

A Two-Stage Finite Element Approach for High-precision Guaranteed Lower Eigenvalue Bounds

Xuefeng LIU

`xfliu@lab.twcu.ac.jp`

Tokyo Woman's Christian University

Michael Plum

Karlsruhe Institute of Technology

Research Article

Keywords: Eigenvalue bound, Crouzeix–Raviart finite element method, Guaranteed computation, Laplacian eigenvalue problem, Steklov eigenvalue problem, Lehmann–Goerisch’s theorem

Posted Date: October 21st, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11155910/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

A Two-Stage Finite Element Approach for High-precision Guaranteed Lower Eigenvalue Bounds

Xuefeng Liu^{1*} and Michael Plum²

^{1*}School of Arts and Sciences, Tokyo Woman's Christian University,
2-6-1 Zenpukuji, Suginami-ku, Tokyo, 167-8585, Japan.

²Institute for Analysis, Karlsruhe Institute of Technology, Englerstraße
2, Karlsruhe, 76131, Germany.

*Corresponding author(s). E-mail(s): xfliu@lab.twcu.ac.jp;
Contributing authors: michael.plum@kit.edu;

Abstract

Obtaining high-precision guaranteed lower eigenvalue bounds remains difficult, even though the standard high-order conforming finite element (FEM) easily yields extremely sharp upper bounds. Recently developed rigorous approaches using such as Crouzeix–Raviart or linear conforming elements do not extend well to high-order FEM. Some non-standard FEM approaches can provide sharp eigenvalue bounds but are technically involved. This persistent gap between accurate upper bounds and equally sharp rigorous lower bounds via standard high-order conforming FEMs makes the problem technically demanding and highly competitive. In this paper, we propose a new two-stage rigorous algorithm that closes this gap by employing high-order FEM on graded meshes and producing *rigorous lower eigenvalue bounds as sharp as the corresponding high-order upper bounds*, as demonstrated in our numerical examples. Numerical experiments for the Laplacian and Steklov eigenvalue problems on square, dumbbell and L-shaped domains in 2D, and for the Dirichlet Laplacian on a regular tetrahedron in 3D, show the accuracy and efficiency of the method, particularly on graded or highly nonuniform meshes. In the three-dimensional regular-tetrahedron test, a direct application of the Crouzeix–Raviart bounds is predicted to require approximately 10^6 times more degrees of freedom than our Lehmann–Goerisch approach to achieve bounds of comparable quality. These results confirm that the proposed approach provides a practical and competitive solution to the long-standing difficulty of obtaining sharp, reliable lower eigenvalue bounds.

Keywords: Eigenvalue bound, Crouzeix–Raviart finite element method, Guaranteed computation, Laplacian eigenvalue problem, Steklov eigenvalue problem, Lehmann–Goerisch’s theorem

MSC Classification: 65N30 , 65N25

1 Introduction

Obtaining high-precision guaranteed lower eigenvalue bounds remains difficult, even though standard high-order conforming finite element methods (FEM) easily yield extremely sharp upper bounds. To take advantage of high-order FEMs, Carstensen et al. have developed several non-standard FEM approaches to obtain high-precision lower bounds [1–3].

In this paper, we instead apply the classical Lehmann–Goerisch theorem in combination with standard high-order FEM to obtain lower eigenvalue bounds of quality comparable to the well-known Rayleigh–Ritz upper bounds. Generally, the Lehmann–Goerisch theorem by itself is not straightforward to use because it requires substantial spectral pre-information, namely a rough lower bound for some higher eigenvalue. To address this difficulty, a homotopy method was proposed by the second author (see [4, 5] for the initial ideas, and further developments in [6, Chapter 10]), and applied to many examples, but this approach may involve substantial additional processes and numerical effort. In contrast, we introduce a much simpler approach: the projection-based lower eigenvalue bounds obtained via, e.g., the Crouzeix–Raviart FEM are used to provide the necessary spectral pre-information for the Lehmann–Goerisch theorem, thereby avoiding the homotopy method; consequently, the use of standard high-order FEM within the Lehmann–Goerisch framework produces high-precision, fully guaranteed lower eigenvalue bounds.

Eigenvalue problems are among the most fundamental subjects in numerical analysis, with a long history of methods for upper and lower bound estimation; see [7] for foundational theory and [8] for a survey of FEM-based eigenvalue techniques. Below, we review the literature on rigorously validated eigenvalue bounds that do not rely on additional assumptions and place our method in the broader context.

The homotopy method developed by the second author [5] sets up a family of eigenvalue problems that connects the target eigenvalue problem to a base problem with known spectral bounds, and then rigorously estimates the variation of the eigenvalues in this homotopy process. This rather general method is applicable to self-adjoint eigenvalue problems also with essential spectrum, and hence, particularly to partial differential equation eigenvalue problems also on unbounded domains. To apply the homotopy method, one needs to construct a base problem, which is often available through theoretical techniques. The homotopy method takes advantage of the Lehmann–Goerisch theorem along with rough eigenvalue bounds obtained from the respective previous homotopy step to produce sharp eigenvalue bounds (see early papers [4, 5, 9–13] and a systematic introduction in [6, Chapter 10]).

The projection-based method is another useful approach to provide explicit eigenvalue bounds when the projection error can be explicitly estimated [14–21]; see also the survey in [22]. In [14], Birkhoff et al. obtained eigenvalue bounds for smooth Sturm–Liouville systems by using projection to piecewise-cubic polynomials. In [23] and [17], the idea of Birkhoff was extended to propose guaranteed two-sided bounds for the Laplacian eigenvalue problem by using the Galerkin projection associated to conforming finite element methods. The proposed eigenvalue bounds can naturally handle problems defined over bounded polygonal domains of arbitrary shape. The projection-error based method is further considered for general self-adjoint differential operators with compact resolvent [18] and eigenvalue problems formulated by positive semidefinite bilinear forms [24, 25], where essential spectrum may appear. Meanwhile, the nonconforming finite element method is utilized to provide easy-to-obtain estimation of the projection error. Under efforts from different research groups, the projection-based method proposed has been successfully applied to various eigenvalue problems, including the Laplacian [18, 21], the Biharmonic operator [20, 26, 27], the Stokes operator [28, 29], the Steklov operator [24, 30] and the Maxwell operator [31]. It is worth pointing out that, since the projection-based method cannot take advantage of high-order finite element methods or graded meshes, it is costly to apply this kind of method with mesh refinement for the purpose of high-precision eigenvalue bounds. Such a defect urges us to turn to the Lehmann–Goerisch theorem for sharper eigenvalue bounds. Note that the above methods, together with the Rayleigh–Ritz method providing upper eigenvalue bounds, give two-sided eigenvalue bounds with the information about the indices of the enclosed eigenvalues. Hence, eigenvalue-excluding ranges are automatically obtained.

Another interesting approach is to verify existence and non-existence ranges of eigenvalues in the case that no eigenvalue index information is available, for example, for non-self-adjoint problems. Nakao et al. [32], Lahmann et al. [33] and Watanabe et al. [34–36] developed methods that provide eigenvalue bounds by identifying ranges where eigenvalues exist and cannot exist. The eigenvalue problem is transformed into an investigation of solution existence and uniqueness for certain nonlinear differential equations, for which techniques as described in [6, 37, 38] can be used. In Nagatou et al. [39], a Birman–Schwinger approach is used for identifying eigenvalue exclusion ranges. See also the general description in [6, Section 9.3].

In the field of FEMs, determining two-sided eigenvalue bounds for differential operators is also a significant and difficult task. For special domains with well-constructed meshes, it has been proved that approximate eigenvalues obtained by the mass-lumping method provide lower eigenvalue bounds directly [40]. Many nonconforming finite element methods (FEMs) also yield asymptotic lower bounds for eigenvalues when the eigenfunctions exhibit singularities. That is, as the mesh size tends to zero, the computed eigenvalues converge to the exact values from below. See the early work [41] and the surveys in [42, 43]. However, it is still not clear how to verify the required precondition, i.e., the mesh size being small enough, to ensure the asymptotic lower bounds. Recently, Carstensen et al. developed non-standard FEM approaches to provide direct eigenvalue bounds that can take advantage of non-uniform (graded) meshes [1–3]. The utilization of such approaches leads to a significant increase in degrees of

freedom and complexity of the finite element spaces as compared to standard finite element methods.

Contributions of the paper

This paper proposes a new two-stage algorithm to provide high-precision guaranteed eigenvalue bounds for differential operators. Different from the projection-based eigenvalue bounds, the utilization of Lehmann–Goerisch’s theorem in the proposed algorithm takes advantage of high-order FEMs and graded meshes. The numerical results include two-dimensional Laplacian and Steklov examples and a three-dimensional Dirichlet Laplacian computation on the regular tetrahedron. In the latter case, a degree-8 conforming FEM combined with the Lehmann–Goerisch method gives sharp certified intervals on a small 8-tetrahedron mesh, and the predicted direct Crouzeix–Raviart cost for matching the same precision is up to about 2.3×10^6 times larger.

In the first stage of the proposed algorithm, in order to deal with the eigenvalue problem formulated by positive semidefinite bilinear forms, e.g., the Steklov eigenvalue problem, a new concise proof over more general function space settings is provided for the lower eigenvalue bounds proposed in [24, 25] (see also [15, 17, 19, 21]):

$$\lambda_k \geq \frac{\lambda_{k,h}}{1 + C_h^2 \lambda_{k,h}}. \quad (1.1)$$

Here, $\lambda_{k,h}$ is the exact k th eigenvalue of the discretized matrix eigenvalue problem obtained from the finite element method (see §3.2 for details), and C_h is a quantity that can be evaluated explicitly. Here, the nonconforming FEM is helpful, since the quantity C_h can then easily be determined by the error constant for the local interpolation operator associated to the FEM space. The new proof simplifies the one of [24], which considers a complicated space decomposition with respect to the kernel space of the operator. Another feature of the proof is that the assumption of positive definiteness of $a(\cdot, \cdot)$ on $V(h)$ (see definition in §3.1) is not needed any more; see Remark 3.2.

In the second stage, based on the rough eigenvalue bound (1.1) obtained from the first stage with low computational intensity, high-precision eigenvalue bounds are provided by utilizing the Lehmann–Goerisch theorem along with high-order conforming FEMs. The Lehmann–Goerisch theorem is a general theoretical framework that provides sharp and optimal eigenvalue bounds, given rough *a priori* eigenvalue bounds and high-quality approximate eigenfunctions. Sharp eigenvalue bounds derived from the Lehmann–Goerisch theorem in combination with FEM have been studied in previous works [44–46]. The application of the Lehmann–Goerisch theorem to the Laplacian and the Steklov eigenvalue problems are discussed in Sections 4 and 5.

Advantages of proposed method

We summarize the advantages of the proposed two-stage algorithm for the purpose of high-precision eigenvalue bounds.

- (a) The utilization of high-order FEM spaces leads to high-precision eigenvalue bounds even for coarse meshes. For example, for the unit square domain, by using the second-order Lagrange FEM over an 8×8 mesh, we *prove* by our two-stage algorithm that the first Steklov eigenvalue is bounded as follows:

$$0.240079083 \leq \lambda_1 \leq 0.240079091 .$$

In our numerical examples in §5.1, we consider a problem with nearby eigenvalues, where the high-precision bounds based on the Lehmann–Goerisch theorem are accurate enough to separate the enclosing intervals and thus prove strict inequality between these eigenvalues, while the rough projection based eigenvalue bounds computed with the same domain triangulation are far too inaccurate for such a separation.

- (b) The Lehmann–Goerisch theorem can take advantage of graded meshes, which is not possible in the projection-based lower eigenvalue bounds in (1.1).
- (c) According to numerical evidence, the utilization of Lehmann–Goerisch’s theorem provides lower eigenvalue bounds of the optimal convergence rate for the eigenvalues of interest, compared with the possible sub-optimal bounds from the projection-error based method. For example, the explicit lower eigenvalue bound (1.1) when applied to the Steklov eigenvalue problem has a convergence rate $O(h)$ since $C_h = O(\sqrt{h})$ (h : the mesh size) even for convex domains, which is suboptimal compared with the convergence rate of $\lambda_{k,h}$ itself; see details in Remark 3.4. On the contrary, the eigenvalue bound using the linear Lagrange FEM in the second stage of the proposed algorithm has the convergence rate $O(h^2)$.

The rest of the paper is organized as follows. In §2, we introduce the abstractly formulated eigenvalue problems and their basic properties. In §3, as the first stage of the proposed algorithm, the projection-based eigenvalue bounds are introduced along with a new and concise proof. There are details on the application of this eigenvalue bounds to the Laplacian and the Steklov eigenvalue problems. In §4, as the second stage, high-precision eigenvalue bounds based on the Lehmann–Goerisch theorem are discussed. In §5, computational results are presented to demonstrate the efficiency of our proposed algorithm for bounding eigenvalues. Finally, in §6, we summarize the results of this paper.

2 Variational eigenvalue problem

First, let us formulate the assumptions for the target eigenvalue problem to be discussed.

Let V and W be separable Hilbert spaces endowed with the norms $\|\cdot\|_a$ and $\|\cdot\|_b$ induced by the inner products $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, respectively. Let $\gamma : V \rightarrow W$ be a compact linear operator and consider the following eigenvalue problem.

Find $u \in V \setminus \{0\}$ and $\lambda \in \mathbb{R}$ such that

$$a(u, v) = \lambda b(\gamma u, \gamma v) \quad \forall v \in V. \quad (2.1)$$

To study this eigenvalue problem, let us introduce the solution operator $G : W \rightarrow V$ such that for any $f \in W$, $Gf \in V$ uniquely solves the following equation:

$$a(Gf, v) = b(f, \gamma v) \quad \forall v \in V.$$

It is easy to see that G is bounded and hence $\mathcal{K} := G\gamma : V \rightarrow V$ is a compact self-adjoint positive semidefinite operator. The eigenvalue problem for the operator $\mathcal{K}$ is to find eigenfunctions $u_k \in V \setminus \{0\}$ and corresponding eigenvalues $\mu_k \geq 0$ such that

$$b(\gamma u_k, \gamma v) = \mu_k a(u_k, v) \quad \forall v \in V. \quad (2.2)$$

Remark 2.1 We now illustrate the above framework with two concrete eigenvalue problems. For the Laplacian eigenvalue problem on a bounded domain $\Omega \subset \mathbb{R}^k$, we take the following setting.

$$\begin{cases} V = H_0^1(\Omega), & W = L^2(\Omega); \\ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, & b(u, v) = \int_{\Omega} uv \, dx; \\ \gamma : \text{embedding operator.} \end{cases} \quad (2.3)$$

For the Steklov eigenvalue problem on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^k$, we use the following setting.

$$\begin{cases} V = H^1(\Omega), & W = L^2(\partial\Omega); \\ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + uv \, dx, & b(u, v) = \int_{\partial\Omega} uv \, ds; \\ \gamma : \text{trace operator.} \end{cases} \quad (2.4)$$

Spectrum of $\mathcal{K}$

Let us summarize standard results based on the spectral theory of compact symmetric operators. In the case where $\text{Ker}(\mathcal{K}) = \{0\}$, the eigenvalues of $\mathcal{K}$ are given by a finite (when $\dim(V) < \infty$) or countable (when $\dim(V) = \infty$) sequence of nonzero μ_k 's such that

$$\mu_k \geq \mu_{k+1} (> 0) \text{ for } k \geq 1; \quad \lim_{k \rightarrow \infty} \mu_k = 0 \text{ if } \dim(V) = \infty.$$

Note that these eigenvalues are repeated according to their multiplicities. In the case where $\dim(\text{Ker}(\mathcal{K})) > 0$, $b(\gamma \cdot, \gamma \cdot)$ is only positive semidefinite on V , and in addition to the nonzero μ_k 's, 0 is also an eigenvalue of $\mathcal{K}$, possibly of infinite multiplicity.

