_t = \Lambda(qv_\tau + v_\xi X_t - U_t), \quad u_x = \Lambda(v_\xi X_x - U_x).$$

Only differentiation of the qv_τ term can produce $v_{\tau\tau}$ in u_{tt} . Therefore, its coefficient is Λq^2 .

For the spatial derivatives, the second identity gives $\partial u_x / \partial v_\xi = \Lambda L$ when the base variables (t, x, u) are fixed. Each further spatial differentiation contributes a factor L to the highest derivative. Thus, the coefficient of $v_{\xi\xi\xi}$ in u_{xxx} is ΛL^3 and the coefficient of $v_{\xi\xi\xi\tau}$ in βu_{xxxt} is $\beta \Lambda L^3 q$.

No other source term produces $v_{\xi\xi\xi\tau}$. Also, no term other than u_{tt} produces $v_{\tau\tau}$, since spatial differentiation does not involve τ and u_{xxxt} contains only one time differentiation. Both

transformed and target equations can therefore be solved for $v_{\tau\tau}$. Equivalence requires the resulting expressions to agree. Dividing by Λq^2 and comparing the mixed fourth-order coefficient gives

$$\tilde{\beta}(T(t)) = \frac{\beta(t)L^3}{q(t)}. \quad (4.3)$$

The left-hand side is independent of u_x . Differentiating the right-hand side with respect to u_x gives

$$0 = \frac{3\beta L^2}{q} X_u.$$

Since $\beta L q \neq 0$, we obtain $X_u = 0$. Now, $L = X_x$ and (4.3) gives $X_x^3 = q\tilde{\beta}(T)/\beta$ which depends only on t . Taking the real cube root shows that $X_x = K(t) \neq 0$ and then integration with respect to x yields $X = K(t)x + \chi(t)$. ■

Lemma 7. *Every transformation in Lemma 6 has the more restricted form*

$$u = H(t)v(\xi, \tau) + G(t), \quad \xi = \ell x + \chi(t), \quad \tau = T(t), \quad (4.4)$$

where ℓ is a nonzero constant, $HT' \neq 0$, and $2\chi' + \ell\alpha G = 0$.

Proof. By Lemma 6, $\tau = T(t)$ and $\xi = K(t)x + \chi(t)$. Write $q = T'$. Since the point transformation is invertible and $qK \neq 0$, the field transformation can be inverted locally as $u = V(t, x, v)$ with $V_v \neq 0$.

The term in u_{xxx} containing the highest spatial derivative of v is $K^3 V_v v_{\xi\xi\xi}$. Its time derivative contributes $\beta q K^3 V_{vv} v_\tau v_{\xi\xi\xi}$ to $\beta u_{xxx\tau}$. Neither u_{tt} nor the nonlinear flux produces this product and the purely spatial terms contain no v_τ . The product is also absent from the target equation. Hence, $\beta q K^3 V_{vv} = 0$. Since $\beta q K \neq 0$, $V_{vv} = 0$, and thus $u = H(t, x)v + G(t, x)$ with $H \neq 0$.

For this affine field transformation,

$$u_{xxx} = HK^3 v_{\xi\xi\xi} + 3H_x K^2 v_{\xi\xi} + 3H_{xx} K v_\xi + H_{xxx} v + G_{xxx}.$$

Therefore, the coefficient of $v_{\xi\xi\tau}$ in the source equation is $3\beta q K^2 H_x$. No other source term produces this derivative and it is absent from the target. Thus, $H_x = 0$ and $H = H(t)$.

The time derivative of the spatial coordinate is $K'x + \chi'$. Thus, the coefficients of $v_{\xi\xi\xi\xi}$ and $v_{\xi\xi\xi\tau}$ are $\beta H K^3 (K'x + \chi' - bsK)$ and $\beta H K^3 q$, respectively. Their ratio must equal the corresponding target ratio:

$$\frac{K'x + \chi' - bsK}{q} = -\tilde{b}\tilde{s}.$$

The right-hand side depends only on time. Differentiating this identity with respect to x gives $K'/q = 0$. Hence, $K' = 0$ and write $K = \ell \neq 0$.

It remains to determine G . The linear $v_{\xi\tau}$ term receives a contribution $2Hq\chi'$ from u_{tt} and a contribution $\alpha H G \ell q$ from the time derivative of the nonlinear flux. Thus, its total coefficient is $Hq(2\chi' + \alpha\ell G)$. The target contains the nonlinear term $vv_{\xi\tau}$ but no linear $v_{\xi\tau}$ term. Thus, $2\chi' + \alpha\ell G = 0$ and since α and χ' depend only on time and $\alpha\ell \neq 0$, we have $G = -2\chi'/(\alpha\ell)$. Therefore, $G = G(t)$. This completes the reduction (4.4). ■

Lemma 8. *For an equivalence of the form (4.4),*

$$T'' = 0, \quad H' = 0, \quad G'' = 0, \quad (\alpha G)' = 0.$$

Proof. Use the form (4.4) and write $q = T'$. The coefficient of $v_{\tau\tau}$ is Hq^2 . After dividing the transformed equation by this coefficient, (4.3) gives

$$\tilde{\beta}(T(t)) = \frac{\ell^3 \beta(t)}{q(t)}.$$

We must also match the coefficient of $v_{\xi\xi\xi}$. Since $u_{xxx} = H\ell^3 v_{\xi\xi\xi}$, the source coefficient after normalization is $\ell^3(\beta H)'/(Hq^2)$. In the target equation, this coefficient is $\tilde{\beta}_\tau$ because $\tilde{\beta}$ lies inside the time derivative. Using $d/d\tau = q^{-1}d/dt$, we obtain

$$\frac{\ell^3(\beta H)'}{Hq^2} = \frac{1}{q} \left(\frac{\ell^3\beta}{q} \right)'.$$

Expanding both sides and multiplying by q^2/ℓ^3 gives

$$\beta' + \beta \frac{H'}{H} = \beta' - \beta \frac{q'}{q}.$$

Since $\beta \neq 0$, this reduces to $H'/H = -q'/q$.

A second relation comes from the linear v_τ term. Indeed,

$$u_t = H'v + Hqv_\tau + H\chi'v_\xi + G',$$

so its coefficient in u_{tt} is $2H'q + Hq'$. No other source term contributes a linear v_τ term and none is present in the target. Therefore, $2H'/H + q'/q = 0$. Combining this with $H'/H = -q'/q$ gives $H' = q' = 0$. Since $q = T'$, we also have $T'' = 0$.

