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Research Article

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Posted Date: October 29th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11153320/v1>

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Additional Declarations: No competing interests reported.

Merging nonlinear finite element models and the floating frame of reference formulation

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Abstract

The current state of the art floating frame of reference formulation is used for flexible multibody dynamics simulations. In this formulation, the flexibility is modelled using linear finite element models with respect to a moving frame. Nevertheless, taking into account higher order terms locally can be important for the purpose of modelling moderate to large strains within a body. Nonlinear finite element models contain such terms. In this work, it is explained how the floating frame of reference formulation and the nonlinear finite element formulation can be merged to obtain the best of both worlds. A new nonlinear finite element formulation is developed relative to a body's floating frame. This is used to find expressions for the internal elastic loads and tangent stiffness matrix. Both expressions contain all possible higher order terms and can be used in the incremental solution procedures for static and dynamic simulations. The new formulation is developed such that when these higher order terms are neglected, a floating frame formulation is obtained in terms of the absolute interface coordinates. Hence, this work enables inclusion of carefully selected higher order terms in the floating frame formulation for specific applications. In this way, the floating frame of reference formulation allows for higher order terms while at the same time model order reduction techniques can still be applied.

Keywords: *nonlinear finite element method, floating frame of reference formulation, multibody dynamics, large strains, tangent stiffness matrix*

1. Introduction

In the field of flexible multibody dynamics, systems of flexible bodies are considered that are subjected to arbitrary large rigid body motions but small elastic deformations. The *floating frame formulation* is a commonly applied formulation to describe this [9,10]. In this formulation, the body's floating frame is used to define how the undeformed body moves along with the deformed body. This motion can be interpreted as the gross motion or rigid body motion relative to the inertial frame. Elastic displacements are described with respect to this floating frame. Because the floating frame is positioned and oriented near the deformed body, the elastic displacements are relatively small. Therefore, floating frame formulation can assume a linear strain definition and a linear finite element model is sufficient to describe the body's local deformation.

Due to the application of the linear finite element technique, well-developed model order reduction techniques can be applied to decrease the number of degrees of freedom and optimize the computation times [4,5,6,7,8]. For instance, the Craig-Bampton method is a simple and powerful reduction method commonly applied in the floating frame formulation. Because the used boundary Craig-Bampton modes can be interpreted as the body's shape functions, the resulting flexible multibody model, is also referred to as a superelement.

In general, the floating frame formulation is numerically very efficient when simulating a flexible multibody system. However, assuming local linear elastic behavior is sometimes not sufficient to accurately describe the dynamics of a system. This is for instance the case when slender bodies are subjected to high axial loads. Stretching-bending coupling effects are a result of the higher order terms in the strain definition and these geometric nonlinear effects are not included in the superelement of the classical floating frame formulation. If required, ad-hoc methods are available to include them afterwards, but without a systematic approach.

In nonlinear solid mechanics formulations, these problems do not arise. The resulting equations of equilibrium then are most often solved with two well-known methods: the Total Lagrange method and the Updated Lagrange method [1,2,3].

Both methods apply a strain definition with non-linear terms in the displacements. At the one hand, the Total Lagrange method describes the deformation of the body with a displacement field relative to an undeformed body which is fixed in space. These displacements and rotations then can become very large. To correctly describe the large rotations, the higher order terms in the strain definition are required. On the other hand, the Updated Lagrange method solves the equations based on an incremental displacement field relative to the currently deformed configuration of the body. The displacements then are small but still the higher order terms in the strain definition are taken into account. Both methods leads to *nonlinear finite element formulations*. Unfortunately, for some applications in flexible multibody dynamics, it yields unnecessary large models with considerable computation times that are not easy to reduce.

In this work it is shown that it is possible to combine the floating frame of reference method with the nonlinear finite elements methods. To that end, a floating frame is introduced in a nonlinear continuum mechanics formulation. The floating frame will be located near the body and, as a consequence, the displacements relative to the floating frame remain relatively small. However, like in the Updated Lagrange methods, the strains are allowed to be arbitrarily large. At this stage, a linear elastic material behavior is assumed. With specific kinematic relations and carefully selected degrees of freedom, the presented new floating frame formulation becomes, in a way, an alternative nonlinear finite element formulation, combining the Total Lagrange and the Updated Lagrange methods.

The purpose of this work is to show how the nonlinear finite element methods and the floating frame formulations can be merged and in this way:

- Demonstrate that the floating frame formulation can be obtained directly as a simplification of the nonlinear continuum mechanics, instead of assuming linear continuum mechanics initially. This can be understood as a unification of all different formulations mentioned above.
- Identify the exact forms of the higher order terms that are ignored in the traditional floating frame formulation. When assuming linear strain-displacement relations a priori, this is often not clear. With this it can be understood how to add relevant nonlinear terms to the floating frame formulation.

In order to achieve this, the starting point is the set of standard equations of equilibrium from nonlinear continuum mechanics, in which the undeformed reference body is fixed in an inertial frame. These equations are reformulated, such that they allow for a moving undeformed body. Subsequently applying the nonlinear finite element method, a description of the body in discretized form is obtained where both methods are merged. Expressions for the internal elastic loads and tangent stiffness matrix are obtained which can be used to solve the equations of equilibrium incrementally.

This nonlinear finite element formulation with a moving undeformed body, enables to position and orient the undeformed body where it might be useful. Suppose the floating frame defines the undeformed body. An interesting option is then to choose the floating frame of reference in a material point that is attached to the moving deformed body. In this way, the undeformed body is near to the deformed body and the elastic displacements relative to the floating frame remain relatively small which is numerically beneficial. However, a disadvantage is that the floating frame and the elastic displacements becomes implicitly defined in terms of the finite element degrees of freedom. Such a problem does not occur in the floating frame formulation due to the different set of degrees of freedom that is applied. Interestingly, it is found to be possible to express the position and orientation of the floating frame and elastic displacements incrementally in the finite element degrees of freedom in a novel explicit way.

In standard nonlinear finite element formulations, the absolute nodal coordinates are used as degrees of freedom [1]. In standard floating frame formulations, the absolute floating frame coordinates and local elastic coordinates are used as degrees of freedom [16,17]. Using Craig-Bampton modes to describe local elastic deformations, it is possible to define coordinate transformations between these two sets of generalized coordinates. In earlier work [14], the authors have derived an explicit relation that expresses incremental change in the floating frame coordinates in terms of the incremental change in absolute interface coordinates. Additionally, it has been demonstrated that also the incremental change in local elastic deformations can be expressed in terms of the incremental change of absolute

interface coordinates. By applying these two coordinate transformations, the floating frame formulation can be reformulated using absolute interface coordinates. This approach facilitates solving the implicit definition of the floating frame and elastic displacements, allowing for a floating frame formulation expressed in the degrees of freedom of the nonlinear finite element form.

By combining the nonlinear finite element formulation with respect to a moving undeformed body and the coordinate transformations between the degrees of freedom in the floating frame formulation and nonlinear finite element formulation, one merges the nonlinear finite element formulation and the floating frame formulation. Because no additional assumptions on the elastic displacements are used, the newly obtained floating frame formulation still contains all higher order terms in the local displacement field in the strain definition. With the expressions for internal elastic loads and tangent stiffness matrix in this new formulation, it can be explained which higher order terms are neglected in the traditional floating frame formulation. The other way around, it is now also known which higher order terms can be added to the standard floating frame formulation, should they be of importance for improved simulation performance.

The outline of this work is as follows:

- In Section 2, the equations of equilibrium from nonlinear continuum mechanics are formulated relative to an undeformed body that is fixed with respect to the inertial frame.
- In Section 3, the equations of equilibrium are reformulated relative to a undeformed body that moves with respect to the inertial frame.
- In Section 4, the equations of equilibrium are discretized in a nonlinear finite element formulation, using absolute nodal coordinates. Expressions for the internal elastic loads and tangent stiffness matrix in this nonlinear finite element formulation are derived.
- In Section 5, the transformation from the incremental change in absolute floating frame coordinates to the incremental change in nonlinear finite element coordinates is derived. Expressions for the internal elastic loads and tangent stiffness matrix in the newly obtained floating frame formulation are derived. It is shown that by ignoring higher order terms in the newly

obtained floating frame formulation, the standard floating frame formulation is obtained.

- In section 6, the applicability of the newly developed method is demonstrated by solving the deformation of a clamped beam subjected to a substantial force. In the non-linear range, the contribution of the higher order terms in the displacement field in the resulting equations are presented. Additionally, they are compared with the floating frame formulation and the Total Lagrange method.
- The paper is finalized by the conclusions.

2. Deformation in the inertial frame

Let B denote a flexible body in an deformed state. See Figure 1. Consider the inertial frame O and let B_O denote the undeformed body in frame O . As body B moves and deforms under the influence of external loads, the position and orientation of body B_O remains constant.

