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Riemann–Liouville Derivative Approach of the Fractional X-ray Diffraction Theory: the Cauchy nonlocal problem

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Abstract

Towards to the theory of the fractional X-ray diffraction crystal optics, the Riemann–Liouville derivative approach has been developed and analyzed for arbitrary fractional-order-parameter α , $\alpha \in (0, 1]$. Using the Green function technique, fundamentals of the fractional matrix differential theory of the X-ray diffraction scattering by the imperfect crystals, has been formulated in the terms of solving the Cauchy nonlocal problem. In a general case of an arbitrary displacement imperfect-crystal-lattice function $\mathbf{f}(\mathbf{r})$, one has used the Liouville–Neumann-type series formalism to build up the resolvent-function-solution of the Cauchy problem rewritten in the form of the matrix integral Fredholm–Volterra equation of the second kind. One has made of comprising the theoretical results obtained in the frame of the Riemann–Liouville derivative operator approach with the corresponding ones using the fractional derivatives in the Gerasimov–Caputo's sense. In the case when the displacement field function $\mathbf{f}(\mathbf{r})$ is the lateral coordinate $\mathbf{x}$ is the 1D linear function, $\mathbf{f}(\mathbf{r}) = \mathbf{a}\mathbf{x}$, $\mathbf{a} = \text{const}$, one has obtained the rigorous Cauchy problem solutions expressed through the Mittag-Leffler-type and generalized trigonometric functions. One has numerically calculated, built up and in details, analyzed the graph-patterns of the amplitudes of the transmitted in details $\mathbf{d}$ and diffracted waves along the crystal depth $\mathbf{t}$ for different values of the fractional-order-parameter α in the both cases of the theoretical Riemann–Liouville derivative and Gerasimov–Caputo's

one approaches. Based on Mittag-Leffler-type asymptotic expressions (for large values of t), the decay behavior of the X-ray wave amplitude was determined for both the Riemann–Liouville and Gerasimov–Caputo derivative approaches.

Keywords: Diffraction optics system of fractional partial differential equations, the Riemann–Liouville fractional derivative, the Cauchy nonlocal problem, the matrix resolvent-function-solution, the Fredholm–Volterra integral equation, the generalized trigonometric functions, the generalized Euler’s formula.

1 Introduction

As is known, in recent years, to model and analyze the real-world phenomena in the terms of the mathematical physics, fractional calculus has become widespread of allowing to go beyond the integer differential equations solutions. In particular, the latter is important to taking into account and adequately considering such the effects as memory and no locality of the quantum radiation-medium interactions (see [1]–[5] and huge cited references therein).

Recently [5], [6], the theory of the fractional X-ray diffraction crystal optics has been developed with using the system of the fractional Takagi–Taupin (TT) differential equations in the terms of the Gerasimov–Caputo (GC) derivatives of an arbitrary fractional order α , $\alpha \in (0, 1]$, along the crystal depth. In [7], the integral formalism of the 3D fractional X-ray diffraction crystal optics has been done, and the matrix resolvent solution of the fractional GC-Cauchy problem has been constructed has been found out as well.

Accordingly, in the case when the lattice-crystal displacement field function $f(r)$ vs the lateral coordinate x is the 1D linear function, $f(r) = ax$, $a = \text{const}$, the rigorous analytical solution of the fractional in GC-Cauchy problem with an arbitrary fractional-order-parameter (FOP) α , $\alpha \in (0, 1]$ has been found out and analyzed.

As was pointed out in [7], various kinds of real problems-in-physics such as signal processing, the radiation propagation, atom diffusion in material media, are currently modeled by the aid of fractional derivative by highlighting in particular the Riemann–Liouville (RL) derivative and GC derivative approaches as the fundamental formulations in fractional calculus that generalizes the concept of derivatives to non-integer.

Based on the fractional TT differential equations, a goal of the present study is to push ahead the theory of the fractional X-ray diffraction crystal optics in the terms of the RL derivative concept beyond the frame of the GC derivative constructive scheme [5], [6].