Space decomposition of V

Note that $\text{Ker}(\mathcal{K}) (\subset V)$ can be characterized by ¹

$$\text{Ker}(\mathcal{K}) = \{u \in V \mid b(\gamma u, \gamma u) = 0\} = \{u \in V \mid \gamma u = 0\}.$$

¹Note that for u such that $b(\gamma u, \gamma u) = 0$, we have $\gamma u = 0$, since $b(\cdot, \cdot)$ is the inner product on W . Hence, $\mathcal{K}u = G\gamma u = 0$. On the other side, for u such that $\mathcal{K}u = 0$, we have $b(\gamma u, \gamma v) = a(\mathcal{K}u, v) = 0 \, \forall v \in V$, and hence $b(\gamma u, \gamma u) = 0$.

Then V admits an a -orthogonal decomposition: $V = \text{Ker}(\mathcal{K}) \oplus \text{Ker}(\mathcal{K})^\perp$.

We introduce the Rayleigh quotient $R(\cdot)$ on V : for $v \in V$, $v \neq 0$,

$$R(v) := \frac{b(\gamma v, \gamma v)}{a(v, v)}. \quad (2.5)$$

The eigenvalues μ_k are the stationary values of R on V . The following principles play an important role in studying eigenvalue problems (see, e.g., [32, 47, 48]).

$$\mu_k = \max_{H_k \subset V} \min_{v \in H_k \setminus \{0\}} R(v) = \min_{S_{k-1} \subset V} \max_{v \perp S_{k-1}, v \neq 0} R(v). \quad (2.6)$$

Here, H_k denotes a k -dimensional subspace of V , and S_k denotes a subspace of V with $\dim(S_k) \leq k$.

Let $\{u_1, u_2, \dots\}$ be an a -orthonormal system of eigenfunctions associated to the μ_k 's. Define $E_k := \text{span}\{u_1, \dots, u_k\}$. Then the eigenvalue μ_k is characterized by

$$\mu_k = \min_{v \in E_k \setminus \{0\}} R(v) = \max_{v \in V \setminus \{0\}, v \perp E_{k-1}} R(v), \quad (2.7)$$

3 First stage: projection-based explicit eigenvalue bounds

As the first stage of our proposed algorithm, we present the projection-based method that produces easy-to-implement lower bounds for the target eigenvalues. Such a method has been well studied in a series of papers by the first author [17, 18, 25]. See also [14, 15, 19, 21].

Let us recall the target eigenvalue problem: Find $u \in V$ and $\mu \geq 0$ such that

$$b(\gamma u, \gamma v) = \mu a(u, v) \quad \forall v \in V. \quad (3.1)$$

Eigenvalue problems for linear partial differential operators are usually formulated for the reciprocals of the nonzero eigenvalues μ_k . Denote $\lambda_k = 1/\mu_k$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{and} \quad a(u_k, v) = \lambda_k b(\gamma u_k, \gamma v) \quad \forall v \in V. \quad (3.2)$$

3.1 Discretized eigenvalue problems and lower eigenvalue bounds

Let V_h be a finite-dimensional space for the purpose of approximating V ; $\dim(V_h) = d(< \infty)$. It is assumed that V_h and V inherit the same addition and scalar multiplication from a certain larger vector space so that $V(h) := V + V_h$ is properly defined. Let $\hat{a}(\cdot, \cdot)$ be a positive semidefinite bilinear form on $V(h)$ and whose restriction to V_h is an inner product on V_h . It is supposed that $\hat{a}(\cdot, \cdot)$ is an extension of $a(\cdot, \cdot)$ to $V(h)$ in the sense that

$$\hat{a}(u, v) = a(u, v) \quad \text{for any } u, v \in V.$$

Note that $\widehat{a}(\cdot, \cdot)$ is not necessarily positive definite on $V(h)$. An example of $\widehat{a}(\cdot, \cdot)$ being not positive definite on $V(h)$ is as follows ²: $V = \{(x, 0) | x \in \mathbb{R}\}$, $V_h = \{(0, y) | y \in \mathbb{R}\}$, $V(h) = V + V_h = \mathbb{R}^2$, $\widehat{a}(u, v) = u^\top \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} v$ for $u, v \in V(h)$. The seminorm or norm induced by $\widehat{a}$ is still denoted by $\|\cdot\|_a$.

Assume that the operator γ is extended to a compact linear operator $\widehat{\gamma} : V(h) \rightarrow W$, i.e., $\widehat{\gamma}u = \gamma u$ for $u \in V$. In the rest of the paper, we henceforth write $a(\cdot, \cdot)$ and γ for $\widehat{a}(\cdot, \cdot)$ and $\widehat{\gamma}$ when no confusion can arise.

Discretized eigenvalue problem

The approximation of the eigenvalue problem (3.1) on V_h is given by: Find $\mu_h \geq 0$ and $u_h \in V_h \setminus \{0\}$, such that

$$b(\gamma u_h, \gamma v_h) = \mu_h a(u_h, v_h) \quad \forall v_h \in V_h. \quad (3.3)$$

The approximate eigenvalues are denoted as

$$\mu_{1,h} \geq \mu_{2,h} \geq \cdots \geq \mu_{d',h} > \mu_{d'+1,h} = \cdots = \mu_{d,h} = 0.$$

Here, $(d - d')$ is the dimension of the kernel of $b(\gamma \cdot, \gamma \cdot)$ on V_h . Let $\{u_{k,h}\}_{k=1}^d$ be a -orthonormal eigenfunctions associated to the eigenvalues $\mu_{k,h}$. Define $E_{k,h} := \text{span}\{u_{1,h}, \dots, u_{k,h}\}$.

Corresponding to (3.2), we also consider its reciprocal version: Find $\lambda_h > 0$ and $u_h \in V_h, \gamma u_h \neq 0$ such that

$$a(u_h, v_h) = \lambda_h b(\gamma u_h, \gamma v_h) \quad \forall v_h \in V_h. \quad (3.4)$$

The above problem has d' positive eigenvalues

$$\lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{d',h}; \quad \lambda_{k,h} = 1/\mu_{k,h} \quad (1 \leq k \leq d'). \quad (3.5)$$

The max-min and min-max principles for eigenvalues tell that

$$\mu_{k,h} = \max_{H_{k,h} \subset V_h} \min_{v_h \in H_{k,h} \setminus \{0\}} R(v_h) = \min_{S_{k-1,h} \subset V_h} \max_{\substack{v_h \in V_h \setminus \{0\}, \\ v_h \perp S_{k-1,h}}} R(v_h). \quad (3.6)$$

Here, $H_{k,h}$ denotes a k -dimensional subspace of V_h , and $S_{k-1,h}$ denotes a subspace of V_h such that $\dim(S_{k-1,h}) \leq k-1$. With the eigen-subspace $E_{k,h}$, we also have

$$\mu_{k,h} = \min_{v_h \in E_{k,h} \setminus \{0\}} R(v_h) = \max_{\substack{v_h \in V_h \setminus \{0\} \\ v_h \perp E_{k-1,h}}} R(v_h). \quad (3.7)$$

²Thanks to Ryoki Endo for the construction of this example.

Define the projection $P_h : V(h) \rightarrow V_h$ such that for any $u \in V(h)$,

$$\widehat{a}(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h. \quad (3.8)$$

Note that P_h is well defined because $\widehat{a}(\cdot, \cdot)$ is an inner product on V_h and $\widehat{a}(u, \cdot)$ is a bounded linear functional on V_h . Furthermore, it can easily be confirmed that $P_h^2 = P_h$, i.e., P_h is a projection. According to (3.8), we have a decomposition of $u \in V(h)$ with respect to the norm $\|\cdot\|_a$:

$$\|P_h u\|_a^2 + \|(I - P_h)u\|_a^2 = \|u\|_a^2. \quad (3.9)$$

Unlike [18], we do not assume that P_h is an orthogonal projection on $V(h)$; in general it need not be, since $\widehat{a}$ may fail to be an inner product on $V(h)$.

Let us provide the theorem of explicit eigenvalue bounds using the projection P_h .

Theorem 3.1 *Suppose that the following relation holds for the projection error estimate of P_h :*

$$\|\gamma(I - P_h)v\|_b \leq C_h \|(I - P_h)v\|_a \quad \forall v \in V(h), \quad (3.10)$$

with a certain computable quantity C_h . An explicit upper bound for μ_k is given by

$$\mu_k \leq \mu_{k,h} + C_h^2, \quad k = 1, 2, \dots, d. \quad (3.11)$$

For $\mu_k \neq 0$, the above estimate yields the following lower bound for λ_k :

$$\lambda_k \geq \frac{\lambda_{k,h}}{1 + C_h^2 \lambda_{k,h}}, \quad k = 1, 2, \dots, d'. \quad (3.12)$$

Proof Let us start with the case that there exists a nonzero $v \in E_k(\subset V)$ such that $P_h v = 0$. Then the error estimate for P_h implies that

$$\|\gamma v\|_b = \|\gamma(I - P_h)v\|_b \leq C_h \|(I - P_h)v\|_a = C_h \|v\|_a, \quad \text{i.e., } R(v) \leq C_h^2.$$

Thus, property (2.7) yields that $\mu_k \leq R(v) \leq C_h^2 \leq \mu_{k,h} + C_h^2$.

Now consider the case that for any nonzero $v \in E_k$, $P_h v \neq 0$, which implies $\dim(P_h(E_k)) = k$. With the special choice $H_{k,h} := P_h(E_k)$, the characterization of $\mu_{k,h}$ in (3.6) leads to

$$\min_{v_h \in P_h(E_k) \setminus \{0\}} R(v_h) \leq \mu_{k,h}.$$

Suppose that $v_h = P_h \phi$ ($\phi \in E_k$) minimizes R in $P_h(E_k)$. Therefore,

$$\|\gamma P_h \phi\|_b \leq \sqrt{\mu_{k,h}} \|P_h \phi\|_a.$$

By further utilizing the error estimate for P_h and the decomposition of ϕ with respect to the $\|\cdot\|_a$ norm: $\|P_h \phi\|_a^2 + \|(I - P_h)\phi\|_a^2 = \|\phi\|_a^2$, we have

$$\begin{aligned} \|\gamma \phi\|_b &\leq \|\gamma P_h \phi\|_b + \|\gamma(I - P_h)\phi\|_b \\ &\leq \sqrt{\mu_{k,h}} \|P_h \phi\|_a + C_h \|(I - P_h)\phi\|_a \\ &\leq \sqrt{\mu_{k,h} + C_h^2} \|\phi\|_a. \end{aligned}$$

Since $\phi \in E_k$, the property of μ_k in (2.7) gives the estimate $\mu_k \leq R(\phi) \leq \mu_{k,h} + C_h^2$. $\square$

Remark 3.2 We highlight two features of [Theorem 3.1](#) and its proof. In Theorem 4.3 of [\[18\]](#), it is assumed that $\widehat{a}(\cdot, \cdot)$ is positive definite over $V(h)$. The current theorem eliminates the need for this assumption, making it applicable to more complicated eigenvalue problems, such as the Maxwell eigenvalue problem, where the argument about the positive definiteness of $\widehat{a}(\cdot, \cdot)$ on $V(h)$ is delicate. Another feature is that, by working with μ rather than λ , the proof here avoids the complicated discussion on the space decomposition $V = \text{Ker}(\mathcal{K}) \oplus \text{Ker}(\mathcal{K})^\perp$, which has been used in [\[24\]](#) to handle the lack of positive definiteness of $b(\gamma \cdot, \gamma \cdot)$ appearing in the Rayleigh quotient for λ .

3.2 Application of projection-based eigenvalue bounds

The lower eigenvalue bounds provided by [Theorem 3.1](#) have been applied to various differential operators. In this section, as a preparation for high-precision eigenvalue bounds, we recall several known results for two model eigenvalue problems: the Laplacian eigenvalue problem and the Steklov eigenvalue problem.

The analysis is undertaken within the framework of Sobolev spaces. Let Ω be a bounded domain in $\mathbb{R}^k$ with Lipschitz boundary. The $L^2(\Omega)$ function space is the set of real square-integrable functions on Ω , with inner product denoted by $(\cdot, \cdot)_\Omega$ or $(\cdot, \cdot)$. We shall use the standard notation for the Sobolev spaces $W^{k,p}(\Omega)$ and their associated norms $\|\cdot\|_{k,p,\Omega}$ and seminorms $|\cdot|_{k,p,\Omega}$ (see, e.g., [\[49\]](#)). For $p = 2$, define $H^k(\Omega) = W^{k,2}(\Omega)$, $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$, $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$.

For convenience of error analysis and computations, from now on we assume $\Omega \subset \mathbb{R}^2$ is bounded and polygonal. Let $\mathcal{T}^h$ be a triangular subdivision of Ω . Given an element $K \in \mathcal{T}^h$, h_K denotes the length of the longest edge of K . Let $h_{max} := \max_{K \in \mathcal{T}^h} h_K$. That is, h_{max} is the maximal edge length of the triangulation. Let us introduce the Crouzeix–Raviart finite element space V_h^{CR} defined on $\mathcal{T}^h$:

$$V_h^{CR} := \{v \mid v \text{ is a linear function on each } K \in \mathcal{T}^h \text{ and continuous at the midpoints of interior edges}\}. \quad (3.13)$$

Since $V_h^{CR} \not\subset H^1(\Omega)$, we introduce the discrete gradient operator ∇_h , which takes the gradient element-wise for $v_h \in V_h^{CR}$. For simplicity, $\|\nabla_h v_h\|_{L^2(\Omega)}$ is still denoted by $\|\nabla v_h\|_{L^2(\Omega)}$ or $|v_h|_{1,\Omega}$.

3.2.1 Laplacian eigenvalue problem

We consider the Laplacian eigenvalue problem associated with the homogeneous Dirichlet boundary condition:

Find $u \in H_0^1(\Omega) \setminus \{0\}$ and $\lambda \in \mathbb{R}$ such that

$$(\nabla u, \nabla v)_\Omega = \lambda(u, v)_\Omega \quad \forall v \in H_0^1(\Omega). \quad (3.14)$$

To apply Theorem 3.1, we take the following setting for (3.14) (see Remark 2.1):

$$\begin{cases} V = H_0^1(\Omega), \quad W = L^2(\Omega), \quad a(u, v) = (\nabla u, \nabla v)_\Omega, \quad b(u, v) = (u, v)_\Omega, \\ \gamma : \text{the compact embedding operator } H_0^1(\Omega) \hookrightarrow L^2(\Omega). \end{cases}$$

Define V_h by

$$V_h = \{v_h \in V_h^{\text{CR}} \mid v_h = 0 \text{ at midpoints of boundary edges of } \mathcal{T}^h\}.$$

Define $\gamma_h : V_h \rightarrow L^2(\Omega)$ as the embedding operator, and define $\hat{\gamma} : V(h) \rightarrow L^2(\Omega)$ via $\hat{\gamma}(v + v_h) := \gamma v + \gamma_h v_h$ ($v \in V, v_h \in V_h$)³.

Since V_h is finite dimensional, γ_h is compact. Hence, using the compactness of γ , we have the compactness of $\hat{\gamma}$. Note that v_h may be discontinuous across interior edges and does not necessarily vanish on $\partial\Omega$. The bilinear form $\hat{a}(\cdot, \cdot)$ for $V(h)$ is defined by

$$\hat{a}(u_h, v_h) := \sum_{K \in \mathcal{T}^h} \int_K \nabla u_h \cdot \nabla v_h \, dx \quad \forall u_h, v_h \in V(h). \quad (3.15)$$

For any $u, v \in V$, we have $\hat{a}(u, v) = a(u, v)$.

