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Interaction between a crack and punches in an orthotropic strip

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Abstract

A drive has been made to find the stress intensity factor (SIF) of a Griffith crack situated in an orthotropic strip with punches on both faces of the strip. The Fourier transform technique is used to reduce the problem to a set of simultaneous singular integral equations with Cauchy-type singularities, which are finally solved using an approximate series solution with Chebyshev polynomials. The approximate analytical expression of the SIF at the tip of the crack is obtained for a large value of the depth of the strip. Numerical values of the SIF have been presented graphically for various depths of the strip. The effect of punches on the SIF for various depths of the orthotropic strip is presented graphically for different particular cases.

Keywords: Orthotropic Material, Crack, Punch, Singular Integral Equations, Chebyshev Polynomial, Stress Intensity Factor

1 Introduction

The subject Fracture mechanics was developed during the time of first World War by the English aeronautical engineer Alan Arnold Griffith and the terminology Griffith crack was according to his name, which is use to explain the failure of brittle materials. He noticed that the stress needed to fracture bulk glass is around 100 MPa while the

theoretical stress needed for breaking atomic bonds of glass is approximately 10,000 MPa. After observing contradictory fact, he developed a new theory to reconcile those.

It was suggested by Griffith [1] that due to the presence of microscopic flaws in the bulk material, the low fracture strength exists in experiments as well as the size-dependence of strength. During experimental glass specimens during verification of the flaw hypothesis, He introduced an artificial flaw, which was in the form of a surface crack which was much larger as compared to the other flaws in a specimen. It was showed that the product of the stress at fracture (σ_f) and the square root of the flaw length ($\sqrt{a}$) is nearly constant, which is expressed by the relation

$$\sigma_f \sqrt{a} \sim C.$$

In linear elasticity theory, the stress at the tip of a sharp flaw in a elastic material is infinite. To avoid it, Griffith developed thermodynamic approach to explain the relation that he observed. The growth of the surfaces on either side of the crack, requires an increase in the surface energy. Griffith found an expression for the constant C in terms of the surface energy of the crack by solving the elasticity problem of a finite crack in an elastic plate.

In recent years, the advancement of space research demands a deep inside into the character of materials particularly of the anisotropic materials which are habitually used in missiles, high speed aircraft, aerospace structures, rocket engines, tube blades etc. This is due to the advent and increased use of high modulus, high strength low weight materials in the various field of modern technology. When a crack is found in such an anisotropic material, anisotropic theory of elasticity should be used to determine the stresses and other physical quantities. This leads to an increasing interest in the analysis of elasto-static and elasto-dynamic crack problems in anisotropic media. But due to the inbuilt complicacy in the stress-strain relations and in equations of equilibrium or in equations of motion [2, 3], the study of the problems of fracture mechanics in such anisotropic media is comparatively difficult.

In solid mechanics, the orthotropic materials have their mechanical or thermal properties unique and independent in three mutually perpendicular directions. As a particular type of anisotropic materials, the properties of orthotropic materials change when measured from different directions, while for Isotropic materials it is found to have homogeneous or non-homogeneous microscopic structures. There are a number of orthotropic materials viz., Alpha-uranium, Beech wood, Pine wood, Beryllium, E-Glass epoxy, Graphite epoxy, etc. These materials are useful while dealing with problems of fracture in composite materials, probably due to the advancement of space research which demands a deep insight into the character of materials. Also the orthotropic composite materials are used in missiles, high speed aircraft, aerospace structures, rocket engines, turbine blades etc.

It is considered that two-dimensional Griffith crack problems represented by a line segment. It is in reality a long flat ribbon-shaped cavity in a solid and is stressed in such a way that the stress pattern remains unaltered while passing in a direction parallel to the plane of the crack. The fracture process zone is a small region surrounding the crack where fracture develops through the successive stages of inhomogeneous slip, void growth, coalescence, and bond breaking on the atomic scale. Obviously, a macroscopic

theory (classical continuum mechanics) is unable to explain what happens in this zone. The inverse statement can be used to define the extent of the process zone, and the surface separating this zone from the continuum region will be viewed as the crack surface. This anticipates an assumption which is essential for the analysis to be limited to a continuum, i.e., that the size of the process zone is negligible compared with the macroscopic dimensions of the body including the crack length. Therefore, the fact that plastic strain develops at the tips of Griffith cracks is not considered in the present analysis. Irwin [4] supported through physical arguments that this simplification does not result in a significant loss of accuracy in the calculation of relevant quantities in fracture mechanics.