In the paper, using the standard Green’s function technique, a key of working is to develop the matrix Fredholm–Volterra integral equation formalism earlier proposed by authors [5], [6], and builds up the matrix resolvent-function solution of the RL-Cauchy nonlocal problem.

In the case of the lattice-crystal displacement field function $f(r)$ is the 1D linear function, namely: $f(r) = ax$, $a = \text{const}$, for an arbitrary FOP α , $\alpha \in (0, 1]$ of the

RL derivative, the exact analytical solution of the RL-Cauchy nonlocal problem has been found out and analyzed of comparing it with the GC concept-Cauchy problem solution.

Thus, following up the theoretical formalism earlier developed in [5], [6] the system of the original TT differential equations in the frame of the RL concept of the fractional X-ray diffraction crystal optics takes the form

$$\begin{pmatrix} D_{0t}^\alpha - \frac{\partial}{\partial x} & 0 \\ 0 & D_{0t}^\alpha + \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} E_0(x, t) \\ E_h(x, t) \end{pmatrix} = \\ = i \begin{pmatrix} \gamma & \sigma \exp[if(x, t)] \\ \sigma \exp[-if(x, t)] & \gamma \end{pmatrix} \begin{pmatrix} E_0(x, t) \\ E_h(x, t) \end{pmatrix}, \quad (1.1)$$

with the Cauchy problem's condition

$$\lim_{t \rightarrow 0} D_{0t}^{\alpha-1} \begin{pmatrix} E_0(x, t) \\ E_h(x, t) \end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_h(x) \end{pmatrix}, \quad -\infty < x < \infty, \quad (1.2)$$

where $\varphi_0(x)$ and $\varphi_h(x)$ are the given real-valued functions.

The RL fractional integro-differential operator of order $\alpha < 0$ [1] is equal to

$$D_{at}^\alpha g(t) = \frac{\text{sgn}(t-a)}{\Gamma(-\nu)} \int_a^t \frac{g(s)ds}{|t-s|^{\alpha+1}},$$

and for $\alpha \in (0, 1]$ the operator D_{at}^ν is determined by following relation

$$D_{at}^\alpha g(t) = \text{sgn}(t-a) \frac{d}{dt} D_{at}^{\alpha-1} g(t), \quad (1.3)$$

here $\Gamma(z)$ is the gamma-function.

2 Integral fundamentals of the RL Cauchy non-local problem

Let us convert the Cauchy non-local problem in the 'differential form' (1.1)–(1.2) to the matrix Fredholm–Volterra-type integral equation of the second kind. The latter takes a special sense to build up the resolvent-function solution of the Cauchy problem, which in turn is very important to computer-aided modelling and recovering the crystal displacement field function $f(\mathbf{R})$ from the microtomography data.

The system of differential TT-type equations (1.1) may be rewritten into the form

$$\begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} = i\sigma \mathbf{K} \mathbf{E}, \quad (2.1)$$

where

$$O_+^\alpha = D_{0t}^\alpha + \frac{\partial}{\partial x}, \quad O_-^\alpha = D_{0t}^\alpha - \frac{\partial}{\partial x},$$

$$\mathbf{E} \equiv \mathbf{E}(x, t) = \begin{pmatrix} E_0(x, t) \\ E_h(x, t) \end{pmatrix}, \quad \mathbf{K} \equiv \mathbf{K}(x, t) = \begin{pmatrix} 0 & e^{if(x, t)} \\ e^{-if(x, t)} & 0 \end{pmatrix}.$$