With H and q constant, the coefficient of the linear v_ξ term is $H[\chi'' + \ell(\alpha G)']$. The field-independent term is G'' . Both are absent from the target equation, so

$$\chi'' + \ell(\alpha G)' = 0, \quad G'' = 0.$$

Lemma 7 already gives $2\chi' + \ell\alpha G = 0$. Differentiating this relation yields

$$2\chi'' + \ell(\alpha G)' = 0.$$

Subtracting twice $\chi'' + \ell(\alpha G)' = 0$ gives $-\ell(\alpha G)' = 0$. Since $\ell \neq 0$, we conclude $(\alpha G)' = 0$. ■

Theorem 9. *Every regular local point transformation between nondegenerate members of (2.1) has the form*

$$\tau = q_0 t + \tau_0, \quad \xi = \ell x - \frac{\ell\delta}{2}t + \xi_0, \quad u = h_0 v + g_1 t + g_0, \quad (4.5)$$

where all parameters are real constants, $q_0 h_0 \neq 0$, and

$$\alpha(t)(g_1 t + g_0) = \delta. \quad (4.6)$$

The coefficient relations, evaluated at $\tau = q_0 t + \tau_0$, are

$$\tilde{\alpha} = \frac{\ell h_0}{q_0} \alpha, \quad \tilde{\beta} = \frac{\ell^3}{q_0} \beta, \quad (4.7)$$

$$\tilde{s}^2 = \frac{\ell^2}{q_0^2} \left(s^2 + as\delta + \frac{\delta^2}{4} \right), \quad (4.8)$$

$$\tilde{a}\tilde{s} = \frac{\ell}{q_0} (as + \delta/2), \quad \tilde{b}\tilde{s} = \frac{\ell}{q_0} (bs + \delta/2). \quad (4.9)$$

Conversely, these formulas define an equivalence wherever $\tilde{s}^2 > 0$, with $\tilde{s}$ its positive square root. Any additional restrictions on the coefficient domain must also hold for the target.

Proof. Lemma 8 gives $T'' = 0$, $H' = 0$, $G'' = 0$, and $(\alpha G)' = 0$. Integration on the connected comparison interval gives

$$T = q_0 t + \tau_0, \quad H = h_0, \quad G = g_1 t + g_0, \quad \alpha G = \delta,$$

where all displayed parameters are real constants. The nonvanishing conditions in the preceding lemmas give $q_0 \ell h_0 \neq 0$. Using $2\chi' + \ell \alpha G = 0$, we obtain $\chi' = -\ell \delta / 2$ and hence $\chi = -\ell \delta t / 2 + \xi_0$. These identities prove (4.5) and (4.6).

We now determine the target coefficients. When acting on v , the derivative operators are $\partial_x = \ell \partial_\xi$ and $\partial_t = q_0 \partial_\tau + \chi' \partial_\xi$. After division by $h_0 q_0^2$, the coefficients of $\partial_\tau(vv_\xi)$ and $v_{\xi\xi\xi\tau}$ are $\frac{\ell h_0 \alpha}{q_0}$ and $\frac{\ell^3 \beta}{q_0}$, respectively. Matching these with the target gives (4.7). The accompanying coefficients of vv_ξ and $v_{\xi\xi\xi}$ also agree, because

$$\tilde{\alpha}_\tau = \frac{\ell h_0 \alpha'}{q_0^2}, \quad \tilde{\beta}_\tau = \frac{\ell^3 \beta'}{q_0^2}.$$

The coefficients of $\partial_\xi(vv_\xi)$ and $v_{\xi\xi\xi\xi}$ give

$$-\tilde{a}\tilde{s}\tilde{\alpha} = \frac{\ell h_0 \alpha}{q_0^2}(\chi' - a s \ell), \quad -\tilde{b}\tilde{s}\tilde{\beta} = \frac{\ell^3 \beta}{q_0^2}(\chi' - b s \ell).$$

Substituting (4.7) and dividing by the nonzero coefficients yields

$$\tilde{a}\tilde{s} = \frac{a s \ell - \chi'}{q_0}, \quad \tilde{b}\tilde{s} = \frac{b s \ell - \chi'}{q_0}.$$

Since $\chi' = -\ell \delta / 2$, these are exactly (4.9).

For the linear $v_{\xi\xi}$ term, the contribution from $u_{tt} - s^2 u_{xx}$ is $h_0[(\chi')^2 - \ell^2 s^2]$. The background G in the nonlinear flux contributes $h_0 \ell \alpha G(\chi' - a s \ell)$. After normalization, their sum is

$$\frac{(\chi')^2 + \ell \alpha G \chi' - \ell^2 s^2 - a s \alpha G \ell^2}{q_0^2} = -\frac{\ell^2}{q_0^2} \left(s^2 + a s \delta + \frac{\delta^2}{4} \right).$$

Matching this with $-\tilde{s}^2$ proves (4.8). This completes the necessity argument.

Conversely, take constants and coefficient functions satisfying the stated formulas. Then H and q are constant, $G'' = 0$, $(\alpha G)' = 0$, and $2\chi' + \ell \alpha G = 0$. These identities cancel the additional terms found in the preceding lemmas. Therefore, the coefficient comparisons above give the exact residual identity

$$\mathcal{E}[u](x, t) = h_0 q_0^2 \tilde{\mathcal{E}}[v] \left(\ell x - \frac{\ell \delta}{2} t + \xi_0, q_0 t + \tau_0 \right),$$

where $\tilde{\mathcal{E}}$ denotes the target differential expression.

The point Jacobian is $q_0 \ell / h_0 \neq 0$, the Jacobian of the independent coordinates is $q_0 \ell \neq 0$, and the residual multiplier is $h_0 q_0^2 \neq 0$. Thus, the transformation is regular and identifies the two equations. The condition $\tilde{s}^2 > 0$ supplies a positive target speed while (4.7) preserves the nonvanishing of the nonlinear and dispersive coefficients. Any additional coefficient-domain restrictions are imposed as stated in the theorem. $\blacksquare$

The background restriction depends on α : generically $G = 0$ while constant α permits constant nonzero G . A nonconstant affine G requires α to be a nonzero constant divided by G on an interval where $G \neq 0$. Thus, restricting backgrounds to constants would omit some admissible transformations.

Corollary 10. *On a connected comparison interval, a nondegenerate member of (2.1) is point-equivalent on the regular charts above to an autonomous member of the same family if and only if α, β, s, a, b are constant on that interval.*

Proof. Suppose a regular point transformation maps (2.1) to an autonomous member of the same family. Then $\tilde{\alpha}, \tilde{\beta}, \tilde{s}, \tilde{a}, \tilde{b}$ are constant. Their values remain constant after the time change $\tau = q_0 t + \tau_0$.

By Theorem 9, the transformation parameters are constants and $q_0 \ell h_0 \neq 0$. Solving (4.7) for the original coefficients gives

$$\alpha = \frac{q_0}{\ell h_0} \tilde{\alpha}, \quad \beta = \frac{q_0}{\ell^3} \tilde{\beta}.$$

Both right-hand sides are constant, so α and β are constant.

Similarly, (4.9) gives

$$as = \frac{q_0}{\ell} \tilde{a} \tilde{s} - \frac{\delta}{2}, \quad bs = \frac{q_0}{\ell} \tilde{b} \tilde{s} - \frac{\delta}{2}.$$

Thus, the products as and bs are constant. To determine s separately, rearrange (4.8) as

$$s^2 = \frac{q_0^2}{\ell^2} \tilde{s}^2 - \delta(as) - \frac{\delta^2}{4}.$$

Every term on the right-hand side is constant. Hence, s^2 is constant. Since $s > 0$, s is its fixed positive square root. Dividing the constant products as and bs by this positive constant shows that a and b are constant. This proves necessity.