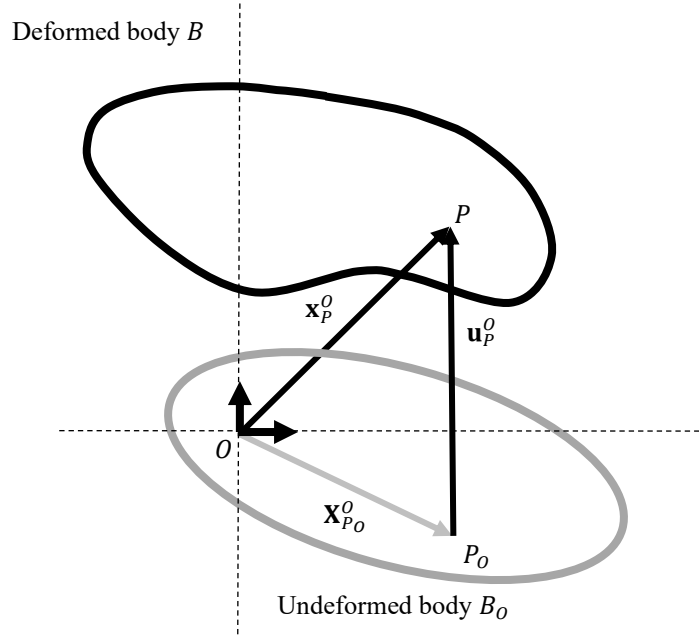


Figure 1: Geometric description of a deformed body B relative to a fixed body B_O in the inertial reference frame

Now consider an arbitrary point P on the deformed body. Let P_O identify the same material point on the undeformed body B_O . The position vector $\mathbf{x}_P^O$ of point P on the deformed body in frame O can be expressed as:

$$\mathbf{x}_P^O = \mathbf{x}^O(\mathbf{X}_{P_O}^O) = \mathbf{X}_{P_O}^O + \mathbf{u}_P^O, \quad (2.1)$$

In this, $\mathbf{X}_{P_O}^O$ is the position of P_O on the undeformed body B_O and $\mathbf{u}_P^O$ the displacement of that point. The displacement field is a function of the coordinates on the undeformed body:

$$\mathbf{u}_P^O = \mathbf{u}^O(\mathbf{X}_{P_O}^O), \quad (2.2)$$

The deformation tensor $\mathbf{F}_O^O$ is a measure for the change in position on the deformed body with a change in position on the undeformed body. The sub- and superscripts

are used to explicitly denote that both the positions are expressed relative to frame O :

$$\mathbf{F}_O^O \equiv \nabla_O \mathbf{x}^O = \mathbf{I} + \nabla_O \mathbf{u}^O, \quad (2.3)$$

in which ∇_O denotes the gradient operator relative to the inertial frame O . Hence, the deformation tensor can be interpreted as a tensor that transforms an infinitesimally small material fiber $d\mathbf{X}_{P_O}^O$ on the undeformed body to an infinitesimally small material fiber $d\mathbf{x}_P^O$ on the deformed body:

$$d\mathbf{x}_P^O = \mathbf{F}_O^O d\mathbf{X}_{P_O}^O, \quad (2.4)$$

To cope with large strains and or large displacements and rotations, the Green-Lagrange strain tensor and the second Piola-Kirchhof stress tensor are used. The Green-Lagrange strain tensor $\mathbf{L}_O^O$ is defined in terms of the deformation tensor $\mathbf{F}_O^O$ as follows:

$$\mathbf{L}_O^O \equiv \frac{1}{2} ([\mathbf{F}_O^O]^T \mathbf{F}_O^O - \mathbf{I}), \quad (2.5)$$

in which $[\dots]^T$ denotes the transposed matrix of its argument.

The second Piola-Kirchhof stress tensor $\mathbf{S}_O^O$ is conjugated to the Green-Lagrange strain tensor $\mathbf{L}_O^O$. Here, it is related to $\mathbf{L}_O^O$ by means of Hooke's law, describing linear elastic material behavior. Introducing $\mathbf{C}_O^O$ as the fourth order material stiffness tensor, this relation can be written, using the Einstein summation convention, as:

$$[\mathbf{S}_O^O]_{mn} = [\mathbf{C}_O^O]_{mnpq} [\mathbf{L}_O^O]_{pq}, \quad (2.6)$$

For a system of bodies, the equations of equilibrium or equations of motion can be derived from the principle of virtual work, relating virtual changes in internal work (internal elastic energy), virtual changes in external work (performed by external loads) and, in case of dynamics, virtual changes in kinetic energy (or inertial forces). As this work is concerned with the derivation of the internal elastic loads and the tangent stiffness matrix, only the expression of the virtual internal elastic work δW_{int} is required for the present purposes:

$$\delta W_{int} = \int_{V_O} [\delta \mathbf{L}_O^O]_{mn} [\mathbf{S}_O^O]_{mn} dV_O, \quad (2.7)$$

Note that the integral is taken with respect to the undeformed body in frame O .

3. Deformation in a moving frame

Consider an undeformed body B_j that may have an arbitrary rigid body motion with respect to the inertial frame O . See Figure 2. The undeformed body B_0 is added for reference. Let j denote the body's floating frame of reference. The position of frame j relative to frame O is denoted by the position vector $\mathbf{r}_j^O$ and the rotation of frame j relative to frame O is denoted by the rotation matrix $\mathbf{R}_j^O$.

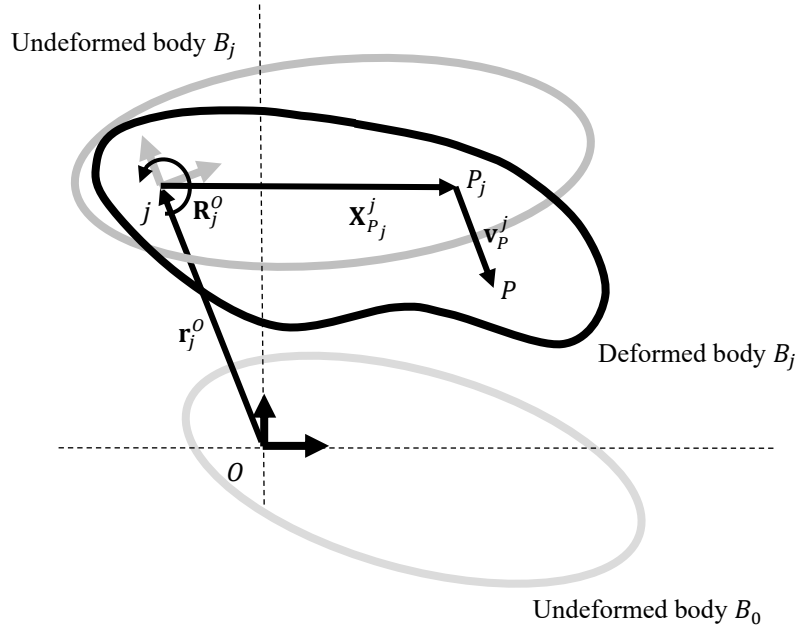


Figure 2: Description of the deformed body B relative to a reference frame connected to a moving undeformed body B_j .

Instead of expressing the deformation and strain in frame O , it can also be expressed in frame j . Reconsider the arbitrary point P on the deformed body. For the position vector $\mathbf{x}_{P_j}^j$ of P relative to frame j holds:

$$\mathbf{x}_{P_j}^j = \mathbf{X}_{P_j}^j + \mathbf{v}_P^j, \quad (3.1)$$

In this, $\mathbf{X}_{P_j}^j$ is the position vector of point P_j relative to frame j and $\mathbf{v}_P^j$ is the displacement of point P relative to P_j in frame j . Point P_j identifies point P on the undeformed body B_j . Note that $\mathbf{X}_{P_j}^j$ remains constant if body B_j is moving. Also note that $\mathbf{v}$ is used for the displacement field instead of $\mathbf{u}$ to denote explicitly that the displacement field is expressed relative to a moving frame: these are really distinct vectors. Finally note that because body B_0 and B_j represent the same

undeformed body, it holds for an arbitrary point P on the deformed body that the relating coordinates of P_0 and P_j on the undeformed bodies B_0 and B_j in the body frames that:

$$\mathbf{X}_{P_j}^j = \mathbf{X}_{P_0}^0, \quad (3.2)$$

Introduce the deformation tensor $\mathbf{F}_j^j$ relating the change in location of P , on the deformed body, due to a change in location of P_j , on the undeformed body B_j if both vectors are expressed in frame j :

$$\mathbf{F}_j^j = \nabla_j \mathbf{x}^j = \mathbf{I} + \nabla_j \mathbf{v}^j, \quad (3.3)$$

The deformation tensor $\mathbf{F}_j^j$ can be interpreted as a tensor that transforms an infinitesimally small material fiber $d\mathbf{X}_{P_j}^j$ on the undeformed body B_j to an infinitesimally small material fiber $d\mathbf{x}_P^j$ on the deformed body, expressed in frame j :

$$d\mathbf{x}_P^j = \mathbf{F}_j^j d\mathbf{X}_{P_j}^j, \quad (3.4)$$

The Green-Lagrange strain tensor $\mathbf{L}_j^j$ defined in terms of the deformation tensor $\mathbf{F}_j^j$ is as follows:

$$\mathbf{L}_j^j \equiv \frac{1}{2} \left([\mathbf{F}_j^j]^T \mathbf{F}_j^j - \mathbf{I} \right), \quad (3.5)$$

These relations can be recognized as the standard derivation of the deformation tensor and Green-Lagrange strain tensor, but in frame j instead of frame O . The second Piola-Kirchhof stress tensor $\mathbf{S}_j^j$ is related to the Green-Lagrange strain tensor by means of Hooke's law, describing linear elastic material behavior:

$$[\mathbf{S}_j^j]_{mn} = [\mathbf{C}_j^j]_{mnpq} [\mathbf{L}_j^j]_{pq}, \quad (3.6)$$

Note that because body B_0 and B_j represent the same undeformed body, except for a rigid body motion, their local material behavior is represented by the same material stiffness tensor:

$$\mathbf{C}_j^j = \mathbf{C}_0^0 = \mathbf{C}, \quad (3.7)$$

Consider a point P on the deformed body. The related points, P_0 on body B_0 and P_j on B_j , have similar coordinates, when expressed in their local body frame. As a consequence the component vectors $d\mathbf{X}_{P_0}^0$ of a fiber starting in P_0 on body B_0 expressed in frame 0, and $d\mathbf{X}_{P_j}^j$ starting in P_j on body B_j expressed in frame j , are the same:

$$d\mathbf{X}_{P_j}^j = d\mathbf{X}_{P_O}^O, \quad (3.8)$$

The corresponding fibers on the deformed body are the same vector. Only, the vector components are either defined in body frame O or body frame j . Therefore, as $\mathbf{R}_0^j$ relates both coordinate frames, is found:

$$R_j^O d\mathbf{x}_{P_j}^j = d\mathbf{x}_{P_O}^O, \quad (3.9)$$

Writing this relation in terms of the deformation tensors, and using (3.8) this means that

$$R_j^O \mathbf{F}_j^j d\mathbf{X}_{P_j}^j = \mathbf{F}_0^O d\mathbf{X}_{P_O}^O = \mathbf{F}_0^O d\mathbf{X}_{P_j}^j, \quad (3.10)$$

This shows that the deformation tensors are related

$$R_j^O \mathbf{F}_j^j = \mathbf{F}_0^O, \quad (3.11)$$

For the displacements fields is then found:

$$\mathbf{I} + \nabla_0 \mathbf{u}^i = R_j^O (\mathbf{I} + \nabla_j \mathbf{v}^j), \quad (3.12)$$

Hence, the deformation tensor $\mathbf{F}_0^O$ is related to the deformation tensor $\mathbf{F}_j^j$ by a pre-multiplication with a single rotation matrix R_j^O . Note that a standard coordinate transformation of a second order tensor involves pre- and post-multiplication with a rotation matrix. However, this is not a coordinate transformation, as the undeformed body B_j is a different undeformed body than B_O .