Acting onto both sides of (2.1) by the operator $\text{diag}(O_+^\alpha - i\gamma, O_-^\alpha - i\gamma)$, one obtains

$$\begin{aligned} & \begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} = \\ & = i\sigma \begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \mathbf{K} \mathbf{E} \end{aligned} \quad (2.2)$$

and after some straightforward routine calculations, one finds out

$$\begin{aligned} & \begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \mathbf{K} \mathbf{E} = \left[D_{0t}^\alpha + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} - i\gamma \right] (\mathbf{K} \mathbf{E}) = D_{0t}^\alpha (\mathbf{K} \mathbf{E}) + \\ & + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\left(\frac{\partial}{\partial x} \mathbf{K} \right) \mathbf{E} + \mathbf{K} \left(\frac{\partial}{\partial x} \mathbf{E} \right) \right] + \mathbf{K} (D_{0t}^\alpha \mathbf{E}) - \mathbf{K} (D_{0t}^\alpha \mathbf{E}) - i\gamma \mathbf{K} \mathbf{E} = \\ & = \mathbf{K} \begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} + D_{0t}^\alpha (\mathbf{K} \mathbf{E}) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{\partial}{\partial x} \mathbf{K} \right) \mathbf{E} - \mathbf{K} (D_{0t}^\alpha \mathbf{E}) = \\ & = \mathbf{K} \begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} + (D_{0t}^\alpha + if'_x) (\mathbf{K} \mathbf{E}) - \mathbf{K} (D_{0t}^\alpha \mathbf{E}). \end{aligned} \quad (2.3)$$

Taking into account that the $\mathbf{E} = \mathbf{E}(x, t)$ is the solution of Eq. (2.1) and $\mathbf{K}^2$ is equal to unit matrix, from Eq. (2.3) it directly follows

$$i\sigma \begin{pmatrix} O_+^\alpha - i\gamma & 0 \\ 0 & O_-^\alpha - i\gamma \end{pmatrix} \mathbf{K} \mathbf{E} = -\sigma^2 \mathbf{E} + i\sigma (D_{0t}^\alpha + if'_x) (\mathbf{K} \mathbf{E}) - i\sigma \mathbf{K} (D_{0t}^\alpha \mathbf{E}).$$

The last allows us to rewrite down Eq. (2.2) into the form

$$\begin{aligned} & \begin{pmatrix} (O_+^\alpha - i\gamma)(O_-^\alpha - i\gamma) + \sigma^2 & 0 \\ 0 & (O_-^\alpha - i\gamma)(O_+^\alpha - i\gamma) + \sigma^2 \end{pmatrix} \mathbf{E} = \\ & = i\sigma (D_{0t}^\alpha + if'_x) (\mathbf{K} \mathbf{E}) - i\sigma \mathbf{K} (D_{0t}^\alpha \mathbf{E}). \end{aligned} \quad (2.4)$$

Further, we denote the Fourier transform of the function $f(x)$ by $(f(x))_k$, the Laplace transform of the function $g(t)$ by $(g(t))_p$, and respectively, the double Fourier–Laplace transform of the function $h(x, t)$ by $(h(x, t))_{k, p}$.

By using the following formula

$$[D_{0t}^\alpha H(x, t)]_p = p^\alpha [H(x, t)]_p - D_{00}^{\alpha-1} H,$$

where

$$D_{00}^{\alpha-1} H = \lim_{t \rightarrow 0} D_{0t}^{\alpha-1} H(x, t),$$

one can get

$$[O_\pm^\alpha H(x, t)]_{k, p} = (p^\alpha \pm ik) [H(x, t)]_{k, p} - [(D_{00}^{\alpha-1} H)(x)]_k. \quad (2.5)$$