Conversely, if α, β, s, a, b are constant, the original equation is already autonomous. Therefore, the identity transformation $\tau = t$, $\xi = x$, and $v = u$ gives the required equivalence. ■

When $c(t)$ is nonconstant, the coefficients $\alpha = 3(c - 1)$ and $\beta = (c + 1)/4$ in (3.15) vary. Therefore, Corollary 10 excludes autonomous reduction within this family on that interval. The affine-amplitude example has the same property. Both conclusions concern the whole PDE.

5 Linear propagation and the parameter boundary

The reconstruction and point-equivalence results concern exact pulses and whole equations. A separate linear calculation is needed to determine whether the exceptional pulse parameters also produce repeated background frequencies. We therefore consider the constant-coefficient linearization of (2.1) about $u = 0$:

$$w_{tt} - s^2 w_{xx} + \beta w_{xxx} - bs\beta w_{xxxx} = 0. \quad (5.1)$$

For a mode $w = e^{i(\kappa x - \omega t)}$, the real spatial wavenumber $\kappa \neq 0$ has wavelength $2\pi/|\kappa|$, and ω is the temporal frequency. Substitution gives

$$\omega^2 + \beta\kappa^3\omega - s^2\kappa^2 + bs\beta\kappa^4 = 0, \quad \omega_{\pm} = \frac{-\beta\kappa^3 \pm \sqrt{\Delta}}{2}, \quad (5.2)$$

where

$$\Delta = \kappa^2[(\beta\kappa^2 - 2bs)^2 + 4s^2(1 - b^2)]. \quad (5.3)$$

When ω is real, the mode oscillates in time without exponential growth or decay. If $\omega = \omega_R + i\omega_I$, then

$$e^{i(\kappa x - \omega t)} = e^{\omega_I t} e^{i(\kappa x - \omega_R t)}.$$

Thus, a positive imaginary part gives exponential growth whereas a negative imaginary part gives exponential decay.

Proposition 11. Let $s > 0$, $\beta \neq 0$, and b be the fixed real coefficients in (5.2). Define $\sigma = b \operatorname{sgn} \beta$, where $\operatorname{sgn} \beta = \beta/|\beta|$. For real $\kappa \neq 0$, the two frequencies in (5.2) satisfy

$$\begin{cases} \text{They are real and distinct for every } \kappa \neq 0, & \sigma < 1, \\ \text{They are real for every } \kappa \neq 0, \text{ coinciding only at } \kappa^2 = 2s/|\beta|, & \sigma = 1, \\ \text{They are nonreal precisely in the open band (5.4),} & \sigma > 1. \end{cases}$$

For $\sigma > 1$, this band of nonreal frequencies is given by

$$\frac{2s}{|\beta|}(\sigma - \sqrt{\sigma^2 - 1}) < \kappa^2 < \frac{2s}{|\beta|}(\sigma + \sqrt{\sigma^2 - 1}). \quad (5.4)$$

At either boundary of this band, the frequencies coincide. Outside the closed band, they are real and distinct.

Consequently, the frequencies are real for every nonzero real wavenumber if and only if $\sigma \leq 1$.

Proof. The dispersion relation is a quadratic equation in ω with real coefficients. Set

$$y = \frac{|\beta|\kappa^2}{2s}.$$

Since $s > 0$, $\beta \neq 0$, and $\kappa \neq 0$, we have $y > 0$. The discriminant can be written as

$$\Delta = 4s^2\kappa^2(y^2 - 2\sigma y + 1).$$

The factor $4s^2\kappa^2$ is positive. Therefore, the sign of Δ is determined by the quadratic $y^2 - 2\sigma y + 1$.

Suppose first that $\sigma < 1$. Then,

$$y^2 - 2\sigma y + 1 = (y - 1)^2 + 2(1 - \sigma)y.$$

The first term is nonnegative and the second is strictly positive because $y > 0$ and $\sigma < 1$. Hence, $\Delta > 0$ for every $\kappa \neq 0$, so the two frequencies are real and distinct.

If $\sigma = 1$, the quadratic becomes $(y - 1)^2$. It is nonnegative for every $y > 0$ and vanishes only at $y = 1$. Thus, both frequencies are real and they coincide exactly when $\kappa^2 = 2s/|\beta|$.

Finally, suppose that $\sigma > 1$. The quadratic has two distinct roots,

$$y_- = \sigma - \sqrt{\sigma^2 - 1}, \quad y_+ = \sigma + \sqrt{\sigma^2 - 1}.$$

Both roots are positive because $0 < \sqrt{\sigma^2 - 1} < \sigma$. The factorization

$$y^2 - 2\sigma y + 1 = (y - y_-)(y - y_+)$$

shows that the quadratic is negative precisely when $y_- < y < y_+$, which gives (5.4). At the endpoints the discriminant vanishes and outside the closed band it is positive.

The band is nonempty whenever $\sigma > 1$, so in this case, some nonzero real wavenumbers give nonreal frequencies. Together with the first two cases, this proves the stated if-and-only-if condition. $\blacksquare$

Inside (5.4), a Fourier mode grows at rate $\sqrt{-\Delta}/2$. At a repeated frequency ω_* , the temporal solution is $(C_1 + C_2 t)e^{-i\omega_* t}$. Thus, real frequencies need not give bounded modes. At zero wavenumber, $\hat{w}_{tt} = 0$ likewise permits linear growth. These modal statements give neither a uniform function-space estimate nor nonlinear pulse stability.

The strip $|b| < 1$ ensures distinct real frequencies for every nonzero wavenumber with the exact condition $b \operatorname{sgn} \beta < 1$ for a fixed sign of β . Neither a nor α enters (5.1). For $\beta > 0$, the pulse exception $(1, -1)$ has distinct frequencies whereas $(-1, 1)$ has a repeated frequency at

$\kappa^2 = 2s/\beta$. Their roles reverse for $\beta < 0$. Both pairs have $P \equiv 0$, so pulse compatibility and frequency coincidence are different properties.

Figure 1(a) locates these cases in the optional square $[-1, 1]^2$. Orange circles mark opposite-sign pulse exceptions and blue squares mark same-sign common-factor cases for constant s . For $\beta > 0$, the upper edge $b = 1$ permits a repeated nonzero frequency while the lower edge does not. Panel (b) compares the sorted frequency branches for $s = \beta = 1$ and $\kappa \geq 0$. The $b = 1$ branches meet at $\kappa = \sqrt{2}$. Both choices also have a meeting at $\kappa = 0$.

