

Supplementary Materials: Multicomponent Magnetic Domain Walls in Rhombohedral Graphene

This Supplementary Material is organized as follows. In Sec. A, we present the microscopic Hartree–Fock (HF) theory of the uniform quarter-metal ground state, including the phase diagram and the transition between the two- and four-component intervalley-coherent (IVC) phases. Sec. B describes the HF formulation for spin–valley domain walls in a one-dimensional magnetic supercell, together with the numerical convergence and microscopic characterization of the domain-wall solutions. Section C presents the long-wavelength CP^3 theory and compares its predictions with the microscopic HF results.

We first summarize the microscopic continuum model and its uniform HF solution. The domain-wall calculations presented in the main text are performed within the spin–valley polarized quarter-metal (SVQM) phase, where two time-reversal-related ferromagnetic ground states are exactly degenerate and serve as the boundary conditions for the domain-wall calculations.

A. HARTREE–FOCK MEAN-FIELD THEORY OF UNIFORM GROUND STATE

We consider rhombohedral bilayer graphene (rBG) in the presence of an interlayer potential U_D controlled by a perpendicular displacement field, which isolates the low-energy bands near charge neutrality. The electronic structure is described within the Slonczewski–Weiss–McClure (SWMc) continuum model obtained by expanding the tight-binding Hamiltonian around the two graphene valleys, K and K' . Throughout this work, we use the same microscopic parameters as in the main text.

The graphene lattice has lattice constant $a = 2.46$ Å, nearest-neighbor carbon bond length $a_{CC} = 1.42$ Å, and interlayer spacing $d_{\text{layer}} = 3.35$ Å. Together with the SWMc hopping amplitudes listed in Table I, these structural parameters determine the non-interacting band structure and the interaction-driven phase diagram discussed below.

We consider a single p_z orbital on each carbon atom. In the continuum description, the low-energy states are labeled by valley $\tau = \pm 1$, spin $s = \uparrow, \downarrow$, layer $l = 1, 2$, and sublattice $\sigma = A, B$. In the layer–sublattice basis, we define the four-component annihilation spinor

$$c_{\tau,s}(\mathbf{k}) = \begin{pmatrix} c_{\tau,s,A_1}(\mathbf{k}) \\ c_{\tau,s,B_1}(\mathbf{k}) \\ c_{\tau,s,A_2}(\mathbf{k}) \\ c_{\tau,s,B_2}(\mathbf{k}) \end{pmatrix}, \quad (\text{S1})$$

where $\mathbf{k}$ is measured relative to the center of valley τ .

The mean-field Hamiltonian is

$$\hat{H}^{\text{MF}}(\mathbf{k}) = \hat{H}^{\text{SWMc}}(\mathbf{k}) + \hat{H}^{\text{SOC}} + \hat{\Sigma}^{\text{HF,C}}(\mathbf{k}) + \hat{\Sigma}^{\text{HF,Hund}}, \quad (\text{S2})$$

where $\hat{H}^{\text{SWMc}}(\mathbf{k})$ is the SWMc band Hamiltonian, $\hat{H}^{\text{SOC}}$ is the intrinsic Kane–Mele spin–orbit coupling (SOC), $\hat{\Sigma}^{\text{HF,C}}(\mathbf{k})$ is the Hartree–Fock self-energy arising from the long-range Coulomb interaction, and $\hat{\Sigma}^{\text{HF,Hund}}$ is the Hartree–Fock self-energy arising from the intervalley Hund’s interaction. The individual terms are defined in the following subsections.

Band Hamiltonian and Spin-orbit Interaction

The continuum band Hamiltonian is diagonal in the spin and valley degrees of freedom and can be written as

$$\hat{\mathcal{H}}^{\text{SWMc}} = \sum_{\mathbf{k}, \tau, s} c_{\tau s}^\dagger(\mathbf{k}) h_\tau(\mathbf{k}) c_{\tau s}(\mathbf{k}), \quad (\text{S3})$$

where the 4×4 matrix $h_\tau(\mathbf{k})$ is the valley-resolved SWMc Hamiltonian acting on the layer–sublattice space. Explicitly, the valley-resolved SWMc Hamiltonian is given by

$$h_\tau(\mathbf{k}) = \begin{bmatrix} t(\mathbf{k}) + U_1 & t_{12}(\mathbf{k}) \\ t_{12}^\dagger(\mathbf{k}) & t(\mathbf{k}) + U_2 \end{bmatrix}_{4 \times 4} \quad (\text{S4})$$

The Hamiltonian consists of intra-layer hopping (t), nearest-layer hopping (t_{12}), and layer-dependent onsite potentials U_i arising from externally applied gate voltages and/or broken symmetries. Explicitly,

$$t(\mathbf{k}) = \begin{bmatrix} 0 & v_0 \pi^\dagger \\ v_0 \pi & 0 \end{bmatrix}, \quad t_{12}(\mathbf{k}) = \begin{bmatrix} -v_4 \pi^\dagger & v_3 \pi \\ \gamma_1 & -v_4 \pi^\dagger \end{bmatrix} \quad (\text{S5})$$

where $\pi = \tau k_x + i k_y$ is a linear momentum, and γ_i ($i = 0, \dots, 4$) are hopping amplitudes of the tight-binding model with corresponding velocity parameters $v_i = (\sqrt{3}/2)a\gamma_i/\hbar$. The U_i terms are

$$U_1 = \begin{bmatrix} \frac{\Delta + \delta + U_D}{2} & 0 \\ 0 & \frac{\Delta + U_D}{2} \end{bmatrix}, \quad U_2 = \begin{bmatrix} \frac{\Delta - U_D}{2} & 0 \\ 0 & \frac{\Delta + \delta - U_D}{2} \end{bmatrix} \quad (S6)$$

We use the model parameters listed in Table I, which are chosen to match quantum oscillation frequency signatures in Ref. [1]. The tunable parameter U_D in Eq. S6 is the interlayer potential, and is proportional to out-of-plane electric displacement field.

Table I. Tight-binding parameters (in meV) for rhombohedral graphene, see also Refs. [1].

γ_0	γ_1	γ_2	γ_3	γ_4	Δ	δ	U_D
3160	380	-15	-290	141	-23	-10.5	25

In the large-displacement-field regime considered here, the low-energy electronic states are predominantly polarized onto the non-dimer A_1 and B_2 sublattices, suppressing the leading effect of Rashba SOC. The dominant SOC contribution is therefore the momentum-independent intrinsic Kane–Mele term

$$\hat{H}^{\text{SOC}} = -\frac{\lambda_{\text{soc}}}{2} \hat{\sigma}_z \hat{\tau}_z \hat{s}_z. \quad (S7)$$

Here, $\hat{\sigma}_z$, $\hat{\tau}_z$, and $\hat{s}_z$ are Pauli matrices acting in sublattice, valley, and spin space, respectively. We take $\sigma_z = +1$ (-1) on the A (B) sublattice, $\tau_z = +1$ (-1) in the K (K') valley, and $s_z = +1$ (-1) for spin $\uparrow$ ($\downarrow$). For the sign of U_D and the hole-doped regime considered here, the relevant valence-band states are predominantly polarized onto the B_2 sublattice. With the convention $\lambda_{\text{soc}} > 0$, the SOC term above therefore favors the $K \uparrow$ and $K' \downarrow$ hole flavors. We use $\lambda_{\text{soc}} = 0.1$ meV, consistent with the experimentally estimated intrinsic SOC strength in rhombohedral graphene [2]. The opposite sign convention would simply interchange the two SOC-favored spin–valley pairs, $\{K \uparrow, K' \downarrow\} \leftrightarrow \{K \downarrow, K' \uparrow\}$. In the absence of an external Zeeman field, this relabeling does not affect the existence, structure, or transition between the two- and four-component domain walls studied here.

Electron–electron interactions

The electron–electron interaction consists of two contributions: a long-range screened Coulomb interaction and a lattice-scale intervalley Hund’s interaction. The former drives exchange ferromagnetism, while the latter favors parallel spin alignment between opposite valleys and is responsible for the transition between the two- and four-component intervalley-coherent states discussed below. Both interactions are treated within the Hartree–Fock approximation.

The long-range Coulomb interaction is described by

$$\hat{\mathcal{H}}^{\text{C}} = \frac{1}{2A} \sum_{\mathbf{q}} V_C(\mathbf{q}) : \hat{n}(\mathbf{q}) \hat{n}(-\mathbf{q}) :, \quad \hat{n}(\mathbf{q}) = \sum_{\mathbf{k}, \tau, s, l, \sigma} c_{\tau, s, l, \sigma}^\dagger(\mathbf{k} + \mathbf{q}) c_{\tau, s, l, \sigma}(\mathbf{k}). \quad (S8)$$

Here $::$ denotes normal ordering and A is the sample area. The screened Coulomb potential is taken to be $V_C(\mathbf{q}) = 2\pi k_e \tanh(|\mathbf{q}|d)/(\epsilon_r |\mathbf{q}|)$, where $k_e = 1.44$ eV nm is the Coulomb constant, $\epsilon_r = 15$ is the effective dielectric constant, and $d = 50$ nm is the gate distance.

