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Research Article

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Numerical investigation of homogeneous and stratified turbulence under 45° inclined shear via the coupled SSG-CL second-order model: Effects of Froude and Shear numbers

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Abstract

This study investigates effects of the Froude and Shear numbers on the thermal and dynamic characteristics of homogeneous and stratified turbulence subjected to an inclined shear fixed at $\theta = 45^\circ$ via the coupled SSG-CL second-order model. The model combines the Speziale–Sarkar–Gatski (SSG) Reynolds-stress model with the Craft–Launder (CL) scalar-flux model. Froude number is varied from 0.35 to 1.11, covering strongly to weakly stratified conditions, while Shear number ranges from 2 to 20.

Firstly, the asymptotic behavior of dimensionless kinematic and scalar parameters governing the turbulent dynamics is systematically examined for different Froude and Shear numbers. Results demonstrate that the SSG-CL model predicts well-defined asymptotic equilibrium states for the investigated thermal and dynamic quantities, including Reynolds stress anisotropy (b_{12}), density flux $\overline{\rho u_1}$, normalized dissipation rate (ε/KS), non-dimensional ratio (η), and the ratios of kinetic energy to total energy (K/E) and potential energy to total energy (K_p/E). Secondly, present predictions are compared with Direct Numerical Simulation results reported by Jacobitz et al. (DNSJ) [22–24]. Overall, good agreement is obtained between the SSG-CL predictions and DNSJ findings, particularly for particularly principal components of anisotropic stress (b_{12} and b_{13}), normalized turbulence dissipation (ε/KS), and the normalized production term (P/KS). These results demonstrate the capability of the coupled SSG-CL model to reproduce the asymptotic behavior of homogeneous and stratified turbulence under inclined shear over a broad range of stratification and shear conditions.

Keywords: Homogeneous and stratified turbulence; inclined shear; coupled second-order models; Froude number; shear number; Direct Numerical Simulation (DNS) of Jacobitz.

Nomenclature

b	Anisotropic tensor of Reynolds
C_p	Specific heat at constant pressure
g	Acceleration due to gravity
P	Pressure
p	Fluctuation of the pressure
Fr	Froude number
SK/ε	Shear number
S	Uniform mean shear
S_ρ	Uniform stable stratification
K	Turbulent kinetic energy
K_p	Potential energy
$\overline{T}_{,i}$	Gradient of the scalar
t	Time
τ	Non-dimensional time
u_i	i-th component of the fluctuating velocity
U_i	i-th component of mean velocity
$\overline{u_i u_j}$	Reynolds stress tensor
$\overline{u_i \rho}$	Turbulent flux of the scalar
$\overline{U}_{p,q}$	Gradient of mean speed
x_i	Component of an Orthonormal Cartesian coordinate system

μ	Dynamic viscosity
λ	Viscosity ratio
ν	Kinematic viscosity
α	Thermal diffusivity
τ	Non dimensional time
δ_{ij}	Kronecker Symbol
ρ	Fluctuation of the scalar
ρ_0	Density of reference
$\overline{\rho^2}$	Density variance
ε	Terms of dissipation of turbulent kinetic energy
$\varepsilon_{\rho\rho}$	Terms of dissipation of variance of the scalar
ε_{ij}	Terms of dissipation of tensor of Reynolds
$\varepsilon_{i\rho}$	Terms of dissipation of the scalar flux turbulent
ϕ_{ij}	Terms of pressure-strain correlation
$\phi_{i\rho}$	Terms of pressure- scalar gradient correlation
θ	Angle between the shear and the vertical gradient of stratification
SSG	Speziale, Sarkar and Gatski model
CL	Craft and Launder model
DNSJ	Direct Numerical Simulation of Jacobitz

1. Introduction

Stratified and homogeneous turbulent flows subjected to different shear orientations have been extensively investigated because of their relevance to a wide range of geophysical and engineering applications. Particular attention has been devoted to the evolution of turbulence under inclined shear, motivated by its occurrence in atmospheric and oceanic flows, as well as in lakes, estuaries, and rivers. Among these flows, stably stratified turbulence is of particular interest because of its fundamental role in understanding complex turbulent processes in geophysical systems and industrial applications [1-4].

Research on the modeling of homogeneous and stratified turbulence in geophysical flows has played an essential role in verifying turbulence theories and confirming predictions through second-order models such as DNS, SSG, CL, LRR, and SL [5-9]. This research leads to engineering applications with significant benefits for society, such as the development of high-resolution climate and space weather models. Rohr et al. [10] and Piccirillo et al. [11] examined homogeneous and stratified turbulence subjected to inclined shear and demonstrated that turbulent parameters are influenced by variations in the Froude number. Holt et al. [12] focused on the coupling effects between stable stratification [13] and shear in homogeneous turbulence. Gerz et al. [14] conducted direct numerical simulations (DNS) of homogeneous turbulence under vertical shear. Jeffrey [15] investigated equilibrium states of turbulent parameters in both homogeneous and stably stratified turbulence at higher Froude numbers. Mohammadreza et al. [16] used DNS to study the influence of the Froude number on the motion of settling bidisperse inertial particles in isotropic turbulence. Ben Abdallah et al. [17] studied asymptotic equilibrium states of stably stratified and homogeneous turbulence under non-vertical shear, comparing the results with those from DNS simulations by Jacobitz. Blumen [18] examined vertical stable-stratified flow under non-vertical shear. Thamri et al. [19, 20] investigated stratified turbulence with inclined shear using second-order models, including the coupled SSG-CL, SSG-SL, and SSG-LRR models. They demonstrated the significant contribution of coupled models in predicting turbulent characteristics of stratified and homogeneous turbulence. Using the SSG-CL, SSG-SL, and SSG-LRR models, Thamri et al. [21] studied the effect of non-vertical shear on stratified and homogeneous turbulence. They found good agreement between results from the coupled models and predictions from DNS simulations by Jacobitz. Thamri and Naffouti [22] conducted a comparative analysis of three coupled second-order models, namely SSG-CL, SSG-SL, and SSG-LRR, to investigate the effects of the shear and Richardson numbers on homogeneous and stratified turbulence under horizontal shear. Their numerical results demonstrated that the evolution of turbulence strongly depends on the Richardson number for different values of the Shear number. In addition, Thamri and Naffouti [25] evaluated the performance of the SSG-LRR model for homogeneous turbulence subjected to inclined shear under both strongly and weakly stratified

conditions, with particular emphasis on the effect of the Shear number. Results obtained using the SSG-LRR model demonstrated the existence of asymptotic equilibrium states for various thermal and dynamic parameters, particularly under weak stratification conditions.