We know from the classical theory of elasticity that the stress component σ_{yy} at any crack tip of a Griffith crack becomes infinite. In spite of this fact, cracked bodies can sustain some load. This anomaly was resolved by Griffith, who introduced energy considerations to predict crack growth. From the stress analysis of the Griffith problem, we observe that the stress becomes singular at the crack tip. The strength of this singularity is measured by the stress intensity factor, which controls the magnitude of stresses at any point in a small neighbourhood around the crack tip. Stress intensity factor (SIF) constitute crucial parameters in engineering design against brittle fracture. The stress intensity factor depends on both the geometry of the cracked body and the associated loading.

Problems with Griffith cracks in orthotropic elastic materials have already been studied by Erdogan et al.[5], Dhaliwal[6], Piva and Viola[7], Cinar and Erdogan[8], Kassir and Tse[9], Das et al.[10], Das and Debnath[11], Das[12], Singh et al.[13], Trivedi et al.[14], Tanwar et al.[15], Ingraffea[16], Yang et al.[17] etc. Crack growth in orthotropic materials subjected to dynamic loading has been numerically studied using an adaptive phase-field method by He et al.[18]. Sadowski et al.[19] has applied Finite Element Method to compare the mathematical analytical solution and to determine the fields in the vicinity of the crack tips in an elastic orthotropic material. Jain et al.[20] presented an adaptive phase-field method for modeling crack propagation in orthotropic composites. Yang et al.[17] has studied the thermo-viscoelastic analysis of a cracked orthotropic strip. But the problems in orthotropic medium involving crack and punch problems are few in number Tait and Moodie[21], De and Patra[22], Das and Debnath[23].

The present article focuses on the determination of the Mode-I stress intensity factor for a crack situated in an infinite orthotropic strip subjected to boundary loading through punches. The article is organised as follows. The mathematical formulation of the problem is given in Section 2. The formulated boundary value problem is transformed into a pair of singular integral equations, which have been solved using a series expansion technique involving Chebyshev polynomials. An approximate analytical expression for the stress intensity factor at the crack tip is obtained for generalized loading and punch conditions (see Section 3). The Section 4 contains the numerical findings and graphical showcasing of the variations of the stress intensity factor with respect to the depth of the strip for different ratios of applied loading and punch forces

for the orthotropic materials Beryllium and Steel-Mylar composite. The article concludes in Section 5, where a concise summary of the research work, key outcomes and future direction of the research in the concerned area are provided.

2 Mathematical formulation of the problem

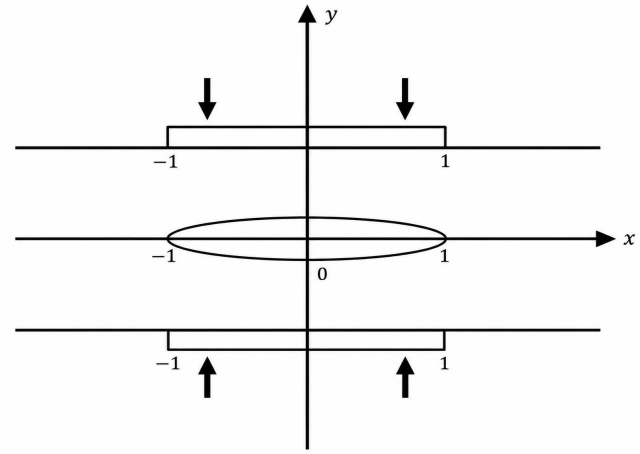


Fig. 1 Geometry of the problem.

Let us consider a plane elastostatic problem of a Griffith crack situated at the mid-plane of an infinite orthotropic strip of thickness $2h$ $(-\infty < x < \infty, -h \leq y \leq h)$ with punches on both faces of the strip (see Figure 1).

Under the assumption of plane strain in an orthotropic medium, the displacement equations of motion are

$$\begin{aligned} c_{11}\frac{\partial^2 u}{\partial x^2} + c_{66}\frac{\partial^2 u}{\partial y^2} + (c_{12}+c_{66})\frac{\partial^2 v}{\partial x \partial y} &= 0, \\ c_{22}\frac{\partial^2 v}{\partial y^2} + c_{66}\frac{\partial^2 v}{\partial x^2} + (c_{12}+c_{66})\frac{\partial^2 u}{\partial x \partial y} &= 0, \end{aligned} \tag{1}$$

where c_{ij}'s are elastic constants, and $u(x, y)$ and $v(x, y)$ are displacements along x and y directions.