The lattice-scale intervalley Hund’s interaction originates from short-range exchange processes between electrons in opposite valleys and is modeled by

$$\hat{\mathcal{H}}^{\text{Hund}} = \frac{g_\perp}{2A} \sum_{\tau} \sum_{\mathbf{k}, \mathbf{k}'} \sum_{\mathbf{q}} \sum_{l, \sigma} \sum_{s, s'} : c_{\bar{\tau} s l \sigma}^\dagger(\mathbf{k}) c_{\tau s l \sigma}(\mathbf{k} + \mathbf{q}) c_{\tau s' l \sigma}^\dagger(\mathbf{k}') c_{\bar{\tau} s' l \sigma}(\mathbf{k}' - \mathbf{q}) :, \quad (S9)$$

where $g_\perp > 0$ is the intervalley Hund’s coupling strength, and $\bar{\tau} = -\tau$ denotes the opposite valley. A positive $g_\perp$ favors ferromagnetic alignment of spins between opposite valleys.

The above intervalley Hund’s coupling $\hat{\mathcal{H}}^{\text{Hund}}$ is one of the many two-body interactions allowed by the lattice symmetries of graphene. Because these interactions originate from processes involving momenta of order the inverse lattice spacing, they are short-ranged on the scale of the continuum theory and can, to leading order be represented by contact interactions. Corrections beyond the zero-range approximation were studied in Ref. [3]. The symmetry classification of these interactions was developed in Ref. [4] for monolayer graphene and in Ref. [5] for bilayer graphene. Their bare coupling constants were calculated in Ref. [6] by approximating the three-dimensional electronic wavefunctions as linear combinations of carbon $2p_z$ orbitals with an effective nuclear charge, and their RG flow was studied in Ref. [7, 8].

The above analyses were originally developed for monolayer graphene and bilayer graphene. Closely related intervalley Hund interactions have also been incorporated into continuum models of rhombohedral multilayer graphene [9, 10]. The intrasublattice component presents no difficulty and can be represented by a momentum-independent local interaction. For the intersublattice component, however, Ref. [9] constructs a C_3 -invariant interaction using the phenomenological kernel $V(\mathbf{q}) = J_H e^{2i\tau\theta_{\mathbf{q}}}$, where $\theta_{\mathbf{q}} = \arg(q_x + iq_y)$. Since it has no direction-independent limit when $\mathbf{q} \rightarrow 0$, the potential is nonanalytic at $\mathbf{q} = 0$. As a result, its Fourier transform has a slow algebraic decay in real space $e^{im\theta_r}/r^2$, and therefore does not correspond to an ultrashort-range potential in the continuum model. For simplicity, we therefore adopt a minimal local description in which the intervalley Hund interaction is diagonal in the microscopic sublattice index and has the same coupling strength on every graphene layer. This approximation is physically motivated when the carbon $2p_z$ orbitals are sufficiently compact compared with the interlayer separation. The leading layer-dependent interaction is the intralayer and interlayer Coulomb energies difference, see Ref. [11] and Section. VI of [12].

Within the Hartree–Fock approximation, the quartic interaction terms are factorized with respect to the single-particle density matrix, giving the Coulomb and intervalley Hund self-energies entering Eq. (S2). The resulting Coulomb and intervalley Hund Hartree–Fock self-energies entering Eq. S2 are given below. Applying the Hartree–Fock decoupling to Eq. S8 gives

$$\hat{\Sigma}^{\text{HF,C}}(\mathbf{k}) = \hat{\Sigma}^{\text{H,C}} + \hat{\Sigma}^{\text{F,C}}(\mathbf{k}),$$

$$\Sigma_{\alpha,\beta}^{\text{H,C}} = \frac{N}{A} \delta_{\alpha,\beta} V_C(0) \quad , \quad \Sigma_{\alpha,\beta}^{\text{F,C}}(\mathbf{k}) = -\frac{1}{A} \sum_{\mathbf{q}} V_C(\mathbf{q}) \rho_{\alpha,\beta}(\mathbf{k} + \mathbf{q}). \quad (\text{S10})$$

Here N is total number of occupied electronic states, and $\rho_{\alpha,\beta}(\mathbf{k}) = \langle c_{\beta}^{\dagger}(\mathbf{k}) c_{\alpha}(\mathbf{k}) \rangle$ is the single-particle density matrix. The composite indices α, β label the combined valley (τ), spin (s), layer (l), and sublattice (σ) degrees of freedom. The Hartree term describes the electrostatic interaction generated by the total charge density, whereas the Fock term gives the exchange interaction responsible for spontaneous symmetry breaking. For the dual-gated Coulomb interaction, the $\mathbf{q} = 0$ Hartree contribution is finite and depends only on the spatially averaged carrier density. At fixed carrier density, it produces a uniform shift of the mean-field potential and contributes equally to the competing states. We therefore omit the $\mathbf{q} = 0$ Hartree term from the self-consistent mean-field Hamiltonian, while retaining all finite- $\mathbf{q}$ Hartree components associated with spatial charge redistribution.

The corresponding intervalley Hund Hartree–Fock self-energy is

$$\hat{\Sigma}^{\text{HF,Hund}} = \hat{\Sigma}^{\text{H,Hund}} + \hat{\Sigma}^{\text{F,Hund}},$$

$$\hat{\Sigma}^{\text{H,Hund}} = \frac{g_{\perp}}{A} \sum_{\tau,l,\sigma} \sum_{\mathbf{k},s} c_{\bar{\tau}sl\sigma}^{\dagger}(\mathbf{k}) c_{\tau sl\sigma}(\mathbf{k}) \sum_{\mathbf{k}',s'} \langle c_{\bar{\tau}s'l\sigma}^{\dagger}(\mathbf{k}') c_{\bar{\tau}s'l\sigma}(\mathbf{k}') \rangle \quad (\text{Hartree: } \mathbf{q} = 0) \quad (\text{S11})$$

$$\hat{\Sigma}^{\text{F,Hund}} = -\frac{g_{\perp}}{A} \sum_{\tau,l,\sigma} \sum_{\mathbf{k},s,s'} c_{\bar{\tau}sl\sigma}^{\dagger}(\mathbf{k}) c_{\bar{\tau}s'l\sigma}(\mathbf{k}) \sum_{\mathbf{k}'} \langle c_{\bar{\tau}s'l\sigma}^{\dagger}(\mathbf{k}') c_{\tau sl\sigma}(\mathbf{k}') \rangle \quad (\text{Fock: } \mathbf{q} = \mathbf{k}' - \mathbf{k}) \quad (\text{S12})$$

The Hartree contribution penalizes spin-preserving intervalley coherence, whereas the Fock contribution lowers the energy through intervalley exchange and favors parallel spin alignment between opposite valleys. The competition between these two terms determines whether the intervalley-coherent state remains confined to two spin-valley flavors or reconstructs into a four-component state.

The self-consistent Hartree–Fock equations are solved over the (n_e, U_D) phase space to determine the uniform quarter-metal ground states. We then examine how increasing intervalley Hund's coupling drives a continuous transition from the two-component to the four-component intervalley-coherent phase, which provides the microscopic origin of the analogous domain-wall transition discussed in the main text.

Uniform Quarter-Metal Ground States

We now solve the self-consistent Hartree–Fock equations for the uniform system as a function of carrier density n_e and interlayer potential U_D . Figure S1(a) shows the resulting phase diagram at $g_{\perp} = 0$. Depending on the carrier density and displacement field, the system realizes several symmetry-broken metallic phases distinguished by their spin and valley polarization. The horizontal dashed line denotes the fixed interlayer potential $U_D = 25$ meV used throughout this work, along which the representative band structures in Figs. S1(b)–(d) are obtained. Throughout the main text, we focus on the spin–valley polarized quarter-metal (SVQM) phase at $n_e = -0.09 \times 10^{12} \text{ cm}^{-2}$, indicated by the blue dot in Fig. S1(a), where two time-reversal-related ferromagnetic