Explicit lower bounds for Laplacian eigenvalues

Let $\lambda_{k,h}^{\text{CR}}$ be the k -th discretized eigenvalue given by the Crouzeix–Raviart FEM. Then, the following lower bound is given by Theorem 3.1.

$$\lambda_k \geq \frac{\lambda_{k,h}^{\text{CR}}}{1 + C_h^2 \lambda_{k,h}^{\text{CR}}} \quad (1 \leq k \leq \dim(V_h)). \quad (3.16)$$

Here, for the triangulation of a 2D domain, C_h can be chosen as $C_h = 0.1893h_{\max}$ [18]. For the analogous three-dimensional setting with tetrahedral meshes, one has $C_h = 0.3804h_{\max}$ [18]. If we would use conforming FEMs instead of the Crouzeix–Raviart FEM, the computation of C_h would be more complicated; see [17]. It is worth pointing out that to have a guaranteed lower bound of λ_k using (3.16), one has to rigorously solve the eigenvalue problem of the discretized matrices from the Crouzeix–Raviart FEM to obtain an exact lower bound of $\lambda_{k,h}^{\text{CR}}$. Note that matrix eigenvalue enclosures are accessible by, for example, the algorithm proposed in [50].

3.2.2 Steklov eigenvalue problem

A specific feature here is that the bilinear form $b(\cdot, \cdot)$ induces a nonzero kernel of $\mathcal{K} = G\gamma$.

³Although the decomposition $v + v_h$ is not unique, the definition is well-posed since $\hat{\gamma}v = \gamma v = \gamma_h v$ for $v \in V \cap V_h$.

The Steklov eigenvalue problem

Find $\lambda > 0$ and $u \in H^1(\Omega)$, $\gamma u \neq 0$ such that

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega = \lambda(\gamma u, \gamma v)_{\partial\Omega} \quad \forall v \in H^1(\Omega). \quad (3.17)$$

Here, $\gamma : H^1(\Omega) \mapsto L^2(\partial\Omega)$ is the trace operator. Since Ω is a bounded Lipschitz domain, γ is a compact operator; see, e.g., [51].

To apply Theorem 3.1, we take the following setting (see Remark 2.1)

$$V := H^1(\Omega), \quad W := L^2(\partial\Omega), \quad a(u, v) := (\nabla u, \nabla v)_\Omega + (u, v)_\Omega, \quad b(u, v) := (u, v)_{\partial\Omega}.$$

For the discretized eigenvalue problem, take $V_h := V_h^{\text{CR}}$ and $V(h) := V + V_h$. Let $\gamma_h : V_h \rightarrow W$ be the piecewise trace operator defined for functions on the boundary elements of $\mathcal{T}^h$. For $v + v_h \in V(h)$ with $v \in V$ and $v_h \in V_h$, $\hat{\gamma}(v + v_h) := \gamma v + \gamma_h v_h$.⁴

Define $\hat{a}(\cdot, \cdot)$ on $V(h)$ by

$$\hat{a}(u_h, v_h) := \sum_{K \in \mathcal{T}^h} \int_K \nabla u_h \cdot \nabla v_h + u_h v_h \, dx \quad \forall u_h, v_h \in V(h). \quad (3.18)$$

It holds that for $u, v \in V$, $\hat{a}(u, v) = a(u, v)$.

Let $P_h : V(h) \mapsto V_h$ be the projection as defined in (3.8). To apply Theorem 3.1, the following projection error estimate is required.

$$\|\gamma(u - P_h u)\|_{L^2(\partial\Omega)} \leq C_h \|u - P_h u\|_{1,\Omega}, \quad \forall u \in V.$$

In [24], by using the computed value of $\lambda_{1,h}$ from (3.4), an explicit value of the quantity C_h is provided as

$$C_h := 0.6711 \max_{K \in \mathcal{T}_b^h} \frac{h_K}{\sqrt{H_K}} + \frac{0.1893}{\sqrt{\lambda_{1,h}}} \max_{K \in \mathcal{T}^h} h_K, \quad (3.19)$$

under the assumption that all elements of $\mathcal{T}_b^h$ have exactly one edge on $\partial\Omega$. Here, $\mathcal{T}_b^h$ is the set of elements of $\mathcal{T}^h$ having an edge on $\partial\Omega$; for $K \in \mathcal{T}_b^h$, H_K denotes the height of K with respect to its boundary edge.

It is worth pointing out that in (3.19), except for the term $\lambda_{1,h}$ determined by the eigenvalue problem on the domain, the other terms are only related to the mesh and are essentially independent of the shape of the domain. Note that, again, conforming FEMs instead of the Crouzeix–Raviart FEM would complicate the computation of C_h ; see [30].

Explicit lower bounds for Steklov eigenvalues

Let $\lambda_{k,h}^{\text{CR}}$ be the k -th discretized eigenvalues given by the Crouzeix–Raviart FEM in V_h . With Theorem 3.1 and the explicit projection error estimate for P_h , we obtain

⁴See the footnote for the case of Laplacian eigenvalue problem.

explicit lower bounds for Steklov eigenvalues of (3.17),

$$\lambda_k \geq \frac{\lambda_{k,h}^{\text{CR}}}{1 + C_h^2 \lambda_{k,h}^{\text{CR}}} \quad (k = 1, 2, \dots, d'). \quad (3.20)$$

Here, $d' = \dim(V_h) - \dim(\text{Ker}(\gamma_h))$, $\text{Ker}(\gamma_h) = \{v_h \in V_h \mid \gamma_h v_h = 0\}$.

Remark 3.3 Because C_h (for nonconforming but also for conforming FEMs) is governed by the global maximal edge length of the triangulation, the projection-based bounds (3.16) and (1.1) work well on nonuniform meshes with moderate grading, but their accuracy deteriorates on strongly graded meshes where element sizes vary greatly (i.e., very small and very large elements coexist). This is one motivation for us to introduce our two-stage method.

Remark 3.4 Note that since $C_h = O(\sqrt{h})$ as $h \rightarrow 0$, the lower eigenvalue bound obtained in (3.20) only converges at rate $O(h)$. This is not optimal when compared with the approximate eigenvalues themselves, which converges at rate $O(h^2)$ for H^2 -regular solutions; see, e.g., [42].

4 Second stage: high-precision eigenvalue bounds using Lehmann–Goerisch’s theorem

So far, in the first stage of studying eigenvalue bounds, we have successfully constructed projection-based lower eigenvalue bounds by utilizing for example the nonconforming Crouzeix–Raviart element space. To provide sharp eigenvalue bounds, a natural idea is to refine the mesh or increase the degree of the polynomials used in the finite element spaces. For nonconforming FEMs, one can refine the mesh to improve the precision, but a high-order Crouzeix–Raviart-like FEM is generally not available.

Instead, in the second stage of our proposed algorithm to obtain high-precision eigenvalue bounds, let us introduce Lehmann–Goerisch’s theorem using high-order conforming FEMs and the rough eigenvalue bounds obtained in the first stage.

The idea of adopting Lehmann–Goerisch’s theorem and conforming FEMs has also been successfully used to bound the Poincaré constants, where the high-order conforming FEM produces quite efficient eigenvalue bounds (see [44]), and to give computer-assisted existence and multiplicity proofs for various non-linear boundary value problems (see, e.g., [6, 38, 52, 53]);

4.1 Lehmann–Goerisch’s theorem

Let us introduce the formulation of the eigenvalue problem to be investigated by applying the Lehmann–Goerisch theorem.

Find $u \in D$ and $\lambda \in \mathbb{R}$ such that,

$$M(u, v) = \lambda N(u, v) \quad \forall v \in D. \quad (4.1)$$

The notation follows the original one used by F. Goerisch. Compared with the eigenvalue problem defined in (2.1), we have

$$D = V, \quad M(\cdot, \cdot) = a(\cdot, \cdot), \quad N(\cdot, \cdot) = b(\gamma \cdot, \gamma \cdot).$$

Lehmann–Goerisch’s theorem first declares the existence of eigenvalues in a certain range. By further utilizing rough eigenvalue bounds, which can be obtained from the first stage of our algorithm, one can obtain high-precision bounds for eigenvalues of (4.1).

Now, let us display the assumptions made for Lehmann–Goerisch’s theorem ⁵.

Assumption:

A1 Let D be a real vector space, $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ be symmetric bilinear forms on D . $M(\cdot, \cdot)$ is positive definite on D .

A2 There exists a sequence $\{(\lambda_i, \varphi_i)\}_{i \in \mathbb{N}}$ in $\mathbb{R} \times D$ of eigenpairs satisfying

$$M(\varphi_i, v) = \lambda_i N(\varphi_i, v) \quad \forall v \in D; \quad M(\varphi_i, \varphi_j) = \delta_{ij} \quad \text{for } i, j \in \mathbb{N}.$$

Here, δ_{ij} is the Kronecker delta. Moreover,

$$N(v, v) = \sum_{i=1}^{\infty} \lambda_i |N(v, \varphi_i)|^2 \quad \forall v \in D. \quad (4.2)$$

A3 Let X be a real vector space and b_G be a symmetric positive semidefinite bilinear form on X . Let $T : D \rightarrow X$ be a linear operator such that

$$b_G(Tu, Tv) = M(u, v) \quad \forall u, v \in D.$$

A4 Given $v_1, v_2, \dots, v_n \in D$, let $w_1, w_2, \dots, w_n \in X$ satisfy

$$b_G(w_i, Tv) = N(v_i, v) \quad \forall v \in D \quad (i = 1, \dots, n). \quad (4.3)$$

A5 Given $\rho \in \mathbb{R}$, $\rho > 0$, define $n \times n$ matrices A_0, A_1, A_2, A, B by

$$A_0 := (M(v_i, v_j))_{i,j=1}^n, \quad A_1 := (N(v_i, v_j))_{i,j=1}^n, \quad A_2 := (b_G(w_i, w_j))_{i,j=1}^n, \\ A := A_0 - \rho A_1, \quad B := A_0 - 2\rho A_1 + \rho^2 A_2. \quad (4.4)$$

Suppose B is positive definite. Let $\nu_1 \leq \nu_2 \leq \dots \leq \nu_n$ be the eigenvalues of $Az = \nu Bz$. Let q be the number of negative eigenvalues among them.

Now, we recall the Lehmann–Goerisch theorem; see, for example, [6, Chapter10] and [22, Chapter5] for its proof.

⁵The assumptions here are the same as in [54, 55]. Note that the assumptions in [54] are formulated with operators, while in a later paper of Goerisch and Albrecht [55], the eigenvalue problem and the assumptions are formulated with bilinear forms.

Theorem 4.1 (Lehmann–Goerisch’s theorem) *Under assumptions A1-A5, for each k ($1 \leq k \leq q$), the interval $[\rho - \rho/(1 - \nu_k), \rho)$ contains at least k eigenvalues of $\{\lambda_i\}_{i \in \mathbb{N}}$, counting by multiplicity.*

Lemma 4.2 *Given $v \in D$, let $\tilde{u} \in D$ be the solution to the equation:*

$$M(\tilde{u}, \phi) = N(v, \phi) \quad \forall \phi \in D. \quad (4.5)$$

For any $w \in X$ satisfying

$$b_G(w, T\phi) = N(v, \phi) \quad \forall \phi \in D,$$

we have $b_G(w, w) \geq b_G(T\tilde{u}, T\tilde{u}) = M(\tilde{u}, \tilde{u})$.

Proof Let $\hat{w} := w - T\tilde{u} \in X$. Since

$$b_G(\hat{w}, T\phi) = b_G(w, T\phi) - M(\tilde{u}, \phi) = N(v, \phi) - N(v, \phi) = 0 \quad \forall \phi \in D,$$

we have

$$b_G(w, w) = b_G(T\tilde{u} + \hat{w}, T\tilde{u} + \hat{w}) = b_G(T\tilde{u}, T\tilde{u}) + b_G(\hat{w}, \hat{w}) \geq b_G(T\tilde{u}, T\tilde{u}).$$

□

Remark 4.3 (Selection of the w_i term) Generally, there is freedom to select w_i under the condition (4.3). A selection of w_i ’s with “small” values of $b_G(w_i, w_i)$ will produce “good” lower eigenvalue bounds. In fact, $b_G(w_i, w_i)$ under the constraint condition (4.3) is minimized by $w_i = T\tilde{u}_i$, where $\tilde{u}_i$ is obtained from the standard setting of Lehmann’s method by solving (4.5). However, this original Lehmann’s equation is generally difficult (or impossible) to solve explicitly. The introduction of the w_i term by Goerisch (called “Goerisch’s XbT-concept”; see, e.g., [6, Lemma 10.25]) makes it flexible to construct w_i to obtain eigenvalue bounds for the target eigenvalue problem. In practical computing, we construct a finite dimensional space $X_h \subset X$ and compute the w_i term by solving the minimization problem (see [44, Remark 5] and [6, Remark 10.26 (c)]):

$$\min_{w_i \in X_h} b_G(w_i, w_i) \text{ subject to condition (4.3)}. \quad (4.6)$$

To have a rigorous eigenvalue bound, the minimization problem (4.6) does nevertheless not need to be solved exactly, while the condition (4.3) has to be satisfied strictly.

From now on, suppose A_1 is positive definite. Let Λ_n be the maximal eigenvalue of $A_0x = \Lambda A_1x$.

Effort has been paid to validate the positive definiteness of matrix B in assumption A5, see, e.g., [6, Theorem 10.10, 10.21, 10.29]. The following Lemma provides a direct condition $\rho > \Lambda_n$ (see also [6, (10.28) and (10.92)]) for this aim.

Lemma 4.4 *In the case where $\rho > \Lambda_n$, the matrix B in equation (4.4) is indeed positive definite, as required in assumption A5.*

Proof Let $v (\neq 0)$ be a linear combination of $v_1, v_2, \dots, v_n \in D$. Let $\tilde{u}$ be the solution of equation (4.5) with v_i replaced by v . That is,

$$M(\tilde{u}, \phi) = N(v, \phi), \quad \forall \phi \in D. \quad (4.7)$$

Let $\hat{w} := T\tilde{u}$. We have $b_G(\hat{w}, Tv) = b_G(T\tilde{u}, Tv) = M(\tilde{u}, v) = N(v, v)$, and

$$0 \leq b_G(Tv - \rho\hat{w}, Tv - \rho\hat{w}) = M(v, v) - 2\rho N(v, v) + \rho^2 b_G(\hat{w}, \hat{w}).$$

Moreover, $b_G(Tv - \rho\hat{w}, Tv) \neq 0$, because otherwise,

$$M(v, v) = b_G(Tv, Tv) = b_G(\rho\hat{w}, Tv) = \rho N(v, v).$$

However, $M(v, v) - \rho N(v, v) = 0$ contradicts the assumption that $\rho > \Lambda_n$.

From the confirmed fact that $b_G(Tv - \rho\hat{w}, Tv) \neq 0$, we have ⁶

$$b_G(Tv - \rho\hat{w}, Tv - \rho\hat{w}) > 0. \quad (4.8)$$

Given any $w \in X$ such that

$$b_G(w, T\phi) = N(v, \phi) \quad \forall \phi \in D,$$

we have $b_G(w, w) \geq b_G(\hat{w}, \hat{w})$ by Lemma 4.2. By further using the inequality (4.8),

$$\begin{aligned} M(v, v) - 2\rho N(v, v) + \rho^2 b_G(w, w) &\geq M(v, v) - 2\rho N(v, v) + \rho^2 b_G(\hat{w}, \hat{w}) \\ &= b_G(Tv - \rho\hat{w}, Tv - \rho\hat{w}) > 0. \end{aligned}$$

This implies that the matrix B is positive definite. $\square$

The following theorem gives an application of Theorem 4.1 to obtaining lower bounds for n eigenvalues: $\lambda_{m-n+1}, \dots, \lambda_m$, by using a lower bound (denoted by ρ) of λ_{m+1} ; see Figure 1 for an illustration of the involved eigenvalues.