If this relation is substituted in the definition of the Green-Lagrange strain tensor, it becomes:

$$\mathbf{L}_0^O = \frac{1}{2} ([\mathbf{F}_0^O]^T \mathbf{F}_0^O - \mathbf{I}) = \frac{1}{2} ([\mathbf{F}_j^j]^T \mathbf{F}_j^j - \mathbf{I}) = \mathbf{L}_j^j = \mathbf{L}, \quad (3.13)$$

This is a known property of the Green-Lagrange strain tensor that it is invariant under rigid body motions. By the above derivation it is shown that it is also invariant when the undeformed body is allowed to undergo rigid body motions and the displacements are expressed relative to this moving body. This is an important observation for future purposes.

Because the Green-Lagrange strain tensors $\mathbf{L}_0^O$ and $\mathbf{L}_j^j$ are identical, using (3.7), it follows that the second Piola-Kirchhoff stress tensors are identical too:

$$\mathbf{S}_j^j = \mathbf{S}_0^O = \mathbf{S}, \quad (3.14)$$

Now reconsider the expression for virtual internal work Eq. (2.7):

$$\delta W_{int} = \int_{V_0} [\delta \mathbf{L}_O^O]_{mn} [\mathbf{S}_O^O]_{mn} dV_0, \quad (3.15)$$

Substituting the expressions for $[\mathbf{S}_O^O]$ and $[\mathbf{L}_O^O]$ in terms of $[\mathbf{S}_j^j]$ and $[\mathbf{L}_j^j]$ gives:

$$\delta W_{int} = \int_{V_j} [\delta \mathbf{L}_j^j]_{mn} [\mathbf{S}_j^j]_{mn} dV_j, \quad (3.16)$$

This is a result that might be expected because the internal energy is independent of rigid body motions, so a variation of the internal energy must also be independent of rigid body motions.

To simplify derivations, in the remainder of this paper, the Voight notation is used. The second order strain and stress tensors then are noted as (6x1) vectors and the fourth order tensor with material properties as a (6x6) matrix. Additionally, it is noted that in the reminder of this paper body B_j is considered as a reference. Therefore, the super en subscript j are omitted. The internal elastic energy then becomes:

$$\delta W_{int} = \int_{V_j} \delta L_a S_a dV, \quad (3.17)$$

where

$$\{\mathbf{L}\} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{Bmatrix} = \begin{Bmatrix} [\mathbf{L}_j^j]_{11} \\ [\mathbf{L}_j^j]_{22} \\ [\mathbf{L}_j^j]_{33} \\ [\mathbf{L}_j^j]_{23} + [\mathbf{L}]_{32} \\ [\mathbf{L}]_{31} + [\mathbf{L}]_{13} \\ [\mathbf{L}]_{12} + [\mathbf{L}]_{21} \end{Bmatrix} ; \quad \{\mathbf{S}\} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \begin{Bmatrix} [\mathbf{S}_j^j]_{11} \\ [\mathbf{S}_j^j]_{22} \\ [\mathbf{S}_j^j]_{33} \\ [\mathbf{S}_j^j]_{23} \\ [\mathbf{S}_j^j]_{31} \\ [\mathbf{S}_j^j]_{12} \end{Bmatrix}, \quad (3.18)$$

And the constitutive law,

$$\{\mathbf{S}\} = [\mathbf{C}]\{\mathbf{L}\}, \quad S_a = C_{ab} L_b, \quad (3.19)$$

Note that for convenience, a particular notation is used: $\{\dots\}$ indicates a column vector and $[\dots]$ either a matrix or a row vector. This will be clear from the context. The notation will be used throughout the remainder of the paper.

4. FEM formulation

A system is usually discretized in many bodies and a body usually consists of many finite elements. In this chapter, a body is considered to be discretized in one finite element. This is no restriction, but it simplifies the derivation. Subsequently, the expressions for the generalized internal elastic forces $\{\mathbf{Q}\}$ and tangent stiffness matrix $[\mathbf{K}_t]$ are derived for a body.

Let $\{\mathbf{q}\}$ denote the vector of finite element nodal degrees of freedom. The virtual internal work of the body can be expressed as:

$$\delta W_{int} = \{\delta \mathbf{q}\}^T \{\mathbf{Q}\}, \quad (4.2)$$

in which $\{\mathbf{Q}\}$ is the vector of generalized internal elastic loads, which can be written as:

$$\{\mathbf{Q}\} = [\mathbf{K}]\{\mathbf{q}\}, \quad (4.3)$$

in which $[\mathbf{K}] = [\mathbf{K}(\mathbf{q})]$ is the stiffness matrix. To obtain an expression for the tangent stiffness matrix $[\mathbf{K}_t]$, the variation of the internal elastic load vector must be calculated. The result is of the form:

$$\{\delta \mathbf{Q}\} = [\mathbf{K}_t]\{\delta \mathbf{q}\}, \quad (4.4)$$

Let the displacement field $\{\mathbf{v}\}$ be approximated by a superposition of interpolation functions, each of which is multiplied with the nodal degrees of freedom vector $\{\mathbf{q}\}$:

$$\{\mathbf{v}\} = [\mathbf{N}]\{\mathbf{q}\}, \quad (4.5)$$

in which $[\mathbf{N}]$ is the matrix of spatial interpolation functions. The dimensions are:

$$\{\mathbf{v}\}: 3 \times 1, \quad [\mathbf{N}]: 3 \times (M \cdot N), \quad \{\mathbf{q}\}: (M \cdot N) \times 1, \quad (4.6)$$

in which N is the number of nodes of the element and M is the number of nodal degrees of freedom. Consider the gradient of the displacement field $\nabla \mathbf{v}$, its mn -th component is:

$$(\nabla \mathbf{v})_{mn} = \frac{\partial \mathbf{v}_m}{\partial \mathbf{X}_n} = \frac{\partial \mathbf{N}_m}{\partial \mathbf{X}_n} \mathbf{q} = [\mathbf{N}_{m,n}]\{\mathbf{q}\}, \quad m, n = 1, 2, 3 \quad (4.7)$$

in which $[\mathbf{N}_{m,n}]$ denotes the m -th row of the matrix of interpolation functions that are differentiated with respect to the n -th coordinate. The dimensions are:

$$(\nabla \mathbf{v})_{mn}: 1 \times 1, \quad [\mathbf{N}_{m,n}]: 1 \times M \cdot N, \quad \{\mathbf{q}\}: (M \cdot N) \times 1, \quad (4.8)$$

To utilize the Voight notation , introduce the following matrices:

(4.11)

$$\begin{aligned}
[\mathbf{B}_1] &= [\mathbf{N}_{1,1}]; & [\mathbf{D}_1] &= [\mathbf{N}_{k,1}]^T [\mathbf{N}_{k,1}] & &= [\mathbf{D}_1]^T \\
[\mathbf{B}_2] &= [\mathbf{N}_{2,2}]; & [\mathbf{D}_2] &= [\mathbf{N}_{k,2}]^T [\mathbf{N}_{k,2}] & &= [\mathbf{D}_2]^T \\
[\mathbf{B}_3] &= [\mathbf{N}_{3,3}]; & [\mathbf{D}_3] &= [\mathbf{N}_{k,3}]^T [\mathbf{N}_{k,3}] & &= [\mathbf{D}_3]^T \\
[\mathbf{B}_4] &= [\mathbf{N}_{2,3}] + [\mathbf{N}_{3,2}]; & [\mathbf{D}_4] &= [\mathbf{N}_{k,2}]^T [\mathbf{N}_{k,3}] + [\mathbf{N}_{k,3}]^T [\mathbf{N}_{k,2}] & &= [\mathbf{D}_4]^T \\
[\mathbf{B}_5] &= [\mathbf{N}_{3,1}] + [\mathbf{N}_{1,3}]; & [\mathbf{D}_5] &= [\mathbf{N}_{k,3}]^T [\mathbf{N}_{k,1}] + [\mathbf{N}_{k,1}]^T [\mathbf{N}_{k,3}] & &= [\mathbf{D}_5]^T \\
[\mathbf{B}_6] &= [\mathbf{N}_{1,2}] + [\mathbf{N}_{2,1}]; & [\mathbf{D}_6] &= [\mathbf{N}_{k,1}]^T [\mathbf{N}_{k,2}] + [\mathbf{N}_{k,2}]^T [\mathbf{N}_{k,1}] & &= [\mathbf{D}_6]^T
\end{aligned}$$

The matrices $[\mathbf{B}_a]$ are row vectors of dimension $(1 \times (M \cdot N))$ and $[\mathbf{D}_a]$ are symmetric square matrices of dimension $((M \cdot N) \times (M \cdot N))$. When using the Voight notation, the components of the Green-Lagrange strain tensor become,

$$L_a = \left([\mathbf{B}_a] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_a] \right) \{\mathbf{q}\}, \quad (4.14)$$

The virtual change of the Green-Lagrange tensor equals

$$\delta L_a = \left([\mathbf{B}_a] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_a] \right) \{\delta \mathbf{q}\}, \quad (4.15)$$

Finally, the second Piola-Kirchhoff stress tensor is given by

$$S_a = C_{ab} L_b = C_{ab} \left([\mathbf{B}_a] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_a] \right) \{\mathbf{q}\}, \quad (4.16)$$

4.1 Internal elastic forces $\{\mathbf{Q}\}$

Substitution of the virtual Green-Lagrange strain Eq. (4.15) and the second Piola-Kirchhoff stress Eq. (4.16) in the expression for the virtual internal work Eq. (3.17) yields:

$$\delta W_{int} = \{\delta \mathbf{q}\}^T \int_V \left([\mathbf{B}_a] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_a] \right)^T C_{ab} \left([\mathbf{B}_a] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_a] \right) \{\mathbf{q}\} dV, \quad (4.17)$$