The stress displacement relations are

$$\begin{aligned} \sigma_{xx} &= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}, \\ \sigma_{yy} &= c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}, \\ \sigma_{xy} &= c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \end{aligned} \tag{2}$$

The problem is symmetric with respect to $y = 0$, so it is sufficient to consider the problem for the half strip $0 \le y \le h$. It is assumed that the crack defined by $|x| \le 1$, $y = 0$ is opened by the normal traction $-p(x)$ and punches are given on the boundaries of the strip of an amount $-q(x)$ in such a way as to guarantee the symmetry requirements.

Therefore, the boundary conditions on $y = 0$ are

$$\begin{aligned} \sigma_{yy}(x,0) &= -p(x), && |x| < 1, && (3)\\ v(x,0) &= 0, && |x| \ge 1, && (4)\\ \sigma_{xy}(x,0) &= 0, && -\infty < x < \infty. && (5) \end{aligned}$$

The boundary conditions on $y = h$ are

$$\begin{aligned} \sigma_{yy}(x,h) &= -q(x), && |x| < 1, && (6)\\ v(x,h) &= 0, && |x| \ge 1, && (7)\\ \sigma_{xy}(x,h) &= 0, && -\infty < x < \infty. && (8) \end{aligned}$$

In addition, all components of the stress and displacement vanish at the remote distances from the crack.

3 Solution of the problem

The solutions of the equations (1) are considered as

$$\begin{aligned} u(x,y) &= \int_0^\infty A(s,y)\sin sx\, ds,\\ v(x,y) &= \int_0^\infty B(s,y)\cos sx\, ds, \end{aligned} \tag{9}$$

where the arbitrary functions $A(s,y)$ and $B(s,y)$ satisfy the equations

$$\begin{aligned} c_{11}s^2A - c_{66}\frac{d^2A}{dy^2} + (c_{12}+c_{66})s\frac{dB}{dy} &= 0,\\ c_{66}s^2B - c_{22}\frac{d^2B}{dy^2} - (c_{12}+c_{66})s\frac{dA}{dy} &= 0, \end{aligned} \tag{10}$$

whose solutions are considered as

$$\begin{aligned} A(s,y) &= A_1(s)e^{\gamma_1 sy} + A_2(s)e^{-\gamma_1 sy} + C_1(s)e^{\gamma_2 sy} + C_2(s)e^{-\gamma_2 sy},\\ B(s,y) &= -\frac{\alpha_1}{\gamma_1}A_1(s)e^{\gamma_1 sy} - \frac{\alpha_2}{\gamma_2}A_2(s)e^{-\gamma_1 sy} - \frac{\alpha_1}{\gamma_1}C_1(s)e^{\gamma_2 sy} - \frac{\alpha_2}{\gamma_2}C_2(s)e^{-\gamma_2 sy}, \end{aligned} \tag{11}$$

where γ_1 and γ_2 ($\gamma_1 > \gamma_2$) are the positive roots of the equation

$$c_{66}c_{22}\gamma^4 + \left\{(c_{12}+c_{66})^2 - c_{11}c_{22} - c_{66}^2\right\}\gamma^2 + c_{11}c_{66} = 0, \tag{12}$$

where

$$\alpha_1 = \frac{c_{11} - c_{66}\gamma_1^2}{c_{12} + c_{66}}, \qquad \alpha_2 = \frac{c_{11} - c_{66}\gamma_2^2}{c_{12} + c_{66}}.$$

The expressions of stresses are

$$\sigma_{yy}(x,y) = \int_0^\infty \left[\eta_1 A_1(s)e^{\gamma_1 sy} + \eta_1 A_2(s)e^{-\gamma_1 sy} + \eta_2 C_1(s)e^{\gamma_2 sy} + \eta_2 C_2(s)e^{-\gamma_2 sy}\right] s\cos sx\, ds, \tag{13}$$

$$\sigma_{xy}(x,y) = \int_0^\infty \left[\mu_1 A_1(s)e^{\gamma_1 sy} - \mu_1 A_2(s)e^{-\gamma_1 sy} + \mu_2 C_1(s)e^{\gamma_2 sy} - \mu_2 C_2(s)e^{-\gamma_2 sy}\right] s\cos sx\, ds, \tag{14}$$

where

$$\eta_j = c_{12} - c_{22}\alpha_j, \qquad \mu_j = c_{66}\left(\gamma_j + \frac{\alpha_j}{\gamma_j}\right), \qquad j = 1, 2.$$

The boundary conditions (5) and (8) give

$$\begin{aligned} C_1(s) &= -\frac{\mu_1}{\mu_2}\frac{e^{\gamma_1 sh} - e^{-\gamma_2 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}A_1(s) + \frac{\mu_1}{\mu_2}\frac{e^{-\gamma_1 sh} - e^{-\gamma_2 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}A_2(s), \\ C_2(s) &= \frac{\mu_1}{\mu_2}\frac{e^{\gamma_2 sh} - e^{\gamma_1 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}A_1(s) - \frac{\mu_1}{\mu_2}\frac{e^{\gamma_2 sh} - e^{-\gamma_1 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}A_2(s). \end{aligned} \tag{15}$$