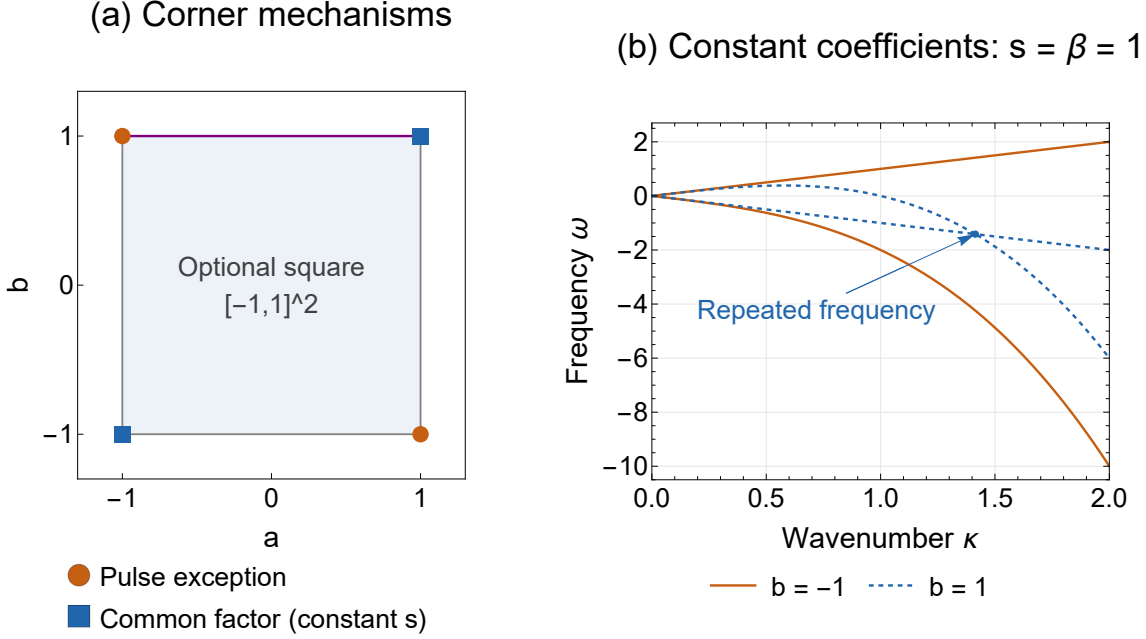


Figure 1: Parameter cases and zero-background dispersion. (a) Pulse exceptions and common-factor cases. (b) Frequency branches for $s = \beta = 1$.

For time-dependent coefficients, use the Fourier convention $\partial_x \mapsto i\kappa$. The zero-background linearization gives

$$\hat{w}_{tt} - i\beta(t)\kappa^3\hat{w}_t + [s(t)^2\kappa^2 - b(t)s(t)\beta(t)\kappa^4 - i\beta'(t)\kappa^3]\hat{w} = 0. \quad (5.5)$$

The β' term comes from differentiating the dispersive coefficient. Proposition 11 applies to constant coefficients and therefore does not determine this nonautonomous evolution. Linearization about a pulse u_* also introduces the spatially varying term $(\partial_t - as\partial_x)[\alpha(u_*w_x + u_{*x}w)]$. The response of such a pulse is examined next through an exact initial calculation and full-equation simulations.

6 Exact benchmarks and perturbation response

6.1 Exact benchmarks and illustrations

The two exact pulses from Section 3 illustrate amplitude and speed modulation. We use the periodic-speed pulse as the reference solution for the response calculations. With $F(x) = \text{sech}^2 x$,

the initial fields for both examples are

	$u(x, 0)$	$u_t(x, 0)$
affine amplitude	$F(x)$	$F(x) - 8F'(x)$
periodic speed	$F(x)$	$-2F'(x)$

(6.1)

Both pulses have full width at half maximum $2 \operatorname{arcosh} \sqrt{2}$. All figures use model variables without physical units.

For (3.18), the peak $1+t$ increases while the centre $X(t) = 8t/(1+t)$ moves with decreasing positive speed $X'(t) = 8/(1+t)^2$. The spatial width remains constant, in agreement with Proposition 1. Figure 2 illustrates this behaviour on $0 \leq t \leq 0.9$ where $s > 0$. Panel (a) compares the profiles at $t = 0, 0.45$, and 0.9 . Panel (b) shows the space-time field with the exact centre $x = X(t)$ marked by a white dashed curve. Panel (c) compares the normalized amplitude and speed, showing their opposite trends over this interval.

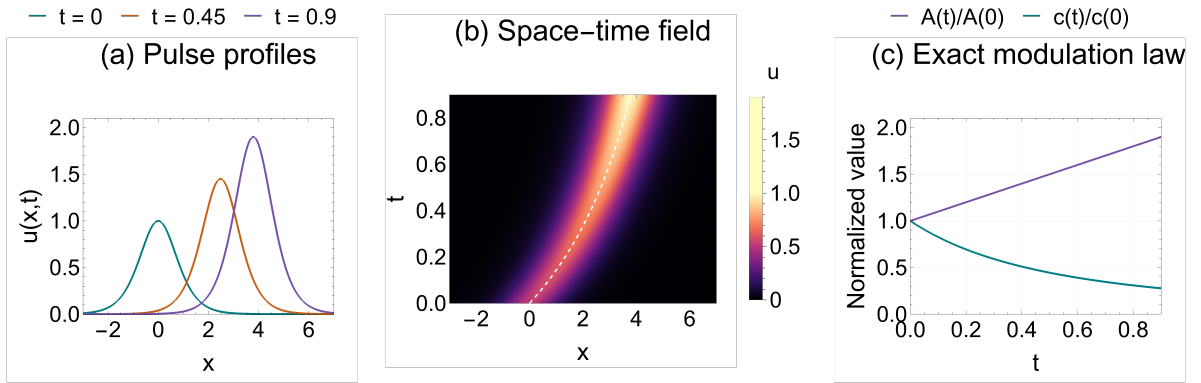


Figure 2: Affine-amplitude pulse (3.18): profiles, space-time field, and normalized amplitude and speed.

For (3.17), the centre is $X(t) = 2t + \frac{1}{2}(1 - \cos t)$ and the speed ranges from 1.5 to 2.5 while the peak and width remain constant. Figure 3 shows this pulse over two modulation periods. Panel (a) displays the field in the original coordinate. Panel (b) uses the moving coordinate $y = x - 2t$ in which the centre oscillates between 0 and 1. These panels show the same single pulse with the common colour range $0 \leq u \leq 1$. White dashed curves mark the exact centre in each coordinate. Panel (c) shows the periodic speed and its mean value 2, indicated by the horizontal dashed line. The moving coordinate makes the oscillation about the mean motion visible. It does not make the equation autonomous, consistent with Corollary 10.

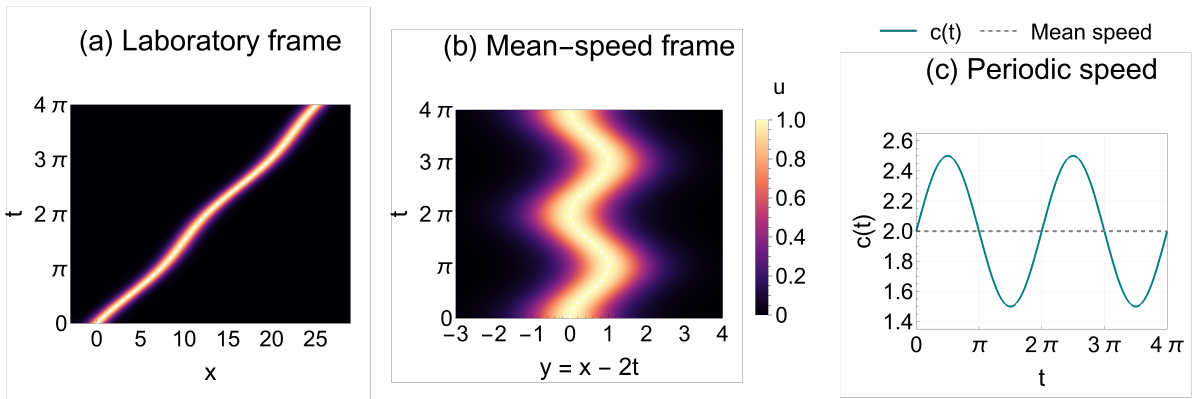


Figure 3: Periodic-speed pulse (3.17): the field in the original and moving coordinates, and its speed.