The vector of internal elastic loads $\{\mathbf{Q}\}$ is then:

$$\{\mathbf{Q}\} = \int_V \left([\mathbf{B}_a] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_a] \right)^T C_{ab} \left([\mathbf{B}_b] + \frac{1}{2} \{\mathbf{q}\}^T [\mathbf{D}_b] \right) dV \{\mathbf{q}\}, \quad (4.18)$$

This result can be rewritten as,

$$\{\mathbf{Q}\} = \left[[\mathbf{K}_I] + \frac{1}{2}[\mathbf{K}_{II}] + [\mathbf{K}_{III}] + \frac{1}{2}[\mathbf{K}_{IV}] \right] \{\mathbf{q}\}, \quad (4.19)$$

Where the stiffness matrices are

- $[\mathbf{K}_I] = C_{ab} \int_V [\mathbf{B}_a]^T [\mathbf{B}_b] dV$
- $[\mathbf{K}_{II}] = C_{ab} \int_V [\mathbf{B}_a]^T \{\mathbf{q}\}^T [\mathbf{D}_b] dV \quad a, b = 1, \dots, 6, \quad (4.20)$
- $[\mathbf{K}_{III}] = C_{ab} \int_V [\mathbf{D}_a] \{\mathbf{q}\} [\mathbf{B}_b] dV$
- $[\mathbf{K}_{IV}] = C_{ab} \int_V [\mathbf{D}_a] \{\mathbf{q}\} \{\mathbf{q}\}^T [\mathbf{D}_b] dV$

From the symmetry of the material properties C_{ab} , it is shown that

$$\begin{aligned} [\mathbf{K}_I] &= [\mathbf{K}_I]^T \\ [\mathbf{K}_{II}] &= [\mathbf{K}_{III}]^T, \\ [\mathbf{K}_{IV}] &= [\mathbf{K}_{IV}]^T \end{aligned} \quad (4.21)$$

4.2 Tangent stiffness matrix $[\mathbf{K}_t]$

To obtain an expression for the tangent stiffness matrix, the variation of the internal elastic force vector $\{\mathbf{Q}\}$ in Eq (4.19) is calculated:

$$\begin{aligned} \{\delta\mathbf{Q}\} &= \left[[\mathbf{K}_I] + \frac{1}{2}[\mathbf{K}_{II}] + [\mathbf{K}_{III}] + \frac{1}{2}[\mathbf{K}_{IV}] \right] \delta\{\mathbf{q}\} \\ &\quad + \left[\delta[\mathbf{K}_I] + \frac{1}{2}\delta[\mathbf{K}_{II}] + \delta[\mathbf{K}_{III}] + \frac{1}{2}\delta[\mathbf{K}_{IV}] \right] \{\mathbf{q}\}, \end{aligned} \quad (4.22)$$

To rewrite the virtual changes of the stiffness matrices, consider:

- $[\mathbf{K}_I]$ is a constant matrix, thus

$$\delta[\mathbf{K}_I] = 0, \quad (4.23)$$

- $[\mathbf{D}_b]$ is symmetric, therefore $\{\delta\mathbf{q}\}^T [\mathbf{D}_b] \{\mathbf{q}\} = \{\mathbf{q}\}^T [\mathbf{D}_b] \{\delta\mathbf{q}\}$. As a consequence

$$\delta[\mathbf{K}_{II}]\{\mathbf{q}\} = [\mathbf{K}_{II}]\{\delta\mathbf{q}\}, \quad (4.24)$$

- $[\mathbf{B}_b]\{\mathbf{q}\}$ is a scalar and can be switched with $\{\delta\mathbf{q}\}$. Further C_{ab} is symmetric, so that

$$\delta[\mathbf{K}_{III}]\{\mathbf{q}\} = C_{ab} \int_V ([\mathbf{B}_b]\{\mathbf{q}\}) [\mathbf{D}_a] dV \{\delta\mathbf{q}\}, \quad (4.25)$$

- $\{\mathbf{q}\}^T [\mathbf{D}_a] \{\mathbf{q}\}$ is a scalar and thus can be switched with $\{\delta\mathbf{q}\}$. Combining this with $\{\delta\mathbf{q}\}^T [\mathbf{D}_b] \{\mathbf{q}\} = \{\mathbf{q}\}^T [\mathbf{D}_b] \{\delta\mathbf{q}\}$ yields

(4.26)

$$\delta[\mathbf{K}_{IV}]\{\mathbf{q}\} = C_{ab} \left(\int_V (\{\mathbf{q}\}^T [\mathbf{D}_b] \{\mathbf{q}\}) [\mathbf{D}_a] dV + \int_V [\mathbf{D}_a] \{\mathbf{q}\} \{\mathbf{q}\}^T [\mathbf{D}_b] dV \right) \{\delta\mathbf{q}\}$$

From Eq. (4.25), Eq. (4.26) and Eq. (4.14), it is shown that

$$([\mathbf{B}_a]\{\mathbf{q}\})[\mathbf{D}_a] + \frac{1}{2}(\{\mathbf{q}\}^T [\mathbf{D}_b] \{\mathbf{q}\})[\mathbf{D}_a] = L_b[\mathbf{D}_a]$$

So that

$$\delta[\mathbf{K}_{III}]\{\mathbf{q}\} + \frac{1}{2} \delta[\mathbf{K}_{IV}]\{\mathbf{q}\} = C_{ab} \int_V [\mathbf{D}_a] L_b dV \{\delta\mathbf{q}\} + \frac{1}{2} [\mathbf{K}_{IV}]\{\delta\mathbf{q}\}, \quad (4.27)$$

Substituting Eq. (4.23), Eq (4.24) and Eq. (4.27) in Eq. (4.22) yields the tangent stiffness matrix $[\mathbf{K}_t]$:

$$\{\delta\mathbf{Q}\} = [\mathbf{K}_t] \{\delta\mathbf{q}\}, \quad [\mathbf{K}_t] = [\mathbf{K}_I] + [\mathbf{K}_{II}] + [\mathbf{K}_{III}] + [\mathbf{K}_{IV}] + [\mathbf{K}_V], \quad (4.28)$$

Where

$$[\mathbf{K}_V]\{\mathbf{q}\} = C_{ab} \int_V L_b [\mathbf{D}_a] dV \{\mathbf{q}\}, \quad (4.29)$$

Note that $[\mathbf{K}_I]$, $[\mathbf{K}_{IV}]$ and $[\mathbf{K}_V]$ are symmetric matrices. Because $[\mathbf{K}_{III}] = [\mathbf{K}_{II}]^T$, the sum $[\mathbf{K}_{II}] + [\mathbf{K}_{III}]$ is also symmetric. Consequently, the tangent stiffness matrix is symmetric.

Recall that for all expressions in this chapter, the moving frame j was used as a reference. Hence, Eq. (4.19) and Eq. (4.28) represent the expressions for the internal elastic loads and the tangent stiffness matrix with respect to frame j . Note that these expressions are similar to expressions one would obtain using frame O .

5. Floating frame formulation

The expressions for internal elastic loads and tangent stiffness matrix are defined in moving frame j , that can be chosen anywhere. Assume the moving frame, as the floating frame of reference, is located in a material point on the deformed body. The undeformed body then is near to the deformed body and the elastic displacements relative to the floating frame, in general are small. This is computational beneficial. Also the incremental rotations are small and can be used as generalize coordinates as in linear finite elements. In this chapter, it is explained how this choice can be exploited to express incremental changes in the absolute floating frame degrees of freedom and the incremental changes in the degrees of freedom of the nonlinear finite element formulation, which are introduced in the previous chapter. A detailed explanation is found in earlier work of the authors [14,15,20].

In Figure 3 in black, the degrees of freedom of point P in the floating frame of reference formulation are shown and in grey in the non-linear finite element formulation, i.e. the Total Lagrange method.

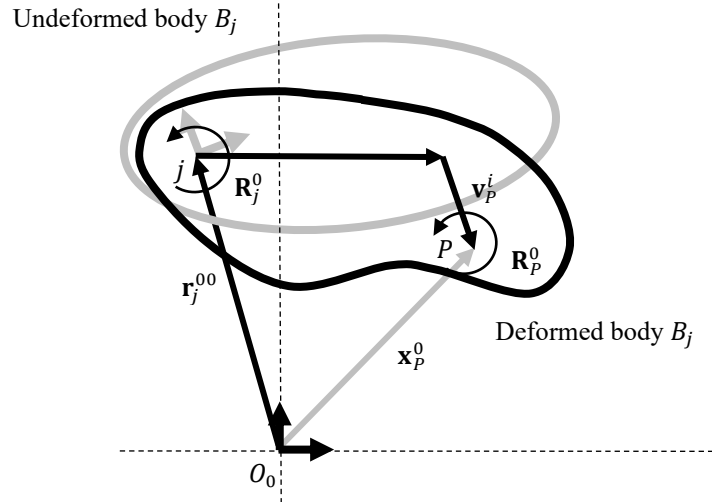


Figure 3: Relation between the degrees of freedom in a floating frame formulation (black) and in the nonlinear finite element formulation (grey).

Consider a body with N interface points that have M degrees of freedom each. Then, the number of interface degrees of freedom is $M \cdot N$. In case the nodal degrees of freedom are only the displacements $M = 3$, and if the nodal degrees of freedom are both displacements and rotations $M = 6$. In both cases, Craig-Bampton modes are used to describe the body's local deformation. Note that if the body consists of only

one finite element, the interface coordinates are in fact identical to the nodal degrees of freedom and the Craig-Bampton modes are the element's interpolation functions. The degrees of freedom of any point P can be expressed as:

$$\mathbf{q}_P^{j,j} = [\Phi_P] \mathbf{q}^{j,j}, \quad (5.1)$$

where the $(M \times M \cdot N)$ matrix $[\Phi_P]$ contains all the Craig-Bampton modes, evaluated at P and the $(M \cdot N \times 1)$ vector $\mathbf{q}^{j,j}$ contains all local interface degrees of freedom:

$$[\Phi_P] \equiv [\Phi_1(\mathbf{x}_P^{j,j}) \quad \dots \quad \Phi_N(\mathbf{x}_P^{j,j})], \quad \mathbf{q}^{j,j} \equiv \begin{bmatrix} \mathbf{q}_1^{j,j} \\ \vdots \\ \mathbf{q}_N^{j,j} \end{bmatrix}, \quad (5.2)$$

A well know property of these Craig-Bampton modes $[\Phi_P]$ is that they still can describe a rigid body motion.