The boundary condition (4) gives

$$\int_0^\infty [A_1(s) - A_2(s)]\cos sx\, ds = 0, \qquad x > 1. \tag{16}$$

The boundary condition (7), with the aid of equations (15), we obtain

$$\int_0^\infty [\delta_1(s)A_1(s) + \delta_2(s)A_2(s)]\cos sx\, ds = 0, \qquad x > 1, \tag{17}$$

where

$$\begin{aligned} \delta_1(s) =& -\frac{\alpha_1}{\gamma_1}e^{\gamma_1 sh} + \frac{\alpha_2\mu_1}{\gamma_2\mu_2}e^{\gamma_2 sh}\left(\frac{e^{\gamma_1 sh} - e^{-\gamma_2 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right) \\ &+ \frac{\alpha_2\mu_1}{\gamma_2\mu_2}e^{-\gamma_2 sh}\left(\frac{e^{\gamma_2 sh} - e^{\gamma_1 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right), \\ \delta_2(s) =& \frac{\alpha_1}{\gamma_1}e^{-\gamma_1 sh} - \frac{\alpha_2\mu_1}{\gamma_2\mu_2}e^{\gamma_2 sh}\left(\frac{e^{-\gamma_1 sh} - e^{-\gamma_2 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right) \\ &- \frac{\alpha_2\mu_1}{\gamma_2\mu_2}e^{-\gamma_2 sh}\left(\frac{e^{\gamma_2 sh} - e^{\gamma_1 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right). \end{aligned}$$

Setting

$$A_1(s) - A_2(s) = \frac{1}{s}\int_0^1 f_1(t)\sin st\,dt, \tag{18}$$

and

$$\delta_1(s)A_1(s) + \delta_2(s)A_2(s) = \frac{1}{s}\int_0^1 f_2(t)\sin st\,dt, \tag{19}$$

we see that the equations (16) and (17) are identically satisfied, and the unknown functions $f_1(t)$ and $f_2(t)$ are odd functions of t. The equations (18) and (19) can be rewritten as

$$\begin{aligned} A_1(s) =& \frac{\delta_2(s)}{s[\delta_1(s)+\delta_2(s)]}\int_0^1 f_1(t)\sin st\,dt + \frac{1}{s[\delta_1(s)+\delta_2(s)]}\int_0^1 f_2(t)\sin st\,dt, \\ A_2(s) =& -\frac{\delta_1(s)}{s[\delta_1(s)+\delta_2(s)]}\int_0^1 f_1(t)\sin st\,dt + \frac{1}{s[\delta_1(s)+\delta_2(s)]}\int_0^1 f_2(t)\sin st\,dt. \end{aligned} \tag{20}$$

The boundary condition (3), with the help of the expressions $C_1(s)$ and $C_2(s)$ obtained from equation (15), gives

$$\int_0^\infty [\delta_3(s)A_1(s) + \delta_4(s)A_2(s)]\, s\cos sx\,ds = -p(x), \quad 0 < x < 1, \tag{21}$$

where

$$\begin{aligned} \delta_3(s) =& \eta_1 - \frac{\mu_1\eta_2}{\mu_2}\left(\frac{e^{\gamma_1 sh} - e^{-\gamma_2 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right) + \frac{\mu_1\eta_2}{\mu_2}\left(\frac{e^{\gamma_2 sh} - e^{\gamma_1 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right), \\ \delta_4(s) =& \eta_1 + \frac{\mu_1\eta_2}{\mu_2}\left(\frac{e^{-\gamma_1 sh} - e^{-\gamma_2 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right) - \frac{\mu_1\eta_2}{\mu_2}\left(\frac{e^{\gamma_2 sh} - e^{-\gamma_1 sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right). \end{aligned}$$

Now, substituting the expressions of $A_1(s)$ and $A_2(s)$ from equations (20) in the equation (21), we get

$$\begin{aligned} &\int_0^1 f_1(t)dt\left(\int_0^\infty \frac{\delta_3(s)\delta_2(s) - \delta_4(s)\delta_1(s)}{\delta_1(s)+\delta_2(s)}\sin st\cos sx\,ds\right) \\ &+ \int_0^1 f_2(t)dt\left(\int_0^\infty \frac{\delta_3(s)+\delta_4(s)}{\delta_1(s)+\delta_2(s)}\sin st\cos sx\,ds\right) = -p(x), \quad 0 < x < 1. \end{aligned} \tag{22}$$