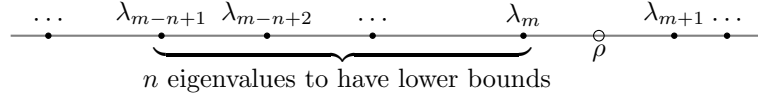


Fig. 1 Bounding eigenvalues by using Lehmann–Goerisch’s theorem.

Theorem 4.5 Suppose ρ is a lower bound to the $(m+1)$ th eigenvalue λ_{m+1} for some $m \geq n$ ($m \in \mathbb{N}$), i.e.,

$$\rho \leq \lambda_{m+1}. \quad (4.9)$$

If assumptions A1-A5 are satisfied, then the following lower eigenvalue bound holds.

$$\lambda_{m+1-k} \geq \rho - \frac{\rho}{1 - \nu_k} \quad (1 \leq k \leq q). \quad (4.10)$$

In particular, if $\rho > \Lambda_n$, then $\nu_n < 0$, i.e., $q = n$, and

$$\lambda_{m+1-k} \geq \rho - \frac{\rho}{1 - \nu_k} \quad (1 \leq k \leq n). \quad (4.11)$$

⁶Note that, for any symmetric and positive semidefinite bilinear form $b_G(\cdot, \cdot)$, $b_G(f, f) = 0$ implies $b_G(f, g) = 0$ for all $g \in X$ due to the Cauchy–Schwarz inequality.

We remark that in the case that $m = n$, under the essential condition (see also [6, Theorem 10.31])

$$\Lambda_n < \rho \leq \lambda_{n+1}, \quad (4.12)$$

the right-hand side of (4.11) provides lower bounds for $\lambda_1, \dots, \lambda_n$.

Remark 4.6 It is crucial to point out that in our proposed two-stage algorithm, the projection-based lower eigenvalue bounds in the first stage serve to compute ρ satisfying (4.9) or (4.12).

Remark 4.7 The lower bound given by Lehmann–Goerisch’s theorem is monotonically increasing with respect to the a priori lower bound ρ ($\leq \lambda_{m+1}$) (Theorem 4.1 of [56]). In practice, even with a rough ρ , the eigenvalue bounds in (4.10) are still sharp if v_i is a good approximation to the exact eigenfunction.

4.2 Application of Lehmann–Goerisch’s theorem to concrete eigenvalue problems

In this section, we show the application of the Lehmann–Goerisch theorem to two concrete eigenvalue problems: the Dirichlet Laplacian eigenvalue problem and the Steklov eigenvalue problem. Here, to simplify the argument, we consider lower bounds for the first n eigenvalues. Generally, the application of Lehmann–Goerisch’s theorem does not need to start from the first eigenvalue.

As a feature of Lehmann–Goerisch’s method, it takes advantage of graded meshes (i.e., meshes with refinement around possible singular parts) to recover the convergence rate of the approximate solution, when the objective solution has singularities. Different from the lower eigenvalue bound of Theorem 3.1 based on interpolation or projection-based error estimation, the lower eigenvalue bound from Lehmann–Goerisch’s theorem is not affected by the size of rough elements when graded meshes are used. In the last section on numerical computations, an example using a graded mesh for the L-shaped domain is provided to demonstrate this feature.

As a preparation for the implementation of Lehmann–Goerisch’s theorem, let us introduce several FEM spaces for the currently considered second-order eigenvalue problems. Note that here the degree of a polynomial is defined by the summation of exponents for each variable. For example, xy has the degree 2. The order p of the following FEM spaces will be taken as $p \geq 1$.

- Let CG_h^p be the Lagrange FEM space consisting of continuous piecewise polynomials of degree up to p .
- Let $\text{RT}_h^p(\subset H(\text{div}; \Omega))$ be the Raviart–Thomas FEM space of degree p :

$$\text{RT}_h^p := \left\{ v : v = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + c_3 \begin{pmatrix} x \\ y \end{pmatrix} \text{ on each element } K, \text{ where the } c_i\text{'s} \right. \\ \left. \text{are polynomials on } K \text{ of degree up to } p; \right.$$

$v \cdot \vec{n}$ is continuous across each interior edge e . $\left. \vphantom{\int} \right\}$.

- Let DG_h^p be the space of piecewise polynomials with degree up to p . The property of the Raviart–Thomas space implies that $\text{div}(\text{RT}_h^p) = \text{DG}_h^p$.

4.3 The Laplacian eigenvalue problem

We consider the eigenvalue problem defined in (3.14), and aim to obtain high-precision eigenvalue bounds for $\lambda_1, \dots, \lambda_n$. It is assumed that λ_n and λ_{n+1} are well separated from each other. It is worth pointing out that to apply Theorem 4.5 for obtaining the lower bounds for the first n eigenvalues, the projection-error based lower bound from the first stage is important and will be used in the selection of ρ such that $\Lambda_n < \rho \leq \lambda_{n+1}$ (see Remark 4.6).

Introduce the following setting for Lehmann–Goerisch’s theorem.

$$D := H_0^1(\Omega); \quad M(u, v) := (\nabla u, \nabla v)_\Omega, \quad N(u, v) := (u, v)_\Omega \quad \forall u, v \in D.$$

Goerisch’s XbT setting is taken as

$$X := (L^2(\Omega))^2; \quad b_G(u, v) := (u, v)_\Omega \quad \forall u, v \in X; \quad T := \nabla.$$

Take the eigenpairs $\{\lambda_i, u_i\}_{i=1}^\infty$ of (3.14) for the ones in Assumption A2. It is easy to confirm Assumption A3, i.e., $b_G(\nabla u, \nabla v) = M(u, v)$ for all $u, v \in D$.

Now, let us describe the selection of functions required by Lehmann–Goerisch’s theorem.

Selection of v_i

Let us consider the discretized eigenvalue problem: Find $u_h (\neq 0)$ in CG_h^p and $\lambda_h \in \mathbb{R}$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in \text{CG}_h^p. \quad (4.13)$$

Denote the exact eigenpairs for the first n eigenvalues by $\{\lambda_{i,h}, u_{i,h}\}_{i=1}^n$. The approximate eigenfunctions $v_1, \dots, v_n$ required in A4 of Lehmann–Goerisch’s theorem are selected as $v_i \approx u_{i,h}$. Note that v_i does not need to be exactly equal to $u_{i,h}$, since any conforming approximation is allowed in the setting of Lehmann–Goerisch’s theorem.

Calculation of w_i

For each $v_i (\approx u_{i,h})$, the constraint condition (4.3) for $w_i \in X = (L^2(\Omega))^2$ becomes

$$(w_i, \nabla v) = (v_i, v) \quad \forall v \in H_0^1(\Omega). \quad (4.14)$$

By the distributional definition of div , the condition (4.14) means that w_i belongs to $H(\operatorname{div}; \Omega)$ and has to satisfy ⁷

$$\operatorname{div} w_i + v_i = 0 \quad \text{in } \Omega . \quad (4.15)$$

Since $v_i \in \operatorname{CG}_h^p (\subset \operatorname{DG}_h^p = \operatorname{div}(\operatorname{RT}_h^p))$, one can find a $w_i \in \operatorname{RT}_h^p$ that solves (4.15) exactly. That is, $\operatorname{div} w_i + v_i = 0$ holds pointwise. In numerical computation, equation (4.15) is asserted by using test functions from DG_h^p , i.e., by requiring

$$(\operatorname{div} w_i + v_i, \psi_h) = 0 \quad \forall \psi_h \in \operatorname{DG}_h^p .$$

The solution w_i to (4.15) is not unique. To have a “good” selection of w_i to obtain a tight eigenvalue bound, let us follow Remark 4.3 to consider the minimization problem:

$$\min_{w_i \in \operatorname{RT}_h^p} \|w_i\| \quad \text{subject to } \operatorname{div} w_i + v_i = 0 . \quad (4.16)$$

The saddle point formulation of (4.16) is to find $(w_i, \phi_{i,h}) \in \operatorname{RT}_h^p \times \operatorname{DG}_h^p$ s.t.

$$\begin{cases} (w_i, q_h) + (\phi_{i,h}, \operatorname{div} q_h) = 0 & \forall q_h \in \operatorname{RT}_h^p , \\ (\operatorname{div} w_i, \psi_h) + (v_i, \psi_h) = 0 & \forall \psi_h \in \operatorname{DG}_h^p . \end{cases} \quad (4.17)$$

The system (4.17) admits a unique solution $(w_i, \phi_{i,h})$ (c.f., e.g., [57], §IV.1 Prop. 1.1 of [58]). As noted in Remark 4.3, to have rigorous eigenvalue bounds, w_i does not necessarily have to be the exact minimizer of (4.16), while the constraint condition (4.15) must hold exactly. Such a w_i can be obtained as an interval function by solving (4.17) with interval arithmetic, which however requires extra computing cost compared to an approximate solver for (4.17). For an efficient way of computation of w_i , see Remark 4.8 for the technique to introduce a shift to the eigenvalue problem.

Selection of ρ

In the first stage, we have applied the projection-based method in §3 to obtain an exact lower bound ρ for eigenvalue λ_{n+1} , i.e., $\rho \leq \lambda_{n+1}$. In the case that the CR FEM is used in the eigenvalue lower bound (3.16), ρ is taken as

$$\rho := \frac{\lambda_{n+1,h}^{\operatorname{CR}}}{1 + C_h^2 \lambda_{n+1,h}^{\operatorname{CR}}} .$$

The matrices A_1 , A_1 , A_2 , A and B in A5 of Lehmann–Goerisch’s theorem are constructed by using the above v_i ’s and w_i ’s ($i = 1, \dots, n$). Let Λ_n be the maximal eigenvalue of the eigenvalue problem with the matrices A_0 and A_1 :

$$A_0 x = \Lambda A_1 x . \quad (4.18)$$

⁷By taking test functions v from $C_0^\infty(\Omega)$ in (4.14), we can see that the distributional $\operatorname{div} w_i$ is actually in $L^2(\Omega)$, and $\operatorname{div} w_i + v_i = 0$ holds.

With a proper selection of n such that λ_n and λ_{n+1} are separated from each other, one can calculate v_i using refined meshes and ρ to make sure the following condition holds

$$(\lambda_n \leq) \Lambda_n < \rho \leq \lambda_{n+1} .$$

Note that the above selection of ρ implies that B is positive definite; see Lemma 4.4.

To have an easy-to-follow description of Lehmann–Goerisch’s theorem, let us summarize the computation procedure in Algorithm 1.

Algorithm 1 Guaranteed eigenvalue bounds by Lehmann–Goerisch’s theorem

Objective: Lower bounds for the first n eigenvalues. It is assumed that λ_n and λ_{n+1} are well separated from each other so that the condition (4.19) can be satisfied.

Procedure:

- Step 1: Solve the eigenvalue problem (4.13) approximately to obtain $v_i (\approx u_{i,h}) \in \text{CG}_h^p$. Solve the eigenvalue problem (4.18) rigorously to obtain Λ_n , which provides an upper bound of $\lambda_{n,h}$.
- Step 2: Solve the eigenvalue problem (3.14) in V_h^{CR} in a verified way by using interval arithmetic to calculate $\lambda_{n+1,h}^{\text{CR}}$. With refined meshes for V_h^{CR} and CG_h^p , obtain a ρ such that

$$\lambda_{n,h} \leq \Lambda_n < \rho := \frac{\lambda_{n+1,h}^{\text{CR}}}{1 + C_h^2 \lambda_{n+1,h}^{\text{CR}}} \leq \lambda_{n+1} . \quad (4.19)$$

- Step 3: For each v_i ($1 \leq i \leq n$), let w_i be the solution of (4.17) that satisfies (4.15) exactly.
- Step 4: Define A_0, A_1, A_2, A and B with the functions v_i, w_i ($1 \leq i \leq n$). Let the eigenvalues of $Ax = \nu Bx$ be $\nu_1 \leq \dots \leq \nu_n$. The relation $\lambda_{n,h} < \rho$ makes sure that B and $-A$ are positive definite and hence $\nu_n < 0$. The lower bound of λ_k in (3.14) is given by

$$\lambda_k \geq \rho - \rho / (1 - \nu_{n+1-k}) \quad (1 \leq k \leq n) .$$

Remark 4.8 Let us introduce a well-used technique that avoids heavy computation in rigorously solving the linear system (4.17) when seeking the w_i terms. We did not use this technique for the examples in Section 5, but refer to the technique here for future purposes. Let us consider the eigenvalue problem with a shift $\hat{\lambda}(> 0)$, the setting of which is given by

$$D := H_0^1(\Omega); \quad M(u, v) := (\nabla u, \nabla v)_\Omega + \hat{\lambda}(u, v)_\Omega, \quad N(u, v) := (u, v)_\Omega \quad \forall u, v \in D .$$

Goerisch’s XbT setting is taken as $X := \left(L^2(\Omega) \right)^3$, $Tu := \{\nabla u, u\} \in X$ for $u \in D$. Rewrite each $u \in X$ by $u = \{u^{(1)}, u^{(2)}\}$, $u^{(1)} \in \left(L^2(\Omega) \right)^2$, $u^{(2)} \in L^2(\Omega)$. For $u, v \in X$,

$$b_G(u, v) := (u^{(1)}, v^{(1)})_\Omega + \hat{\lambda}(u^{(2)}, v^{(2)})_\Omega .$$

For each approximate eigenfunction v_i , the condition (4.3) for $w_i = \{w_i^{(1)}, w_i^{(2)}\} \in X$ becomes

$$\text{div } w_i^{(1)} + \hat{\lambda} w_i^{(2)} + v_i = 0 \quad \text{in } \Omega. \quad (4.20)$$

For given $v_i \approx u_{i,h}$, calculate $w_i^{(1)} \in \text{RT}_h^p$ by solving (4.17) with an approximate solver. Then, calculate $w_i^{(2)}$ as $w_i^{(2)} = (-\text{div}(w_i^{(1)}) - v_i)/\hat{\lambda}$ in a rigorous way. The resulting $w_i = \{w_i^{(1)}, w_i^{(2)}\}$ satisfies the condition (4.20) exactly. By applying Lehmann–Goerisch’s theorem to the current setting, one can directly obtain lower bounds for the shifted eigenvalues, and thus bounds for the original eigenvalues after re-subtracting the shift $\hat{\lambda}$.

4.4 The Steklov eigenvalue problem

In this subsection, we consider high-precision bounds for the first n eigenvalues for the Steklov eigenvalue problem. Let us take the following setting for the Lehmann–Goerisch theorem.

- (a) Let $D = H^1(\Omega)$ and define the bilinear forms $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ by

$$M(u, v) := \int_{\Omega} \nabla u \cdot \nabla v + uv \, dx, \quad N(u, v) := \int_{\partial\Omega} \gamma u \gamma v \, ds \quad \forall u, v \in D.$$

Here, γ is the trace operator.