Let $\mathbf{q}_j^{o,o}$ denote the (6×1) vector of absolute floating frame coordinates. It contains both the displacements and the rotations defining the motion of the floating frame. To simplify the explanation, the rotation is parametrized with three variables. As outlined in the introduction, the position and orientation of the floating frame are anchored at a material point in the deformed body. This results in a rigid body motion of the structure due to the movement of the floating frame. However, since the floating frame moves with the material point, any internal elastic displacement or rotation at this point also induces a rigid body motion. To resolve this ambiguity, six conditions are imposed on the elastic deformation at the material point where the floating frame is attached. Specifically, let the arbitrary point P in Eq. (5.1) align with the material point P_j , and ensure no local elastic displacement or rotation. This can be formulated as:

$$[\Phi_j] \mathbf{q}^{j,j} = \mathbf{0}, \quad (5.3)$$

in which $[\Phi_j]$ are evaluated at P_j , where the floating frame is attached. The six reference conditions in Eq. (5.3) will be referred to as the floating frame constraints and are expressed in terms of the local interface coordinates. These local coordinates depend nonlinearly on the absolute interface coordinates $\mathbf{q}^{o,o}$ and the absolute floating frame coordinates $\mathbf{q}_j^{o,o}$. Hence, the floating frame constraints Eq. (5.3) cannot be solved analytically on position level.

The equations of equilibrium in the nonlinear finite element formulation are a set of nonlinear equations in terms of the absolute interface coordinates. The nonlinear nature of these equations prevents them from being solved analytically as well. For that reason, incremental solution techniques are used. Also Eq. (5.3) can be solved in this process, in incremental form:

$$[\Phi_j] \delta \mathbf{q}^{j,j} = \mathbf{0}, \quad (5.4)$$

Using standard kinematic relations, the incremental change in local interface coordinates $\delta \mathbf{q}^{j,j}$ can be expressed analytically in terms of the incremental change in absolute interface coordinates $\delta \mathbf{q}^{o,o}$ minus the incremental change in rigid body motions, described by $\delta \mathbf{q}_j^{o,o}$:

$$\delta \mathbf{q}^{j,j} = [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o} - [\Phi_{rig}] [\mathbf{R}_o^j] \delta \mathbf{q}_j^{o,o}, \quad (5.5)$$

in which $[\bar{\mathbf{R}}_o^j]$ is a $((M \cdot N) \times (M \cdot N))$ block diagonal rotation matrix. $[\Phi_{rig}]$ is the $(M \cdot N \times 6)$ matrix of rigid body modes of the deformed body: each column of this matrix describes the degrees of freedom in each interface node when the floating frame is displaced along one of its axis or is rotated about one of its axis.

Substitution of Eq. (5.5) in Eq. (5.4) allows to express the incremental change in the floating frame coordinates in terms of the incremental change of the absolute interface coordinates. This can be written in the following form:

$$\delta \mathbf{q}_j^{o,o} = [\mathbf{R}_j^o] [\mathbf{Z}_j] [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o}, \quad [\mathbf{Z}_j] \equiv ([\Phi_j] [\Phi_{rig}])^{-1} [\Phi_j], \quad (5.6)$$

Back substitution in Eq. (5.5) yields an expression for the incremental change of the local interface coordinates in terms of the incremental change of the absolute interface coordinates:

$$\delta \mathbf{q}^{j,j} = [\mathbf{T}_j] [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o}, \quad [\mathbf{T}_j] \equiv \mathbf{1} - [\Phi_{rig}] [\mathbf{Z}_j], \quad (5.7)$$

Equations (5.6) and (5.7) define the transformations that can be used to rewrite the equation of equilibrium in the standard floating frame formulation in terms of the absolute interface coordinates. An elaborated derivation of the transformation matrices $[\mathbf{Z}_j]$ and $[\mathbf{T}_j]$, including all relevant kinematic relations and physical interpretations is given in [14,15,20].

The coordinate transformations in Equations (5.6) and (5.7) form the basis to merge the nonlinear finite element formulation of the previous chapter and the floating frame of reference formulation. To this end, the coordinate transformations are used to derive expressions for the internal elastic loads and tangent stiffness matrix relative to a floating frame, but derived from the full nonlinear finite element formulation. In this way, all local higher order terms in the displacement field of a body are taken into account when formulating the internal elastic loads and tangent stiffness matrix. This, is in contrast to the traditional floating frame formulation.

Reconsider the expression for the virtual elastic work, Eq. (4.2). It is explicitly noted that both the virtual displacement field $\delta \mathbf{q}$ and the internal elastic loads $\mathbf{Q}$, are expressed locally, relative to the moving frame j :

$$\delta W_{int} = (\delta \mathbf{q}^{j,j})^T \mathbf{Q}^j, \quad (5.8)$$

The expression for $\mathbf{Q}^j$ has been derived in Eq. (4.28) and is based on the nonlinear finite element formulation. Substitution of Eq. (5.7) in Eq. (5.8) yields:

$$\delta W_{int} = (\delta \mathbf{q}^{O,O})^T [\bar{\mathbf{R}}_j^O] [\mathbf{T}_j]^T \mathbf{Q}^j, \quad (5.9)$$

In this, the product $[\bar{\mathbf{R}}_j^O] [\mathbf{T}_j]^T \mathbf{Q}^j$ form the internal elastic loads expressed in the inertial frame. The expression for the virtual internal work Eq. (5.9) is equated to the virtual work by external loads. From this, the equations of equilibrium are found as follows:

$$[\bar{\mathbf{R}}_j^O] [\mathbf{T}_j]^T \mathbf{Q}^j = \mathbf{Q}_{ext}^O, \quad (5.10)$$

in which $\mathbf{Q}_{ext}^O$ are the externally applied loads at the interface points, expressed in the inertial frame. Since $[\mathbf{T}_j]$ and $\mathbf{Q}^j$ depend on the elastic deformation of the body and $[\bar{\mathbf{R}}_j^O]$ depends on the orientation of the floating frame, the final set of Eq. (5.10) are nonlinear. These equations are solved incrementally, for which the variation of Eq. (5.10) is required:

$$[\bar{\mathbf{R}}_j^O] [\mathbf{T}_j]^T \delta \mathbf{Q}^j + [\bar{\mathbf{R}}_j^O] \delta [\mathbf{T}_j]^T \mathbf{Q}^j + \delta [\bar{\mathbf{R}}_j^O] [\mathbf{T}_j]^T \mathbf{Q}^j = \delta \mathbf{Q}_{ext}^O, \quad (5.110)$$

The terms $\delta \mathbf{Q}^j$, $\delta [\mathbf{T}_j]$ and $\delta [\bar{\mathbf{R}}_j^O]$ can all be expressed in terms of the incremental change in the absolute interface coordinates $\delta \mathbf{q}^{O,O}$. This is shown next. Consider

the variation of the virtual internal loads in the first term in Eq. (5.10). Using Eq. (4.52) and the coordinate transformation Eq. (5.7), this can be rewritten as:

$$\delta \mathbf{Q}^j = [\mathbf{K}_t] \delta \mathbf{q}^{j,j} = [\mathbf{K}_t] [\mathbf{T}_j] [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o}, \quad (5.12)$$

With this, the first term in Eq. (5.11) becomes:

$$[\bar{\mathbf{R}}_j^o] [\mathbf{T}_j]^T \delta \mathbf{Q}^j = [\bar{\mathbf{R}}_j^o] [\mathbf{K}_1] [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o}, \quad [\mathbf{K}_1] \equiv [\mathbf{T}_j]^T [\mathbf{K}_t] [\mathbf{T}_j], \quad (5.13)$$

In [14] it was derived how the second term in Eq. (5.11) can be rewritten to:

$$[\bar{\mathbf{R}}_j^o] \delta [\mathbf{T}_j]^T \mathbf{Q}^j = [\bar{\mathbf{R}}_j^o] [\mathbf{K}_2] [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o}, \quad [\mathbf{K}_2] \equiv -[\mathbf{Z}_j]^T [\hat{\mathbf{F}}^j]^T [\mathbf{T}_j], \quad (5.14)$$

and that the third term in Eq. (5.10) can be rewritten to:

$$\delta [\bar{\mathbf{R}}_j^o] [\mathbf{T}_j]^T \mathbf{Q}^j = [\bar{\mathbf{R}}_j^o] [\mathbf{K}_3] [\bar{\mathbf{R}}_o^j] \delta \mathbf{q}^{o,o}, \quad [\mathbf{K}_3] \equiv -([\hat{\mathbf{F}}^j] + [\hat{\mathbf{M}}^j]) [\mathbf{Z}_j], \quad (5.15)$$

Note that the matrices $[\mathbf{K}_2]$ and $[\mathbf{K}_3]$ depend on the internal elastic loads at the interface points by means of the matrices $[\hat{\mathbf{F}}^j]$ and $[\hat{\mathbf{M}}^j]$. For the detailed derivation of Eq. (5.14) and (5.15) and exact expressions for $[\hat{\mathbf{F}}^j]$ and $[\hat{\mathbf{M}}^j]$, the reader is referred to [14]. Substitution of Eq. (5.13), (5.14) and (5.15) in (5.11), yields the following equations of equilibrium in incremental form:

$$[\mathbf{K}_t^o] \delta \mathbf{q}^{o,o} = \delta \mathbf{Q}_{ext}^o, \quad (5.16)$$

where $[\mathbf{K}_t^o]$ is the total tangential stiffness matrix in the inertial frame:

$$[\mathbf{K}_t^o] = [\bar{\mathbf{R}}_j^o] ([\mathbf{K}_1] + [\mathbf{K}_2] + [\mathbf{K}_3]) [\bar{\mathbf{R}}_o^j], \quad (5.17)$$

with:

$$[\mathbf{K}_1] = [\mathbf{T}_j]^T ([\mathbf{K}_I] + [\mathbf{K}_{II}] + [\mathbf{K}_{III}] + [\mathbf{K}_{IV}] + [\mathbf{K}_V]) [\mathbf{T}_j], \quad (5.18)$$

From Eq. (5.17), it can be seen that the tangent stiffness matrix in the inertial frame $[\mathbf{K}_t^o]$ consists of local tangent stiffness matrix, rotated to the inertial frame. In this local tangent stiffness matrix, the higher order terms $[\mathbf{K}_2]$ and $[\mathbf{K}_3]$ originate from the coordinate transformation from floating frame coordinates to absolute interface coordinates. These terms represent the stiffening effects due to the internal elastic loads at the interface points and might be relevant even though the internal strains in a body remain small. The effect of potentially large strains in a body on the tangent stiffness matrix is accounted for in matrix $[\mathbf{K}_1]$ by means of the higher order terms $[\mathbf{K}_{II}]$, $[\mathbf{K}_{III}]$, $[\mathbf{K}_{IV}]$ and $[\mathbf{K}_V]$.