We next examine departures from the periodic-speed pulse (3.17). Two experiments perturb either the initial pair or the modal parameters while keeping $c = 2 + \frac{1}{2} \sin t$, $\alpha = 3(c - 1)$, $\beta = (c + 1)/4$, and $s = 1$ as the reference coefficient laws. An exact initial-response calculation is followed by simulations on a large periodic domain. Their numerical accuracy is assessed through refinement and domain-size checks.

6.2 An exact initial deformation law

Keep the reference coefficient laws for α , β , and s from the periodic-speed example and set $a = 1 - \varepsilon$ and $b = -1 + \varepsilon$, where $0 < \varepsilon < 1$. This places (a, b) inside the square $[-1, 1]^2$. Let $\mathcal{E}^\varepsilon$ denote the differential expression in (2.1) with these modified parameters. For the reference pulse $u_* = F(x - X(t))$, substitution gives

$$\begin{aligned}\mathcal{E}^\varepsilon[u_*] &= \varepsilon R(F, t), \\ R(F, t) &= -4(c + 1)F + (54c + 6)F^2 - 60cF^3.\end{aligned}\tag{6.2}$$

Thus, the reference pulse is no longer an exact solution after the parameter change. Its residual allows the initial response to be computed directly from the equation.

Theorem 12. *Assume $0 \in I$ and fix $0 < \varepsilon < 1$. Suppose a sufficiently smooth solution u^ε of $\mathcal{E}^\varepsilon[u^\varepsilon] = 0$ satisfies*

$$u^\varepsilon(x, 0) = F(x), \quad u_t^\varepsilon(x, 0) = -2F'(x).$$

Set $x_0 = \operatorname{arcosh} \sqrt{2}$. Also, assume that u^ε and u_x^ε are C^2 in (x, t) near $(0, 0)$ and $(\pm x_0, 0)$.

Let $C^\varepsilon(t)$ be the position of the local maximum continuing from $x = 0$, and let $H^\varepsilon(t) = u^\varepsilon(C^\varepsilon(t), t)$ be its height. Let $W^\varepsilon(t)$ be the distance between the two half-height points continuing from $\pm x_0$. These quantities are well defined and C^2 for sufficiently small $|t|$, and

$$\begin{aligned}(H^\varepsilon)''(0) &= 18\varepsilon, \\ (C^\varepsilon)''(0) &= \frac{1}{2}, \\ (W^\varepsilon)''(0) &= -33\sqrt{2}\varepsilon.\end{aligned}\tag{6.3}$$

Consequently,

$$\begin{aligned}H^\varepsilon(t) &= 1 + 9\varepsilon t^2 + o(t^2), \\ C^\varepsilon(t) - X(t) &= o(t^2), \\ W^\varepsilon(t) &= 2 \operatorname{arcosh} \sqrt{2} - \frac{33\sqrt{2}}{2}\varepsilon t^2 + o(t^2).\end{aligned}\tag{6.4}$$

The remainders refer to $t \rightarrow 0$ with ε fixed.

Proof. 1. *The peak and half-height points.* The initial profile satisfies

$$F(0) = 1, \quad F'(0) = 0, \quad F''(0) = -2.$$

Apply the implicit function theorem to $u_x^\varepsilon(x, t) = 0$ at $(0, 0)$. The assumed regularity and the nonzero derivative $u_{xx}^\varepsilon(0, 0) = F''(0)$ give a unique nearby C^2 branch $x = C^\varepsilon(t)$ with $C^\varepsilon(0) = 0$. By continuity, u_{xx}^ε remains negative along this branch for sufficiently small $|t|$. Therefore, it continues the strict local maximum. The height $H^\varepsilon(t) = u^\varepsilon(C^\varepsilon(t), t)$ is also C^2 .

At the initial half-height points,

$$F(\pm x_0) = \frac{1}{2}, \quad F'(\pm x_0) = \mp \frac{1}{\sqrt{2}}.$$

Apply the implicit function theorem to $u^\varepsilon(x, t) - \frac{H^\varepsilon(t)}{2} = 0$ at each point $(\pm x_0, 0)$. This gives two C^2 branches $x_\pm^\varepsilon(t)$ with $x_\pm^\varepsilon(0) = \pm x_0$. Their ordering persists for sufficiently small time, so $W^\varepsilon(t) = x_+^\varepsilon(t) - x_-^\varepsilon(t)$ is well defined and C^2 .

We next determine their initial velocities. Differentiating the peak identity once gives

$$u_{xt}^\varepsilon + (C^\varepsilon)' u_{xx}^\varepsilon = 0$$

along the peak. At $t = 0$, the initial data imply $u_{xt}^\varepsilon(x, 0) = -2F''(x)$. Hence, $F''(0)((C^\varepsilon)'(0) - 2) = 0$, and thus $(C^\varepsilon)'(0) = 2$. Along the peak, $u_x^\varepsilon = 0$, so

$$(H^\varepsilon)'(0) = u_t^\varepsilon(0, 0) = -2F'(0) = 0.$$

Now, differentiating the half-height identities once gives

$$F'(\pm x_0)((x_\pm^\varepsilon)'(0) - 2) = \frac{(H^\varepsilon)'(0)}{2} = 0.$$

Since $F'(\pm x_0) \neq 0$, both half-height points have initial velocity 2. It follows that $(W^\varepsilon)'(0) = 0$.

2. *The difference in initial acceleration.* Changing a and b contributes

$$\varepsilon\{\alpha(FF')' - \beta F''''\}$$

to the residual of the reference pulse. For $F = \text{sech}^2 \zeta$, the profile identities give

$$(FF')' = 8F^2 - 10F^3, \quad F'''' = 16F - 120F^2 + 120F^3.$$

Using $\alpha = 3(c - 1)$ and $\beta = (c + 1)/4$, we obtain

$$\begin{aligned} \alpha(FF')' - \beta F'''' &= 3(c - 1)(8F^2 - 10F^3) - \frac{c + 1}{4}(16F - 120F^2 + 120F^3) \\ &= -4(c + 1)F + (54c + 6)F^2 - 60cF^3. \end{aligned}$$

This proves (6.2).