However, despite these investigations, a systematic analysis of homogeneous and stratified turbulence subjected to an inclined shear at ($\theta = 45^\circ$) using the coupled SSG-CL model remains lacking. In particular, the combined effects of the Froude and shear numbers on the asymptotic behavior of both thermal and dynamic turbulence parameters have not yet been systematically investigated within the SSG-CL framework. The present study therefore aims to address this gap by investigating effects of the Froude and Shear numbers on homogeneous and stratified turbulence subjected to a fixed inclined shear at $\theta = 45^\circ$, using the coupled SSG-CL second-order model. This model combines the Speziale-Sarkar-Gatski (SSG) model [8], which describes the dynamic field, with the Craft-Launder (CL) model [7], which characterises the thermal field. The objective is to analyze temporal evolution and asymptotic behavior of thermal and dynamic parameters characterizing homogeneous and stratified turbulence for different Froude and Shear numbers. The Froude number is varied from 0.35 to 1.11, while the Shear number takes the values 2, 6, 10, 14, and 20. The numerical simulations are performed using a MATLAB code developed for the present study. In addition, the predictions of various turbulence parameters obtained with the SSG-CL model are compared with the corresponding results from the Direct Numerical Simulations of Jacobitz et al. (DNSJ) [24–26].

2. Mathematical consideration

2.1 Equation of motion

This study is based on the continuity equation for incompressible turbulent shear flow, the three-dimensional unsteady Navier–Stokes equations in the Boussinesq approximation, and a transport equation for density. In the following, x_i ($i = 1, 2, 3$) denotes the i -th component of an orthonormal Cartesian coordinate system. According to the classic Reynolds decomposition [27], the Navier–Stokes equations and the associated

scalar transport equation can be decomposed into a mean part (denoted by an overbar) and a fluctuating part (denoted by lowercase letters):

$$U_i = \overline{U}_i + u_i, \quad P = \overline{P} + p, \quad \rho = \overline{\rho} + \rho$$

Where U_i the total value of velocity components, P is the total pressure and ρ is the total density.

We note that the inclined shear is fixed at $\theta = 45^\circ$ where θ is the inclination angle between vertical stratification and shear. The decomposition of dependent variables is introduced into equations of motion and following equations for fluctuating parts are obtained [28]:

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1)$$

$$\begin{aligned} \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \left(S \frac{\sqrt{2}}{2} x_2 + S \frac{\sqrt{2}}{2} x_3 \right) \frac{\partial u_i}{\partial x_1} + \left(S \frac{\sqrt{2}}{2} u_2 + S \frac{\sqrt{2}}{2} u_3 \right) \delta_{i1} = \\ - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - \frac{g}{\rho_0} \rho \delta_{i3} \end{aligned} \quad (2)$$

$$\frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} + \left(S \frac{\sqrt{2}}{2} x_2 + S \frac{\sqrt{2}}{2} x_3 \right) \frac{\partial \rho}{\partial x_1} + S_\rho u_3 = \alpha \frac{\partial^2 \rho}{\partial x_k \partial x_k} \quad (3)$$

g is the acceleration due to gravity, ν the kinematic viscosity, α the scalar diffusivity, ρ is the fluctuation of the scalar and ρ_0 is the density of reference.

$S_\rho = \frac{\partial \overline{\rho}}{\partial x_3}$ is the stable density stratification and S is the shear rate.

2.2 Transport equations for an inclined shear

From equations of the motion, transport equation for the velocity correlation $\overline{u_i u_j}$, transport equation for the density flux $\overline{u_i \rho}$, transport equation for the turbulent kinetic

energy $K = \frac{\overline{u_i u_i}}{2}$ and transport equation for the density fluctuation $\overline{\rho^2}$ are obtained from equations (1), (2) and (3).

$$\begin{aligned} \frac{d \overline{u_i u_j}}{dt} = & -S \frac{\sqrt{2}}{2} \overline{u_j u_2} \delta_{i1} - S \frac{\sqrt{2}}{2} \overline{u_j u_3} \delta_{i1} - S \frac{\sqrt{2}}{2} \overline{u_i u_2} \delta_{j1} - S \frac{\sqrt{2}}{2} \overline{u_i u_3} \delta_{j1} - \\ & \frac{g}{\rho_0} (\overline{u_i \rho} \delta_{j3} + \overline{u_j \rho} \delta_{i3}) + \frac{1}{\rho_0} p \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - 2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{d \overline{u_i \rho}}{dt} = & -S \frac{\sqrt{2}}{2} \overline{u_2 \rho} \delta_{i1} - S \frac{\sqrt{2}}{2} \overline{u_3 \rho} \delta_{i1} - S_\rho \overline{u_i u_3} - \frac{g}{\rho_0} \overline{\rho^2} \delta_{i3} \\ & + \frac{1}{\rho_0} p \frac{\partial \rho}{\partial x_i} - (\alpha + \nu) \frac{\partial \rho}{\partial x_k} \frac{\partial u_i}{\partial x_k} \end{aligned} \quad (5)$$

$$\frac{dK}{dt} = -S \frac{\sqrt{2}}{2} \overline{u_1 u_2} - S \frac{\sqrt{2}}{2} \overline{u_1 u_3} - \frac{g}{\rho_0} \overline{u_3 \rho} - \nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \quad (6)$$

$$\frac{d \overline{\rho^2}}{dt} = -2S_\rho \overline{\rho u_3} + 2\nu \frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_k} \quad (7)$$

In the next part, a coupling between the Speziale, Sarkar, and Gatski (SSG) model [8], which is used for the kinematic field, and the Craft and Launder (CL) model [7], which is applied to the scalar field, is proposed. This coupling forms the basis of the present numerical investigation.

2.3 Second-order modeling

In this section, the details of the coupled second-order model of SSG-CL (Speziale, Sarkar, and Gatski (SSG) model [8] and Craft and Launder (CL) model [7, 29]) are discussed. The second-order models of SSG and CL are used to close the transport equations (4-7). The pressure-strain correlation terms ϕ_{ij} of the SSG model [8] are used to model the dynamic turbulent parameters of stratified and homogeneous turbulent flows

subjected to inclined shear for various Froude and Shear numbers. The pressure–scalar gradient correlation $\phi_{i\rho}$ associated with the CL model [7, 29] is applied to characterize the thermal turbulent characteristics of the problem. The terms for the pressure–strain correlation ϕ_{ij} and the pressure–scalar gradient correlation $\phi_{i\rho}$ are given as follows:

$$\phi_{ij} = \frac{1}{\rho_0} \overline{p \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} \quad \text{and} \quad \phi_{i\rho} = \frac{1}{\rho_0} \overline{p \frac{\partial \rho}{\partial x_i}}$$

Usually, in a stratified turbulence, ϕ_{ij} and $\phi_{i\rho}$ are classified into three parts [27]:

$$\phi_{ij} = \phi_{ij}^1 + \phi_{ij}^2 + \phi_{ij}^3 \quad \text{and} \quad \phi_{i\rho} = \phi_{i\rho}^1 + \phi_{i\rho}^2 + \phi_{i\rho}^3$$

ϕ_{ij}^1 and $\phi_{i\rho}^1$ are terms of turbulence-turbulence, ϕ_{ij}^2 and $\phi_{i\rho}^2$ are terms related to the interaction between turbulent and mean flows and ϕ_{ij}^3 and $\phi_{i\rho}^3$ are terms due to buoyancy effects.