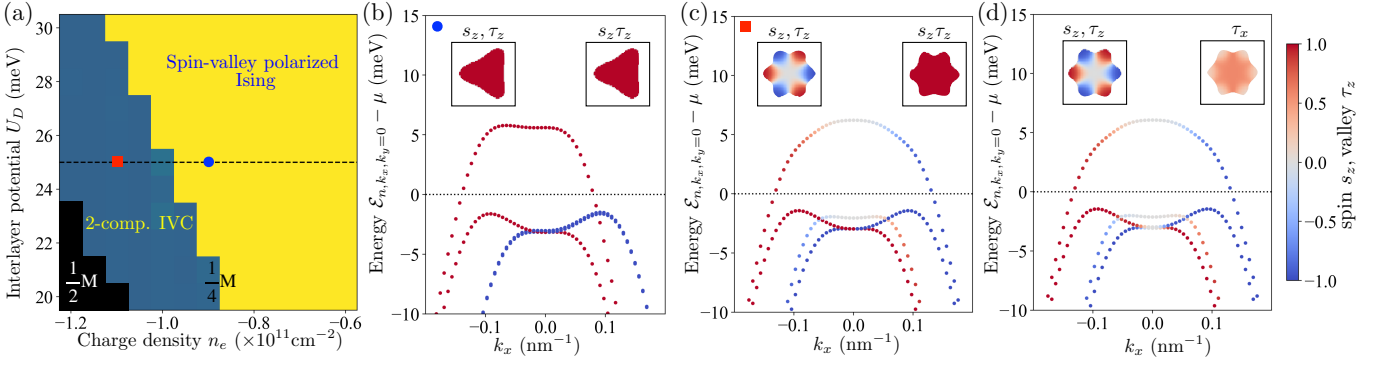


Figure S1. (a) Hartree-Fock phase diagram as a function of carrier density n_e and interlayer potential U_D at $g_\perp = 0$. The horizontal dashed line marks the fixed interlayer potential $U_D = 25$ meV used throughout this work. The system realizes spin-valley polarized and intervalley-coherent (IVC) quarter-metal phases, while the black region denotes the half-metal ($1/2M$) phase. The blue dot marks the parameter set used for the domain-wall calculations in the main text, and the red square indicates the density used to study the Hund-driven transition within the IVC quarter metal. (b) Quasiparticle band structure of the spin-valley quarter metal (SVQM) at $n_e = -0.09 \times 10^{12} \text{ cm}^{-2}$ along $k_y = 0$. The quasiparticle energies are measured relative to the chemical potential μ , and the horizontal dashed line marks the Fermi level. (c) Quasiparticle band structure of the two-component IVC quarter metal at $n_e = -0.11 \times 10^{12} \text{ cm}^{-2}$ and $g_\perp = 0$. The insets show the spin and valley order parameters evaluated over the occupied Fermi sea. The IVC order is confined to the $K \uparrow$ and $K' \downarrow$ flavors, giving $s_z \tau_z = 1$ throughout the occupied Fermi sea. (d) Quasiparticle band structure at the same carrier density for $g_\perp |n_e| / \lambda_{\text{soc}} = 5.5$, corresponding to the four-component IVC quarter metal. The left inset shows the spin and valley polarizations, while the right inset shows the intervalley coherence τ_x evaluated over the occupied Fermi sea.

ground states are exactly degenerate. These degenerate states provide the boundary conditions for the domain-wall calculations discussed in the main text.

Representative quasiparticle band structures of the metallic phases are shown in Fig. S1(b)–(d). Figure S1(b) shows the SVQM band structure at this carrier density. The band colors indicate the spin and valley polarizations, s_z and τ_z , and the horizontal dashed line indicates the Fermi level. Figures S1(c) and (d) instead consider a slightly higher hole density at $n_e = -0.11 \times 10^{12} \text{ cm}^{-2}$ indicated by the red square in Fig. S1(a), where the ground state is an intervalley-coherent (IVC) quarter metal. In the absence of intervalley Hund's coupling, the IVC order resides within two spin-valley flavors, $K \uparrow$ and $K' \downarrow$, giving rise to the two-component IVC state shown in Fig. S1(c). The inset shows the momentum distribution of order parameters inside the Fermi surface. For 2-component IVC phase, $s_z \tau_z$ order parameter is 1 throughout the Fermi surface, while s_z and τ_z oscillates around the Fermi surface. Upon increasing the Hund's coupling to $g_\perp |n_e| / \lambda_{\text{soc}} = 5.5$, the quasiparticle spectrum evolves into the four-component IVC state shown in Fig. S1(d).

To make the distinction between the two IVC states explicit, an occupied HF quarter-metal state can be written as

$$|\psi_{\mathbf{k}}\rangle = \sum_{\tau,s} c_{\tau s}^\dagger(\mathbf{k}) u_{\tau s}(\mathbf{k}) |\text{CN}\rangle, \quad (\text{S13})$$

where $|\text{CN}\rangle$ denotes the charge-neutral reference state and each $u_{\tau s}(\mathbf{k})$ is a four-component layer-sublattice spinor. Here, $c_{\tau s}^\dagger(\mathbf{k})$ creates the relevant electron or hole carrier relative to $|\text{CN}\rangle$. The product $c_{\tau s}^\dagger u_{\tau s} = \sum_{l,\sigma} c_{\tau s l \sigma}^\dagger z_{\tau s l \sigma}$ implicitly contracts the layer-sublattice indices. The many-body HF ground state is obtained by occupying all such states within the Fermi sea.

The transverse spin polarization and spin-preserving intervalley coherence are

$$s_x = \frac{1}{N_e} \sum_{\mathbf{k}, \tau} \left[\langle c_{\tau \uparrow}^\dagger(\mathbf{k}) c_{\tau \downarrow}(\mathbf{k}) \rangle + \text{c.c.} \right], \quad (\text{S14})$$

$$\tau_x = \frac{1}{N_e} \sum_{\mathbf{k}, s} \left[\langle c_{Ks}^\dagger(\mathbf{k}) c_{K's}(\mathbf{k}) \rangle + \text{c.c.} \right], \quad (\text{S15})$$

where $N_e = |n_e|A$, and the layer-sublattice contraction is implicit. In terms of the spinor components,

$$s_x(\mathbf{k}) \propto 2 \text{Re} \left[u_{K\uparrow}^*(\mathbf{k}) u_{K\downarrow}(\mathbf{k}) + u_{K'\uparrow}^*(\mathbf{k}) u_{K'\downarrow}(\mathbf{k}) \right], \quad (\text{S16})$$

$$\tau_x(\mathbf{k}) \propto 2 \text{Re} \left[u_{K\uparrow}^*(\mathbf{k}) u_{K'\uparrow}(\mathbf{k}) + u_{K\downarrow}^*(\mathbf{k}) u_{K'\downarrow}(\mathbf{k}) \right]. \quad (\text{S17})$$

For the two-component IVC state, $u_{K'\uparrow} = u_{K\downarrow} = 0$, so that $s_x = \tau_x = 0$. Its coherence between $K \uparrow$ and $K' \downarrow$ instead lies in a mixed spin-valley channel. In the self-consistent four-component state, both $u_{K'\uparrow}$ and $u_{K\downarrow}$ become nonzero, allowing finite s_x

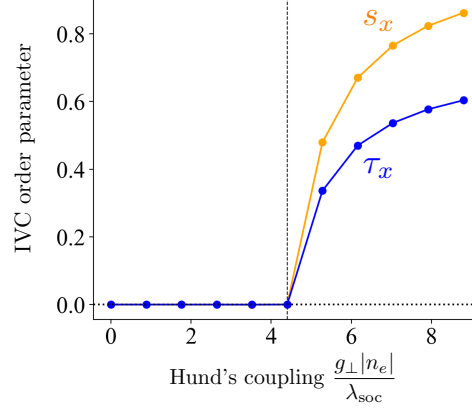


Figure S2. Evolution of the spin and valley order parameters as a function of the dimensionless intervalley Hund's coupling $g_{\perp}|n_e|/\lambda_{\text{soc}}$ at $n_e = -0.11 \times 10^{12} \text{ cm}^{-2}$. The in-plane spin and valley components develop continuously above the critical coupling $g_{\perp}^*|n_e|/\lambda_{\text{soc}} = 4.5$, signaling a second-order transition from the two-component to the four-component IVC quarter-metal phase.

and τ_x , as seen in the insets of Fig. S1(d). Thus, these transverse order parameters diagnose the reconstruction out of the original $\{K \uparrow, K' \downarrow\}$ subspace. Throughout this evolution, the system remains a metallic quarter metal, while its internal spin–valley structure changes continuously.

To characterize the continuous onset of these additional spin–valley components, Fig. S2 shows the evolution of the spin and valley order parameters as a function of the dimensionless Hund's coupling $g_{\perp}|n_e|/\lambda_{\text{soc}}$ at $n_e = -0.11 \times 10^{12} \text{ cm}^{-2}$. Below the critical coupling, the in-plane spin polarization and intervalley coherence vanish, corresponding to the two-component IVC state. At the critical coupling, $g_{\perp}^*|n_e|/\lambda_{\text{soc}} = 4.5$, both order parameters develop continuously from zero, signaling a second-order transition into the four-component IVC phase. The continuous onset of these additional order parameters demonstrates that the competition between the long-range Coulomb interaction and the intervalley Hund's coupling naturally drives a reconstruction of the IVC order.

This result provides the microscopic context for the domain-wall calculations presented in the main text. Since the center of the domain wall locally realizes an intervalley-coherent state, the continuous transition between the two- and four-component IVC phases in the uniform system provides a natural basis for understanding the analogous reconstruction between the two- and four-component domain walls induced by increasing intervalley Hund's coupling.

B. HARTREE–FOCK MEAN–FIELD THEORY OF SPIN–VALLEY DOMAIN WALLS

In this section, we extend the uniform Hartree–Fock theory to describe spin–valley domain walls. We first formulate the one-dimensional magnetic-supercell geometry, then derive the Hartree–Fock self-consistent equations in the folded momentum basis, and finally discuss the numerical convergence and microscopic properties of the resulting domain-wall solutions.