Now, the boundary condition (6), with the help of equation (15), gives

$$\int_0^\infty [\delta_9(s)A_1(s) + \delta_{10}(s)A_2(s)]\, s\cos sx\,ds = -q(x), \quad 0 < x < 1, \tag{23}$$

where

$$\delta_9(s) = \eta_1 e^{\gamma_1 sh} - \frac{\mu_1\eta_2}{\mu_2}\left(\frac{e^{(\gamma_1+\gamma_2)sh} - 1}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right) + \frac{\mu_1\eta_2}{\mu_2}\left(\frac{1 - e^{(\gamma_1-\gamma_2)sh}}{e^{\gamma_2 sh} - e^{-\gamma_2 sh}}\right),$$

$$\delta_{10}(s) = \eta_1 e^{-\gamma_1 sh} + \frac{\mu_1\eta_2}{\mu_2}\left(\frac{e^{-(\gamma_1-\gamma_2)sh}-1}{e^{\gamma_2 sh}-e^{-\gamma_2 sh}}\right) - \frac{\mu_1\eta_2}{\mu_2}\left(\frac{1-e^{-(\gamma_1+\gamma_2)sh}}{e^{\gamma_2 sh}-e^{-\gamma_2 sh}}\right).$$

Now equation (22), with the aid of the equations (20), gives

$$\int_0^1 f_1(t)dt\left(\int_0^\infty \frac{\delta_9(s)\delta_2(s)-\delta_{10}(s)\delta_1(s)}{\delta_1(s)+\delta_2(s)}\sin st\cos sx\,ds\right) + \int_0^1 f_2(t)dt\left(\int_0^\infty \frac{\delta_9(s)+\delta_{10}(s)}{\delta_1(s)+\delta_2(s)}\sin st\cos sx\,ds\right) = -q(x). \tag{24}$$

After a lengthy mathematical calculation, we get

$$\begin{aligned}
\frac{\delta_3(s)+\delta_4(s)}{\delta_1(s)+\delta_2(s)} &= M_1(s),\\
\frac{\delta_3(s)\delta_2(s)-\delta_4(s)\delta_1(s)}{\delta_1(s)+\delta_2(s)} &= -(\mu_2\eta_1-\mu_1\eta_2)/\mu_2 - M_2(s),\\
\frac{\delta_9(s)+\delta_{10}(s)}{\delta_1(s)+\delta_2(s)} &= \frac{\mu_2\eta_1-\mu_1\eta_2}{\frac{\alpha_2\mu_1}{\gamma_2}-\frac{\alpha_1\mu_2}{\gamma_1}} + M_3(s),\\
\frac{\delta_9(s)\delta_2(s)-\delta_{10}(s)\delta_1(s)}{\delta_1(s)+\delta_2(s)} &= -M_4(s),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
M_1(s) =& \frac{1}{\left(\frac{\alpha_2\mu_1}{\gamma_2}-\frac{\alpha_1\mu_2}{\gamma_1}\right)(1-e^{-2\gamma_1 sh})}\Bigg\{\mu_2\eta_1 e^{-\gamma_1 sh} + (\mu_2\eta_1-\mu_1\eta_2)e^{-\gamma_1 sh}\\
&- \mu_1\eta_2\left[\frac{2e^{-2\gamma_2 sh}+e^{-(\gamma_1+2\gamma_2)sh}-2e^{-(2\gamma_1+\gamma_2)sh}-e^{-\gamma_1 sh}}{1-e^{-2\gamma_2 sh}}\right]\Bigg\}.
\end{aligned} \tag{26}$$

$$\begin{aligned}
M_2(s) =& \frac{1}{\mu_2\left(1-e^{-2\gamma_1 sh}\right)}\Bigg\{(\mu_2\eta_1-\mu_1\eta_2)e^{-2\gamma_1 sh} + \mu_2\eta_1 e^{-2\gamma_1 sh}\\
&- \mu_1\eta_2\left[\frac{e^{-(\gamma_1+2\gamma_2)sh}-e^{-(2\gamma_1+\gamma_2)sh}-e^{-2\gamma_1 sh}}{1-e^{-2\gamma_2 sh}}\right]\\
&+ 2\mu_1\eta_2\left[\frac{e^{-(\gamma_1+\gamma_2)sh}-e^{-2\gamma_2 sh}}{1-e^{-2\gamma_2 sh}}\right]\Bigg\}.
\end{aligned} \tag{27}$$