In the specific case that higher order strains in a body remain small, $[\mathbf{K}_{II}]$, $[\mathbf{K}_{III}]$, $[\mathbf{K}_{IV}]$ and $[\mathbf{K}_V]$ can be ignored. In that case, the expression for the tangent stiffness matrix reduces to the tangent stiffness matrix that was previously derived based on a floating frame formulation in which a linear finite element model is used [15]. A more detailed study of $[\mathbf{K}_{II}]$, $[\mathbf{K}_{III}]$, $[\mathbf{K}_{IV}]$ and $[\mathbf{K}_V]$ could reveal which higher order terms have to be taken into account in particular cases. As an example, this is discussed for a beam in the next section.

6. Application of the merged ‘FF – NLFEM’ method

For verification purposes, both the nonlinear finite element method (NFE) for the Total Lagrange description explained in Chapter 4 and the nonlinear floating frame formulation (NFF) as presented in Chapter 5 have been implemented in Python. Especially, accuracy of the methods are addressed, stability issues are not considered yet.

The beam problem described in [14] is used as benchmark. The problem consist of a static cantilever beam. Two cases are considered: 1) the beam is subjected to a large vertical tip force and 2) a follower force is applied at the tip. The cantilever beam has a circular cross section and was modelled with 10 bodies. The total length of the beam is 1 m. The outer radius of the cross-section is 0.01 m with a wall thickness of 0.001 m. The Young’s modulus is 70.0E9 Pa. The beam is incrementally loaded. The results have been compared with results from the linear floating frame formulation (LFF) and MSC/Nastran [14]. MSC/Nastran sol400 is a well-known commercially available non-linear finite element code[22].

NFE is applied in conjunction with the two-dimensional plane stress rectangular 8-node quadrilateral element. Each node has two in-plane displacements as degrees of freedom. As a consequence, each element has 16 degrees of freedom. The element height is chosen 0.05 m. To have the same second moment of area as the cantilever beam has, for the in plane bending, the thickness is tuned accordingly.



Figure 4: Model of a 1m beam with 10 two-dimensional plane stress rectangular 8-node quadrilateral elements

NFF is implemented utilizing two distinct beam-element definitions. The first element is again the two-dimensional plane stress rectangular 8-node quadrilateral element and is referred to as NFF/2D. The second element is a Timoshenko beam element and is referred to as NFF/3D. This element applies a set of linear shape functions where both displacements and rotations are the degrees of freedom in a node. Each node then has 6 degrees of freedom.

To obtain the mapping from the floating frame degrees of freedom to the absolute nodal coordinates, 6 constraint equations must be set. These 6 constraints eliminate the redundant 6 rigid body motions. In case of the Timoshenko beam, the 3

displacements and the 3 rotational degrees of freedom at the position of the floating frame are set to 0, like is shown by the equations (5.4):

$$[\Phi_j] \delta \mathbf{q}^{j,j} = \mathbf{0}$$

The first three rows in $[\Phi_j]$ multiplied with $\mathbf{q}^{j,j}$ define the local elastic displacements of the floating frame and the next three rows multiplied with $\mathbf{q}^{j,j}$ are the local the elastic rotations. The 6 constraints state that the elastic deformation must be 0.

In case of the NLFF/2D, this is less obvious as the element has no rotational degrees of freedom. As a consequence, the 6 constraints must be defined slightly different. The translational rigid body modes are still removed by setting the displacements at the location of the floating frame to 0. But the rotational rigid body modes are removed enforcing that the curl operator applied at the displacement field evaluated at the floating frame location are set to 0. To set the local displacements to 0 is found:

$$[\Phi_j^d] \delta \mathbf{q}^{j,j} = \mathbf{0}$$

Where for the elastic rotation is enforced:

$$\nabla \times [\Phi_j^d] \delta \mathbf{q}^{j,j} = \mathbf{0}$$

This again yields 6 constraints and the transformation matrices (5.6) and (5.7) can be derived.

The first demo case, see figures 5 and 6, is where the cantilever beam is loaded at the tip in the global vertical direction with increasing force. It shows that the non-linear FEM method applying the Total Lagrange algorithm and the nonlinear floating frame method show exact the same results, also for very high loads. Both deformation plots coincide. This is expected as both methods include all higher order terms.

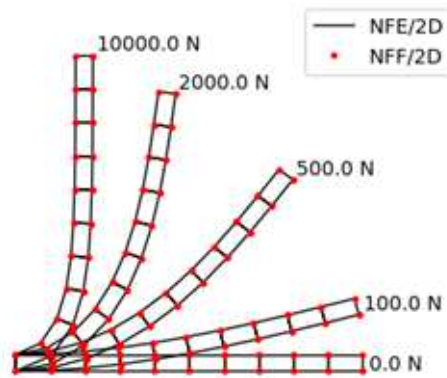


Figure 5: Results NFE/2D model compared with results NFF/2D model of a 1m beam with 10 two-dimensional plane stress rectangular 8-node quadrilateral elements

If both result are compared with results from NFF/2D, the non-linear module SOL400 from the commercial software MSC/Nastran and the ‘linear’ floating frame LFF/3D also then almost no differences are observed. The latter two are both presented before in [14]. Note that for the quadrilateral elements (NFE/2D and NFF/2D) only the centerlines are plotted.

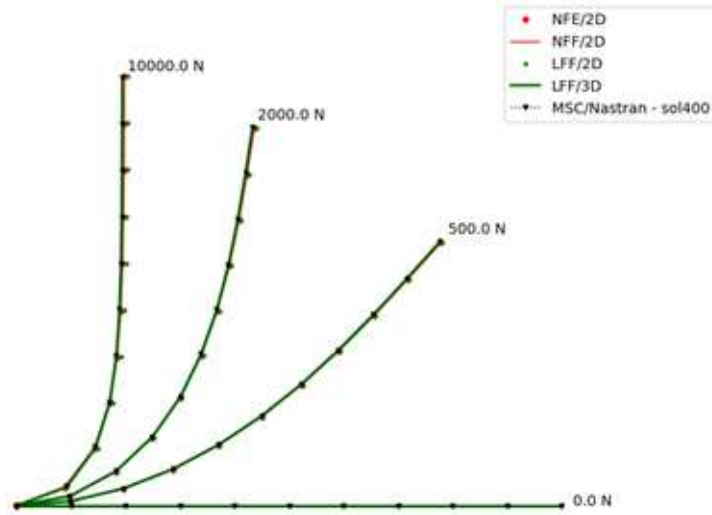


Figure 6: Results NFE and NFF/2D models compared with the NFF/3D model, MSC/Nastran and the LFF model

Apparently, in this load case, the non-linearity is already well described by the LFF method. The terms from the non-linear strain expression are not required. Even not near the clamping, although the deformation seems large. To observe this better, the behaviour of the higher order terms is studied in more detail for two load cases: 1) a follower force and 2) a compressive force (buckling) combined with a very small vertical force (1N) to initiate the deformation. Again the applied loads are incrementally increased. The follower force is pointing in the direction perpendicular to the line connecting the tip and the clamping and buckling force is pointing in the negative X-direction. See figure 7 for plots of the deformation for some load values.

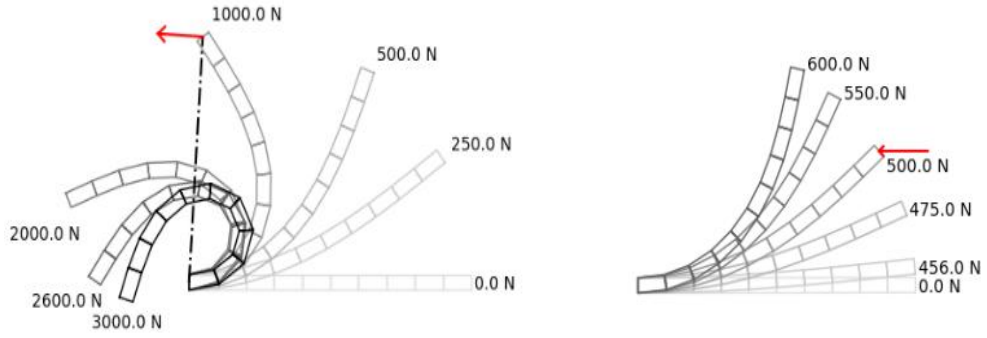


Figure 7: Deformation of the beam for an increasing followers force (left) and the buckling force (right) obtained with the NFE/2D method.

The follower force shows a steady deformation increasing with the load. In case of the buckling load initially the beam hardly deforms, but near a buckling force of about 456 N, the deformation is suddenly changes for an increasing load.

The behavior of the higher order terms in the internal load vector $\mathbf{Q}$ and the stiffness matrix $[\mathbf{K}]$ is studied with the Frobenius norm. The norm is calculated for the bodies (elements) 1 and 10 for all the terms in $\mathbf{Q}$ and $[\mathbf{K}]$. See equations (4.19), (4.29) and (5.18). Note that the stiffness-matrices are first transformed to the global frame.

Figures 8,9,10,11 and 12 show the results for an increasing follower force. Figure 8 presents the norms of the terms in $\mathbf{Q}$ when the NFE method is applied and figure 9 when the NFF method is used.