Applying the double Fourier–Laplace transform to Eq. (2.4) and keeping in mind Eq. (2.5), one can obtain

$$\begin{aligned} [(O_+^\alpha - i\gamma)(O_-^\alpha - i\gamma)E_0(x, t)]_{k,p} &= (p^\alpha - i\gamma + ik)[(O_-^\alpha - i\gamma)E_0(x, t)]_{k,p} - \\ - \{D_{00}^{\alpha-1}[(O_-^\alpha - i\gamma)E_0(x, t)]\}_k &= (p^\alpha - i\gamma + ik)(p^\alpha - i\gamma - ik)[E_0(x, t)]_{k,p} - \\ - \left\{ (p^\alpha - i\gamma + ik)[D_{00}^{\alpha-1}E_0(x, t)] + i\sigma D_{00}^{\alpha-1}[e^{if(x,t)}E_h(x, t)] \right\}_k, \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} [(O_-^\alpha - i\gamma)(O_+^\alpha - i\gamma)E_h(x, t)]_{k,p} &= (p^\alpha - i\gamma - ik)(p^\alpha - i\gamma + ik)[E_h(x, t)]_{k,p} - \\ - \left\{ (p^\alpha - i\gamma - ik)D_{00}^{\alpha-1}E_h(x, t) + i\sigma D_{00}^{\alpha-1}[e^{-if(x,t)}E_0(x, t)] \right\}_k. \end{aligned} \quad (2.7)$$

Here, one has used the relationships (cf. (2.1))

$$D_{00}^{\alpha-1}[(O_-^\alpha - i\gamma)E_0(x, t)] = i\sigma D_{00}^{\alpha-1}[e^{if(x,t)}E_h(x, t)],$$

$$D_{00}^{\alpha-1}[(O_+^\alpha - i\gamma)E_h(x, t)] = i\sigma D_{00}^{\alpha-1}[e^{-if(x,t)}E_0(x, t)].$$

Thus from Eqs. (2.4) – (2.7) it directly follows

$$\begin{aligned} (\mathbf{E}(x, t))_{k,p} &= \frac{i\sigma}{W} \left\{ (D_{0t}^\alpha + if'_x)(\mathbf{K}\mathbf{E}) - \mathbf{K}D_{0t}^\alpha \mathbf{E} \right\}_{k,p} + \\ + \frac{1}{W} \left\{ \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix} (D_{00}^{\alpha-1}\mathbf{E})_k + i\sigma (D_{00}^{\alpha-1}\mathbf{K}\mathbf{E})_k \right\}, \end{aligned} \quad (2.8)$$

where

$$W = (p^\alpha - i\gamma)^2 + k^2 + \sigma^2, \quad W^\pm = p^\alpha - i\gamma \pm ik.$$

In [6] the inverse double Fourier–Laplace transform

$$\left(\frac{1}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} \right)_{x,t} = G_{\alpha,\gamma}(x, t)$$

has been taking out for the Green function [12]

$$G_{\alpha,\gamma}(x, t) = \frac{1}{2} \int_{|x|}^{\infty} e^{i\gamma\tau} J_0 \left(\sigma \sqrt{\tau^2 - x^2} \right) \frac{1}{t} \phi \left(-\alpha, 0; -\frac{\tau}{t^\alpha} \right) d\tau,$$

here $\phi(\beta, \rho; z)$ is the Wright function [11]

$$\phi(\beta, \rho; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\beta k + \rho)}, \quad \beta > -1, \quad \rho \in \mathbb{C}.$$

As a result, from (2.8) one obtains the matrix integral equation

$$\mathbf{E}(x, t) = (\mathbf{A}^{\alpha, \gamma} \mathbf{E}(x, t))(x, t) + (\mathbf{B}^{\alpha, \gamma} \mathbf{E}(x, 0))(x, t), \quad (2.9)$$

where the following notations are introduced

$$\begin{aligned} (\mathbf{A}^{\alpha, \gamma} \mathbf{E}(x, t))(x, t) &= i\sigma \int_0^t dv \int_{-\infty}^{\infty} \mathbf{K}_1(x, t; u, v) \mathbf{E}(u, v) du - \\ &+ i\sigma \int_{-\infty}^{\infty} G_{\alpha, \gamma}(x - u, t) \mathbf{K}(u, 0) D_{00}^{\alpha-1} \mathbf{E}(u, t) du, \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} \mathbf{K}_1(x, t; u, v) &= D_{vt}^{\alpha-1} \left[G_{\alpha, \gamma}(x - u, t - v) \frac{\partial}{\partial v} \mathbf{K}(u, v) \right] + \\ &- i f'_u(u, v) \cdot G_{\alpha, \gamma}(x - u, t - v) \mathbf{K}(u, v). \end{aligned}$$