$$\begin{aligned}
M_3(s) =& \frac{1}{\left(\frac{\alpha_2\mu_1}{\gamma_2}-\frac{\alpha_1\mu_2}{\gamma_1}\right)(1-e^{-2\gamma_1 sh})}\Bigg\{\mu_2\eta_1 e^{-2\gamma_1 sh} + (\mu_2\eta_1-\mu_1\eta_2)e^{-2\gamma_1 sh}\\
&+ \mu_1\eta_2\left[\frac{e^{-(\gamma_1+\gamma_2)sh}-e^{-2\gamma_2 sh}}{1-e^{-2\gamma_2 sh}}\right]\Bigg\}.
\end{aligned} \tag{28}$$

$$M_4(s) = \frac{1}{\mu_2\,(1-e^{-2\gamma_1 sh})}\Bigg\{(\mu_2\eta_1-\mu_1\eta_2)e^{-\gamma_1 sh}+\mu_2\eta_1 e^{-\gamma_1 sh} + \mu_1\eta_2\left[\frac{2e^{-(2\gamma_1+\gamma_2)sh}-e^{-2\gamma_2 sh}}{1-e^{-2\gamma_2 sh}}\right] + \mu_1\eta_2\left[\frac{e^{-(2\gamma_1+\gamma_2)sh}-e^{-(\gamma_1+2\gamma_2)sh}+e^{-\gamma_1 sh}-2e^{-2\gamma_2 sh}+e^{-(\gamma_1+2\gamma_2)sh}}{1-e^{-2\gamma_2 sh}}\right]\Bigg\}. \tag{29}$$

Now the equations (22) and (23), together with the expressions of $\delta_i(s)$ given in equations (24) and (25), lead to the following singular integral equations for unknown functions $f_i(t)$ $(i=1,2)$ as

$$\frac{1}{\mu_2}(\mu_2\eta_1-\mu_1\eta_2)\int_{-1}^{1}\frac{f_1(t)dt}{t-x}+\int_{-1}^{1}\left[K_{11}(x,t)f_1(t)+K_{12}(x,t)f_2(t)\right]dt=2p(x), \qquad 0<x<1, \tag{30}$$

$$\left(\frac{\mu_2\eta_1-\mu_1\eta_2}{\frac{\alpha_2\mu_1}{\gamma_2}-\frac{\alpha_1\mu_2}{\gamma_1}}\right)\int_{-1}^{1}\frac{f_2(t)dt}{t-x}+\int_{-1}^{1}\left[K_{21}(x,t)f_1(t)+K_{22}(x,t)f_2(t)\right]dt=-2q(x), \qquad 0<x<1, \tag{31}$$

where

$$\begin{aligned} K_{11}(x,t) &= \int_0^\infty M_2(s)\sin s(t-x)\,ds,\\ K_{12}(x,t) &= -\int_0^\infty M_1(s)\sin s(t-x)\,ds,\\ K_{21}(x,t) &= -\int_0^\infty M_4(s)\sin s(t-x)\,ds,\\ K_{22}(x,t) &= \int_0^\infty M_3(s)\sin s(t-x)\,ds. \end{aligned} \tag{32}$$

The solutions of the integral equations (30) and (31) are assumed as

$$f_k(t)=\frac{1}{\sqrt{1-t^2}}\sum_{n=0}^{\infty}d_{kn}T_{2n+1}(t), \qquad k=1,2, \tag{33}$$

where $T_n(t)$ is the Chebyshev polynomial of first kind. Now, using the relation

$$\int_{-1}^{1}\frac{T_i(t)\,dt}{\sqrt{1-t^2}(t-x)}=\begin{cases}0, & i=0,\\ \pi U_{i-1}(x), & i>0,\end{cases} \tag{34}$$

and orthogonality relation

$$\int_{-1}^{1} U_i(x)U_j(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & i \neq j, \\ \dfrac{\pi}{2}, & i = j, \end{cases} \tag{35}$$

the equations (30) and (31) will be reduced to the following infinite system of algebraic equations in unknown functions d_{kn} with the Chebyshev polynomials of second kind $U_i(x)$ as

$$\frac{\pi^2}{2\mu_2}(\mu_2\eta_1 - \mu_1\eta_2)d_{1j} + \sum_{n=0}^{\infty} d_{1n}l_{1nj} + \sum_{n=0}^{\infty} d_{2n}l_{2nj} = f_{1j}, \tag{36}$$