- (b) Let $\{\lambda_i, u_i\}_{i \in \mathbb{N}}$ be the eigenpairs of the eigenvalue problem (3.2). Note that $\{u_i\}_{i \in \mathbb{N}}$ is a complete orthonormal system of $\text{Ker}(\mathcal{K})^\perp (\subset H^1(\Omega))$, where the orthogonal complement is taken with respect to $M(\cdot, \cdot)$. Given $v \in D$, decompose v by $v = v_0 + v'$, $v_0 \in H_0^1(\Omega)$, $v' \in \text{Ker}(\mathcal{K})^\perp$ and

$$v' = \sum_{i=1}^{\infty} M(v', u_i) u_i. \quad (4.21)$$

Since $v_0 \in H_0^1(\Omega) = \text{Ker}(\mathcal{K})$, we have $M(u_i, v_0) = \lambda_i N(u_i, v_0) = 0$. Hence,

$$M(u_i, v) = M(u_i, v') = \lambda_i N(u_i, v') = \lambda_i N(u_i, v).$$

It is clear that $M(u_i, u_j) = \delta_{ij}$ and $N(u_i, u_j) = \delta_{ij}/\lambda_i$ (δ_{ij} : Kronecker delta). Therefore, using (4.21) and the continuity of $N(\cdot, \cdot)$ with respect to $M(\cdot, \cdot)$,

$$\begin{aligned} N(v, v) &= N(v', v') = \sum_{i=1}^{\infty} N(u_i, u_i) \cdot |M(v', u_i)|^2 \\ &= \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \cdot \lambda_i^2 |N(v', u_i)|^2 = \sum_{i=1}^{\infty} \lambda_i |N(v', u_i)|^2 = \sum_{i=1}^{\infty} \lambda_i |N(v, u_i)|^2. \end{aligned}$$

With the above argument, we have confirmed that $\{\varphi_i\}_{i \in \mathbb{N}}$ in Assumption A2 can be taken as $\{u_i\}_{i \in \mathbb{N}}$.

- (c) Let $X = (L^2(\Omega))^3$ and define $b_G(\cdot, \cdot)$ for $\phi, \psi \in X$ by $b_G(\phi, \psi) := \int_{\Omega} \phi \cdot \psi \, dx$. Let $Tv := \{\nabla v, v\}$ for $v \in D$. Hence, $b_G(Tu, Tv) = M(u, v)$ for all $u, v \in D$.

Now, let us describe the function required by Lehmann–Goerisch’s theorem.

Selection of v_i

Let us consider the discretized eigenvalue problem:

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in \text{CG}_h^p. \quad (4.22)$$

Or, conversely,

$$N(u_h, v_h) = \mu_h M(u_h, v_h) \quad \forall v_h \in \text{CG}_h^p. \quad (4.23)$$

Take $\mu_{i,h}$ ($i = 1, \dots, n$) as the n largest positive eigenvalues of (4.23) and $u_{i,h} \in \text{Ker}(\mathcal{K})^\perp$ ($i = 1, \dots, n$) as the corresponding eigenfunctions. Then, define $\lambda_{i,h} := 1/\mu_{i,h}$ ($i = 1, \dots, n$). The functions $v_1, \dots, v_n$ required by A4 of Lehmann–Goerisch's theorem are selected in CG_h^p as approximations to $u_{1,h}, \dots, u_{n,h}$.

Calculation of w_i

For each $v_i \approx u_{i,h}$, the equation (4.3) to determine $w_i = \{w_i^{(1)}, w_i^{(2)}\} \in X$, $w_i^{(1)} \in (L^2(\Omega))^2$, $w_i^{(2)} \in L^2(\Omega)$ becomes

$$(w_i^{(1)}, \nabla v)_\Omega + (w_i^{(2)}, v)_\Omega = (\gamma v_i, \gamma v)_{\partial\Omega} \quad \forall v \in D. \quad (4.24)$$

It is easy to see that $w_i^{(1)}$ must belong to $H(\text{div}; \Omega)$ and satisfy

$$-\text{div } w_i^{(1)} + w_i^{(2)} = 0 \text{ in } \Omega, \quad w_i^{(1)} \cdot \vec{n} = \gamma v_i \text{ on } \partial\Omega. \quad (4.25)$$

Therefore, $w_i^{(2)} := \text{div } w_i^{(1)}$ has to be chosen, whence

$$b_G(w_i, w_i) = \|w_i^{(1)}\|_\Omega^2 + \|\text{div } w_i^{(1)}\|_\Omega^2.$$

Since $\{(w_h \cdot \vec{n})|_{\partial\Omega} : w_h \in \text{RT}_h^p\}$ contains all piecewise polynomials of degree p on boundary edges of $\mathcal{T}^h$, $w_i^{(1)}$ can be selected from RT_h^p such that the second equation of (4.25) holds. As suggested in Remark 4.3, the w_i term is determined by approximately solving the following minimization problem:

$$\min_{w_i^{(1)} \in \text{RT}_h^p, (w_i^{(1)} \cdot \vec{n})|_{\partial\Omega} = \gamma v_i} \|w_i^{(1)}\|_\Omega^2 + \|\text{div } w_i^{(1)}\|_\Omega^2. \quad (4.26)$$

The minimization problem (4.26) is equivalent to the following variational problem: Find $w_i^{(1)} \in \text{RT}_h^p$ such that $w_i^{(1)} \cdot \vec{n} = \gamma v_i$ on $\partial\Omega$, and

$$(w_i^{(1)}, w_h)_\Omega + (\text{div } w_i^{(1)}, \text{div } w_h)_\Omega = 0 \quad \forall w_h \in \text{RT}_{0,h}^p. \quad (4.27)$$

Here, $\text{RT}_{0,h}^p := \{w_h \in \text{RT}_h^p \mid w_h \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$.

Selection of ρ

Take ρ as the lower eigenvalue bound of λ_{n+1} obtained by the Crouzeix–Raviart finite element method. That is,

$$\rho := \frac{\lambda_{n+1,h}^{\text{CR}}}{1 + C_h^2 \lambda_{n+1,h}^{\text{CR}}}.$$

Define matrices A_0, A_1 by

$$A_0 := (M(v_i, v_j))_{i,j=1}^n, \quad A_1 := (N(v_i, v_j))_{i,j=1}^n. \quad (4.28)$$

Note that A_1 is positive definite according to the choice of $v_1, \dots, v_n$. Let Λ_n be the maximal eigenvalue of the eigenvalue problem: $A_0 x = \Lambda A_1 x$. With a proper selection of the eigenvalue index n and a refined mesh, we can ensure $\Lambda_n < \rho$.

Matrices for the Lehmann–Goerisch theorem

Let ν_k ($k = 1, \dots, n$) be the negative eigenvalues of $Ax = \nu Bx$, where

$$A_2 = (b_G(w_i, w_j))_{i,j=1}^n, \quad A = A_0 - \rho A_1, \quad B = A_0 - 2\rho A_1 + \rho^2 A_2. \quad (4.29)$$

Then we obtain the following lower bounds.

$$\lambda_k \geq \rho - \frac{\rho}{1 - \nu_{n+1-k}} \quad (1 \leq k \leq n). \quad (4.30)$$

Remark 4.9 Instead of taking $D = H^1(\Omega)$ as discussed in this section, we can also obtain the same eigenvalue bounds by taking $D = \text{Ker}(\mathcal{K})^\perp (\subset H^1(\Omega))$ while keeping the setting of M, N, T, X the same as the one in (a) (b), and (c) in this section. Since the approximate eigenvector $v_i \in \text{CG}_h^p$ may not belong to $D = \text{Ker}(\mathcal{K})^\perp$, more efforts are needed to handle this issue. Introduce the orthogonal projection $P : H^1(\Omega) \mapsto \text{Ker}(\mathcal{K})^\perp$ with respect to $M(\cdot, \cdot)$ and decompose v_i by

$$v_i = Pv_i + (I - P)v_i; \quad Pv_i \in \text{Ker}(\mathcal{K})^\perp, \quad (I - P)v_i \in \text{Ker}(\mathcal{K}).$$

Let $\tilde{v}_i := Pv_i$. Actually, we apply the Lehmann–Goerisch theorem with v_i replaced by $\tilde{v}_i$ ($i = 1, \dots, n$). Note that $\tilde{v}_i$ is just for theoretical analysis and does not need to be computed explicitly. Since $v_i - \tilde{v}_i \in \text{Ker}(\mathcal{K})$, we have $\gamma(v_i - \tilde{v}_i) = 0$ on $\partial\Omega$. Hence, $\gamma v_i = \gamma \tilde{v}_i$. Thus, the w_i term resulting from $\tilde{v}_i$ is the same as the one for v_i . In the construction of the matrices A_0, A_1, A_2 , rather than A_0 using v_i in (4.28), we have $\tilde{A}_0 := (M(\tilde{v}_i, \tilde{v}_j))_{i,j=1}^n$. From the orthogonality of the projection P with respect to $M(\cdot, \cdot)$, we have $M(u, u) \geq M(Pu, Pu)$ for all $u \in H^1(\Omega)$. Therefore, $A_0 - \tilde{A}_0$ is positive semidefinite.

Lehmann–Goerisch’s theorem utilizes the following eigenvalue problem:

$$\tilde{A}x = \tilde{\nu}\tilde{B}x \quad (\tilde{A} := \tilde{A}_0 - \rho A_1, \quad \tilde{B} := \tilde{A}_0 - 2\rho A_1 + \rho^2 A_2).$$

Let $\tilde{\Lambda}_n$ be the maximal eigenvalue of $\tilde{A}_0 x = \tilde{\Lambda} A_1 x$. Then $\tilde{\Lambda}_n \leq \Lambda_n$. By taking $(\tilde{\Lambda}_n \leq) \Lambda_n < \rho \leq \lambda_{n+1}$, we have that $-\tilde{A}$ is positive definite. From the positive definiteness of $\tilde{B}$ given by

Lemma 4.4, we have that $\tilde{B} - \tilde{A} = \rho(\rho A_2 - A_1)$ is positive definite as well. With a proper transformation, the eigenvalue problems to determine $\tilde{\nu}$ and ν (all of which are negative) are given by

$$(\tilde{A}_0 - \rho A_1)x = \frac{\tilde{\nu}}{1 - \tilde{\nu}}(\rho A_2 - A_1)x, \quad (A_0 - \rho A_1)x = \frac{\nu}{1 - \nu}(\rho A_2 - A_1)x.$$

From the min-max principle for eigenvalues, we have $\tilde{\nu}_k \leq \nu_k$ ($k = 1, \dots, n$). Thus, we obtain the same lower bound as (4.30) via the theoretical term $\tilde{\nu}_i$. That is,

$$\lambda_k \geq \rho - \frac{\rho}{1 - \tilde{\nu}_{n+1-k}} \geq \rho - \frac{\rho}{1 - \nu_{n+1-k}} \quad (1 \leq k \leq n). \quad (4.31)$$

5 Computation Results

In this section, the Laplacian and the Steklov eigenvalue problems are considered. For each problem, high-precision eigenvalue bounds are obtained in two steps by applying the projection-based eigenvalue bounds in Theorem 3.1 and Lehmann–Goerisch’s Theorem 4.1. The features of the proposed algorithm are illustrated by the numerical results: the adaptivity to non-uniform meshes or graded meshes is shown in §5.1; high-precision eigenvalue bounds and the recovery of the convergence rate for the bounds to Steklov eigenvalues are demonstrated in §5.2. §5.3 and §5.4 show high-precision bounds for Dirichlet eigenvalues on 3D tetrahedra.

To obtain rigorous bounds for eigenvalues, we use interval arithmetic and the method based on Sylvester’s law of inertia [50] to exactly solve eigenvalue problems for matrices.

Remark 5.1 (Reproducibility of the computation results) The source code for the computations in this section is available at <https://github.com/xfliu/LG-Method>: the MATLAB/INTLAB implementation used to produce the reported results, and a Julia implementation of the three-dimensional computations in §5.3 and §5.4 based on the verified finite element library VFEM.jl [59] and the verified matrix eigenvalue solver Veigs [60]. The computations can also be repeated online on the Ganjin platform at <https://ganjin.online/xfliu/LG-Method>.

5.1 Dirichlet Laplacian eigenvalues

Let us consider the Dirichlet Laplacian eigenvalue problem on a dumbbell domain Ω , which consists of two unit squares connected by a bar with a width of 0.40625 (see Figure 3). The approximate values of the eigenvalues are as follows:

$$\begin{aligned} \lambda_1 &\approx 19.515307, & \lambda_2 &\approx 19.515448, & \lambda_3 &\approx 47.510661, & \lambda_4 &\approx 47.519450, \\ \lambda_5 &\approx 49.342953, & \lambda_6 &\approx 49.342955. \end{aligned}$$

The approximate eigenfunctions corresponding to these six eigenvalues are plotted in Fig. 2 in row-wise order.

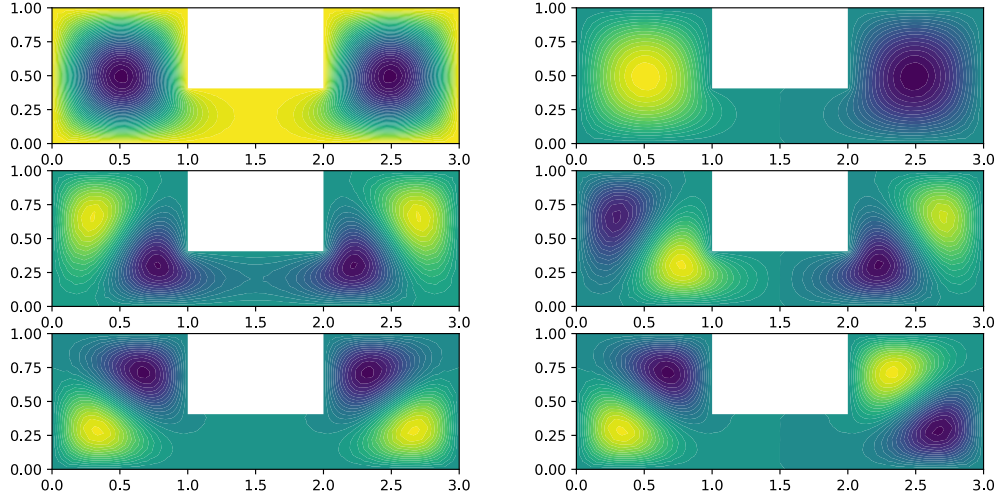


Fig. 2 Approximate eigenfunctions for λ_i 's ($1 \leq i \leq 6$) (dumbbell domain)

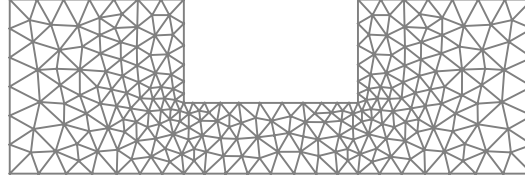


Fig. 3 Graded mesh for the dumbbell domain

A graded mesh is employed for the finite element computations. In accordance with the regularity of the solution, the size of triangular elements is chosen such that

$$h(r) \approx \begin{cases} h_{\max} r^{1/3}, & \text{for } r < 1, \\ h_{\max}, & \text{for } r \geq 1, \end{cases}$$

where r denotes the distance from an element to the two re-entrant corners of the domain, and $h_{\max}$ is the maximum edge length of the mesh.

For a mesh with $h_{\max} = 0.0298$, we employ the Crouzeix–Raviart FEM to compute a rigorous lower bound for λ_7 via the projection error constant $C_h = 0.1893h_{\max}$ and [Theorem 3.1](#):

$$C_h \leq 0.00565, \quad \lambda_{7,h}^{\text{CR}} \in [74.37604, 74.37605], \quad \lambda_7 \geq 74.19986.$$

Thus, for the Lehmann–Goerisch method, the lower bound ρ of λ_7 is set as

$$\rho = 74.0 < \lambda_7.$$

Table 1 Lower and upper bounds for Laplacian eigenvalues (dumbbell domain, $h_{\max} = 0.290$, number of elements=177).