One-dimensional Magnetic Supercell

We consider domain walls separating two time-reversal-related spin–valley polarized quarter-metal (SVQM) ground states. The calculation is performed in a single periodic one-dimensional magnetic supercell of length L . Translational symmetry is preserved along the y direction, while the spin–valley order parameter varies periodically along x . Consequently, each supercell contains two equivalent domain walls. The in-plane coordinate axes are chosen such that $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors parallel and perpendicular, respectively, to the graphene Dirac-point momentum $\mathbf{K}$ measured from the Γ point.

The broken translational symmetry along x couples momentum states differing by reciprocal lattice vectors of the magnetic supercell, $G = \frac{2\pi m}{L}$, where $m \in \mathbb{Z}$. The momentum along the domain wall, k_y , remains a good quantum number, whereas the momentum dependence along x is represented by the discrete reciprocal-vector indices G . The single-particle basis is therefore labeled by (k_y, G) , with the physical momentum entering the continuum band Hamiltonian given by $\mathbf{k}_G = (G, k_y)$.

Mean-Field Hamiltonian in the Magnetic Supercell

In the one-dimensional magnetic-supercell geometry, the single-particle basis states are labeled by (G, α) , where $\alpha = \{s, \tau, l, \sigma\}$ collectively denotes the spin, valley, layer, and sublattice degrees of freedom. The momentum k_y along the domain wall remains conserved, while the momentum along x is represented by the reciprocal supercell index G . In this basis, the mean-field Hamiltonian takes the form given in Eq. ?? of the main text,

$$\hat{H}_{G\alpha, G'\beta}^{\text{MF}}(k_y) = \delta_{GG'} [\hat{H}^{\text{SWMc}}(G\mathbf{e}_x + k_y\mathbf{e}_y) + \hat{H}^{\text{SOC}}]_{\alpha\beta} + \hat{\Sigma}_{G\alpha, G'\beta}^{\text{HF,C}}(k_y) + \hat{\Sigma}_{G\alpha, G'\beta}^{\text{HF,Hund}}(k_y). \quad (\text{S18})$$

The off-diagonal matrix elements with $G \neq G'$ arise from the spatially varying charge and spin–valley order parameters and describe the hybridization between different reciprocal components of the magnetic supercell.

The Coulomb and intervalley Hund self-energies are obtained using the same Hartree–Fock decoupling introduced in Sec. A. In the magnetic supercell, however, the single-particle density matrix also carries reciprocal-vector indices,

$$\rho_{G\alpha, G'\beta}(k_y) = \langle c_{G'\beta}^\dagger(k_y) c_{G\alpha}(k_y) \rangle. \quad (\text{S19})$$

Because translational symmetry is preserved along y , the density matrix is diagonal in k_y but may be off diagonal in G . The Coulomb Hartree and Fock self-energies are

$$\Sigma_{G\alpha, G'\beta}^{\text{H,C}} = \delta_{\alpha\beta} \frac{1}{A} \sum_{k_y', G'', \gamma} V_C[(G - G')\mathbf{e}_x] \rho_{G''\gamma, G''+G-G'\gamma}(k_y'), \quad (\text{S20})$$

$$\Sigma_{G\alpha, G'\beta}^{\text{F,C}}(k_y) = -\frac{1}{A} \sum_{k_y'} \sum_{G_r} V_C[G_r\mathbf{e}_x + (k_y - k_y')\mathbf{e}_y] \rho_{G+G_r, \alpha; G'+G_r, \beta}(k_y'). \quad (\text{S21})$$

Here $G_r = 2\pi m_r/L$ is the reciprocal momentum transferred along the x direction. The Hartree self-energy is independent of k_y because the charge density is uniform along y . The spatially uniform component, corresponding to $G = G'$, is omitted from the mean-field Hamiltonian. In contrast, the finite- $G - G'$ Hartree components describe the electrostatic energy associated with charge redistribution across the domain wall.

Applying the same decoupling to the intervalley Hund interaction gives

$$\begin{aligned} \hat{\Sigma}^{\text{H,Hund}} &= \frac{g_\perp}{A} \sum_{\tau, l, \sigma} \sum_{s, s'} \sum_{G_r} \sum_{k_y, G} c_{\bar{\tau}sl\sigma, G}^\dagger(k_y) c_{\tau sl\sigma, G+G_r}(k_y) \\ &\quad \times \sum_{k_y', G'} \langle c_{\tau s'l\sigma, G'}^\dagger(k_y') c_{\bar{\tau} s'l\sigma, G'-G_r}(k_y') \rangle, \end{aligned} \quad (\text{S22})$$

$$\begin{aligned} \hat{\Sigma}^{\text{F,Hund}} &= -\frac{g_\perp}{A} \sum_{\tau, l, \sigma} \sum_{s, s'} \sum_{G_r} \sum_{k_y, G} c_{\bar{\tau}sl\sigma, G}^\dagger(k_y) c_{\bar{\tau} s'l\sigma, G+G_r}(k_y) \\ &\quad \times \sum_{k_y', G'} \langle c_{\tau s'l\sigma, G'}^\dagger(k_y') c_{\tau sl\sigma, G'-G_r}(k_y') \rangle. \end{aligned} \quad (\text{S23})$$

The Hund Hartree contribution couples spin-preserving intervalley-coherent components, whereas the Hund Fock contribution couples the valley-resolved spin density matrices and favors ferromagnetic spin alignment between opposite valleys. The reciprocal-lattice transfer G_r allows both contributions to vary spatially across the magnetic supercell.

Starting from an initial spin–valley texture satisfying the prescribed domain-wall boundary conditions, we diagonalize Eq. S18, reconstruct the density matrix in Eq. S19, and update the Hartree–Fock self-energies iteratively until self-consistency is reached. The resulting metastable solutions determine the domain-wall profiles and their excess line energies. In the following subsection, we examine the convergence of these quantities with the magnetic-supercell length.

Convergence with Magnetic-Supercell Length

The converged self-consistent solution determines the domain-wall energetics through the Hartree–Fock mean-field energy density,

$$E_{\text{DW}} = \frac{1}{2A} \sum_{k_y} \text{Tr} \left[\hat{\rho}(k_y) \left(\hat{H}^{\text{MF}}(k_y) + \hat{H}^{\text{SWMc}}(k_y) + \hat{H}^{\text{SOC}} \right) \right], \quad (\text{S24})$$

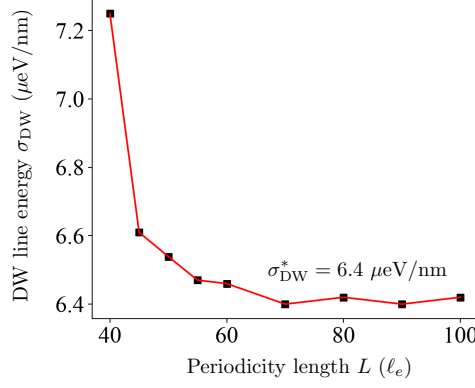


Figure S3. Domain-wall line energy σ_{DW} as a function of the magnetic-supercell length L . The line energy converges to $\sigma_{\text{DW}}^* \simeq 6.4 \mu\text{eV/nm}$ for $L \gtrsim 70 \ell_e$, indicating negligible interaction between the two domain walls contained within the supercell. Throughout this work we use $L = 160 \ell_e$.

where the trace is taken over the reciprocal-vector and internal spin–valley basis indices. The corresponding uniform ground-state energy density is

$$E_0 = \frac{1}{2A} \sum_{\mathbf{k}}^{\text{BZ}} \text{Tr} \left[\hat{\rho}_0(\mathbf{k}) \left(\hat{H}_0^{\text{MF}}(\mathbf{k}) + \hat{H}^{\text{SWMc}}(\mathbf{k}) + \hat{H}^{\text{SOC}} \right) \right], \quad (\text{S25})$$

where $\hat{\rho}_0$ and $\hat{H}_0^{\text{MF}}$ denote the self-consistent density matrix and mean-field Hamiltonian of the corresponding uniform SVQM ground state. The excess domain-wall line energy is then defined as

$$\sigma_{\text{DW}} = \frac{L}{2} (E_{\text{DW}} - E_0), \quad (\text{S26})$$

where the factor of two accounts for the two equivalent domain walls contained within each magnetic supercell. The domain-wall calculations are performed on a discretized one-dimensional Brillouin zone containing $N_{k_y} = 21$ momentum points along the translationally invariant y direction. The reciprocal-space expansion retains N_G reciprocal vectors, corresponding to a momentum cutoff $G_{\text{max}} = 3.4 \ell_e^{-1}$. The resulting momentum discretization corresponds to an effective supercell area of $A = 10^4 \ell_e^2$. Figure S3 shows the convergence of the domain-wall line energy with increasing magnetic-supercell length. As the separation between the two domain walls increases, their mutual interaction becomes negligible and the line energy rapidly approaches a constant value. Throughout this work we use $L = 160 \ell_e$, which lies well within the converged regime.