In view of the following formula for the composition of fractional integration operators is valid as [1]

$$D_{0t}^{\nu} D_{0t}^{\delta} g(t) = D_{0t}^{\nu+\delta} g(t), \quad \nu < 0, \quad \delta < 0, \quad (2.11)$$

the term $(\mathbf{B}^{\alpha, \gamma} \mathbf{E}(x, 0))(x, t)$ may be written as follows

$$\begin{aligned} (\mathbf{B}^{\alpha, \gamma} \mathbf{E}(x, 0))(x, t) &= - \int_{-\infty}^{\infty} [D_{0t}^{\alpha-1} - i\mu D_{0t}^{-1}] G_{\alpha, \gamma}(x - u, t) \cdot D_{00}^{\alpha-1} \mathbf{E} du - \\ &- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int_{-\infty}^{\infty} D_{0t}^{-1} G_{\alpha, \gamma}(x - u, t) \cdot D_{00}^{\alpha-1} \mathbf{E} du. \end{aligned} \quad (2.12)$$

Taking into account Eqs. (2.10), (2.12), the matrix integral equation (2.9) may be reduce to

$$\mathbf{E}(x, t) - i\sigma \int_0^t dv \int_{-\infty}^{\infty} \mathbf{K}_1(x, t; u, v) \mathbf{E}(u, v) du = \mathbf{F}(x, t), \quad (2.13)$$

where the vector $\mathbf{F}(x, t)$ is equal to

$$\mathbf{F}(x, t) = (\mathbf{B}^{\alpha, \gamma} \mathbf{E}(x, 0))(x, t) - i\sigma \int_{-\infty}^{\infty} G_{\alpha, \gamma}(x - u, t) \mathbf{K}(u, 0) (D_{00}^{\alpha-1} \mathbf{E})(u) du.$$

Note that the uniqueness of the solution to the matrix integral equation (2.13) can be proved in the same way as in [6], namely: for the kernel $\mathbf{K}_1(x, t; u, v)$ of integral equation (2.13) and its iterates $\mathbf{K}_n(x, t; u, v)$, the following estimates

$$|\mathbf{K}_n(x, t; \xi, \eta)|_* \leq CC_{n, \beta, \theta} |x - \xi|^{-\theta} (t - \eta)^{n\beta-1}, \quad (2.14)$$

$$|\mathbf{K}_n(x, t; \xi, \eta)|_* \leq CC_{n, \beta, \theta} (t - \eta)^{n\beta-1} \exp \left[-\frac{\sigma_0 |x - \xi|^\varepsilon}{(t - \eta)^{\alpha\varepsilon}} \right], \quad (2.15)$$

hold [6], where $C_{n, \beta, \theta} = \Gamma^n(\beta) C_{1-\theta}^{n-1} / \Gamma(n\beta)$.

These estimates imply the resolvent series convergence

$$\mathbf{R}(x, t; \xi, \eta) = \sum_{n=0}^{\infty} (-i\sigma)^n \mathbf{K}_{n+1}(x, t; \xi, \eta) \quad (2.16)$$

and the unique solution existence as well, i.e.,

$$\mathbf{E}(x, t) = \mathbf{F}(x, t) - i\sigma \int_0^t \int_{-\infty}^{+\infty} \mathbf{R}(x, t; \xi, \eta) \mathbf{F}(\xi, \eta) d\xi d\eta$$

to equation (2.13).