λ_i	Lower bound: $\underline{\lambda}_i$			Upper bound: $\bar{\lambda}_i$	
	CR	LG(p=1)	LG(p=2)	CG(p=1)	CG(p=2)
λ_1	18.5595	19.2865	19.5080	20.1365	19.5265
λ_2	18.5596	19.2867	19.5082	20.1366	19.5266
λ_3	41.9488	41.1170	47.2888	51.0975	47.6105
λ_4	41.9554	41.1189	47.2978	51.1087	47.6192
λ_5	43.7222	41.5923	49.2050	53.3572	49.4143
λ_6	43.7222	41.5926	49.2050	53.3572	49.4143

Here we deliberately take ρ well below the lower bound for λ_7 based on $\lambda_{7,h}^{\text{CR}}$ to show that even a coarse estimate of λ_7 still yields very sharp bounds for $\lambda_1, \dots, \lambda_6$.

The rigorous lower and upper bounds with different algorithms are displayed in Table 1. In this computation, we employ the Crouzeix–Raviart finite element method (abbreviated as CR), the Lagrange finite element method (CG), and the Lehmann–Goerisch method (LG) to compute these lower and upper eigenvalue bounds shown in the table. A sample mesh with maximum edge length (i.e., mesh size) $h_{\max} = 0.194$ is shown in Fig. 3. By using the numerical evaluation of eigenvalues obtained on a highly refined mesh as reference eigenvalues (denoted by $\tilde{\lambda}_i$), we compute the errors of the lower and upper bounds, as shown in Table 2.

It is important to point out that the projection-based lower bounds λ_i^{CR} obtained in the first stage are far too inaccurate to separate λ_3 and λ_4 . While λ_1 and λ_2 are known to be distinct, it is not a priori known whether λ_3 and λ_4 are different. Here, we utilize a quite refined mesh with 1654 elements, for which the conforming space CG_h^3 has $\text{DOF}=30044$, to rigorously confirm their separation by computing validated eigenvalue bounds. By applying Lehmann–Goerisch’s theorem to the cluster of eigenvalues $\lambda_3, \dots, \lambda_6$ and the lower bound $\rho = 74 \leq \lambda_7$, we obtain

$$\lambda_3 \leq 47.51181 < 47.51504 \leq \lambda_4. \quad (5.1)$$

Note that, even with a configuration that successfully separates λ_3 and λ_4 , the results are still too coarse to confirm the separation of λ_5 and λ_6 . Additional effort is needed to distinguish these eigenvalues (if they are different, which is not known). The shifting technique in Remark 4.8 may offer a more efficient approach, as it avoids the need to rigorously solve the large linear system in (4.17).

For recursively refined meshes, the error convergence of various bounds for the eigenvalues is shown in Fig. 4. From the numerical results, the following features of the proposed algorithm are observed.

1. For the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_4$, which may lack full H^2 -regularity, the use of a graded mesh enables the associated eigenvalue bounds computed with linear finite elements to achieve the optimal convergence rate of $O(h^2)$.

Table 2 Errors of eigenvalue bounds shown in Table 1 (dumbbell domain, w.r.t. $\tilde{\lambda}_i$)

	Error of lower bound: $\tilde{\lambda}_i - \underline{\lambda}_i$			Error of upper bound: $\bar{\lambda}_i - \tilde{\lambda}_i$	
	CR	LG(p=1)	LG(p=2)	CG(p=1)	CG(p=2)
λ_1, λ_2	4.9e-02	1.2e-02	3.7e-04	3.2e-02	5.7e-04
λ_3, λ_4	1.2e-01	1.3e-01	4.7e-03	7.5e-02	2.1e-03
λ_5, λ_6	1.1e-01	1.6e-01	2.8e-03	8.1e-02	1.4e-03

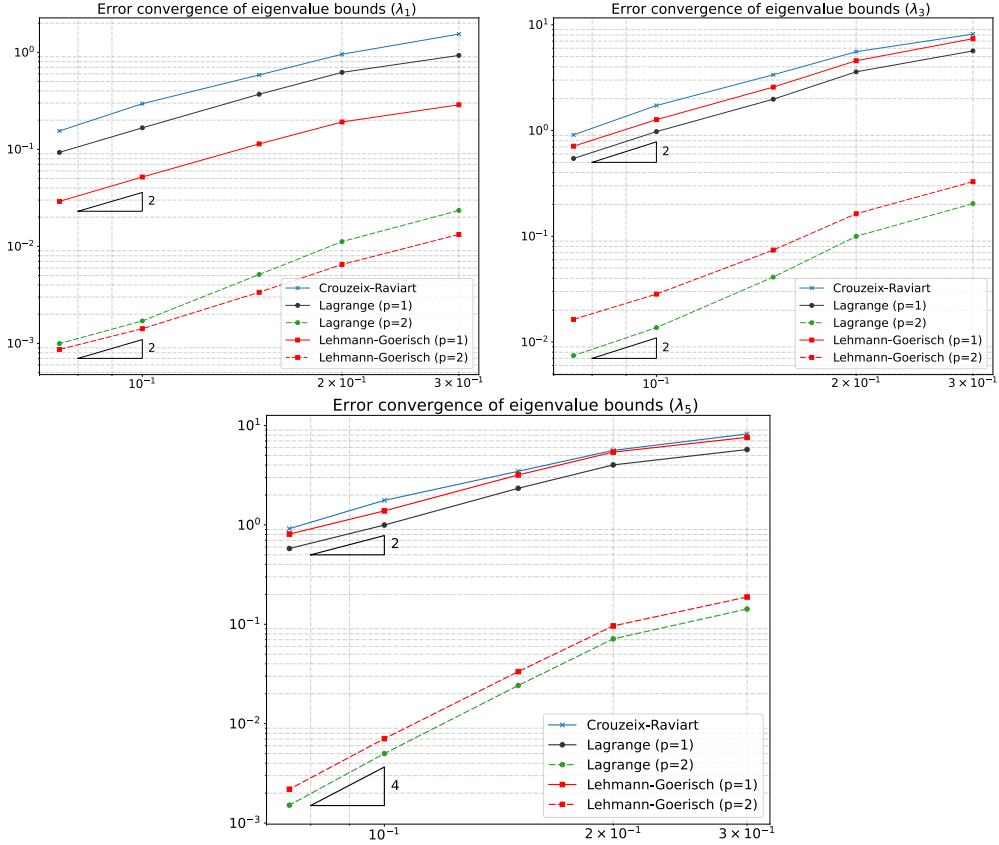


Fig. 4 Convergence of Laplacian eigenvalue bounds for dumbbell domain (x -axis: mesh size $h_{\max}$; y -axis: magnitude of the error).

- When the eigenfunctions possess higher regularity (as one might guess for the eigenfunctions corresponding to λ_5 and λ_6 based on the numerical plots in Fig. 2), the lower bounds obtained via the Lehmann-Goerisch method and the upper bounds from the Lagrange finite element method exhibit optimal convergence rates of $O(h^2)$ for $p = 1$ and $O(h^4)$ for $p = 2$.

3. The Lehmann–Goerisch method using the conforming finite element method with polynomial degree $p = 2$ (i.e., CG_h^2) yields significantly sharper lower bounds for the eigenvalues compared to those obtained using the Crouzeix–Raviart finite element method.

5.2 Steklov eigenvalues

In this section we consider two model Steklov eigenvalue problems: one on the unit square

$$\Omega = (0, 1) \times (0, 1),$$

and one on the L-shaped domain

$$\Omega = (0, 2) \times (0, 2) \setminus [1, 2) \times [1, 2).$$

For each example we obtain a rough lower bound from [Theorem 3.1](#) and a high-precision lower bound via Lehmann–Goerisch’s theorem.

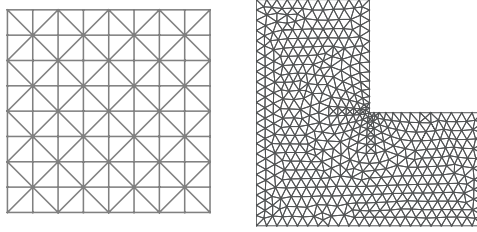


Fig. 5 Uniform mesh for the square domain and graded mesh for the L-shaped domain

Square domain

By separation of variables, this eigenvalue problem can be solved in a closed form, up to the solution of a system of two transcendental equations with two unknowns. These closed-form solutions show that $\lambda_2 = \lambda_3$.

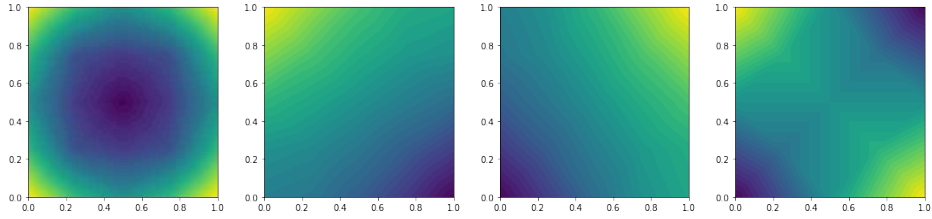


Fig. 6 Approximate eigenfunctions for Steklov eigenvalues $\lambda_i (1 \leq i \leq 4)$ (square domain).

We triangulate Ω uniformly with an $N \times N$ grid; see Figure 5 for $N = 8$. The mesh size $h_{\max}$ as the maximum edge length is $h_{\max} = \sqrt{2}/8$ when $N = 8$. Using conforming FEM with CG_h^2 and $N = 32$, we obtain numerical values for the first four Steklov eigenvalues (see also, e.g., [61]) :

$$\lambda_1 \approx 0.24007909, \quad \lambda_2 = \lambda_3 \approx 1.49230326, \quad \lambda_4 \approx 2.08264706.$$

The eigenfunctions for these 4 eigenvalues are plotted in Figure 6.

For a mesh with $N = 64$, we employ the Crouzeix–Raviart FEM to compute a lower bound for λ_4 via the projection error constant in (3.19) and Theorem 3.1:

$$C_h \leq 0.12717, \quad \lambda_{4,h}^{\text{CR}} \in [2.08216, 2.08217], \quad \lambda_4 \geq 2.01433.$$

Hence, in applying Lehmann–Goerisch’s theorem we take $n = 3$ and

$$\lambda_3 < 1.49246 \leq \rho = 2.0 < \lambda_4. \quad (5.2)$$

Figure 7 shows the convergence of the computed eigenvalue bounds with respect to high-precision reference values. Note that the results for λ_2 and λ_3 are almost the same. We observe that, compared with the convergence rate $O(h)$ of the Crouzeix–Raviart method (see also the theoretical discussion in Remark 3.4), the lower bounds from the P_1 conforming FEM combined with Lehmann–Goerisch’s theorem recover the $O(h^2)$ rate, matching the $O(h^2)$ rate of the upper (Rayleigh–Ritz) bounds from the Lagrange FEM.

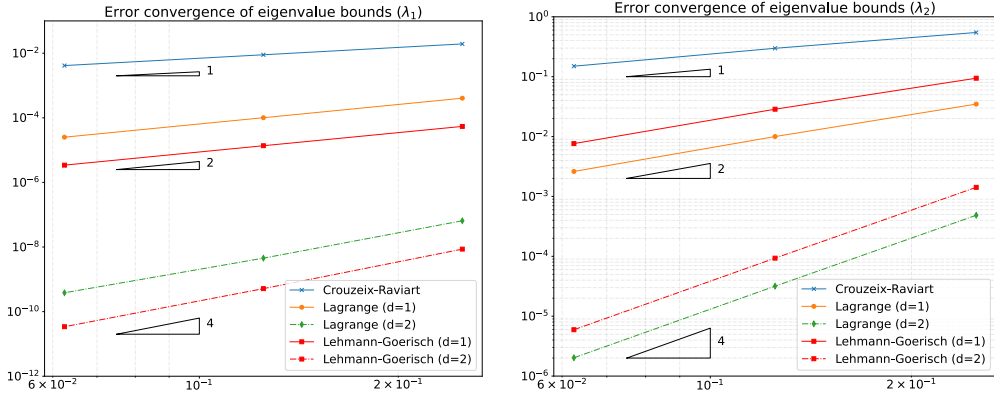


Fig. 7 Error convergence of Steklov eigenvalue bounds on unit square domain (x -axis: mesh size; y -axis: magnitude of the error).

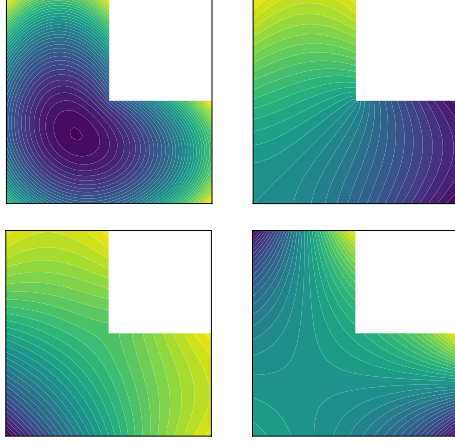


Fig. 8 Approximate eigenfunctions for Steklov eigenvalue λ_i 's ($1 \leq i \leq 4$) (L-shaped domain)

L-shaped domain

Figure 8 displays the computed eigenfunctions for the first four eigenvalues on the L-shaped domain. The first, third, and fourth approximate eigenfunctions are symmetric with respect to reflection across the axis $x_1 = x_2$, whereas the second is antisymmetric.

We use a geometrically graded mesh with $h_K = O(r^{1/3})$, where r is the distance from element K to the re-entrant corner (see Figure 5).

For a graded mesh with $h_{\max} = 0.031$, we employ the Crouzeix–Raviart FEM to obtain the following quantities:

$$C_h \leq 0.14902, \quad \lambda_{4,h} \in [1.69171, 1.69172], \quad \rho = 1.6 < 1.63045 \leq \lambda_4.$$

Figure 9 shows the convergence of the eigenvalue bounds with respect to high-precision reference values. Note that the lower eigenvalue bounds obtained by the Crouzeix–Raviart method exhibit only $O(h)$ convergence, which is due to the projection error term $C_h = O(\sqrt{h})$, as explained in Remark 3.4. The observed convergence rate of $O(h^2)$ for the eigenvalue bounds of λ_2 with $p = 2$ indicates that the eigenfunction associated with λ_2 lacks the higher regularity possessed by those corresponding to λ_1 and λ_3 . On the graded mesh, the P_1 conforming FEM achieves the expected $O(h^2)$ rate for both lower and upper bounds. In contrast, the P_2 conforming FEM exhibits a reduced convergence rate of $O(h^2)$ for the second eigenvalue, while retaining $O(h^4)$ convergence for the first and third eigenvalues. Based on this observation, one might guess that the eigenfunctions corresponding to the first and third eigenvalues have H^3 regularity.