To modelize the third part, the classic model of Zeman and Lumley [30, 31] is adopted.

Terms of ϕ_{ij}^3 and $\phi_{i\rho}^3$ are defined by equations as following:

$$\phi_{ij}^3 = -C_3 \left(\beta_j \overline{u_i \rho} + \beta_i \overline{u_j \rho} - \frac{2}{3} \beta_l \overline{u_l \rho} \delta_{ij} \right)$$

$$\phi_{i\rho}^3 = -C_{3\rho} \beta_i \overline{\rho^2}$$

Where $C_3 = 0,5$, $C_{3\rho} = 0,5$.

$\beta_i = \frac{g_i}{\rho_0}$ is the term of gravity, g_i is the acceleration due to gravity and ρ_0 is the density of reference.

2.3.1 The Speziale, Sarkar and Gatski (SSG) model

The Speziale, Sarkar, and Gatski (SSG) model [8] is adopted to reproduce the pressure-strain correlation in order to predict the dynamic turbulent parameters. The second-order SSG model is based on the following additional conditions, which are coupled with the

previous analysis of invariant dynamical systems. The conditions related to the SSG model are described in the next paragraph.

Firstly, consistency with Rapid Distortion Theory (RDT) for homogeneous strained turbulent flows, initially isotropic, is adopted. Secondly, consistency with equilibrium states of a homogeneous flow subjected to shear, as experimentally calculated by Tavoularis and Corrsin [32], is used. Next, consistency with experimental results on the decrease of isotropic turbulence and the return to isotropy for both homogeneous and anisotropic turbulence is adopted. Asymptotic consistency in the limit of small anisotropies is also considered. For rotating shear flows, consistency with the Rapid Distortion Theory predictions calculated by Bertoglio [33] is retained.

For all computations related to the present problem, we do not focus on the rotation of the turbulent flow.

The SSG model primarily concerns kinematic turbulence. Speziale et al. [8] divided the return-to-isotropy process into a linear part. This model is written in the following form:

$$\phi_{ij} = \phi_{ij}^1 + \phi_{ij}^2 \quad (8)$$

$$\phi_{ij}^1 = -C_1 b_{ij} + 3(C_1 - 2)(b_{ij}^2 + II_b \frac{\delta_{ij}}{3}) \quad \text{Where} \quad II_b = b_{kl} b_{lk} \quad , \quad C_1 = 3,4$$

$$\begin{aligned} \phi_{ij}^2 = & C_2 b_{mn} S_{mn} b_{ij} + \frac{1}{2}(C_3 - \sqrt{b_{mn} b_{mn}} C_{33}) S_{ij} + \frac{1}{2} C_4 (b_{ik} S_{kj} + b_{jk} S_{ki} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij}) \\ & + \frac{1}{2} C_5 (b_{ik} w_{jk} + b_{jk} w_{ik}) \end{aligned}$$

$$C_2 = 1,8 \quad C_3 = 0,8 \quad C_4 = 1,25 \quad C_5 = 0,4 \quad C_{33} = 1,3$$

$$b_{ij} = \frac{\overline{u_i u_j}}{2K} - \frac{\delta_{ij}}{3} \text{ is the dimensionless parameter of the anisotropic tensor where } \delta_{ij} \text{ is the}$$

$$\text{Kronecker Symbol and the kinetic energy is given by: } K = \frac{\overline{u_i u_i}}{2}$$

$$S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \text{ is the rate of strain tensor.}$$

$$w_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i}) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \text{ is the rate of vorticity tensor.}$$

2.3.2 The Craft and Launder (CL) model

Craft and Launder [7, 29] develop second order closure models in order to modelize pressure–strain correlation ϕ_{ij} and pressure–scalar gradient correlation $\phi_{i\rho}$ [7, 29, 34].

These models related to terms of ϕ_{ij} and $\phi_{i\rho}$ satisfy kinematic constraint [38] and exact limits imposed by two components turbulence [34]. Models for ϕ_{ij}^3 and $\phi_{i\rho}^3$ of Craft and Launder [29] are not retained in the present numerical investigation. Classical models of Zeman and Lumley [30, 31] are adopted for representing ϕ_{ij}^3 and $\phi_{i\rho}^3$. Basing on the literature, previous studies show that the CL model has given a good agreement with the values of DNSJ [35, 36] on the problem of a stably stratified sheared turbulence.

Craft and Launder [7] developed expressions of contribution correlations pressure–gradient scalar $\phi_{i\rho}$ into two parts:

$$\phi_{i\rho} = \phi_{i\rho}^1 + \phi_{i\rho}^2 \tag{9}$$

$$\phi_{i\rho}^1 = -3,2 \frac{\varepsilon}{k} r (1 + \Pi^{0.5}) (\overline{u_i \rho} - 2,2 b_{ik} \overline{u_k \rho} + 6,4 b_{ik} b_{kj} \overline{u_j \rho})$$

$$\begin{aligned}
\phi_{i\rho}^2 = & 0,8 \overline{\rho u_k} \frac{\partial U_i}{\partial x_k} - 0,2 \overline{\rho u_k} \frac{\partial U_k}{\partial x_i} + \frac{\varepsilon}{k} \overline{\rho u_i} \left(\frac{P_{kk}}{\varepsilon} \right) - 0,4 \overline{\rho u_k} a_{il} \left[\frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k} \right] \\
& + 0,1 \overline{\rho u_k} a_{ik} a_{ml} \left[\frac{\partial U_m}{\partial x_l} + \frac{\partial U_l}{\partial x_m} \right] - 0,1 \overline{\rho u_k} (a_{im} P_{mk} + 2a_{mk} P_{im}) \\
& - 0,05 a_{ml} \left\{ 7a_{mk} \left[\overline{\rho u_i} \frac{\partial U_k}{\partial x_l} + \overline{\rho u_k} \frac{\partial U_i}{\partial x_l} \right] - \overline{\rho u_k} \left[a_{ml} \frac{\partial U_i}{\partial x_k} + a_{mk} \frac{\partial U_l}{\partial x_i} \right] \right\} \\
& + 0,15 a_{mk} \left[\frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k} \right] (a_{mk} \overline{\rho u_i} - a_{mi} \overline{\rho u_k})
\end{aligned}$$

$$a_{ik} = 2b_{ik}$$

$$P_{ij} = -\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k}$$

Where $r = 0.5$ is the ratio of mechanical to thermal time scales [37] ($r = \frac{\varepsilon_\rho}{\varepsilon} \frac{K}{\rho^2}$).