Quasiparticle Wavefunctions in the Domain Wall

To gain microscopic insight into the domain-wall electronic structure, we examine the quasiparticle wavefunctions obtained from the self-consistent Hartree–Fock Hamiltonian. For a fixed momentum k_y , the n th quasiparticle eigenstate is expanded in the magnetic-supercell basis as

$$|\Psi_n(k_y)\rangle = \sum_{k_x, \alpha} z_{n, k_x, \alpha}(k_y) |k_x, \alpha; k_y\rangle, \quad (\text{S27})$$

where $k_x = 2\pi m/L$ labels the reciprocal momentum of the magnetic supercell and $\alpha = \{s, \tau, l, \sigma\}$ denotes the spin, valley, layer, and sublattice degrees of freedom. We resolve the quasiparticle amplitudes by flavor,

$$|z_{n, \tau s}(k_x, k_y)|^2 = \sum_{l, \sigma} |z_{n, k_x, \tau s l \sigma}(k_y)|^2, \quad (\text{S28})$$

which is the quantity shown in Figs. S4(a) and (b). For this discussion, we consider the 2-component domain wall at $g_{\perp} = 0$.

Figures S4(a) and (b) display the reciprocal-space probability distributions $|z_{n, K\uparrow}(k_x, k_y = 0)|^2$ and $|z_{n, K'\downarrow}(k_x, k_y = 0)|^2$ as functions of k_x and quasiparticle energy $\mathcal{E}_n(k_y = 0) - \mu$. The color scale represents the probability weight of each reciprocal component, while the insets magnify the energy window 5.5–6.0 meV above the Fermi level. The horizontal green and black

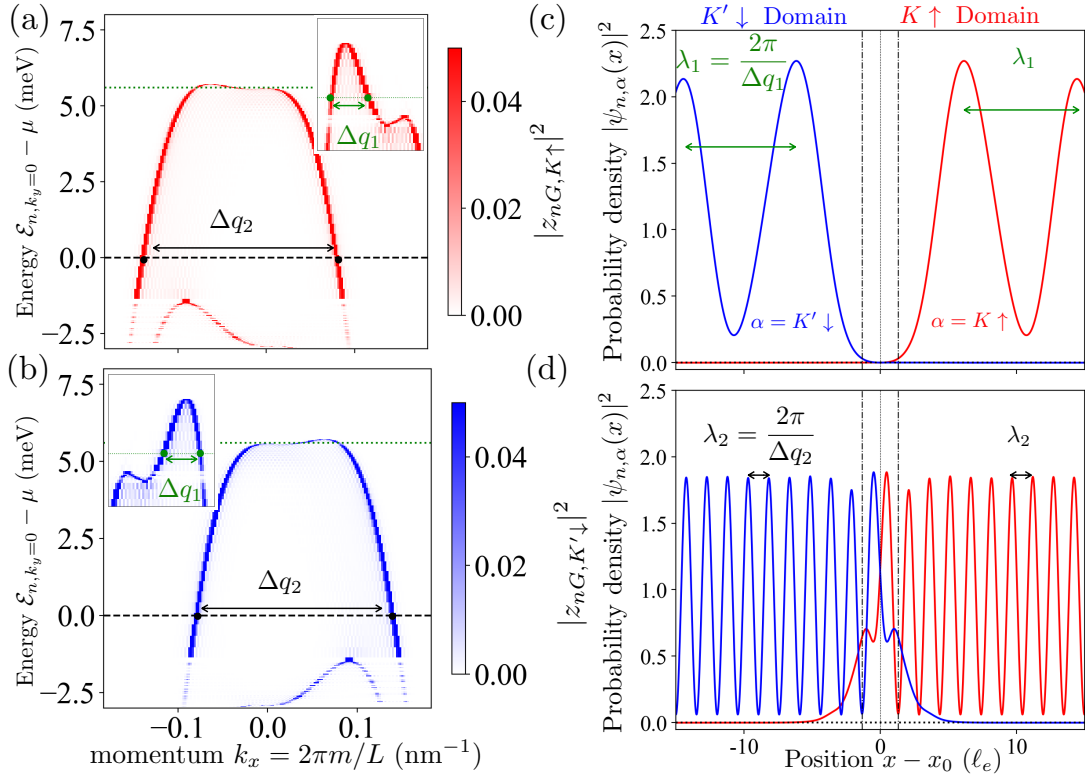


Figure S4. (a),(b) Reciprocal-space probability distributions $|z_{n,K↑}(k_x, k_y = 0)|^2$ and $|z_{n,K'↓}(k_x, k_y = 0)|^2$, respectively, as functions of the discrete momentum $k_x = 2\pi m/L$ and quasiparticle energy $\mathcal{E}_n(k_y = 0) - \mu$. The color scale represents the probability weight of each reciprocal component. Insets magnify the energy window $5.5 \leq \mathcal{E}_n - \mu \leq 6.0$ meV. The horizontal green and black dashed lines indicate the quasiparticle states shown in (c) and (d), respectively. (c),(d) Real-space probability distributions $|\psi_{n,\alpha}(x)|^2$ of the corresponding quasiparticle states at $\mathcal{E}_n(k_y = 0) - \mu = 5.6$ meV and at the Fermi level. The vertical dot-dashed lines denote the domain-wall boundaries. The bulk probability density exhibits interference oscillations with wavelength $\lambda = 2\pi/\Delta q$, where Δq is the momentum separation between the dominant reciprocal components. The oscillation periods satisfy $N = 9$ in (c) and $N = 51$ in (d). While the finite-energy state is suppressed at the domain-wall boundary, the Fermi-level state penetrates into the domain-wall region.

dashed lines indicate the two representative quasiparticle states whose real-space probability distributions are shown in Figs. S4(c) and (d), respectively.

The corresponding real-space probability density is obtained by Fourier transforming the reciprocal components,

$$|\psi_{n,\alpha}(x)|^2 = \left| \sum_{k_x} z_{n,k_x,\alpha} e^{ik_x x} \right|^2. \quad (\text{S29})$$

When two dominant reciprocal components with momenta $k_{x,1}$ and $k_{x,2}$ interfere, the probability density oscillates with wavevector

$$\Delta q = |k_{x,1} - k_{x,2}|, \quad (\text{S30})$$

corresponding to a real-space wavelength

$$\lambda = \frac{2\pi}{\Delta q}. \quad (\text{S31})$$

Because the magnetic supercell contains two equivalent bulk regions, each extending over a distance $L/2$ between neighboring domain walls, the oscillation wavelength satisfies

$$N\lambda = \frac{L}{2}, \quad (\text{S32})$$

where N is the number of oscillation periods between neighboring domain walls. For the representative quasiparticle states shown in Figs. S4(c) and (d), we obtain $N = 9$ at $\mathcal{E}_n(k_y = 0) - \mu = 5.6$ meV and $N = 51$ at the Fermi level, corresponding to a substantially larger momentum separation Δq between the dominant reciprocal components of the latter state.

Figures S4(c) and (d) compare the real-space probability distributions of two representative quasiparticle states. For the state at $\mathcal{E}_n(k_y = 0) - \mu = 5.6$ meV, the probability density oscillates within the corresponding bulk region while remaining strongly suppressed at the domain-wall boundary. In contrast, the quasiparticle state at the Fermi level exhibits appreciable penetration of the $K \uparrow$ and $K' \downarrow$ components into the domain-wall region. This enhanced penetration demonstrates that the domain wall strongly reconstructs the low-energy quasiparticle wavefunctions, whereas higher-energy bulk states remain largely excluded from the domain-wall center.

Domain-Wall Width Estimation

To quantify the spatial extent of the domain wall, we fit the self-consistent order-parameter profiles to the conventional one-dimensional domain-wall form

$$O(x) = O_0 \tanh\left(\frac{x - x_0}{\xi}\right), \quad (\text{S33})$$

where $O(x)$ denotes the corresponding spin or valley polarization, x_0 is the domain-wall center, and ξ is the characteristic width parameter. We define the domain-wall width as

$$w = 2\xi. \quad (\text{S34})$$

For the two-component domain wall, the spin and valley polarizations remain locked throughout the wall, so fitting either $s_z(x)$ or $\tau_z(x)$ yields the same width. In contrast, the four-component domain wall exhibits a partial relaxation of spin–valley locking within the wall due to the enhanced intervalley Hund’s interaction. Consequently, the spin and valley order parameters acquire different characteristic widths, denoted by w_s and w_τ , respectively. We consistently find $w_s > w_\tau$, indicating that the valley polarization varies over a shorter length scale than the spin polarization. This difference reflects the stronger uniaxial anisotropy in the valley sector.

CP³ LONG-WAVELENGTH THEORY

To complement the microscopic Hartree–Fock calculations, we develop a continuum CP³ field theory describing the long-wavelength spin–valley textures of the quarter-metal phase. Similar effective theories have been successfully applied to domain walls in magic-angle twisted bilayer graphene [13] and quantum Hall ferromagnets [14]. The continuum theory provides a transparent description of the domain-wall reconstruction and enables direct comparison with the self-consistent Hartree–Fock results. The exchange stiffness and anisotropy coefficients entering the free-energy functional are determined directly from the microscopic Hartree–Fock calculations, while the intervalley Hund’s coupling is treated as an independent parameter.

A quarter metal is locally described by a normalized four-component spinor $|\psi(x)\rangle$, defined only up to an overall phase. Equivalently, its order parameter is the matrix-valued rank-one projector

$$\hat{P}(x) = |\psi(x)\rangle\langle\psi(x)|, \quad \hat{P}^2 = \hat{P}, \quad \text{Tr } \hat{P} = 1, \quad (\text{S35})$$

whose order-parameter manifold is the complex projective space CP³.