Discussion on rigorous eigenvalue bounds

To control floating-point and rounding errors, we employ interval arithmetic via INTLAB [62] together with Behnke’s method [50], as mentioned at the beginning of this section. This provides fully verified eigenvalue bounds for the resulting generalized

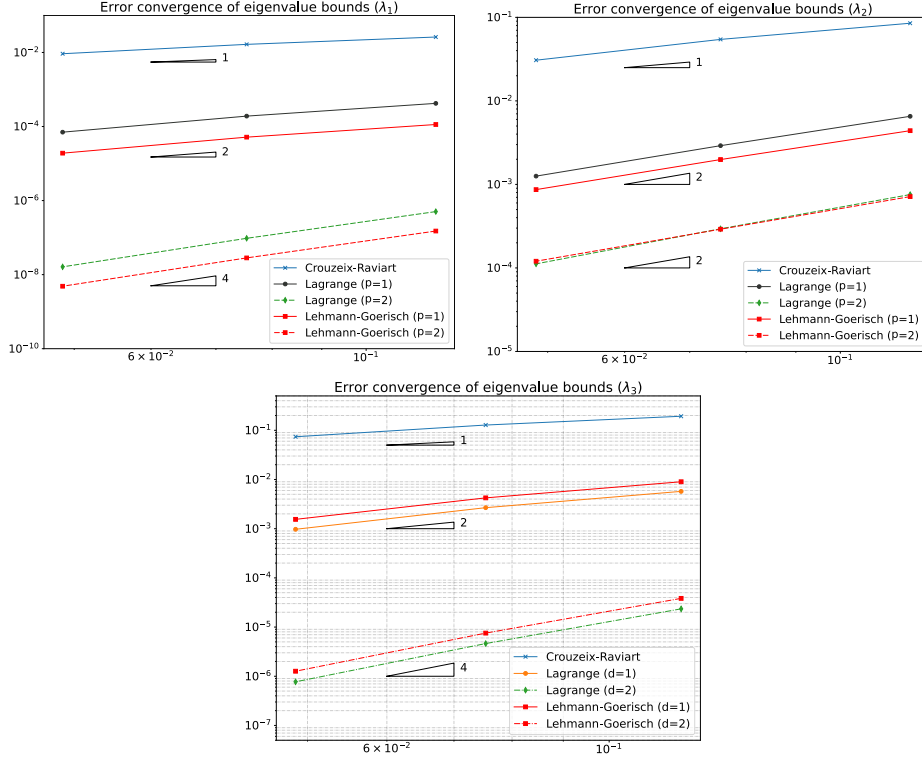


Fig. 9 Error convergence of Steklov eigenvalue bounds on the L-shaped domain (x -axis: mesh size $h_{\max}$; y -axis: magnitude of the error).

matrix eigenproblems. The verified Steklov eigenvalue bounds based on the setting (5.2) are listed in Tables 3 and 4.

Note that the selection of an eigenvalue cluster when applying the Lehmann–Goerisch method affects the accuracy of the lower bounds. For example, in the case of Steklov eigenvalues, by choosing

$$\lambda_1 < \rho = 1.45 \leq \lambda_2,$$

on an 8×8 triangulation of the square domain, and using the second-order Lagrange finite element space CG_h^2 , we obtain the following rigorous bounds:

$$0.240079083 \leq \lambda_1 \leq 0.240079091; \quad \bar{\lambda}_1 - \underline{\lambda}_1 = 8.00\text{E} - 09.$$

This result is significantly sharper than the one presented in Table 3, which was computed using a finer 32×32 triangulation.

Table 3 Rigorous bounds for the Steklov eigenvalues with $p = 2$ and $N = 32$ (square domain)

Rigorous bound	Bound gap: $\bar{\lambda}_i - \underline{\lambda}_i$
$\underline{0.23993698} \leq \lambda_1 \leq \underline{0.24007908546}$	1.42E-04
$\underline{1.49204771} \leq \lambda_2 \leq \underline{1.492303245657}$	2.56E-04
$\underline{1.49204789} \leq \lambda_3 \leq \underline{1.492303278880}$	2.56E-04

Table 4 Rigorous bounds for the Steklov eigenvalues with $p = 2$ and $h = 0.118$ (L-shaped domain)

Rigorous bound	Bound gap: $\bar{\lambda}_i - \underline{\lambda}_i$
$\underline{0.341414120} \leq \lambda_1 \leq \underline{0.34141605885}$	1.93E-06
$\underline{0.616711190} \leq \lambda_2 \leq \underline{0.61698333496}$	2.72E-04
$\underline{0.984274379} \leq \lambda_3 \leq \underline{0.98427919894}$	4.83E-06

5.3 A special tetrahedron with analytic eigenfunctions

We also test the method on a special tetrahedron

$$T_F = \text{conv} \left\{ (0, 0, 0), (0, 0, 1), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}.$$

For this domain, the homogeneous Dirichlet Laplacian has explicit analytic eigenfunctions given by generalized sine functions on the tetrahedral reflection lattice [63]. The first 4 eigenvalues are

$$\lambda_1 = 20\pi^2, \quad \lambda_2 = \lambda_3 = 35\pi^2, \quad \lambda_4 = 40\pi^2.$$

This subsection demonstrates the efficiency of the Lehmann–Goerisch method for 3D problems. Since our focus is on the second stage of the method, we skip the first stage and use the exact eigenvalues directly as the Lehmann–Goerisch separation values:

$$\rho_2 = \lambda_2 = 35\pi^2, \quad \rho_4 = \lambda_4 = 40\pi^2.$$

To study convergence on this analytic benchmark, we computed the bounds on two coarse uniform meshes for both $\text{CG}^2 + \text{RT}^2$ and $\text{CG}^3 + \text{RT}^3$, using the exact separation values above. The level-2 mesh has 64 tetrahedra and the level-3 mesh has 512 tetrahedra. The numerical results are reported in Table 6. The convergence orders of the upper bounds (UBs) and lower bounds (LBs) at level k are computed as $\log_2(\text{Err}_{k-1}/\text{Err}_k)$ where Err_k denotes the error of the corresponding bound on the level- k mesh.

Table 5 Coarse-mesh convergence on T_F . The interval entries are LG lower bound (LB) / CG upper bound (UB). The level-2 and level-3 meshes have 64 and 512 tetrahedra, respectively. The observed order triples (1, 2, 3) are computed from the CG upper-bound errors and LG lower-bound errors after one uniform refinement from level 2 to level 3.

p	level	λ_1 interval	λ_2 interval	λ_3 interval	UB order	LB order
2	2	[171.35, 209.37]	[-, 372.50]	[65.22, 433.87]	–	–
2	3	[193.96, 198.76]	[238.85, 350.12]	[297.07, 355.78]	(3.1, 2.5, 3.1)	(2.9, –, 2.5)
3	2	[194.72, 198.56]	[264.79, 351.38]	[289.12, 353.84]	–	–
3	3	[197.33, 197.42]	[342.23, 345.59]	[343.85, 345.74]	(5.5, 5.3, 4.8)	(5.3, 4.7, 5.1)

The observed orders clearly show the faster convergence of the cubic conforming space in both the Rayleigh–Ritz upper bounds and the Lehmann–Goerisch lower bounds, even on these coarse meshes. The missing CG², level-2 entry for λ_2 means that the corresponding 2×2 Lehmann–Goerisch pencil has only one negative generalized eigenvalue; applying the lower-bound formula to the positive generalized eigenvalue would produce a spurious negative number and is not a certified lower bound. This failure is also predicted by Lemma 4.4: for the CG², level-2 computation, the required separation condition $\Lambda_3 < \rho_4$ does not hold, since the computed third Rayleigh–Ritz value is about 433.87 while $\rho_4 = 40\pi^2 \approx 394.78$.

We then repeat the same test with a higher local polynomial degree, using CG⁶ + RT⁶ on the original tetrahedron and on its first two regular uniform refinements.

Table 6 High-order CG⁶ + RT⁶ bounds on T_F using the exact separation values $\rho_2 = \lambda_2$ and $\rho_4 = \lambda_4$ ($\hat{\lambda} = 200$). The interior CG DOF / RT DOF values for the three mesh rows are 10/280, 165/2016, and 1771/15232, respectively. The order triples (1, 2, 3) are computed from the CG upper-bound errors and LG lower-bound errors between consecutive refinements.

Refinement level	λ_1 interval	λ_2 interval	λ_3 interval	UB order	LB order
0 (#elt=1)	[195.255, 198.586]	[-762.741, 413.561]	[-762.741, 413.561]	–	–
1 (#elt=8)	[197.310, 197.432]	[310.499, 345.975]	[340.704, 349.825]	(4.9, 7.0, 4.0)	(4.7, 5.0, 7.9)
2 (#elt=64)	[197.392, 197.392]	[345.383, 345.437]	[345.424, 345.441]	(10.3, 8.9, 9.7)	(9.5, 9.4, 8.6)

The one-element CG⁶ space is too small to give useful lower bounds for the double eigenvalue; in fact the computed LG lower bounds for λ_2, λ_3 are negative. A plausible reason is that this coarse space does not resolve the double eigenspace and its separation from λ_4 well enough. After one and two regular uniform refinements the certified intervals tighten substantially.

5.4 Dirichlet Laplacian on the regular tetrahedron

We next give a compact three-dimensional test on the unit regular tetrahedron (all six edges have length one). Unlike the special tetrahedron T_F , this domain has no known closed-form eigenvalues. Moreover, although it is convex, solutions of elliptic boundary value problems on polyhedra generally possess only limited Sobolev regularity because of edge and vertex singularities. Thus high-order finite element and spectral

approximations on the regular tetrahedron may display algebraic convergence rather than exponential convergence, as observed numerically by Jia, Li, and Zhang [63].

The rough separation values used by the Lehmann–Goerisch method were first computed by the Crouzeix–Raviart lower-bound method on a level-6 mesh with 127661 tetrahedra and 247130 interior-facet degrees of freedom. If $\lambda_{k,h}^{\text{CR}}$ denotes the CR eigenvalue and $C_h = 0.3804h_{\max}$ [18], the lower bound from Theorem 3.1 is

$$\lambda_k \geq \frac{\lambda_{k,h}^{\text{CR}}}{1 + C_h^2 \lambda_{k,h}^{\text{CR}}}.$$

This gives

$$\rho_2 = 264.522 < \lambda_2, \quad \rho_5 = 386.441 < \lambda_5$$

and these two values are used below. The reported Lehmann–Goerisch lower bounds in this subsection are certified by INTLAB interval arithmetic [62].

We solve the Dirichlet Rayleigh–Ritz problem on a mesh of 8 tetrahedra, obtained by one regular uniform refinement of the regular tetrahedron, using a degree-8 conforming space. The eigenvalue λ_1 is bounded by a scalar LG matrix eigenvalue problem with ρ_2 , while $\lambda_2, \lambda_3, \lambda_4$ are bounded by a 3×3 LG problem with ρ_5 . The auxiliary space is the scalar RT^8 space; the conforming CG^8 Dirichlet space has 455 interior degrees of freedom, the scalar DG space has dimension 1320, the RT^8 space has dimension 3960, and the shifted mixed linear system has dimension $3960 + 1320 = 5280$. The Lehmann–Goerisch small matrices A_0, A_1, A_2 are assembled in interval arithmetic directly on the physical tetrahedra by closed-form Bernstein integrals; in particular the auxiliary RT^8 mass contribution to A_2 is enclosed via a verified solve on each element rather than by enclosing a floating-point matrix.

Table 7 Degree-8 Rayleigh–Ritz and Lehmann–Goerisch bounds from Theorem 4.1 for the Dirichlet Laplacian on the unit regular tetrahedron (8 tetrahedra, $\text{CG}^8 + \text{RT}^8$, $\hat{\lambda} = 200$). All four bounds are certified by INTLAB interval arithmetic.

k	Lower bound	Upper bound	Rel. gap
1	150.9700841736	150.9727752307	0.0018%
2	278.2131313380	278.6994250425	0.17%
3	278.3047808848	278.6994250425	0.14%
4	278.3047808848	278.7253072428	0.15%

Table 7 shows that a very small number of high-order tetrahedra is already effective for this symmetric polyhedron: a degree-8 conforming space on only 8 tetrahedra certifies the isolated eigenvalue λ_1 to a relative gap below 0.002% and the triply-degenerate cluster $\lambda_2, \lambda_3, \lambda_4$ to relative gaps below 0.17%. The CR calculation is used

only to obtain the rough spectral separation values ρ_2 and ρ_5 ; the sharp lower bounds are produced from the degree-8 conforming trial space via the LG method.

Efficiency of the LG lower-bound stage

To estimate what a direct CR computation would require to reach the same accuracy, we used the observed CR convergence from the two finest available regular-tetrahedron CR computations. The CR meshes have $N = 33496$ and $N = 247130$ interior-facet degrees of freedom, respectively. Let $\text{LB}_k^{\text{CR}}(N)$ denote the CR lower bounds. By fitting

$$\text{UB}_k - \text{LB}_k^{\text{CR}}(N) \approx C_k N^{-\alpha_k},$$

where UB_k are the upper bounds in Table 7, we obtain the equivalent mesh-size orders $q_k = 3\alpha_k$ shown below. The target accuracy is taken as the certified interval width $\varepsilon_k^{\text{LG}} = \lambda_k^{\text{UB}} - \lambda_k^{\text{LB}}$, reported in Table 7.

We define the efficiency factor by

$$\eta_k^{\text{eff}} := \frac{N_{\text{CR},k}^{\text{pred}}}{N_{\text{LG}}}, \quad N_{\text{LG}} := \dim \text{RT}^8 + \dim \text{DG}^8 = 5280,$$

where $N_{\text{CR},k}^{\text{pred}}$ is the predicted number of CR degrees of freedom needed to reduce the fitted CR lower-bound error to $\varepsilon_k^{\text{LG}}$.

Table 8 Predicted direct-CR cost for matching the degree-8, 8-tetrahedron $\text{CG}^8 + \text{RT}^8$ LG interval widths on the unit regular tetrahedron.

k	Observed CR order q_k	$\varepsilon_k^{\text{LG}}$	$N_{\text{CR},k}^{\text{pred}}$	η_k^{eff}
1	2.047	2.691×10^{-3}	1.21×10^{10}	2.29×10^6
2	1.949	4.863×10^{-1}	4.44×10^7	8.40×10^3
3	1.948	3.946×10^{-1}	6.14×10^7	1.16×10^4
4	1.948	4.205×10^{-1}	5.57×10^7	1.05×10^4

Thus, even under the observed approximately second-order CR convergence, a direct CR lower-bound computation would require tens of millions to tens of billions of degrees of freedom to match the certified widths obtained here from a 5280-dimensional RT–DG mixed system. For all first four eigenvalues simultaneously, the first eigenvalue is the limiting case and gives

$$\max_{1 \leq k \leq 4} \eta_k^{\text{eff}} \approx 2.29 \times 10^6.$$

This comparison highlights the role of the CR method in the two-stage algorithm: it is efficient for producing the rough separation parameters ρ_2 and ρ_5 , while the

Lehmann–Goerisch stage converts a small high-order conforming computation into sharp certified lower bounds.

Summary

In this paper, a two-stage algorithm is proposed to provide high-precision guaranteed eigenvalue bounds for differential operators. The application of this algorithm to the Dirichlet Laplacian eigenvalue problem and the Steklov eigenvalue problem demonstrates its adaptivity to graded meshes, high-order FEMs and provides high-precision eigenvalue bounds. The algorithm extends naturally to three dimensions, as illustrated by the first rigorous two-sided eigenvalue bounds for the regular tetrahedron in §5.4. It is worth pointing out that in addition to second-order differential operators, the proposed algorithm can also be applied to higher order differential operators, for example, the biharmonic operator. Early approaches for biharmonic operators can be found in [54, 64] and more detailed discussions will be provided in our subsequent work.

Declarations

Funding. The first author is supported by Grant Numbers JP20KK0306, JP24K00538, JP21H00998 and the Japan-China bilateral joint research project (JPJSBP120237407).