ε is the terms of dissipation of turbulent kinetic energy, ε_ρ is the terms of dissipation of variance of the scalar.

2.4 Non-dimensional parameters

All computations for the problem are performed for relevant values of the Froude number and Shear number, ranging from 0.35 to 1.11 and from 2 to 20, respectively. Different values of the Froude number are chosen to cover both strongly stratified and weakly stratified cases.

The dimensionless Froude number related to Richardson number [38] is given as

following: $Fr = \sqrt{\frac{0.5^2}{Ri}}$.

The gradient Richardson number describes the importance of stratification on turbulence

evolution. It is defined by: $Ri = \frac{N^2}{S^2}$.

S is the component of shear and N^2 is the Brunt-Väisälä frequency.

The dimensionless Shear number SK/ε is the ratio of the turbulence time scale K/ε to the shear time scale $1/S$.

ε is the terms of dissipation of turbulent kinetic energy.

The normalized time related to temporal evolution of thermal and dynamic turbulent characteristics of the problem is given as following: $\tau = S.t$.

3. Results and discussions

In this step, the numerical approach involves integrating a closed system of nonlinear dimensionless differential equations using the fourth-order Runge-Kutta method. All simulations are initialized with the same initial conditions as the DNSJ results [24-26].

$$\left((b_{ij})_0 = 0, \left(\frac{\varepsilon}{KS} \right)_0 = 0, (\eta)_0 = 0 \right).$$

Computations with the SSG-CL model are performed for an inclined shear with $\theta = 45^\circ$, using different values of the Froude number ranging from 0.35 to 1.11 and various Shear numbers ($SK/\varepsilon = 2, 6, 10, 14$, and 20). Here, θ defines the angle between the shear and the vertical gradient of stratification.

3.1 Effect of Froude number on turbulent characteristics of the flow

In this section, the influence of the Froude number on the temporal evolution of stratified and homogeneous turbulence under an inclined shear ($\theta = 45^\circ$) is examined for a fixed Shear number of 2. Plots in Figures 1-5 show that both dynamic and thermal turbulent characteristics are affected by variations in the Froude number. Specifically, various dynamic parameters, such as the principal component of anisotropy b_{12} , normalized turbulence dissipation ε/KS and the ratio of kinetic energy to total energy K/E (Figures 1, 3, and 5, corresponding to K/E), demonstrate an initial decay due to the isotropic initial conditions of DNSJ [24-26]. Following this initial phase, the physical parameters decay with normalized time until they reach asymptotic equilibrium states for different Froude numbers. In contrast, Figures 2, 4, and 5, which depict thermal parameters such as density flux, normalized rate η and the ratio of potential energy to total energy K_p/E , show an initial increase due to the isotropic initial conditions of DNSJ [24-26] used in the

computations. Afterward, these parameters increase as time progresses and stabilize at constant values for different Froude numbers.

Figure 1 presents the evolution of the principal component of anisotropy b_{12} as a function of non-dimensional time τ for different Froude numbers, ranging from $Fr = 0.35$ (strongly stratified shear flow) to $Fr = 1.11$ (weaker stratified shear flow). The SSG-CL model confirms the existence of asymptotic equilibrium states for the dynamic parameter b_{12} with a non-dimensional time τ greater than 25. It is observed that the equilibrium values of b_{12} ($b_{12})_{\infty}$ decrease as Fr increases from 0.35 to 1.11, due to the attenuation of the stratification degree of the flow.

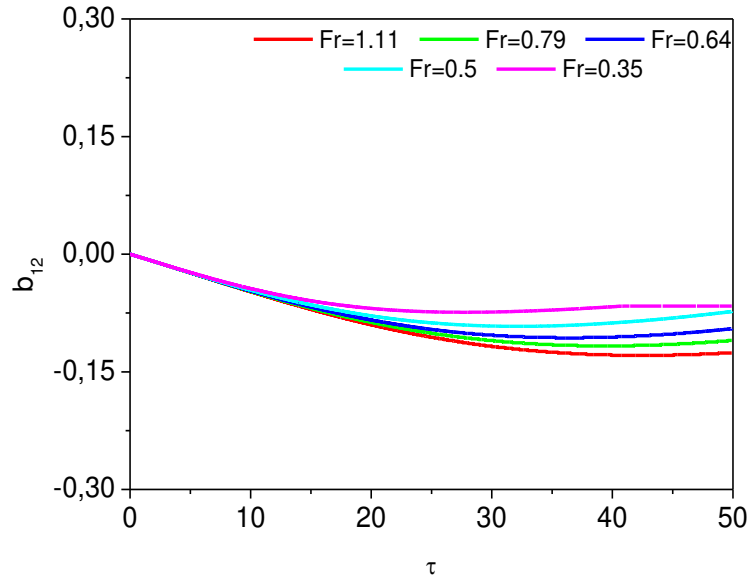


Fig.1. Time evolution of the Reynolds stress anisotropy b_{12} versus Froude number with $SK/\varepsilon = 2$

The dependence of the asymptotic value of the density flux $\overline{\rho u_1}$ on the non-dimensional time for various Froude number ($0.5 \leq Fr \leq 1.11$) is plotted in fig.2. Asymptotic equilibrium states are reached here more quickly at low stratification ($Fr = 1.11, 0.79$ and 0.64). For high stratification corresponding to $Fr = 0.5$, the SSG-CL model does not show a good asymptotic equilibrium state. For this case, the asymptotic value of the density flux is not very sensitive to the initial values of isotropic initial conditions of DNSJ [24-26]. The decrease of Froude number from 1.11 to 0.64 conducts to increase values of

asymptotic equilibrium states of the component $\overline{\rho u_1}$ caused by an intensification of the stratification of the turbulent flow.