Free-Energy Functional

The domain-wall texture is assumed to be translationally invariant along the wall direction (y), so that the continuum theory reduces to a one-dimensional problem. The corresponding excess free energy per unit length of the domain wall, measured relative to the uniform ground state, is

$$\mathcal{F}[\hat{P}] = \int dx \mathcal{H}[\hat{P}], \quad (\text{S36})$$

where the local free-energy density is measured relative to the uniform ground state, so that $\mathcal{H}[\hat{P}_{\text{gs}}] = 0$:

$$\mathcal{H}[\hat{P}] = |n_e| \left(\frac{\rho_s}{2} \text{Tr} (\partial_x \hat{P})^2 + \mathcal{V}_{\text{ani}}[\hat{P}] \right). \quad (\text{S37})$$

Here ρ_s is the spin–valley stiffness arising from the dominant SU(4)-symmetric Coulomb exchange interaction, while $\mathcal{V}_{\text{ani}}$ contains the weaker symmetry-breaking interactions.

Term	Symmetry	Energy scale
(Dominant) Magnetic ordering energies		
Long-range Coulomb interaction	SU(4)	$e^2/(\epsilon_r \ell_e) \sim 5 \text{ meV}$
Single-particle band Hamiltonian	$\text{SU}(2)_K \times \text{SU}(2)_{K'} \times \text{U}(1)_v$	$\epsilon_F \sim 6 \text{ meV}$
(Weak) Magnetic anisotropy energies		
2-body lattice-scale interaction [8]	$\text{SU}(2)_s \times \text{U}(1)_v$	$g_\perp n_e \sim 0.25 \text{ meV}$
Spin–orbit coupling [2]	$\text{U}(1)_s \times \text{U}(1)_v$	$\lambda_{\text{soc}} \sim 0.1 \text{ meV}$

Table II. Hierarchy of approximate internal symmetries of the rhombohedral graphene Hamiltonian. The leading magnetic energy scales arise from the competition between the long-range Coulomb interaction and the single-particle band Hamiltonian. The weaker terms listed below introduce anisotropies that lift the $\text{SU}(2)_K \times \text{SU}(2)_{K'} \times \text{U}(1)_v$ symmetry. The dominant energy scales are estimated for density–displacement field $n_e - U_D = (-0.9 \times 10^{11} \text{ cm}^{-2}, 25 \text{ meV})$ in bilayer graphene.

The hierarchy of interaction scales summarized in Table II is a universal feature of interacting multilayer graphene systems, both twisted and untwisted, and provides the organizing principle for the continuum theory developed here. The dominant long-range Coulomb interaction establishes the spin–valley ferromagnetic order and determines the exchange stiffness, whereas lattice-scale interactions, intrinsic spin–orbit coupling (SOC), and intervalley Hund’s exchange are parametrically weaker and primarily determine the local anisotropy of the order parameter. Guided by this hierarchy, we retain only the leading symmetry-allowed anisotropy terms,

$$\mathcal{V}_{\text{ani}} = \frac{\mathcal{K}}{2} \left\{ 1 - (\text{Tr}[\hat{P}\hat{\tau}_z])^2 \right\} + \frac{\lambda_{\text{soc}}}{2} \left\{ 1 - \text{Tr}[\hat{P}\hat{s}_z\hat{\tau}_z] \right\} - u_\perp \mathbf{S}_K \cdot \mathbf{S}_{K'}. \quad (\text{S38})$$

Here

$$\mathbf{S}_\tau = \text{Tr} \left(\hat{P}\hat{\mathcal{P}}_\tau \hat{\mathbf{s}} \right) \quad (\text{S39})$$

denotes the spin polarization in valley τ , where $\hat{\mathcal{P}}_\tau = (1 + \tau\hat{\tau}_z)/2$ is the valley projection operator.

The first term in Eq. (S38) is the energy cost of reducing the Ising valley polarization from its uniform-ground-state value $|\tau_z| = 1$, where $\tau_z \equiv \text{Tr}[\hat{P}\hat{\tau}_z]$. It therefore favors an Ising valley ferromagnet. The second term is the intrinsic Kane–Mele SOC, which locks the spin and valley quantization axes through the Ising coupling $s_z\tau_z$. The last term describes the intervalley Hund’s interaction, favoring parallel alignment of the valley-resolved spin polarizations.

The continuum coefficients ($\rho_s, \mathcal{K}, \lambda_{\text{soc}}$) are determined directly from constrained microscopic Hartree–Fock calculations, providing a quantitative connection between the microscopic and continuum descriptions. The following subsection also establishes the correspondence between the microscopic intervalley Hund coupling $g_\perp$ and the continuum coupling $u_\perp$, thereby completing the parameterization of the CP^3 free-energy functional.

Microscopic Parameterization

Once the form of the CP^3 free-energy functional is established, its parameters are determined from the microscopic Hartree–Fock (HF) theory. Throughout this subsection, E denotes the HF mean-field energy density, in contrast to the continuum free-energy functional $\mathcal{F}$ introduced in the previous subsection. All energy differences are measured relative to the corresponding uniform ground state at the same carrier density. Dividing the energy-density difference by $|n_e|$ gives the energy difference per charge. The exchange stiffness ρ_s and valley anisotropy coefficient $\mathcal{K}$ are determined independently from constrained microscopic calculations, while the continuum intervalley Hund coupling $u_\perp$ is related analytically to the microscopic interaction strength $g_\perp$. Intrinsic spin–orbit coupling strength λ_{soc} taken directly from experiment rather than extracted from the microscopic calculation.

The exchange stiffness originates from the dominant SU(4)-symmetric component of the long-range Coulomb exchange interaction. It is extracted from the energy cost of a long-wavelength spin spiral imposed on the uniform HF ground state. For sufficiently small spiral wave vector q , the increase in the HF energy density is quadratic,

$$\frac{E(q) - E(0)}{|n_e|} = \frac{\rho_s}{2} q^2 + \mathcal{O}(q^4), \quad (\text{S40})$$

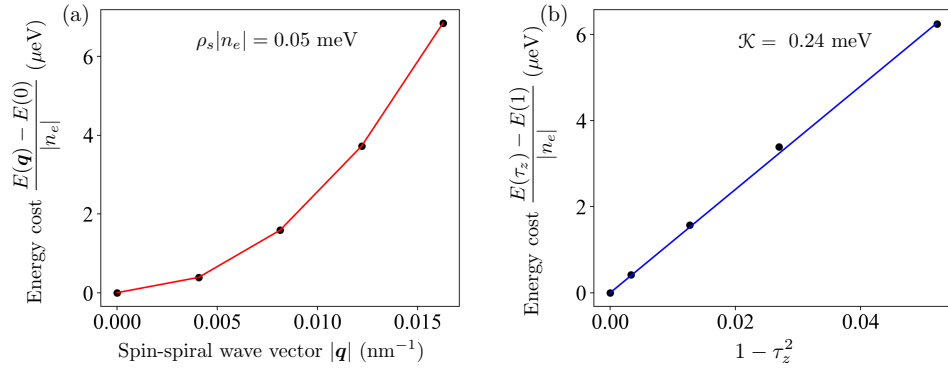


Figure S5. Microscopic extraction of the continuum parameters entering the CP^3 theory. (a) Hartree–Fock energy density of a long-wavelength spin spiral as a function of the spiral wave vector q . The quadratic fit yields the exchange stiffness ρ_s . (b) Uniform Hartree–Fock energy density as a function of $1 - \tau_z^2$, where $\tau_z = \text{Tr}[\hat{P}\hat{\tau}_z]$ is valley polarization. The straight line is a linear fit. According to $\Delta E/|n_e| = (\mathcal{K}/2)(1 - \tau_z^2)$, the fitted slope equals $\mathcal{K}/2$, and hence $\mathcal{K} = 2 \times 0.12 \text{ meV} = 0.24 \text{ meV}$.

from which ρ_s is obtained by fitting the long-wavelength dispersion.

The valley-anisotropy coefficient $\mathcal{K}$ is determined from the energy of spatially uniform Hartree–Fock states with valley polarization $\tau_z \equiv \text{Tr}[\hat{P}\hat{\tau}_z]$, while keeping the carrier density fixed. We subtract the energy of the fully valley-polarized uniform ground state, $\tau_z = 1$, and plot the resulting energy difference per charge against $1 - \tau_z^2$. The anisotropy energy then gives

$$\frac{E(\tau_z) - E(\tau_z = 1)}{|n_e|} = \frac{\mathcal{K}}{2} (1 - \tau_z^2). \quad (\text{S41})$$

The microscopic calculations used to extract ρ_s and $\mathcal{K}$ are summarized in Fig. S5. Panel (a) shows the quadratic dependence of the HF energy density on the spin-spiral wave vector, yielding the exchange stiffness $\rho_s = 0.05 \text{ meV}/|n_e| \sim 50 \text{ meVnm}^2$. Panel (b) shows $[E(\tau_z) - E(\tau_z = 1)]/|n_e|$ as a function of $1 - \tau_z^2$. The data are linear, with slope $\mathcal{K}/2$; therefore the valley-anisotropy energy is $\mathcal{K} = 2 \times (\text{slope})$. The fitted slope is $m = 0.12 \text{ meV}$, giving $\mathcal{K} = 0.24 \text{ meV}$.