References

- [1] Carstensen, C., Zhai, Q., Zhang, R.: A skeletal finite element method can compute lower eigenvalue bounds. *SIAM J. Numer. Anal.* **58**(1), 109–124 (2020) <https://doi.org/10.1137/18M1212276>
- [2] Carstensen, C., Ern, A., Puttkammer, S.: Guaranteed lower bounds on eigenvalues of elliptic operators with a hybrid high-order method. *Numerische Mathematik* **149**(2), 273–304 (2021) <https://doi.org/10.1007/s00211-021-01228-1>
- [3] Carstensen, C., Puttkammer, S.: Direct guaranteed lower eigenvalue bounds with optimal a priori convergence rates for the bi-laplacian. *SIAM J. Numer. Anal.* **61**(2), 812–836 (2023) <https://doi.org/10.1137/21M139921X>
- [4] Plum, M.: Eigenvalue inclusions for second-order ordinary differential operators by a numerical homotopy method. *Zeitschrift für angewandte Mathematik und Physik ZAMP* **41**(2), 205–226 (1990) <https://doi.org/10.1007/BF00945108>
- [5] Plum, M.: Bounds for eigenvalues of second-order elliptic differential operators. *Zeitschrift für Angew. Math. und Phys.* **42**(6), 848–863 (1991) <https://doi.org/10.1007/BF00944567>

- [6] Nakao, M.T., Plum, M., Watanabe, Y.: Numerical Verification Methods and Computer-assisted Proofs for Partial Differential Equations. Springer, Singapore (2019). <https://doi.org/10.1007/978-981-13-7669-6>
- [7] Babuška, I., Osborn, J.: Eigenvalue problems. In: Ciarlet, P.G., Lions, J.L. (eds.) Finite Element Methods (Part 1). Handbook of Numerical Analysis, vol. II, pp. 641–787. Elsevier Science Publishers, North-Holland (1991). [https://doi.org/10.1016/S1570-8659\(05\)80042-0](https://doi.org/10.1016/S1570-8659(05)80042-0)
- [8] Boffi, D.: Finite element approximation of eigenvalue problems. *Acta Numerica* **19**, 1–120 (2010) <https://doi.org/10.1017/S0962492910000012>
- [9] Lehmann, N.J.: Optimale Eigenwerteinschließungen. *Numer. Math.* **5**(1), 246–272 (1963) <https://doi.org/10.1007/BF01385896>
- [10] Behnke, H., Goerisch, F.: Inclusions for eigenvalues of selfadjoint problems. *Topics in Validated Computations* (ed. J. Herzberger), North-Holland, Amsterdam, 277–322 (1994)
- [11] Goerisch, F., He, Z.: The determination of guaranteed bounds to eigenvalues with the use of variational methods I. In: Computer Arithmetic and Self-validating Numerical Methods, pp. 137–153 (1990). <https://doi.org/10.1016/B978-0-12-708245-5.50012-9>. Academic Press Professional, Inc.
- [12] Albrecht, J., Goerisch, F.: Anwendungen des Verfahrens von Lehmann auf Schwingungsprobleme. In: Albrecht, J., Collatz, L., Velte, W., Wunderlich, W. (eds.) Numerical Treatment of Eigenvalue Problems Vol.4 / Numerische Behandlung Von Eigenwertaufgaben Band 4: Workshop in Oberwolfach, 1986 / Tagung in Oberwolfach, 1986, pp. 1–9. Birkhäuser Basel, Basel (1987). https://doi.org/10.1007/978-3-0348-7507-3_1
- [13] Plum, M.: Guaranteed numerical bounds for eigenvalues. In: Spectral Theory & Computational Methods of Sturm-Liouville Problems vol. 191, pp. 313–332. Marcel Dekker, New York (1997)
- [14] Birkhoff, G., Boor, C.D., Swartz, B., Wendroff, B.: Rayleigh–Ritz approximation by piecewise cubic polynomials. *SIAM J. Numer. Anal.*, 188–203 (1966) <https://doi.org/10.1137/0703015>
- [15] Kikuchi, F., Liu, X.: Estimation of interpolation error constants for the P_0 and P_1 triangular finite element. *Comput. Methods Appl. Mech. Eng.* **196**, 3750–3758 (2007) <https://doi.org/10.1016/j.cma.2006.10.029>
- [16] Liu, X., Kikuchi, F.: Analysis and estimation of error constants for P_0 and P_1 interpolations over triangular finite elements. *J. Math. Sci. Univ. Tokyo* **17**, 27–78 (2010)

- [17] Liu, X., Oishi, S.: Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. *SIAM J. Numer. Anal.* **51**(3), 1634–1654 (2013) <https://doi.org/10.1137/120878446>
- [18] Liu, X.: A framework of verified eigenvalue bounds for self-adjoint differential operators. *Appl. Math. Comput.* **267**, 341–355 (2015) <https://doi.org/10.1016/j.amc.2015.03.048>
- [19] Kobayashi, K.: On the interpolation constants over triangular elements. In: *Application of Mathematics 2015*, pp. 110–124. Institute of Mathematics CAS, Prague (2015). <http://eudml.org/doc/287821>
- [20] Carstensen, C., Gallistl, D.: Guaranteed lower eigenvalue bounds for the biharmonic equation. *Numerische Mathematik* **126**(1), 33–51 (2014) <https://doi.org/10.1007/s00211-013-0559-z>
- [21] Carstensen, C., Gedicke, J.: Guaranteed lower bounds for eigenvalues. *Mathematics of Computation* **83**(290), 2605–2629 (2014) <https://doi.org/10.1090/S0025-5718-2014-02833-0>
- [22] Liu, X.: *Guaranteed Computational Methods for Self-Adjoint Differential Eigenvalue Problems*. SpringerBriefs in Mathematics. Springer, Singapore (2024). <https://doi.org/10.1007/978-981-97-3577-8>
- [23] Liu, X., Oishi, S.: Verified eigenvalue evaluation for laplace operator on arbitrary polygonal domain. In: *Proceedings of RIMS Workshop Held on 18–21 October 2010* vol. 1733, pp. 31–39. RIMS Kokyuroku, Kyoto (2011). <http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/1733-03.pdf>
- [24] You, C., Xie, H., Liu, X.: Guaranteed eigenvalue bounds for the Steklov eigenvalue problem. *SIAM J. Numer. Anal.* **57**(3), 1395–1410 (2019) <https://doi.org/10.1137/18M1189592>
- [25] Liu, X.: Explicit eigenvalue bounds of differential operators defined by symmetric positive semi-definite bilinear forms. *J. Comput. Appl. Math.* **371**, 112666 (2020) <https://doi.org/10.1016/j.cam.2019.112666>
- [26] Liu, X., You, C.: Explicit bound for quadratic Lagrange interpolation constant on triangular finite elements. *Applied Mathematics and Computation* **319**, 693–701 (2018) <https://doi.org/10.1016/j.amc.2017.08.020> . Recent Advances in Computing
- [27] Liao, S., Shu, Y., Liu, X.: Optimal estimation for the Fujino–Morley interpolation error constants. *Jpn. J. Ind. Appl. Math.* **36**(2), 521–542 (2019) <https://doi.org/10.1007/s13160-019-00351-9>
- [28] Xie, M., Xie, H., Liu, X.: Explicit lower bounds for Stokes eigenvalue problems

- by using nonconforming finite elements. *Jpn. J. Ind. Appl. Math.* **35**(1), 335–354 (2018) <https://doi.org/10.1007/s13160-017-0291-7>
- [29] Liu, X., Nakao, M.T., You, C., Oishi, S.: Explicit a posteriori and a priori error estimation for the finite element solution of Stokes equations. *Jpn. J. Ind. Appl. Math.* **38**(2), 545–559 (2021) <https://doi.org/10.1007/s13160-020-00449-5>
 - [30] Nakano, T., Li, Q., Sue, M., Liu, X.: Guaranteed lower eigenvalue bounds for Steklov operators using conforming finite element methods. *Comput. Methods Appl. Math.* **24**(2), 495–510 (2024) <https://doi.org/10.1515/cmam-2022-0218>
 - [31] Gallistl, D., Olkhovskiy, V.: Computational lower bounds of the Maxwell eigenvalues. *SIAM J. Numer. Anal.* **61**(2), 539–561 (2023) <https://doi.org/10.1137/21M1461447>
 - [32] Nakao, M.T., Yamamoto, N., Nagatou, K.: Numerical verifications for eigenvalues of second-order elliptic operators. *Jpn. J. Ind. Appl. Math.* **16**(3), 307–320 (1999) <https://doi.org/10.1007/BF03167360>
 - [33] Lahmann, J.R., Plum, M.: A computer-assisted instability proof for the Orr-Sommerfeld equation with Blasius profile. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics* **84**(3), 188–204 (2004) <https://doi.org/10.1002/zamm.200310093>
 - [34] Watanabe, Y., Plum, M., Nakao, M.T.: A computer-assisted instability proof for the Orr-Sommerfeld problem with Poiseuille flow. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics* **89**(1), 5–18 (2009) <https://doi.org/10.1002/zamm.200700158>
 - [35] Watanabe, Y., Nagatou, K., Plum, M., Nakao, M.T.: A computer-assisted stability proof for the Orr-Sommerfeld problem with Poiseuille flow. *Nonlinear Theory and Its Applications, IEICE* **2**(1), 123–127 (2011) <https://doi.org/10.1587/nolta.2.123>
 - [36] Watanabe, Y., Nagatou, K., Plum, M., Nakao, M.T.: Verified computations of eigenvalue enclosures for eigenvalue problems in Hilbert spaces. *SIAM J. Numer. Anal.* **52**(2), 975–992 (2014) <https://doi.org/10.1137/120894683>
 - [37] Nakao, M.T.: A numerical verification method for the existence of weak solutions for nonlinear boundary value problems. *J. Math. Anal. Appl.* **164**(2), 489–507 (1992) [https://doi.org/10.1016/0022-247X\(92\)90129-2](https://doi.org/10.1016/0022-247X(92)90129-2)
 - [38] Plum, M.: Numerical existence proofs and explicit bounds for solutions of nonlinear elliptic boundary value problems. *Computing* **49**(1), 25–44 (1992) <https://doi.org/10.1007/BF02238648>

- [39] Nagatou, K., Plum, M., Nakao, M.T.: Eigenvalue excluding for perturbed-periodic one-dimensional Schrödinger operators. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **468**(2138), 545–562 (2012) <https://doi.org/10.1098/rspa.2011.0159>
- [40] Hu, J., Huang, Y., Shen, H.: The lower approximation of eigenvalue by lumped mass finite element methods. *J. Comput. Math.* **22**, 545–556 (2004)
- [41] Armentano, M.G., Durán, R.G.: Asymptotic lower bounds for eigenvalues by nonconforming finite element methods. *Electron. Trans. Numer. Anal* **17**(2), 93–101 (2004)
- [42] Yang, Y., Zhang, Z., Lin, F.: Eigenvalue approximation from below using non-conforming finite elements. *Sci. China Ser. A Math.* **51**(1), 1–18 (2010) <https://doi.org/10.1007/s11425-009-0198-0>
- [43] Luo, F., Lin, Q., Xie, H.: Computing the lower and upper bounds of Laplace eigenvalue problem: by combining conforming and nonconforming finite element methods. *Sci. China Math.* **55**(5), 1069–1082 (2012) <https://doi.org/10.1007/s11425-012-4382-2>
- [44] Liu, X., Oishi, S.: Guaranteed high-precision estimation for P_0 interpolation constants on triangular finite elements. *Jpn. J. Ind. Appl. Math.* **30**, 635–652 (2013) <https://doi.org/10.1007/s13160-013-0120-6>
- [45] Liu, X., Okayama, T., Oishi, S.: High-Precision Eigenvalue Bound for the Laplacian with Singularities. In: *Computer Mathematics*, pp. 311–323. Springer, Berlin, Heidelberg (2014). https://doi.org/10.1007/978-3-662-43799-5_23
- [46] Vejchodský, T.: Three methods for two-sided bounds of eigenvalues—a comparison. *Numerical Methods for Partial Differential Equations* **34**(4), 1188–1208 (2018) <https://doi.org/10.1002/num.22251>
- [47] Kato, T.: *Perturbation Theory for Linear Operators*. Springer, Berlin, Heidelberg (1966). <https://doi.org/10.1007/978-3-642-53393-8>
- [48] Weinberger, H.F.: *Variational Methods for Eigenvalue Approximation*. Society for Industrial Mathematics, Philadelphia (1974). <https://doi.org/10.1137/1.9781611970531>
- [49] Adams, R.A., Fournier, J.J.F.: *Sobolev Spaces*. Elsevier, Amsterdam (2003)
- [50] Behnke, H.: The calculation of guaranteed bounds for eigenvalues using complementary variational principles. *Computing* **47**(1), 11–27 (1991) <https://doi.org/10.1007/BF02242019>

- [51] Demengel, F., Demengel, G.: Functional Spaces for the Theory of Elliptic Partial Differential Equations. Springer, London (2012). <https://doi.org/10.1007/978-1-4471-2807-6>
- [52] Breuer, B., McKenna, P.J., Plum, M.: Multiple solutions for a semilinear boundary value problem: a computational multiplicity proof. *Journal of Differential Equations* **195**(1), 243–269 (2003) [https://doi.org/10.1016/S0022-0396\(03\)00186-4](https://doi.org/10.1016/S0022-0396(03)00186-4)
- [53] Breuer, B., Horák, J., McKenna, P.J., Plum, M.: A computer-assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam. *Journal of Differential Equations* **224**(1), 60–97 (2006) <https://doi.org/10.1016/j.jde.2005.07.016>
- [54] Goerisch, F., Haunhorst, H.: Eigenwertschranken für Eigenwertaufgaben mit partiellen Differentialgleichungen. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik* **65**, 129–135 (1985) <https://doi.org/10.1515/9783112547168-001>
- [55] Goerisch, F., Albrecht, J.: The convergence of a new method for calculating lower bounds to eigenvalues. In: *Proceedings of the International Conference on Differential Equations and Their Applications Held in Brno, Czechoslovakia, 1985*, pp. 303–308. Springer, Berlin, Heidelberg (1986). <https://doi.org/10.1007/BFb0076085>
- [56] Beattie, C., Goerisch, F.: Methods for computing lower bounds to eigenvalues of self-adjoint operators. *Numerische Mathematik* **72**(2), 143–172 (1995) <https://doi.org/10.1007/s002110050164>
- [57] Arnold, D.N., F. Brezzi: Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. *RAIRO Math. Model. & Numer. Anal.* **19**, 7–32 (1985) <https://doi.org/10.1051/m2an/1985190100071>
- [58] Brezzi, F., Fortin, M.: *Mixed and hybrid finite element methods*. Springer (1991) <https://doi.org/10.1007/978-1-4612-3172-1>
- [59] Liu, X.: VFEM.jl: verified finite element method in 2D and 3D for guaranteed eigenvalue bounds. GitHub repository (2026). <https://github.com/xfliu/vfem>
- [60] Liu, X.: Veigs: a MATLAB and Julia library for verified eigenvalue computation for matrices. GitHub repository (2026). <https://github.com/xfliu/veigs>
- [61] Yang, Y., Li, Q., Li, S.: Nonconforming finite element approximations of the Steklov eigenvalue problem. *Applied Numerical Mathematics* **59**(10), 2388–2401 (2009) <https://doi.org/10.1016/j.apnum.2009.04.005>

- [62] Rump, S.: Intlab-interval laboratory. In: Csendes, T. (ed.) *Developments in Reliable Computing*, pp. 77–104. Kluwer Academic Publishers, Dordrecht (1999). https://doi.org/10.1007/978-94-017-1247-7_7
- [63] Jia, L., Li, H., Zhang, Z.: Sparse spectral-galerkin method on an arbitrary tetrahedron using generalized koornwinder polynomials. *Journal of Scientific Computing* **91**(1), 22 (2022) <https://doi.org/10.1007/s10915-022-01778-y>
- [64] Wiener, C.: Bounds for the N lowest eigenvalues of fourth-order boundary value problems. *Computing* **59**, 29–41 (1997) <https://doi.org/10.1007/BF02684402>