$$\frac{\pi^2}{2}\left(\frac{\mu_2\eta_1 - \mu_1\eta_2}{\dfrac{\alpha_2\mu_1}{\gamma_2} - \dfrac{\alpha_1\mu_2}{\gamma_1}}\right) d_{2j} + \sum_{n=0}^{\infty} d_{1n}l'_{1nj} + \sum_{n=0}^{\infty} d_{2n}l'_{2nj} = q_{1j}, \tag{37}$$

$$j = 0, 1, 2, \ldots,$$

where

$$\begin{aligned}
l_{1nj} &= \int_{-1}^{1} H_{11}(x)U_{2j}(x)\sqrt{1-x^2}\,dx, \qquad l_{2nj} = \int_{-1}^{1} H_{12}(x)U_{2j}(x)\sqrt{1-x^2}\,dx, \\
l'_{1nj} &= \int_{-1}^{1} H_{21}(x)U_{2j}(x)\sqrt{1-x^2}\,dx, \qquad l'_{2nj} = \int_{-1}^{1} H_{22}(x)U_{2j}(x)\sqrt{1-x^2}\,dx, \\
H_{11}(x) &= \int_{-1}^{1} \frac{K_{11}(x,t)T_{2n+1}(t)}{\sqrt{1-t^2}}\,dt, \qquad H_{12}(x) = \int_{-1}^{1} \frac{K_{12}(x,t)T_{2n+1}(t)}{\sqrt{1-t^2}}\,dt, \\
H_{21}(x) &= \int_{-1}^{1} \frac{K_{21}(x,t)T_{2n+1}(t)}{\sqrt{1-t^2}}\,dt, \qquad H_{22}(x) = \int_{-1}^{1} \frac{K_{22}(x,t)T_{2n+1}(t)}{\sqrt{1-t^2}}\,dt,
\end{aligned} \tag{38}$$

with

$$f_{1j} = 2\int_{-1}^{1} p(x)U_{2j}(x)\sqrt{1-x^2}\,dx, \qquad q_{1j} = -2\int_{-1}^{1} q(x)U_{2j}(x)\sqrt{1-x^2}\,dx. \tag{39}$$

The stress intensity factor at the tip $x = 1$ of the crack is found to be

$$\begin{aligned}
K_I &= \lim_{x\to 1^+} \sqrt{2(x-1)}\,\sigma_{yy}(x,0), \\
&= \frac{\pi}{2\mu_2}(\mu_2\eta_1 - \mu_1\eta_2)\sum_{n=0}^{\infty} d_{1n}.
\end{aligned} \tag{40}$$

4 Numerical results and discussion

As particular cases of the problem, two orthotropic materials viz., Beryllium and Steel-Mylar composite have been considered for numerical findings. The elastic moduli (10^{11} dynes cm^{-2}) of the materials are given in Table 1 [24].

Table 1 Material Constants.

Material/Elastic constants	$\mathbf{C_{11}}$	$\mathbf{C_{22}}$	$\mathbf{C_{12}}$	$\mathbf{C_{66}}$
Beryllium	31.48	36.49	8.88	11.24
Steel-Mylar Composite	18.7	2.92	1.3	0.62

Because of its lightweight and strength, Beryllium is of immense interest in spacecraft applications. Pure beryllium material is used in the manufacture of aircraft disc brakes, nuclear weapons and reactors, missile parts, heat shields, X-ray machine parts, mirrors and spacecrafts.
Steel-Mylar composite is an orthotropic material that provides high impact resistance, design flexibility, and superior load-bearing capacity for its weight. Because of its lightweight and strength, budget-friendly maintenance and excellent workability, this material is often used as non-metallic alternative to metals.
Here, the constant tensile load $p(x) = p$ is applied to open the crack face, and a punch of an amount $q(x) = q$ is applied on the strip of the material. The variations of the normalized Mode I SIF, K_I/p vs. h for different values of q/p are drawn for the orthotropic material Beryllium (Fig.2) and for the material Steel–Mylar Composite (Fig.3).
For the first material, when q/p is close to 1, K_I/p initially decreases and then increases as the depth of the strip, h, increases. But for $\frac{q}{p} = 0.2,\ 0.4,\ 0.6$, it initially decreases rapidly and afterward it becomes stable with increasing depth h. For the second material, when q/p is close to 1, K_I/p initially increases and after that it decreases with the increase in h. But for $q/p = 0.2, 0.4, 0.6$, it steadily decreases with increase in h. The decrease in SIF physical justifies that, for sufficiently large depth of the strip, the crack may be completely closed and the fracture tendency is reduced. Again, the increase in K_I/p as q increases is due to the fact that compression opposes crack opening.
For both materials, it is seen from the figures that as the depth of the strip becomes large, K_I/p decreases to 1, which validates the result that the Mode I normalized SIF tends to unity for a crack situated in an infinite orthotropic medium.