The intrinsic SOC strength is taken directly from experiment rather than extracted from the microscopic calculation. Throughout this work we use $\lambda_{\text{soc}} = 0.1 \text{ meV}$, consistent with the experimentally estimated value[2] for rhombohedral graphene.

The remaining continuum parameter, the intervalley Hund coupling $u_{\perp}$, is obtained by matching the microscopic intervalley interaction energy to the corresponding term in the CP^3 free-energy functional. To establish this correspondence, we evaluate the Hartree and Fock contributions to the microscopic mean-field energy for a uniform state and rewrite the result in terms of the continuum order parameters. Using Eqs. (S11) and (S12), the Hartree and Fock contributions to the mean-field energy density are

$$E^{\text{H,Hund}} = \frac{g_{\perp}}{2A^2} \sum_{\tau,l,\sigma} \sum_{\mathbf{k},s} \langle c_{\bar{\tau}sl\sigma}^{\dagger}(\mathbf{k}) c_{\tau sl\sigma}(\mathbf{k}) \rangle \sum_{\mathbf{k}',s'} \langle c_{\bar{\tau}s'l\sigma}^{\dagger}(\mathbf{k}') c_{\bar{\tau}s'l\sigma}(\mathbf{k}') \rangle, \quad (\text{S42})$$

$$E^{\text{F,Hund}} = -\frac{g_{\perp}}{2A^2} \sum_{\tau,l,\sigma} \sum_{\mathbf{k},s,s'} \langle c_{\bar{\tau}sl\sigma}^{\dagger}(\mathbf{k}) c_{\bar{\tau}s'l\sigma}(\mathbf{k}) \rangle \sum_{\mathbf{k}'} \langle c_{\bar{\tau}s'l\sigma}^{\dagger}(\mathbf{k}') c_{\tau sl\sigma}(\mathbf{k}') \rangle. \quad (\text{S43})$$

For the displacement fields considered in this work ($U_D \gg \epsilon_F$), the low-energy carriers are predominantly polarized onto a single layer–sublattice orbital. We therefore suppress the layer and sublattice indices (l, σ) in the following discussion. We first introduce the valley-coherence components,

$$\tau_x = \frac{1}{N_e} \sum_{\mathbf{k},s,\tau} \langle c_{\bar{\tau}s}^{\dagger}(\mathbf{k}) c_{\tau s}(\mathbf{k}) \rangle, \quad (\text{S44})$$

$$\tau_y = -\frac{i}{N_e} \sum_{\mathbf{k},s,\tau} \tau \langle c_{\bar{\tau}s}^{\dagger}(\mathbf{k}) c_{\tau s}(\mathbf{k}) \rangle, \quad (\text{S45})$$

together with the valley occupation and valley-resolved spin polarization,

$$n_{\tau} = \frac{1}{N_e} \sum_{\mathbf{k},s} \langle c_{\tau s}^{\dagger}(\mathbf{k}) c_{\tau s}(\mathbf{k}) \rangle, \quad (\text{S46})$$

$$S_{\tau} = \frac{1}{N_e} \sum_{\mathbf{k},s,s'} \langle c_{\tau s}^{\dagger}(\mathbf{k}) \hat{s}_{ss'} c_{\tau s'}(\mathbf{k}) \rangle, \quad (\text{S47})$$

where $N_e = |n_e|A$ is the total number of charge carriers. The valley-resolved spin density matrix can then be written as

$$\hat{\rho}_\tau = \frac{N_e}{2} \left(n_\tau \hat{\mathbb{I}} + \mathbf{S}_\tau \cdot \hat{\mathbf{s}} \right). \quad (\text{S48})$$

Here n_τ denotes the fraction of carriers occupying valley τ , satisfying $n_K + n_{K'} = 1$. Because the valley-resolved spin polarization is normalized by the total carrier density, its magnitude satisfies $|\mathbf{S}_\tau| \leq 1$.

Expressed in terms of the valley coherence, the Hartree contribution becomes

$$E^{\text{H,Hund}} = \frac{g_\perp N_e^2}{4A^2} \left(\tau_x^2 + \tau_y^2 \right). \quad (\text{S49})$$

Similarly, the Fock contribution can be written as

$$\begin{aligned} E^{\text{F,Hund}} &= -\frac{g_\perp}{A^2} \text{Tr}_s (\hat{\rho}_K \hat{\rho}_{K'}) \\ &= -\frac{g_\perp N_e^2}{2A^2} (n_K n_{K'} + \mathbf{S}_K \cdot \mathbf{S}_{K'}). \end{aligned} \quad (\text{S50})$$

Using the definition of the valley polarization,

$$\tau_z = n_K - n_{K'}, \quad (\text{S51})$$

the valley occupations satisfy

$$n_K n_{K'} = \frac{1}{4} (1 - \tau_z^2), \quad (\text{S52})$$

so that

$$E^{\text{F,Hund}} = -\frac{g_\perp N_e^2}{2A^2} \left[\frac{1}{4} (1 - \tau_z^2) + \mathbf{S}_K \cdot \mathbf{S}_{K'} \right]. \quad (\text{S53})$$

Combining the Hartree and Fock contributions gives

$$E^{\text{Hund}} = \frac{g_\perp N_e^2}{4A^2} \left[\left(\tau_x^2 + \tau_y^2 \right) - \frac{1}{2} (1 - \tau_z^2) \right] - \frac{g_\perp N_e^2}{2A^2} \mathbf{S}_K \cdot \mathbf{S}_{K'}. \quad (\text{S54})$$

We separate the microscopic Hund contribution into a valley-only part and a spin-exchange part. The valley-only contribution is treated as a small correction to the valley-anisotropy energy obtained from the microscopic Hartree–Fock theory and is included in the anisotropy sector of the continuum model. To determine $u_\perp$, we match only the spin-exchange contribution to the corresponding continuum Hund term:

$$\frac{E_{\text{spin}}^{\text{F,Hund}}}{|n_e|} = -\frac{g_\perp |n_e|}{2} \mathbf{S}_K \cdot \mathbf{S}_{K'}. \quad (\text{S55})$$

Comparison with the corresponding term in Eq. (S38) then gives

$$u_\perp = \frac{g_\perp |n_e|}{2}. \quad (\text{S56})$$

Together with the exchange stiffness ρ_s , the valley-anisotropy coefficient $\mathcal{K}$, and the intrinsic SOC strength λ_{soc} obtained from the microscopic Hartree–Fock theory, this coefficient matching specifies the CP^3 energy functional used in this work. The resulting parameters are summarized in Table III.

In the following subsection, we employ this parameterized continuum theory to investigate the reconstruction of magnetic domain walls as a function of the intervalley Hund coupling $u_\perp$.

Parameter	$\rho_s n_e $	$\mathcal{K}$	λ_{soc}	$u_\perp$
Value (meV)	0.05	0.24	0.10	0–0.3

Table III. Parameters entering the CP^3 continuum theory. The exchange stiffness ρ_s and valley anisotropy coefficient $\mathcal{K}$ are extracted from microscopic HF calculations at $n-U_D = (-0.9 \times 10^{11} \text{ cm}^{-2}, 25 \text{ meV})$ in rBG. The SOC strength is taken from experiment[2], while the intervalley Hund coupling $u_\perp$ is treated as the tuning parameter.

Domain-Wall Reconstruction in the CP^3 Theory

We now minimize CP^3 free-energy functional (Eq. (S37)) subject to the same $K \uparrow$ and $K' \downarrow$ boundary conditions imposed in the microscopic Hartree–Fock calculations. Since this functional is invariant under $U(1)_s \times U(1)_v$ rotations about the spin and valley z axes, we choose a symmetry-equivalent representative with $s_y(x) = \tau_y(x) = 0$, so that both polarizations lie in their respective x – z planes. The normalized spinor is initialized as a smooth interpolation between the two bulk spin–valley polarized quarter-metal states and is iteratively relaxed until the free energy converges. Figure S6 summarizes the resulting continuum domain-wall solutions.