3 Cauchy's non-local problem. The case of the FOP $\alpha = 1$

In the case when FOP $\alpha = 1$, the basic system (2.1) reduces to the following

$$\begin{pmatrix} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} E_0(x, t) \\ E_h(x, t) \end{pmatrix} = \\ = i \begin{pmatrix} \gamma & \sigma \exp[i f(x, t)] \\ \sigma \exp[-i f(x, t)] & \gamma \end{pmatrix} \begin{pmatrix} E_0(x, t) \\ E_h(x, t) \end{pmatrix}.$$

Note, in limit of the FOP $\alpha = 1$, the function $G_{\alpha, \gamma}(x, t)$ can be cast into the form

$$\lim_{\alpha \rightarrow 1} G_{\alpha, \gamma}(x, t) = e^{i\gamma|x|} J_0 \left(\sigma \sqrt{t^2 - x^2} \right) \Theta(t - |x|).$$

As was shown earlier [5], one can find out that for $\alpha \rightarrow 1$ integral equation (2.9) can be written down into the form

$$\mathbf{E}(x, t) = (\mathbf{A}^{1, \gamma} \mathbf{E}(x, t))(x, t) + (\mathbf{B}^{1, \gamma} \mathbf{E}(x, 0))(x, t),$$

where

$$\begin{aligned} (\mathbf{A}^{1, \gamma} \mathbf{E}(x, t))(x, t) &= i\sigma \int_0^t dv \int_{x-t+v}^{x+t-v} G_{1, \gamma}(x - u, t - v) \times \\ &\times \begin{pmatrix} 0 & O_+ e^{i f(u, v)} \\ O_- e^{-i f(u, v)} & 0 \end{pmatrix} \mathbf{E}(u, v) du, \\ (\mathbf{B}^{1, \gamma} \mathbf{E}(x, 0))(x, t) &= \int_0^t dv \int_{x-t+v}^{x+t-v} G_{1, \gamma}(x - u, t - v) \times \end{aligned}$$

$$\times \begin{pmatrix} O_+ - i\gamma & i\sigma e^{if(u,0)} \\ i\sigma e^{-if(u,0)} & O_- - i\gamma \end{pmatrix} \delta(v) \mathbf{E}(u,0) du,$$

and ultimately, the Green function takes the form

$$G_{1,\gamma}(x,t) = \frac{1}{2} e^{i\gamma t} J_0 \left(\sigma \sqrt{t^2 - x^2} \right) \Theta[t - |x|]$$

respectively.

4 Cauchy's problem for the displacement field function $f(\mathbf{R}) = ax$

In this Section, we will build up a solution of the fractional Cauchy problem with the crystal displacement field function $f(\mathbf{R}) = ax$. In this case, we obtain an explicit solution of the problem under study in terms of special functions.

After trivial exponential substitutions for the wave amplitudes $E_0(x,t)$, $E_h(x,t)$, the system (1.1) can be written down as (for simplicity, further, the same notations for the wave amplitudes $E_0(x,t)$, $E_h(x,t)$, are to be saved)

$$\begin{aligned} \left(D_{0t}^\alpha - \frac{\partial}{\partial x} \right) E_0(x,t) &= i\gamma E_0(x,t) + i\sigma e^{iax} E_h(x,t), \\ \left(D_{0t}^\alpha + \frac{\partial}{\partial x} \right) E_h(x,t) &= i\sigma e^{-iax} E_0(x,t) + i\gamma E_h(x,t). \end{aligned} \quad (4.1)$$

According to the works [9], [10] the basic solution of the RL Cauchy problem (4.1), (1.2) takes the form

$$\mathbf{E}(x,t) = \int_{-\infty}^{\infty} \mathbf{G}(x,\xi,t) \varphi(\xi) d\xi, \quad (4.2)$$

where the following notations are introduced

$$\mathbf{G}(x,\xi,t) = \frac{1}{2} \int_{|x-\xi|}^{\infty} g(t,\tau) \mathbf{S}(x,\xi,\tau) d\tau + \mathbf{G}_0(x,\xi,t), \quad (4.3)$$