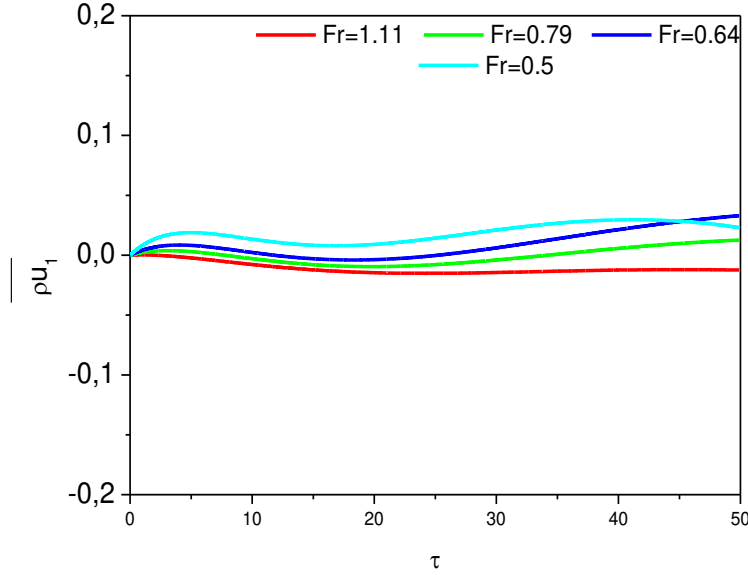


Fig.2. Time evolution of the density flux $\overline{\rho u_1}$ versus Froude number with $SK/\epsilon = 2$

The temporal evolution of the normalized dissipation rate ϵ/KS as a function of Froude number with Shear number $SK/\epsilon = 2$ is shown in fig.3. SSG-CL model confirms the existence of an asymptotic equilibrium states for a long enough time $\tau \geq 35$. From graphs, it is shown that values of equilibrium states related to dynamic parameter of $(\epsilon/KS)_\infty$ decrease as decreasing the Froude number from 1.11 to 0.35. It is due to the growth of the turbulent flow stratification.

The dependence of the asymptotic value of the non-dimensional ratio $\eta = K_\rho/K$ of

potential energy ($K_\rho = \frac{1}{2} \frac{g}{\rho_0 S_\rho} \overline{\rho^2}$) to kinetic energy of the turbulent flow versus

Froude number is presented in fig.4. SSG-CL model proves the presence of an asymptotic equilibrium states for this parameter. For cases of a weak stratification related to $Fr = 1.11, 0.79$ and 0.64 , asymptotic equilibrium states are obtained more rapidly than that the case of an intense stratification corresponding to $Fr = 0.5$ and 0.35 . In addition, profiles demonstrate that decreasing Froude number leads to increase values of

equilibriums states of non-dimensional ratio thus indicating the dominant of the potential energy than the kinetic energy of the turbulent flow.

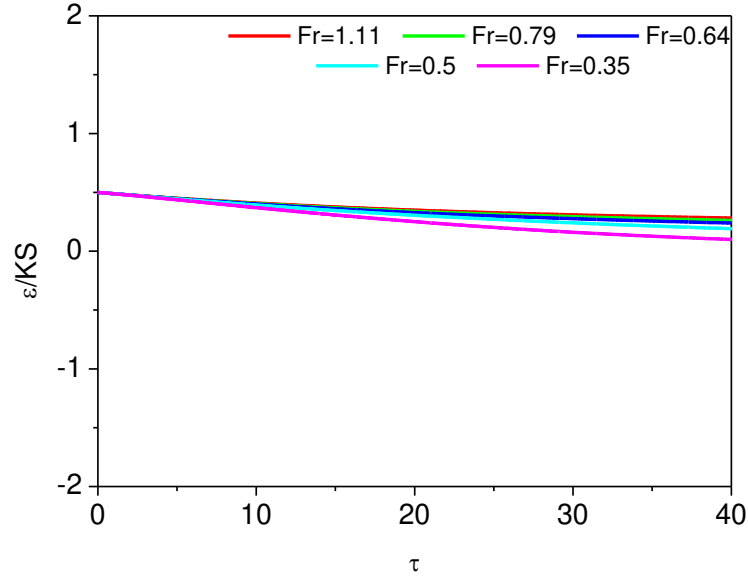


Fig.3. Time evolution of the normalized turbulence dissipation ε/KS versus Froude number with $SK/\varepsilon = 2$

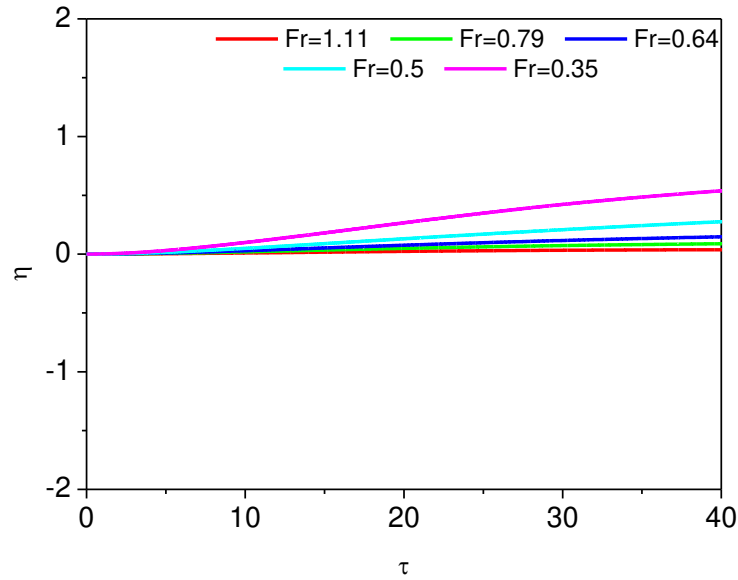


Fig.4. Time evolution of the normalized rate of potential energy to kinetic energy η versus Froude number with $SK/\varepsilon = 2$

In fig.5 is illustrated the time evolution of the ratio of kinetic energy to total energy K/E and the ratio of potential energy to total energy K_p/E for various Froude number. The SSG-CL model proves the existence of an asymptotic equilibrium states for non-dimensional time τ greater than 30. Furthermore, SSG-CL model shows an equal partition of the total energy ($E = K + K_p$) between the turbulent kinetic energy and potential energy for different Froude number. It is shown that the growth of Froude number from 0.35 to 1.11 conducts to increase the ratio of kinetic energy to value about $K/E \approx 1$ and to decrease the ratio of potential energy K_p/E . This result indicates the dominant of Kinetic energy than the potential energy especially for the case of a weaker stratification.

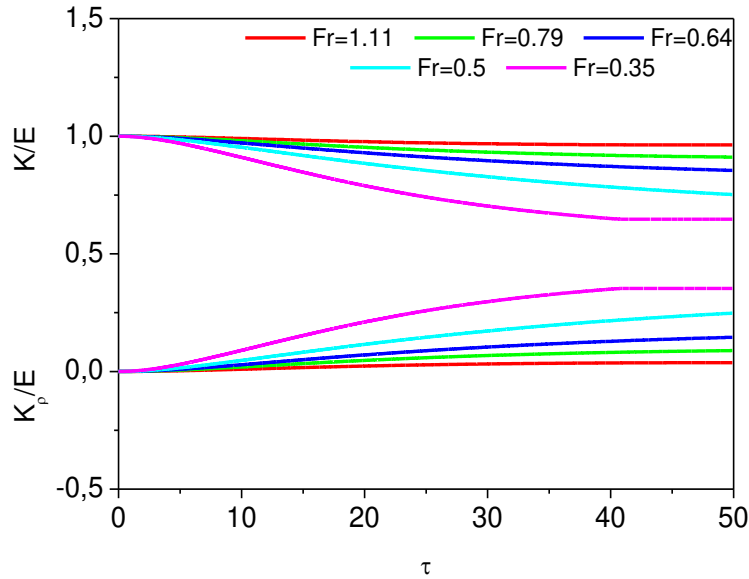


Fig.5. Time evolution of ratios of turbulent kinetic energy to total energy K/E and potential energy to total energy K_p/E versus Froude number with $SK/\varepsilon = 2$