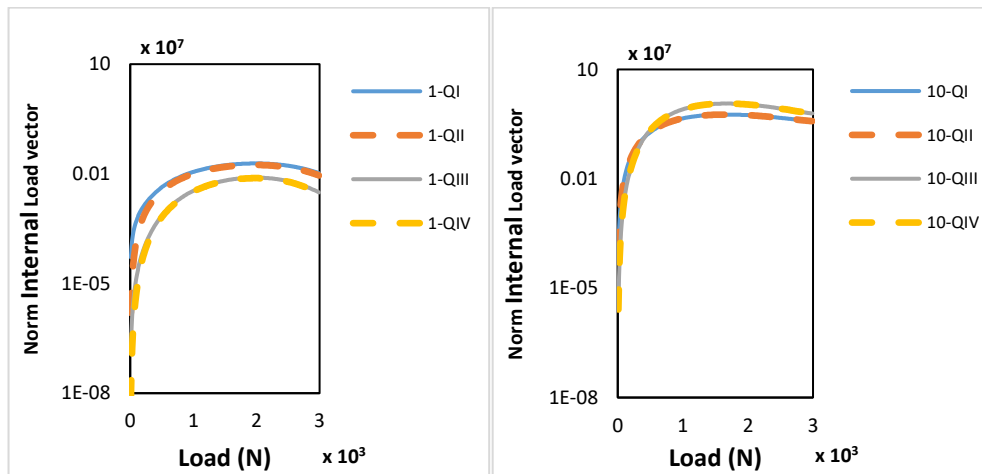


Figure 8: The frobenius norm of the internal load vector terms for bodies 1 and 10 when the NFE/2D method is used (Total Lagrange)

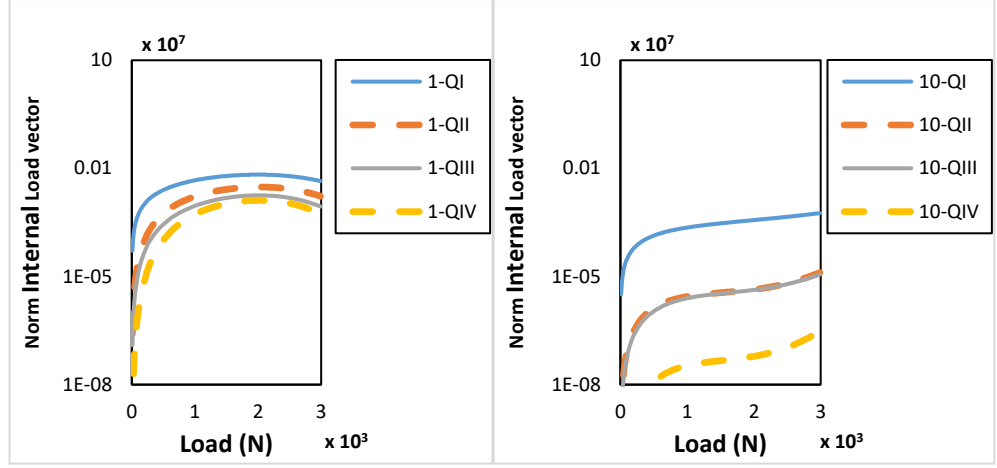


Figure 9: The frobenius norm of the internal load vector terms for bodies 1 and 10 when the NFF/2D method is used

Both figures show that in body 1 the higher order terms become increasingly important. However, while for the NFF method especially the constant term $\mathbf{Q}_I$ is contributing, in case of the NFE method, the higher order terms cannot be neglected. This is even more clear for body 10. This supports the observation that the classical floating frame method (LFF/2D), without higher order terms in $\mathbf{Q}$, can still be used for even large deformations, where this does not hold for the NFE/2D method. The higher order terms are required to compensate for the large geometric deformations. See also figure 12

Figures 10 and 11 show similar results, but then for the higher terms in $[\mathbf{K}]$.

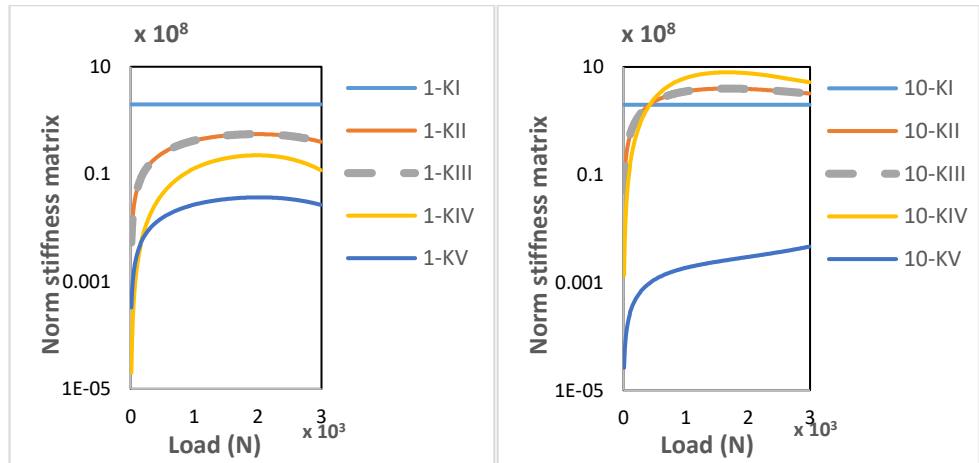


Figure 10: The frobenius norm of the stiffness matrix terms for bodies 1 and 10 when the NFE/2D method is used (Total Lagrange)

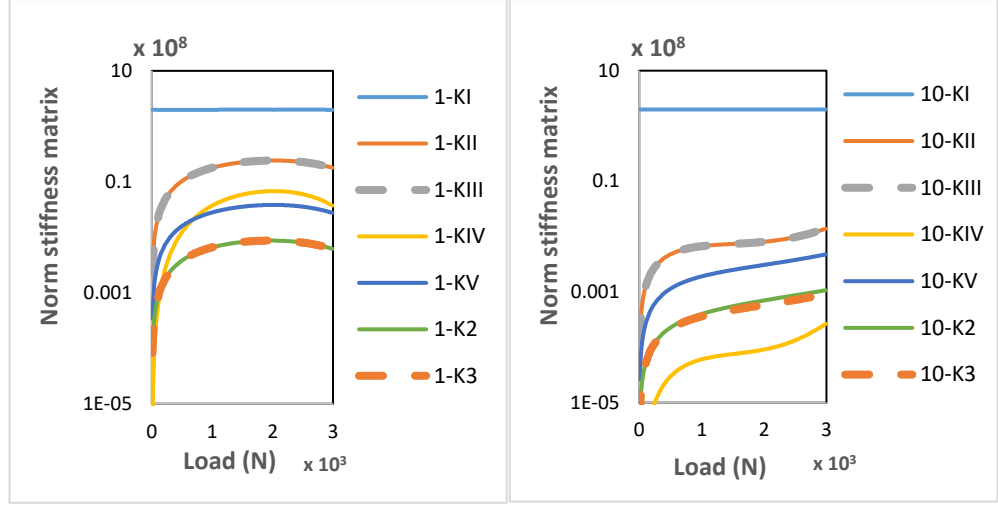


Figure 11: The frobenius norm of the stiffness matrix terms for bodies 1 and 10 when the NFF/2D method is used.

Looking at body 1, in both methods, the constant matrix $[K_I]$ is dominant (note that the axis is logarithmic). Apparently, the strain in body 1 is still not that large, that higher order terms are required. However, considering body 10, in case of method NFE/2D, only $[K_V]$ can be neglected, because $[K_V]$ depends on the internal stresses and they are small in body 1. All the other terms are relevant. For the NFF/2D method, all non-linear terms can be neglected. That means, when applying a modified Newton Raphson technique, the higher order terms in $[K]$ when applying NFF/2D are not needed, but this is definitely not true for the NFE/2D (Total Lagrange method) method.

Figure 12 compares the three methods in terms of deformations. It presents the deformed beam for a few values of the followers force. The NFE/2D, NFF/2D and LFF/2D show similar behavior for a quadrilateral element till about 1000N. For that load the deformation is already substantial. Above that load, the difference between at the one hand NFE/2D and NFF/2D and at the other hand LFF/2D becomes clearly visible.

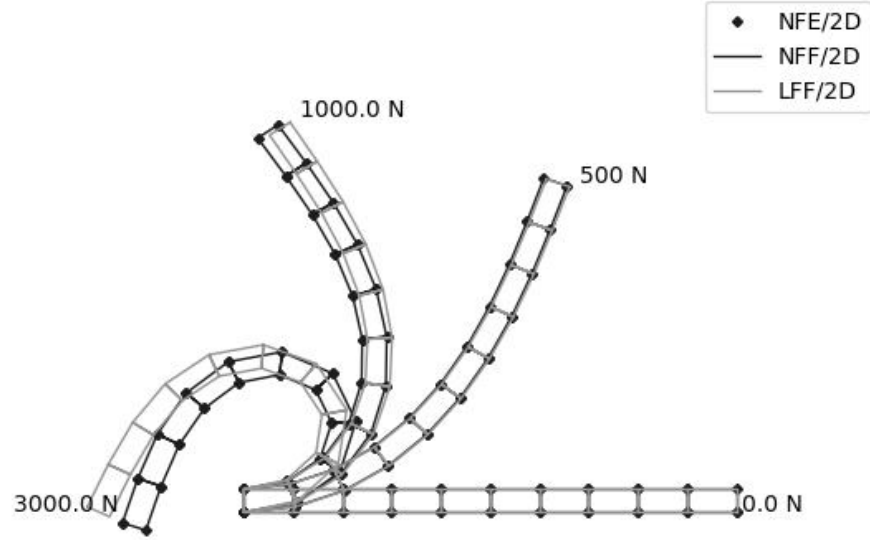


Figure 12: The Deformation of the clamped beam by an increasing followers force

Figures 13,14,15 and 16 present the results for an increasing buckling force. First the terms in $\mathbf{Q}$ are studied and then the terms in $\mathbf{K}$. Again, the methods NFE/2D and NFF/2D are compared.