Define the difference between the second time derivatives at the initial state by

$$\delta u_{tt}(x) = u_{tt}^\varepsilon(x, 0) - u_{*,tt}(x, 0).$$

The two fields have the same initial values of u and u_t and therefore, the same spatial derivatives of those values. Every term in $\mathcal{E}^\varepsilon$ other than u_{tt} depends only on these common initial quantities. Since the coefficient of u_{tt} is one,

$$\mathcal{E}^\varepsilon[u^\varepsilon](x, 0) - \mathcal{E}^\varepsilon[u_*](x, 0) = \delta u_{tt}(x).$$

The first residual is zero and the second is $\varepsilon R(F(x), 0)$. Thus

$$\delta u_{tt}(x) = -\varepsilon R(F(x), 0). \tag{6.5}$$

Since $c(0) = 2$, the polynomial needed below is

$$R(F, 0) = -12F + 114F^2 - 120F^3.$$

In particular,

$$R(1, 0) = -18, \quad R(1/2, 0) = \frac{15}{2}.$$

3. *The peak height.* For a smooth peak branch $x = C(t)$, twice differentiating $H(t) = u(C(t), t)$ gives

$$H'' = u_{tt} + 2C' u_{xt} + (C')^2 u_{xx} + C'' u_x,$$

with all derivatives evaluated along the peak. At $t = 0$, the perturbed and reference fields have the same u_{xt} and u_{xx} , and both peaks have velocity 2. The last term vanishes because $u_x = 0$ at a peak. Thus, subtracting the two height identities leaves only the difference in u_{tt} .

The reference height is identically one. Using (6.5) at $x = 0$ gives

$$(H^\varepsilon)''(0) = \delta u_{tt}(0) = -\varepsilon R(1, 0) = 18\varepsilon.$$

4. *The peak position.* Twice differentiating $u_x(C(t), t) = 0$ gives

$$0 = u_{xtt} + 2C' u_{xxt} + (C')^2 u_{xxx} + C'' u_{xx}.$$

At the initial peak, the two fields have the same u_{xxt} , u_{xxx} , and u_{xx} because their initial values of u and u_t agree. Their peak velocities also agree. Subtracting the reference identity from the perturbed identity yields

$$F''(0)((C^\varepsilon)''(0) - X''(0)) = -\partial_x \delta u_{tt}(0).$$

The function $F(x)$ is even, so $R(F(x), 0)$ is even. Thus, equation (6.5) gives $\partial_x \delta u_{tt}(0) = 0$. Since $F''(0) = -2 \neq 0$, we conclude that

$$(C^\varepsilon)''(0) = X''(0) = c'(0) = \frac{1}{2}.$$

5. *The width and Taylor expansions.* For either half-height branch, twice differentiating $u(x_\pm(t), t) = H(t)/2$ gives

$$u_{tt} + 2x'_\pm u_{xt} + (x'_\pm)^2 u_{xx} + x''_\pm u_x = \frac{H''}{2}.$$

The reference half-height points are $x_\pm^*(t) = X(t) \pm x_0$. At $t = 0$, the perturbed and reference branches start at the same points with the same velocity 2. Subtracting their identities cancels the terms involving u_{xt} and u_{xx} , leaving

$$\delta u_{tt}(\pm x_0) + F'(\pm x_0)((x_\pm^\varepsilon)''(0) - (x_\pm^*)''(0)) = \frac{(H^\varepsilon)''(0)}{2}.$$

Here the reference height has zero second derivative. By (6.5), $\delta u_{tt}(\pm x_0) = -15\varepsilon/2$. Thus,

$$F'(\pm x_0)((x_\pm^\varepsilon)''(0) - (x_\pm^*)''(0)) = 9\varepsilon + \frac{15}{2}\varepsilon = \frac{33}{2}\varepsilon.$$

Using the signs of $F'(\pm x_0)$ gives

$$(x_+^\varepsilon)''(0) - (x_+^*)''(0) = -\frac{33}{\sqrt{2}}\varepsilon, \quad (x_-^\varepsilon)''(0) - (x_-^*)''(0) = \frac{33}{\sqrt{2}}\varepsilon.$$

The reference width is constant. Therefore, subtracting the left endpoint's acceleration from the right endpoint's acceleration gives

$$(W^\varepsilon)''(0) = -\frac{66}{\sqrt{2}}\varepsilon = -33\sqrt{2}\varepsilon.$$

Finally, the initial values and first derivatives are

$$\begin{aligned} H^\varepsilon(0) &= 1, & C^\varepsilon(0) &= X(0) = 0, & W^\varepsilon(0) &= 2x_0, \\ (H^\varepsilon)'(0) &= 0, & (C^\varepsilon)'(0) &= X'(0) = 2, & (W^\varepsilon)'(0) &= 0. \end{aligned}$$

Taylor's formula for the C^2 functions H^ε , $C^\varepsilon - X$, and W^ε now gives (6.4). ■

For each fixed $\varepsilon > 0$, the pulse initially becomes taller and narrower while its centre agrees with the reference motion through second order in time. The unchanged initial centre acceleration follows from the evenness of the residual. The theorem describes only the initial response under smooth evolution. The perturbed solution can change shape, so it is not constrained by the fixed-profile width restriction.

6.3 Full-equation numerical experiments

We now follow the perturbed solutions over one modulation period to examine their response beyond the short-time behaviour described by Theorem 12.

Introduce $v = u_t + \alpha u u_x + \beta u_{xxx}$. Equation (2.1) is equivalent to

$$u_t = v - \alpha u u_x - \beta u_{xxx}, \quad v_t = s^2 u_{xx} + a s \alpha (u u_x)_x + b s \beta u_{xxx}. \quad (6.6)$$

The definition of v retains the effects of α' and β' . In the mean frame $y = x - 2t$, set $U(y, t) = u(y + 2t, t)$ and $V(y, t) = v(y + 2t, t)$, adding $2U_y$ and $2V_y$ to the respective equations. The initial auxiliary field is $v(x, 0) = u_t(x, 0) + \alpha(0)u(x, 0)u_x(x, 0) + \beta(0)u_{xxx}(x, 0)$, with u_t in the original coordinate.

We use N equispaced points on $[-L/2, L/2)$ and project initial fields and quadratic products onto Fourier indices $|m| < N/3$. A Strang-type splitting [15] combines two linear half steps, exact for coefficients frozen at the time-step midpoint, with a classical fourth-order Runge–Kutta nonlinear step at the same coefficient values. The overall method is second order. The nonlinear substep is not itself time-symmetric. No exact-solution forcing is added.

With the Fourier convention $\partial_y \mapsto i\kappa$, the matrix and nonlinear terms in this frame are

$$L_\kappa(t) = \begin{pmatrix} i(2\kappa + \beta\kappa^3) & 1 \\ -s^2\kappa^2 + bs\beta\kappa^4 & 2i\kappa \end{pmatrix}, \quad \begin{pmatrix} -\frac{i\kappa\alpha}{2}\widehat{\Pi_N(U^2)} \\ -\frac{as\alpha\kappa^2}{2}\widehat{\Pi_N(U^2)} \end{pmatrix}, \quad (6.7)$$

where $\kappa = 2\pi m/L$ for integer m , Π_N denotes the retained-mode projection, and hats denote Fourier coefficients. Thus each retained mode evolves through the displayed linear matrix and projected nonlinear terms.