3.2 Effect of Shear number on turbulent characteristics of the flow

In order to better understanding behaviors of thermal and dynamic turbulent characteristics of the present problem, next attention is concentrated on the effect of Shear number (SK/ε) on the temporal evolution of homogeneous and

stratified turbulence under inclined shear with $Fr = 1.29$. Figures 6-9 depicts the effect of this pertinent parameter of Shear number ranging from 2 to 20 on the principal component of anisotropy b_{12} , normalized turbulence production P/KS , normalized rate η , ratios turbulent kinetic energy to total energy K/E and potential energy to total energy K_p/E . In figures 6 and 9 is shown an initial decay of initial states of principal component of anisotropy b_{12} and ratio kinetic energy to total energy K/E owing to isotropic initial conditions of DNSJ [24-26] adopted in the present investigation. Then the initial phase, these dynamic parameters decay as increasing the normalize time in order to reach an asymptotic equilibrium states for $\tau \geq 30$ with various Shear number from 2 to 20.

Figures 7-9 demonstrate an initial grows of initial states of the normalized turbulence production P/KS , normalized rate η and ratio of potential energy to total energy K_p/E due to the isotropic initial conditions of DNSJ [24-26] imposed for all computations. After that, these parameters (P/KS , η and K_p/E) increase as increasing the non-dimensional time to remain at constant values of equilibrium states beyond $\tau = 30$ for different Shear number. Using SSG-CL model, it is found in figures 6 and 7 that equilibrium states of the principal component of anisotropy (b_{12}) $_{\infty}$ and the normalized turbulence production (P/KS) $_{\infty}$ are slightly affected by the variation of Shear number.

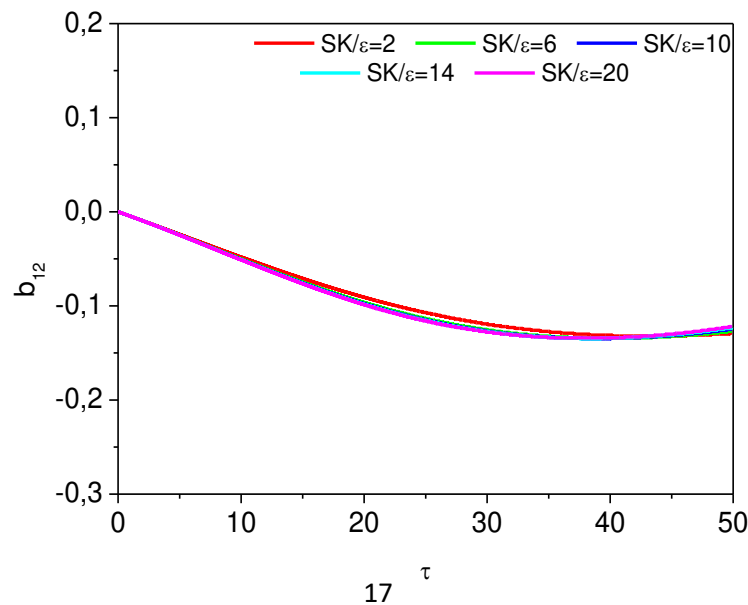


**Fig.6. Time evolution of the Reynolds stress anisotropy b_{12}
for various Shear number SK/ε with $Fr = 1.29$**

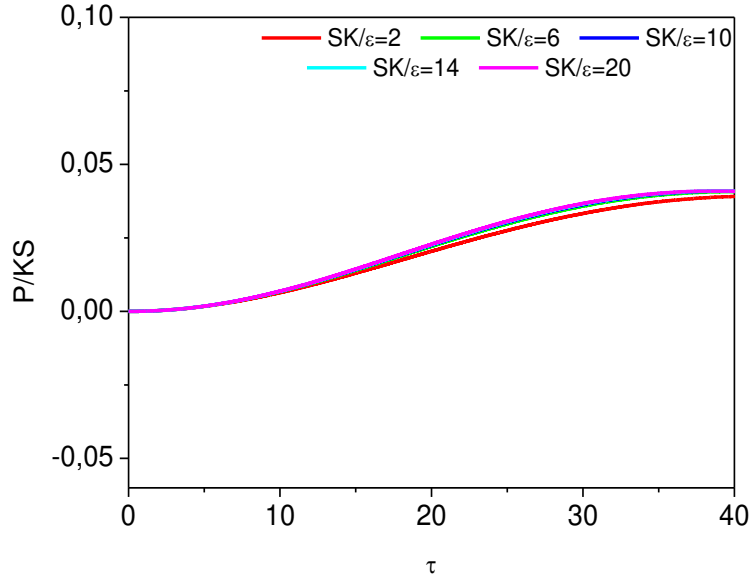


**Fig.7. Time evolution of the normalized turbulence production P/KS
for different Shear number SK/ε with $Fr = 1.29$**

Fig.8 and fig.9 show temporal evolutions of non-dimensional ratio η , K/E and K_p/E versus Shear number with $Fr = 1.29$. SSG-CL model proves the presence of states of an asymptotic equilibrium for these parameters for all values of SK/ε . Furthermore, the second order model of SSG-CL shows that growth of the Shear number from 2 to 20 conducts to an increase the normalized rate η and ratio of potential energy to total energy

K_p/E . These findings via SSG-CL model indicate an intensification of the potential energy than the kinetic energy of the turbulent flow as increasing the Shear number.