$$\mathbf{S}(x,\xi,\tau) = \begin{bmatrix} -\sigma e^{iaX_1/2} \frac{\tau-X_1}{\sqrt{\tau^2-X_1^2}} h_1(\tau, X_1) & i\sigma e^{iaX_2/2} h_0(\tau, X_1) \\ i\sigma e^{-iaX_2/2} h_0(\tau, X_1) & -\sigma e^{-iaX_1/2} \frac{\tau+X_1}{\sqrt{\tau^2-X_1^2}} h_1(\tau, X_1) \end{bmatrix},$$

$$h_0(\tau, X_1) = J_0(\sigma \sqrt{\tau^2 - X_1^2}), \quad h_1(\tau, X_1) = J_1(\sigma \sqrt{\tau^2 - X_1^2}),$$

$$g(t,\tau) = \frac{e^{i(\gamma+a/2)\tau}}{t} \phi(-\alpha, 0; -\tau t^{-\alpha}),$$

$$\mathbf{G}_0(x, \xi, t) = \begin{bmatrix} e^{-i\gamma X_1} \Theta(-X_1) & 0 \\ 0 & e^{i\gamma X_1} \Theta(X_1) \end{bmatrix} \frac{1}{t} \phi(-\alpha, 0; -|X_1|t^{-\alpha}), \quad (4.4)$$

$$X_1 = x - \xi, \quad X_2 = x + \xi.$$

$\Theta(x)$ is the Heaviside function, $J_m(z)$ is the m -order Bessel function of the argument z , $m = 0, 1$.

In the article [6] the basic solution of the Gerasimov-Caputo (GC) Cauchy problem been obtained in form

$$\mathbf{E}(x, t) = \int_{-\infty}^{\infty} D_{0t}^{\alpha-1} \mathbf{G}(x, \xi, t) \psi(\xi) d\xi, \quad (4.5)$$

where

$$\mathbf{E}(x, 0) = \psi(x)$$

is the GC Cauchy problem conditions.

5 Cauchy's non-local problem in the case of the displacement function $f(\mathbf{R}) = ax$ and constant initial conditions

5.1 The generalized Euler's formula

The Stankovic's transformation for the function $v(t)$ is defined as the integral [8]

$$A^{\alpha, \beta} v(t) \equiv \int_0^{\infty} v(\tau) t^{\nu-1} \phi(-\alpha, \beta; -\tau t^{-\alpha}) d\tau.$$

The following formulas (see [8, p. 84] for details)

$$A^{\alpha, \beta} \exp(\lambda t) = t^{\alpha+\beta-1} E_{\alpha, \alpha+\beta}(\lambda t^{\alpha}), \quad (5.1)$$

$$A^{\alpha, \beta} \cos(\lambda t) = t^{\alpha+\beta-1} E_{2\alpha, \alpha+\beta}(-\lambda^2 t^{2\alpha}), \quad (5.2)$$

$$A^{\alpha, \beta} \sin(\lambda t) = \lambda t^{2\alpha+\beta-1} E_{2\alpha, 2\alpha+\beta}(-\lambda^2 t^{2\alpha}), \quad (5.3)$$

take place for any $\lambda \in \mathbb{C}$, $\mu \in (0, 1)$, $\nu \in \mathbb{R}$, where

$$E_{\rho, \mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + \rho k)}$$

is the Mittag-Leffler-type function (see [13, p. 117]).

Using the Eqs. (5.1) – (5.3), and the Euler's formula, one obtains the next generalized Euler's formula

$$t^{\alpha+\beta-1} E_{\alpha, \alpha+\beta}(i\sigma t^{\alpha}) = t^{\alpha+\beta-1} E_{2\alpha, \alpha+\beta}(-\sigma^2 t^{2\alpha}) +$$

$$+it^{2\alpha+\beta-1}E_{2\alpha,2\alpha+\beta}(-\sigma^2t^{2\alpha}), \quad (5.4)$$

which will be useful below. Denoting the parameter $\mu = \alpha + \beta$, equality (5.4) can be rewritten as follows

$$t^{\mu-1}E_{\alpha,\mu}(i\sigma t^\alpha) = t^{\mu-1}E_{2\alpha,\mu}(-\sigma^2t^{2\alpha}) + it^{\alpha+\mu-1}E_{2\alpha,\alpha+\mu}(-\sigma^2t^{2\alpha}). \quad (5.5)$$

Note that for $\mu = 1$, formula (5.5) was obtained in [14].