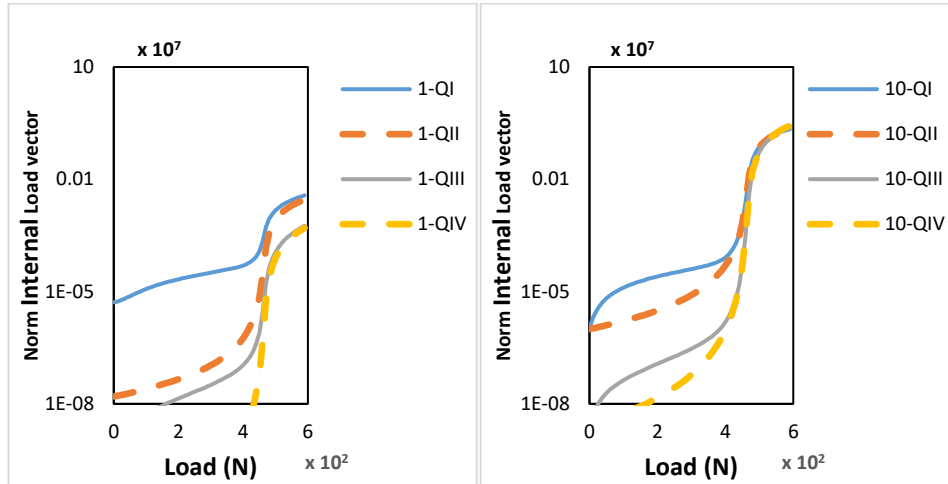


Figure 13: The frobenius norm of the internal load vector terms for bodies 1 and 10 when the NFE/2D method is used (Total Lagrange)

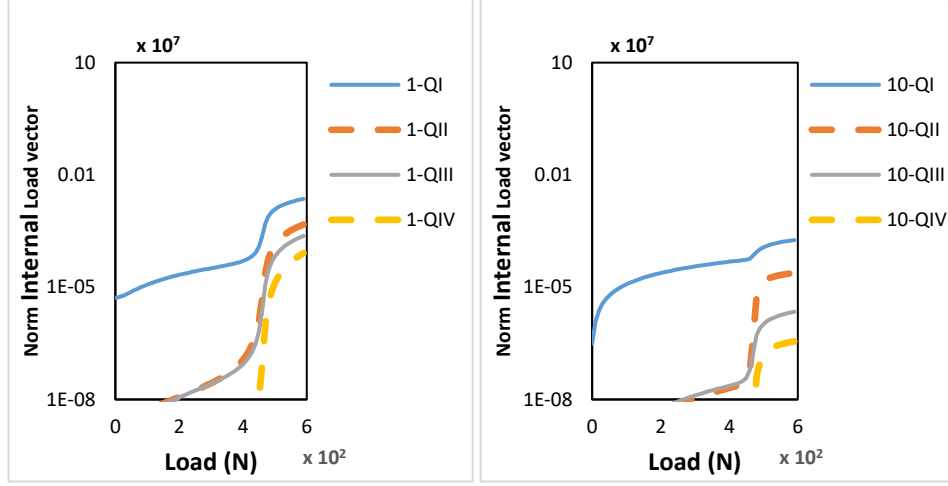


Figure 14: The frobenius norm of the internal load vector terms for bodies 1 and 10 when the NFF/2D method is used

Figure 13 shows that in case of the NFE/2D method all the higher order terms in $\mathbf{Q}$ are needed around the critical buckling load both for body 1 and body 10. However, considering figure 14 for the NFF/2D method it seems that the higher order terms are contributing less. Using the Green-Lagrange strain tensor is may be not required, i.e. having a floating reference frame is sufficient.

Figures 15 and 16 show the Frobenius norms of the higher terms in $[\mathbf{K}]$. Again, in both methods, the constant matrix $[\mathbf{K}_I]$ is most important for body 1. The strain in body 1 is apparently small when buckling starts and higher order terms are not required. However, considering body 10, in case of method NFE/2D, only $[\mathbf{K}_V]$ can be neglected, all the other terms become relevant. $[\mathbf{K}_V]$ depends on the internal stresses and they are small in body 10, so this makes sense. For the NFF/2D method, all non-linear terms in $[\mathbf{K}]$ can be neglected.

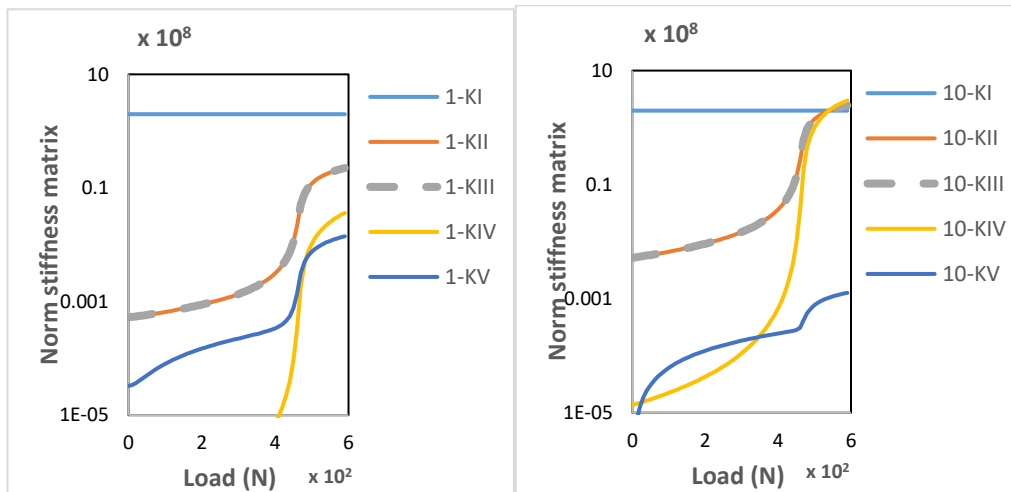


Figure 15: The frobenius norm of the stiffness matrix terms for bodies 1 and 10 when the NFE/2D method is used (Total Lagrange)

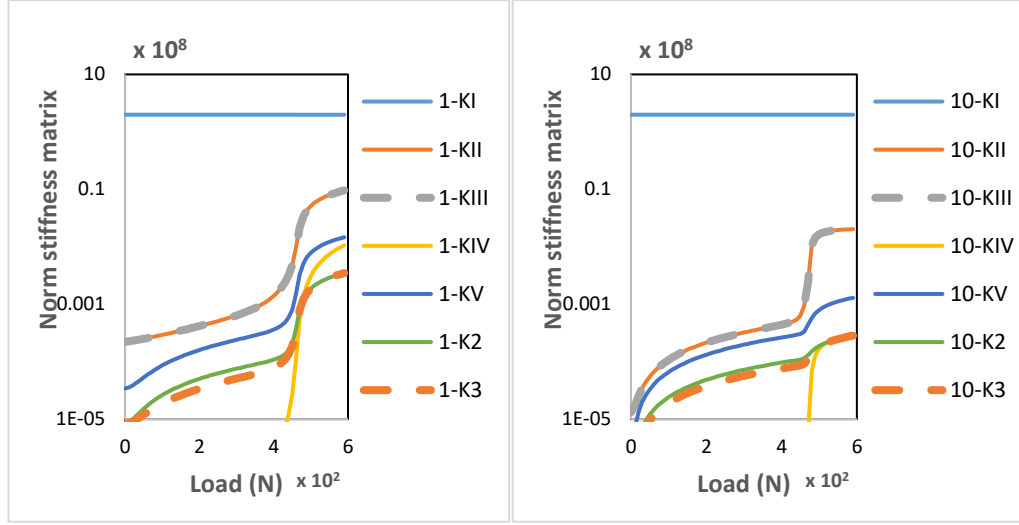


Figure 16: The frobenius norm of the stiffness matrix terms for bodies 1 and 10 when the NFF/2D method is used.

The results suggest that regularly, higher order terms can be neglected in the NFF method. Especially in the stiffness matrix, when applying a modified Newton Raphson method. Moreover, if a higher order term is added, term $\mathbf{Q}_2$ might be sufficient. However, the influence of neglecting higher order terms on numerical stability is part of future work.

7. Conclusion

In this work, an expression for the virtual work in terms of the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor was derived, relative to an undeformed body which might be positioned at an arbitrary location in space. After discretization, a nonlinear finite element formulation is obtained, resulting in expression for the internal elastic loads and tangent stiffness matrix, relative to the frame that defines the location of the undeformed body. This tangent stiffness matrix contains zeroth, first and second order terms in the local elastic displacements.

The frame that defines the location of the undeformed body can be interpreted as the floating frame in a floating frame formulation. It is located at the deformed body in a material point such that no additional rigid body motions occur. Hence, the

material point cannot not have any elastic displacement. By defining the location of the floating frame in this way, it has been shown how to express the generalized coordinates of the floating frame formulation in terms of absolute interface coordinates in the inertial frame.

Using the nonlinear finite element expressions of the internal elastic loads and tangent stiffness matrix and the established coordinate transformations, it is possible to merge a nonlinear finite element model in a floating frame formulation. In this new formulation, it is possible to include higher order terms due to large strains within a body, in the floating frame formulation. When strains within a body remain small, these higher order terms can be ignored and the traditional floating frame formulation is obtained.

To show the performance of the formulation, results obtained from a statically loaded clamped beam are compared for the Total Lagrange method (NFE/2D), the (non-linear) floating-frame method (NFF/2D), based on the Green-Lagrange strain tensor and the (linear) floating frame method (LFF/2D), based on the Cauchy strain tensor. The beam is loaded by a incrementing vertical force in the global frame, a follower force and a buckling force. The results show that the contribution from the higher order stiffness matrices are less important in the floating frame formulation compared to the Total Lagrange method. Also the internal force depends less on the higher order terms in the floating frame method compared to the Total Lagrange method. This might explain why the Total Lagrange method showed instable behavior before the floating frame methods did. A general remark, it seems that the higher order terms in the internal load should be considered earlier then the higher order terms in the stiffness matrix, especially when a modified Newton Raphson scheme is used to solve the non-linear equations.

Finally, it is noted that the higher order terms in the non-linear floating frame method can be calculated with help of the same mesh, this in contrast to the Updated Lagrange method. This is computationally very attractive. As, moreover, the floating frame method can easily be combined with reduction techniques, the non-linear floating frame method seems a new promising formulation to solve problems with large deformations.

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