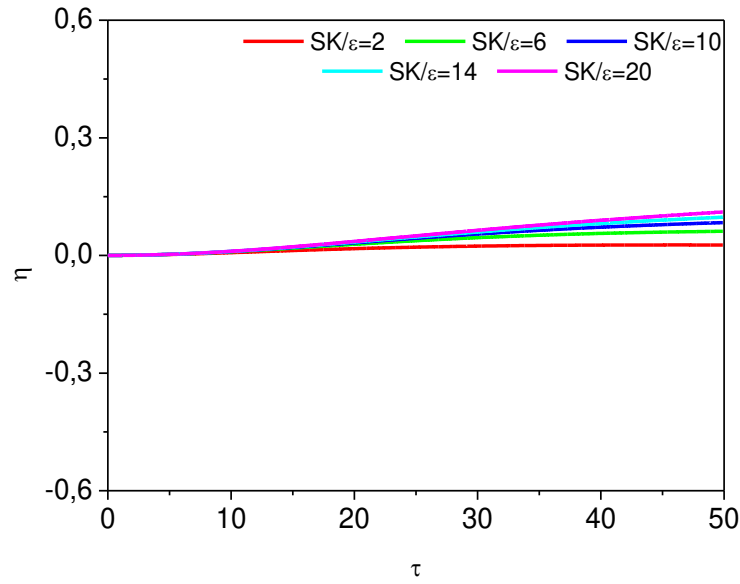


Fig.8. Time evolution of the normalized rate η for several values of Shear number $SK/\epsilon \text{ Fr} = 1.29$

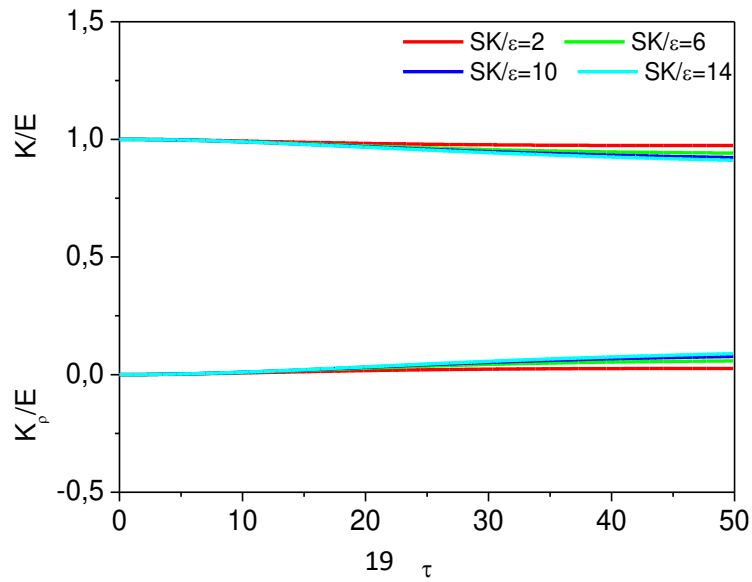


Fig.9. Time evolution of ratios turbulent kinetic energy to total energy K/E and potential energy to total energy K_p/E versus Shear number SK/ε with $Fr = 1.29$

3.3 Comparative analysis of predictions via SSG-CL model and results of DNSJ

In order to check on the accuracy of the coupling SSG and CL models to employed for the solution of homogeneous and stratified turbulence under inclined shear ($\theta = 45^\circ$) with various Froude and Shear numbers, the present numerical MATLAB code is validated with findings of the Direct Numerical Simulation of Jacobitz [24-26]. Figures 10-13 illustrate a comparison between temporal evolutions of principal components of anisotropy b_{12} and b_{13} , normalized turbulence dissipation ε/KS and normalized production P/KS obtained via SSG-CL model and by DNSJ [24-26]. From graphs, various turbulent characteristics of the flow predicted by both approaches show same behaviors. This proved that SSG-CL model with the suggested initial condition of Jacobitz can produces reliable results. In figures 10 and 11 are depicted the variation of principal components of anisotropy b_{12} and b_{13} as a function of the normalized time τ for Froude number equal to 1.11 with Shear number $SK/\varepsilon = 2$. SSG-CL model confirms the existence of asymptotic equilibrium states of turbulent parameters b_{12} and b_{13} at high times $\tau > 10$. From plots, it is found that an excellent agreement between findings of SSG-CL model and values of DNSJ [24-26] for non-dimensional time greater than $\tau = 10$ thus indicating the coupling between SSG and CL models can present a significant method to compute asymptotic equilibrium states of shear and stratified turbulent flows. The comparison of dependence of the normalized turbulence dissipation ε/KS for non dimensional time using SSG-CL model and DNSJ [24-26] is shown in fig.12 for Froude number $Fr = 1.11$ and Shear number $SK/\varepsilon = 2$. The second order model of SSG-CL proves an asymptotic behavior of equilibrium states for the parameter ε/KS . It is clear that SSG-CL model ensures the best agreement with results computed by the numerical approach of DNSJ [24-26] at non-dimensional time beyond $\tau > 10$.

In fig.13 is presented the temporal variation of the normalized production term P/KS using two approaches of SSG-CL model and DNSJ [24-26] for $Fr = 1.29$ and $SK/\varepsilon = 2$. It is found that this component tends towards states of asymptotic equilibrium for normalized time beyond 10. Furthermore, findings predicted by model of SSG-CL present a good agreement with those of DNSJ [24-26].

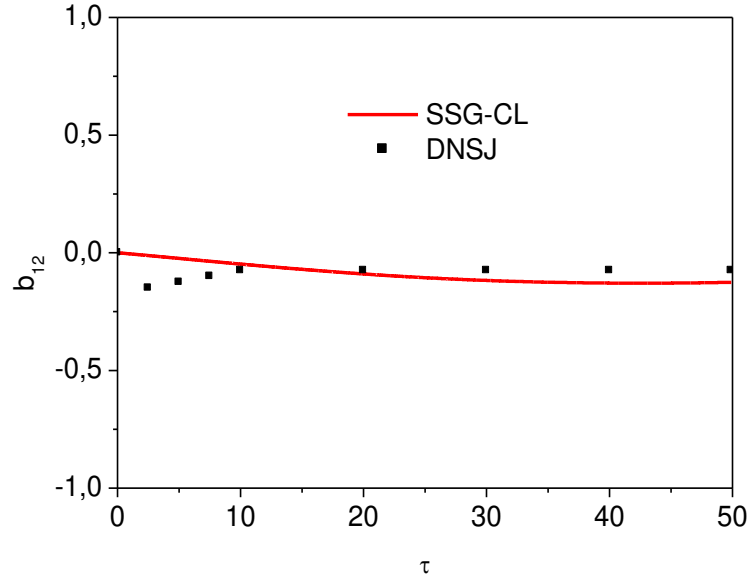


Fig.10. Temporal evolution of the principal component of anisotropy b_{12} for $Fr = 1.11$, $SK/\varepsilon = 2$ and $\theta = 45^\circ$