The experiments use $0 \leq t \leq 2\pi$, $L = 240$, $N = 1536$, and $h = 2\pi/6284$. Initial-data perturbations take

$$u(x, 0) = F + \frac{\eta}{2}F'', \quad u_t(x, 0) = -2F' + \frac{\eta}{8}F''', \quad (6.8)$$

with $\varepsilon = 0$. The parameter experiment takes $\eta = 0$ and the detuning in (6.2). Each uses 0.005, 0.01, and 0.02 for its perturbation parameter. The derivative perturbations have zero integral, preserving the initial values $M(0) = 2$ and $M'(0) = 0$. For periodic computations the zero Fourier mode obeys the same $M'' = 0$ law.

Write $U_*(y, t) = F(y - Y_*(t))$ with $Y_*(t) = \frac{1}{2}(1 - \cos t)$. We report the relative field error $E = \|U - U_*\|_2 / \|U_*\|_2$ and fit $h_p \operatorname{sech}^2(q(y - Y))$ by least squares, with positive height h_p and inverse width q . The relative fitting residual

$$R_{\text{fit}} = \frac{\|U - h_p \operatorname{sech}^2(q(y - Y))\|_2}{\|U\|_2} \quad (6.9)$$

measures the part not captured by the fit. The centre discrepancy is $\Delta Y = Y - \frac{1}{2}(1 - \cos t)$ and the width ratio is q^{-1} . These fitted quantities differ from the exact local height and half-height width in Theorem 12. All norms are discrete spatial L^2 norms over the periodic box.

Least squares starts at $(h_p, Y, q) = (1, Y_*, 1)$, with bounds $0.1 \leq h_p, q \leq 10$ and $|Y - Y_*| \leq 10$. All three stopping tolerances are 10^{-13} . None of the reported fits reaches a bound.

Figure 4(a) checks temporal refinement against the unperturbed benchmark. Halving the time step from approximately 0.004 to 0.002 and 0.001 gives maximum sampled relative errors 2.109×10^{-4} , 5.254×10^{-5} , and 1.313×10^{-5} , with observed orders 2.005 and 2.000. At the smallest step, increasing N from 384 to 768 and 1536 reduces the error from 6.782×10^{-2} to 3.406×10^{-4} and 1.313×10^{-5} .

For the largest perturbation in each experiment, doubling N at fixed L changes the final diagnostics by less than 2×10^{-11} . Halving the time step, or doubling both L and N at fixed grid spacing, changes each final diagnostic $E, R_{\text{fit}}, h_p, Y, q$ by less than 10^{-5} . The conserved spatial integral drifts by less than 2×10^{-15} .

Figure 4(b) compares the initial deformation with Theorem 12. Let $H_{\text{num}}^\varepsilon(t)$ and $W_{\text{num}}^\varepsilon(t)$ denote the numerical local peak height and half-height width for the detuned solution. The corresponding quantities with superscript 0 are obtained from an unperturbed run with $\eta = \varepsilon = 0$, using the same grid and time step. The plotted ratios are

$$\frac{H_{\text{num}}^\varepsilon(t) - H_{\text{num}}^0(t)}{9\varepsilon t^2}, \quad \frac{W_{\text{num}}^\varepsilon(t) - W_{\text{num}}^0(t)}{-\frac{33\sqrt{2}}{2}\varepsilon t^2}.$$

The peak is located by finding a zero of the spatial derivative of the Fourier interpolant. Its height and the two half-height points are then evaluated using this interpolant with a spatial root-finding tolerance of 10^{-14} . For $\varepsilon = 0.02$ and $\eta = 0$, the ratios at $t = 0.001$ are 0.9992 and 0.9998, respectively, and approach one as time decreases. These computations use $L = 80$, $N = 768$, and a time step of 10^{-6} .

Table 2: Pulse diagnostics at $t = 2\pi$ on $L = 240$. The exact pulse has height and width ratio one and centre discrepancy zero.

Perturbation	100E	100R _{fit}	h_p	ΔY	q^{-1}
$\eta = 0.005$	0.410	0.399	0.99965	-0.00096	1.00006
$\eta = 0.01$	0.818	0.797	0.99930	-0.00194	1.00011
$\eta = 0.02$	1.633	1.591	0.99858	-0.00395	1.00022
$\varepsilon = 0.005$	1.401	0.180	0.99944	-0.01534	1.00321
$\varepsilon = 0.01$	2.798	0.360	0.99887	-0.03066	1.00641
$\varepsilon = 0.02$	5.579	0.720	0.99774	-0.06130	1.01282

Table 2 gives the final diagnostics. For initial-data perturbations, the nearly overlapping R_{fit}/η curves in Figure 4(c) have sampled maxima between 0.812 and 0.816, indicating approximately proportional response over the tested amplitudes.

Figure 4(d) compares field error and fitting residual for parameter detuning $\varepsilon = 0.02$. At $t = 2\pi$, these are approximately 5.58% and 0.72%. The fitted centre lags by 0.0613 and the width ratio is 1.0128. Allowing the fitted pulse parameters to change therefore accounts for much of the field discrepancy, with some deformation remaining. The initial narrowing in Theorem 12 concerns the local half-height width near $t = 0$, whereas the broadening reported here concerns the fitted width at $t = 2\pi$. Panels (c,d) cover one modulation period on $L = 240$. These computations describe the tested disturbances on a periodic domain and do not establish nonlinear stability or long-time persistence.