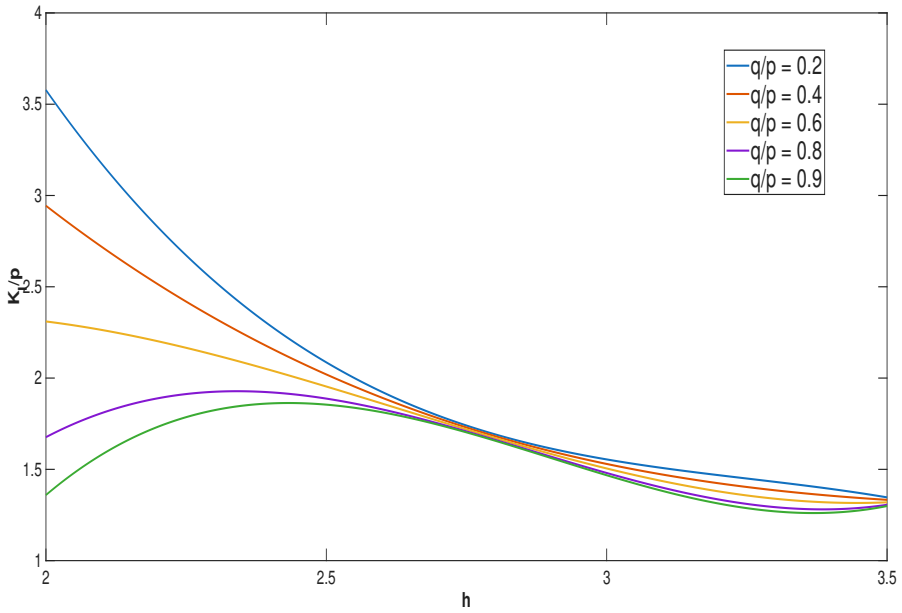


Fig. 2 Variations of K_I/p vs. h for different values of q/p for the material Beryllium.

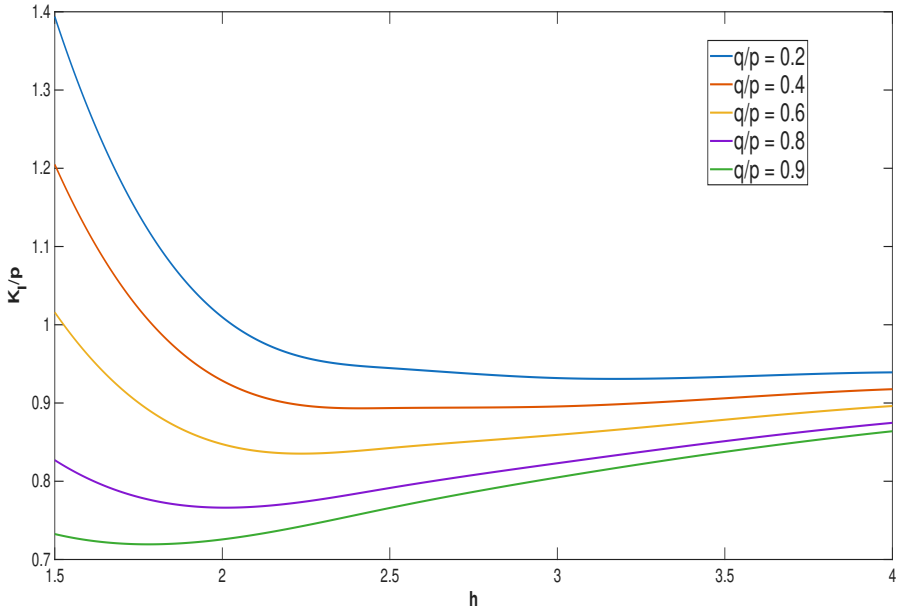


Fig. 3 Variations of K_I/p vs. h for different values of q/p for the material Steel-mylar.

5 Conclusion

The aim of the present research is to develop knowledge about the behaviour of SIF and the possibility of crack arrest and crack propagation due to the interactions between an embedded crack in an infinite orthotropic strip and punches given on the faces of the strip. In this study, the authors have achieved two important goals, viz., the effect on the SIF of the embedded crack for different depths of the cracked orthotropic material and the possibility of crack arrest/crack propagation due to the effect of punches for different depths of the infinite strip. The authors are optimistic that the proposed model will be beneficial for the researchers and engineers working in the field of anisotropic/isotropic materials. The problems of collinear cracks situated in an orthotropic material sandwiched between two different elastic strips with punches under sudden tensile traction/point loading/thermo-mechanical loadings will be the future studies.

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