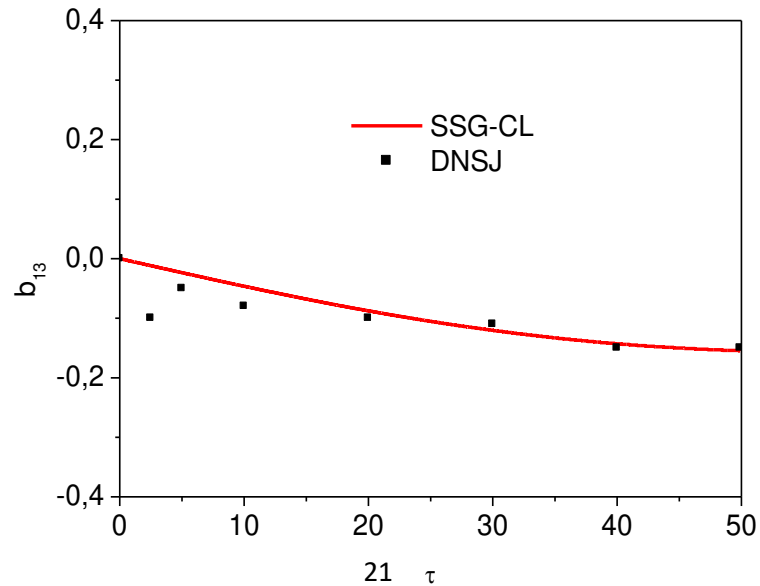


Fig.11. Temporal evolution of the principal component of anisotropy b_{13} for $Fr = 1.11$, $SK/\varepsilon = 2$ and $\theta = 45^\circ$

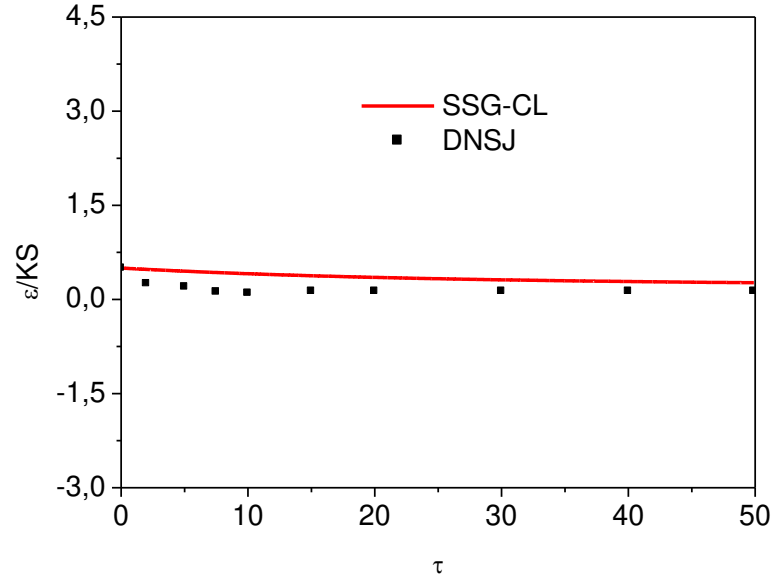


Fig.12. Temporal evolution of the normalized turbulence dissipation ε/KS for $Fr = 1.11$, $SK/\varepsilon = 2$ and $\theta = 45^\circ$

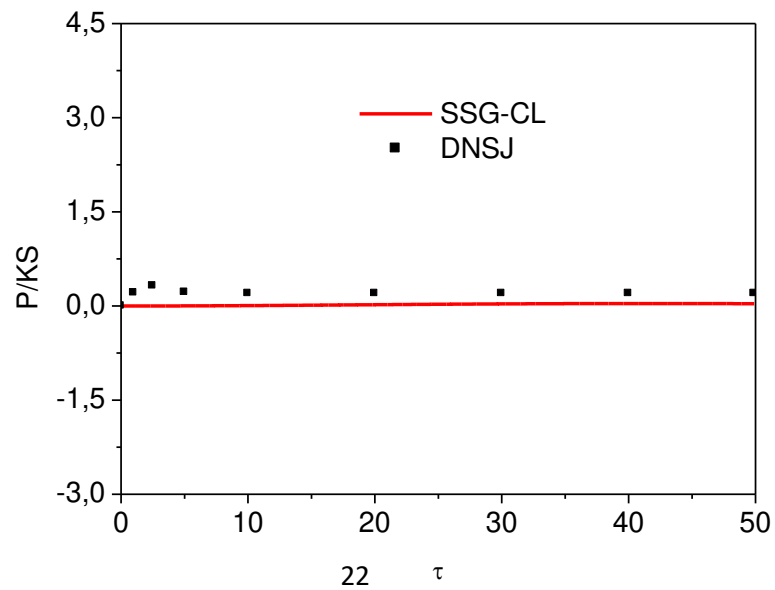


Fig.13. Temporal evolution of the normalized production P/KS

for $Fr = 1.29$, $SK/\varepsilon = 2$ and $\theta = 45^\circ$

4. Conclusion

This study investigates the behavior of stratified and homogeneous turbulence under inclined shear at 45° using the coupled second-order SSG-CL models. Effects of varying the Froude number from 0.35 to 1.11 are examined over a wide range of Shear numbers, ($2 \leq SK/\varepsilon \leq 20$).

Firstly, it is found that at a fixed shear number $SK/\varepsilon = 20$, the variation of the Froude number significantly affects principal anisotropy component b_{12} , density flux $\overline{\rho u_1}$, normalized turbulent dissipation rate ε/KS , the normalized rate of potential energy transfer to kinetic energy η , and ratios of turbulent kinetic energy to total energy K/E and potential energy to total energy K_p/E . According to the SSG-CL model, a decrease of the Froude number leads to an increase of b_{12} , $\overline{\rho u_1}$, η and K/E , while ε/KS decreases. These trends are attributed to the intensification of the stratification as Fr decreases. All simulations show that the SSG-CL model predicts the establishment of asymptotic equilibrium states for various investigated parameters beyond $\tau = 25$. Thus, the coupling of the SSG model for the dynamic field with the CL model for the thermal field makes a significant contribution to the prediction of asymptotic equilibrium states of stratified and homogeneous turbulence with inclined shear.

Secondly, it is shown that the SSG-CL model predicts the existence of equilibrium states for all investigated turbulence parameters as the Shear number varies for $Fr = 1.29$. The equilibrium values of the component of anisotropy $(b_{12})_\infty$ and normalized turbulence production $(P/KS)_\infty$ are only slightly affected by variations of the Shear number. The increase of the Shear number from 2 to 20 conducts to a progressive intensification of the potential energy relative to the kinetic energy.

Finally, a comparison of predictions of the principal components of anisotropy b_{12} and b_{13} , normalized turbulence dissipation ε/KS and normalized production term P/KS obtained using the SSG–CL model with the DNS results reported in [24–26] shows excellent agreement, particularly at large dimensionless times beyond $\tau=10$. This agreement demonstrates the effectiveness of coupling the SSG model with the CL model for simulating homogeneous and stratified turbulence under inclined shear.

Overall, the SSG-CL model provides a reliable and effective framework for modeling shear-stratified turbulent flows. The present study therefore contributes to a better understanding of the complex turbulent processes encountered in geophysical flows, particularly in atmospheric flows.

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CONFLIT OF INTEREST:

The authors declare no conflict of interest.

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