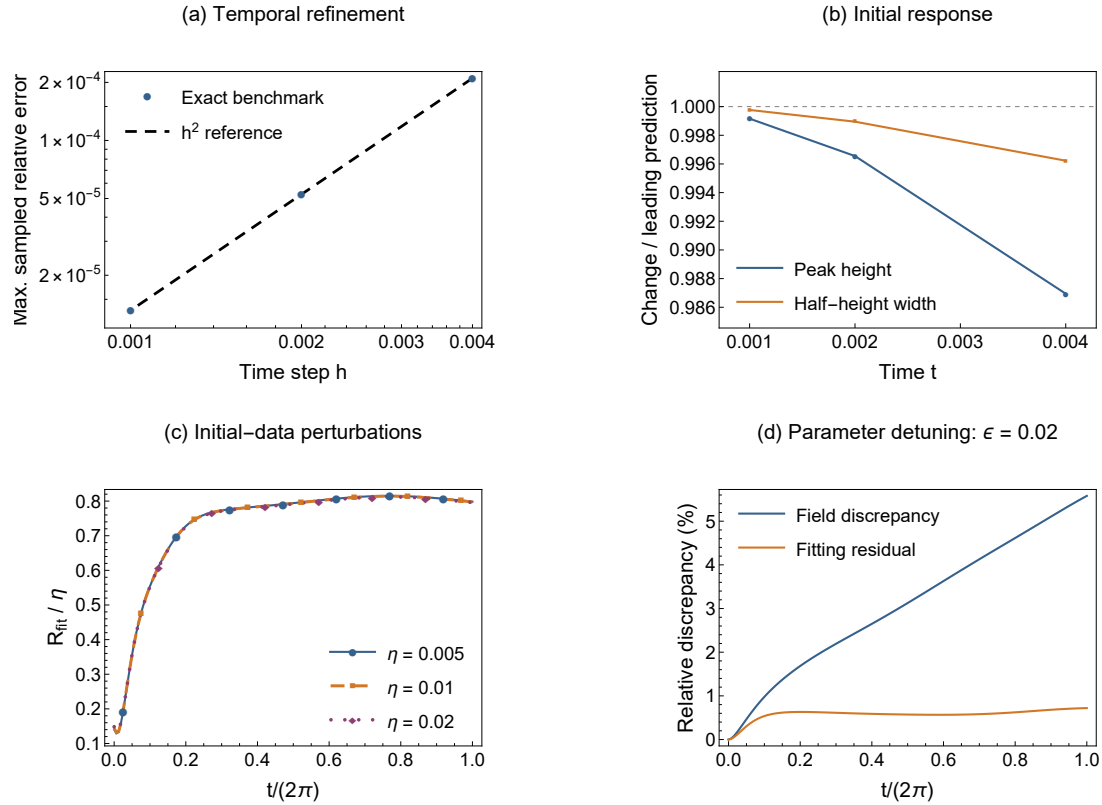


Figure 4: Response of the periodic-speed pulse. (a) Temporal refinement. (b) Normalized initial deformation. (c) Scaled fitting residual for initial-data perturbations. (d) Field error and fitting residual for parameter detuning.

7 Discussion and conclusion

Exact modulation in the temporal flux family is constrained by the prescribed profile. The tail criterion fixes the spatial scale and the quadratic profile gives necessary and sufficient relations between amplitude, phase motion, and coefficients. The reconstruction theorem makes these relations usable in two ways: as a test for a specified equation and as a construction of compatible coefficient laws. The freedom at the opposite-sign pairs belongs to this joint reconstruction. It does not allow the speed to vary freely after the equation has been fixed.

The point-transformation classification establishes that the time dependence of the modulated examples cannot be removed within the same nondegenerate family. The Fourier analysis addresses a separate question. Pulse compatibility, common characteristic factors, and repeated background frequencies need not occur at the same parameters. Background resonance for the periodic specialization is treated separately in the companion study [16].

The deformation theorem connects the classification to a concrete consequence of changing the modal parameters. Under smooth evolution, the periodic-speed pulse initially becomes taller and narrower without a change in centre acceleration. The numerical experiments then distinguish fitted parameter changes from the part of the field not captured by the fit over one modulation period. These local and finite-time results leave nonlinear stability and long-time evolution open.

For more general two-mode equations, the useful sequence is to determine the restrictions imposed by a profile, test whether coefficient variation can be removed at equation level, and then study the response to departures from compatibility. Different fluxes, algebraic tails, or profiles that change shape require new arguments rather than an extension of the present classification without its assumptions.

Declarations

Availability of Data and Materials. The main theoretical results are derived analytically and do not rely on external datasets. Numerical outputs were generated for verification and illustration using the equations and computational procedures described in the manuscript.

Funding Declaration. No funding was received for this research.

Competing Interests. The author has no relevant financial or non-financial interests to disclose.

References

- [1] S. V. Korsunsky, Soliton solutions for a second-order KdV equation, *Phys. Lett. A*. **185**(2) (1994), 174–176. [https://doi.org/10.1016/0375-9601\(94\)90842-7](https://doi.org/10.1016/0375-9601(94)90842-7)
- [2] M. Alquran, A. Jarrah, Jacobi elliptic function solutions for a two-mode KdV equation, *J. King Saud Univ. Sci.* **31**(4) (2019), 485–489. <https://doi.org/10.1016/j.jksus.2017.06.010>
- [3] C.-T. Lee, J.-L. Liu, A Hamiltonian model and soliton phenomenon for a two-mode KdV equation, *Rocky Mountain J. Math.* **41**(4) (2011), 1273–1289. <https://doi.org/10.1216/RMJ-2011-41-4-1273>
- [4] Z.-J. Xiao, B. Tian, H.-L. Zhen, J. Chai, X.-Y. Wu, Multi-soliton solutions and Bäcklund transformation for a two-mode KdV equation in a fluid, *Waves Random Complex Media* **27**(1) (2017), 1–14. <https://doi.org/10.1080/17455030.2016.1185193>
- [5] A.-M. Wazwaz, Multiple soliton solutions and other exact solutions for a two-mode KdV equation, *Math. Methods Appl. Sci.* **40** (2017), 2277–2283. <https://doi.org/10.1002/mma.4138>
- [6] W.-P. Hong, Y.-D. Jung, New non-traveling solitary wave solutions for a second-order Korteweg-de Vries equation, *Z. Naturforsch. A* **54** (1999), 375–378. <https://doi.org/10.1515/zna-1999-6-705>
- [7] M. Ali, M. Alquran, A. BaniKhalid, Symmetric and asymmetric binary-solitons to the generalized two-mode KdV equation: Novel findings for arbitrary nonlinearity and dispersion parameters, *Results Phys.* **45** (2023), 106250. <https://doi.org/10.1016/j.rinp.2023.106250>
- [8] H. I. Abdel-Gawad, Intricate and multiple chirped waves geometric structures solutions of two-mode KdV equation, spectral and stability analysis, *Int. J. Mod. Phys. B* **36**(14) (2022), 2250056. <https://doi.org/10.1142/S0217979222500564>
- [9] C. A. Gómez, A generalized two-mode KdV equation: Exact solutions, *Contemp. Eng. Sci.* **11**(6) (2018), 249–255. <https://doi.org/10.12988/ces.2018.8119>
- [10] C. A. Gómez, Exact solutions for a fifth-order two-mode KdV equation with variable coefficients, *Contemp. Eng. Sci.* **11**(16) (2018), 779–784. <https://doi.org/10.12988/ces.2018.8262>
- [11] C. A. Gómez, Periodic and soliton solutions for a generalized two-mode KdV-Burger’s type equation, *Contemp. Eng. Sci.* **11**(16) (2018), 785–791. <https://doi.org/10.12988/ces.2018.8267>

- [12] M. Alquran, Dynamics of waveforms in the potential KdV equation incorporating time-dependent perturbation coefficient, *Physics Open* **27** (2026), 100401. <https://doi.org/10.1016/j.physo.2026.100401>
- [13] R. O. Popovych, O. O. Vaneeva, More common errors in finding exact solutions of nonlinear differential equations. I, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010), 3887–3899. <https://arxiv.org/abs/0911.1848>
- [14] O. O. Vaneeva, R. O. Popovych, C. Sophocleous, Equivalence transformations in the study of integrability, *Phys. Scr.* **89** (2014), 038003. <https://arxiv.org/abs/1308.5126>
- [15] G. Strang, On the construction and comparison of difference schemes, *SIAM J. Numer. Anal.* **5** (1968) 506–517. <https://doi.org/10.1137/0705041>
- [16] I. Jaradat, Background resonance and transport in a periodically modulated two-mode KdV equation, Unpublished manuscript, 2026.