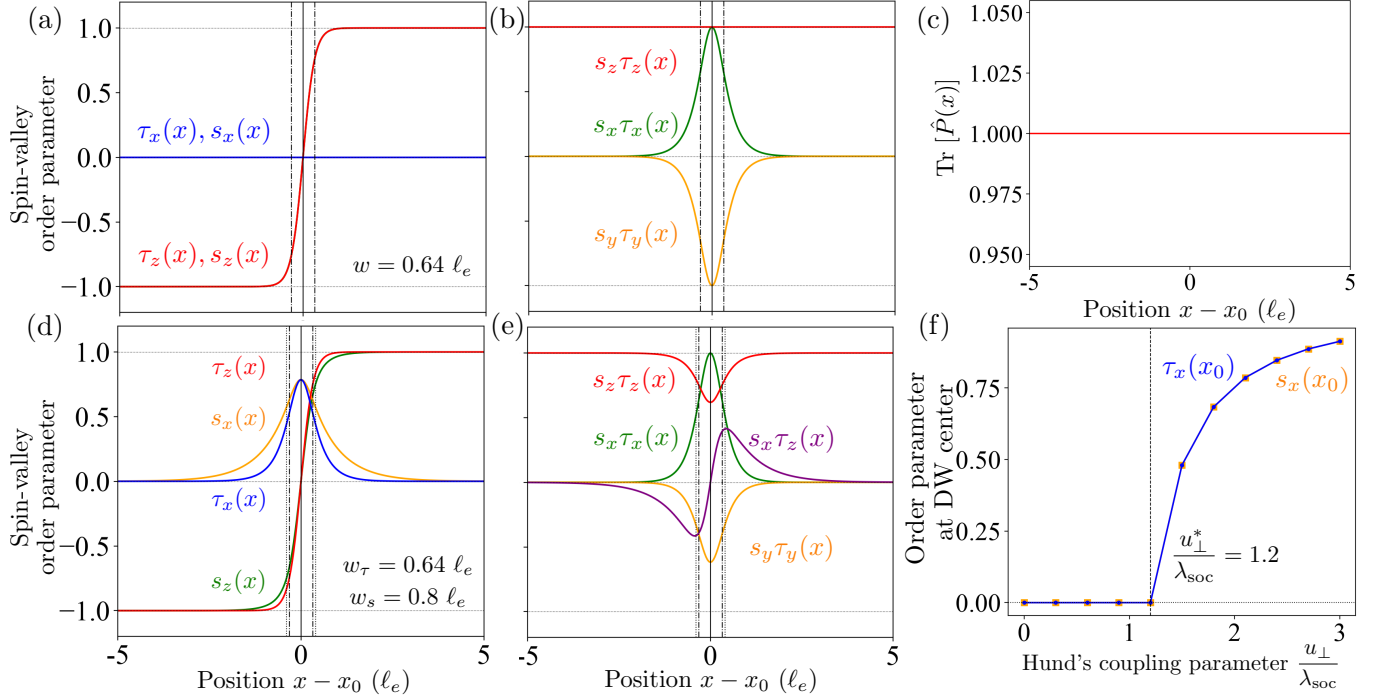


Figure S6. CP^3 continuum domain-wall solutions obtained by minimizing the parameterized free-energy functional. (a) Spatial profiles of the spin and valley order parameters for the two-component domain wall at $u_\perp = 0$. The spin and valley remain locked, satisfying $s_z(x) = \tau_z(x)$ and $s_x(x) = \tau_x(x) = 0$, resulting in a single domain-wall width w . (b) Corresponding composite spin–valley order parameters. The Ising order parameter $s_z \tau_z(x)$ remains 1 throughout the texture, while the transverse components vanish identically. (c) Normalization of the CP^3 spin–valley projector, $\text{Tr}[\hat{P}(x)]$. The continuum theory retains only the normalized internal spin–valley state and assumes a spatially uniform carrier density; charge redistribution such as that found microscopically in ??(c) is therefore excluded. (d) Spatial profiles of the four-component domain wall for $u_\perp / \lambda_{\text{soc}} = 2.1$. Finite transverse spin polarization and intervalley coherence develop inside the wall, leading to distinct spin and valley domain-wall widths, $w_s > w_\tau$. (e) Corresponding composite spin–valley order parameters for the four-component domain wall. (f) Evolution of the domain-wall-center order parameters $s_x(x_0)$ and $\tau_x(x_0)$ as functions of the dimensionless Hund coupling $u_\perp / \lambda_{\text{soc}}$. Above the critical coupling $u_\perp^* / \lambda_{\text{soc}} \approx 1.2$, the transverse order parameters develop continuously from zero, signaling a second-order transition from the two-component to the four-component domain wall. Using the relation $u_\perp = g_\perp |n_e| / 2$, the critical coupling corresponds to $g_{\perp, CP^3}^* |n_e| / \lambda_{\text{soc}} = 2.4$.

For weak intervalley Hund coupling, the minimizing spinor remains confined to the $\{K \uparrow, K' \downarrow\}$ subspace, giving rise to the two-component domain wall shown in Figs. S6(a)–(c). The spin and valley order parameters remain locked throughout the texture, satisfying $s_z(x) = \tau_z(x)$ and $s_x(x) = \tau_x(x) = 0$. Consequently, the Ising order parameter $s_z \tau_z(x)$ remains unity across the entire domain wall, while the intervalley coherence vanishes identically. As expected within the continuum theory, the local charge density remains uniform throughout the texture, as shown in Fig. S6(c). The spin and valley profiles are therefore characterized by the same domain-wall width.

Upon increasing the intervalley Hund coupling, the energy gained by aligning the valley-resolved spin polarizations competes with the spin–valley locking induced by intrinsic SOC. Above the critical coupling $u_\perp^* / \lambda_{\text{soc}} \approx 1.2$, the energy minimum continuously expands into the full CP^3 manifold, producing the four-component domain wall shown in Figs. S6(d) and (e). In this regime, finite transverse spin polarization $s_x(x)$ and intervalley coherence $\tau_x(x)$ develop within the domain wall, accompanied by the composite order parameters $s_x \tau_x(x)$, $s_y \tau_y(x)$, and $s_x \tau_z(x)$. The spin and valley order parameters are no longer locked, resulting in distinct domain-wall widths, with $w_s > w_\tau$.

The continuous nature of this reconstruction is illustrated in Fig. S6(f), which shows the evolution of the domain-wall-center order parameter $s_x(x_0)$ and $\tau_x(x_0)$ as a function of the dimensionless Hund coupling $u_\perp/\lambda_{\text{soc}}$. Below the critical coupling, the transverse order parameter vanishes and the stable solution is the two-component domain wall. Above $u_\perp^*/\lambda_{\text{soc}}$, $s_x(x_0)$ and $\tau_x(x_0)$ develop continuously from zero, signaling a second-order transition into the four-component domain wall. Overall, the parameterized CP^3 theory successfully reproduces the microscopic Hartree–Fock results, including the two distinct classes of domain walls, the spin–valley unlocking inside the four-component domain wall ($w_s > w_\tau$), and the continuous Hund-driven transition between them. These results demonstrate that the domain-wall reconstruction is a generic consequence of the competition between spin–orbit coupling and intervalley Hund exchange, rather than a feature specific to the microscopic details of rhombohedral graphene.

Although the parameterized CP^3 theory successfully captures the qualitative domain-wall reconstruction obtained in the microscopic Hartree–Fock calculations, it is not expected to reproduce every quantitative feature of the microscopic solution. The continuum theory retains only the local spin–valley order parameter through a rank-one projector satisfying $\hat{P}^2 = \hat{P}$, while assuming a uniform local carrier density. By contrast, the microscopic Hartree–Fock theory explicitly incorporates the layer, sublattice, and momentum degrees of freedom. After tracing over the layer–sublattice sector, the resulting local spin–valley density matrix becomes mixed near the domain-wall center, as quantified by the finite value of $\text{Tr}(\hat{P}_{\text{HF}} - \hat{P}_{\text{HF}}^2)$ shown in Fig. ?? of the main text. The self-consistent Hartree–Fock solution also exhibits a weak redistribution of the local charge density [Fig. ??(c) of the main text], which contributes a finite Hartree energy to the domain-wall line energy. The CP^3 theory instead assumes a frozen local carrier density and therefore contains no corresponding Hartree contribution. These additional microscopic degrees of freedom lead to quantitative differences between the two descriptions. In particular, the domain-wall widths predicted by the continuum theory differ from the microscopic Hartree–Fock results by approximately a factor of two. Furthermore, using the microscopic relation $u_\perp = g_\perp|n_e|/2$, the continuum critical coupling $u_\perp^*/\lambda_{\text{soc}} \approx 1.2$ corresponds to

$$\frac{g_{\perp,CP^3}^*|n_e|}{\lambda_{\text{soc}}} \approx 2.4, \quad (\text{S57})$$

whereas the microscopic Hartree–Fock calculation gives $g_{\perp,\text{HF}}^*|n_e|/\lambda_{\text{soc}} \approx 11$. Thus, the continuum estimate of the critical microscopic coupling is smaller by a factor of approximately $11/2.4 \approx 4.6$. Moreover, the rank-one CP^3 pure-state constraint, together with the symmetries of the wall-center solution, yields equal transverse spin polarization and intervalley coherence, $|s_x(x_0)| = |\tau_x(x_0)|$. The microscopic Hartree–Fock solution generally yields different magnitudes for these order parameters because its local reduced spin–valley density matrix is no longer pure.

Nevertheless, the two approaches are in remarkable qualitative agreement. Both predict two distinct classes of spin–valley domain walls connected by a continuous Hund-driven transition, together with the characteristic spin–valley unlocking inside the four-component domain wall. This agreement demonstrates that the reconstruction is governed primarily by the competition between the exchange stiffness, spin–orbit-induced spin–valley locking, and intervalley Hund exchange, rather than by microscopic details of the electronic band structure. The resulting CP^3 theory therefore identifies the minimal collective ingredients required to describe interaction-driven spin–valley domain-wall reconstruction in graphene ferromagnets.

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