Introducing the following notations

$$\exp_{\alpha,\beta}(\sigma, t) = t^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(\sigma t^\alpha),$$

$$\cos_{\alpha,\beta}(\sigma, t) = t^{\alpha+\beta-1}E_{2\alpha,\alpha+\beta}(-\sigma^2t^{2\alpha}),$$

$$\sin_{\alpha,\beta}(\sigma, t) = \sigma t^{2\alpha+\beta-1}E_{2\alpha,2\alpha+\beta}(-\sigma^2t^{2\alpha}),$$

one rewrites formula (5.4) into the form

$$\exp_{\alpha,\beta}(i\sigma, t) = \cos_{\alpha,\beta}(\sigma, t) + i \sin_{\alpha,\beta}(\sigma, t),$$

$$\exp_{\alpha,\beta}(-i\sigma, t) = \cos_{\alpha,\beta}(\sigma, t) - i \sin_{\alpha,\beta}(\sigma, t).$$

From the last two equalities follow the next relations

$$\cos_{\alpha,\beta}(\sigma, t) = \frac{\exp_{\alpha,\beta}(i\sigma, t) + \exp_{\alpha,\beta}(-i\sigma, t)}{2}, \quad (5.6)$$

$$\sin_{\alpha,\beta}(\sigma, t) = \frac{\exp_{\alpha,\beta}(i\sigma, t) - \exp_{\alpha,\beta}(-i\sigma, t)}{2i}. \quad (5.7)$$

5.2 Constant initial conditions

Let us consider the case when the initial conditions for the wavefield amplitudes are constant, and they are equal to

$$\lim_{t \rightarrow 0} D_{0t}^{\alpha-1} E_0(x, t) = \varphi_0(x) \equiv C_0, \quad \lim_{t \rightarrow 0} D_{0t}^{\alpha-1} E_h(x, t) = \varphi_h(x) \equiv C_h. \quad (5.8)$$

Accordingly, formulae (4.2)–(4.4) can be simplified and expressed in the terms of the Mittag–Leffler-type functions.

Bearing in mind Eqs. (4.2), (5.8) and the work [6] calculations details, the vector $\mathbf{E}(x, t)$ can be cast in the form

$$\begin{aligned} \mathbf{E}(x, t) &= \int_{-\infty}^{\infty} \mathbf{G}(x, \xi, t) \begin{pmatrix} C_0 \\ C_h \end{pmatrix} d\xi = \\ &= \int_0^{\infty} t^{-\alpha} \phi(-\alpha, 1-\alpha; -\eta t^{-\alpha}) \left[\mathbf{N}_1(x) e^{i(a_1+k)t} + \mathbf{N}_2(x) e^{i(a_1-k)t} \right] \begin{pmatrix} C_0 \\ C_h \end{pmatrix} d\eta \end{aligned} \quad (5.9)$$

where

$$\mathbf{N}_{1,2}(x) = \frac{1}{2} \begin{pmatrix} 1 \mp \frac{a}{2k} & \pm \frac{\sigma}{k} e^{iax} \\ \pm \frac{\sigma}{k} e^{-iax} & 1 \mp \frac{a}{2k} \end{pmatrix}, \quad a_1 = \gamma + \frac{a}{2}, \quad k = \sqrt{a^2/4 + \sigma^2}.$$

Thus, from Eq. (5.9) for the wave vector $\mathbf{E}(x, t)$ one has obtained

$$\mathbf{E}(x, t) = t^{\alpha-1} \left[\mathbf{N}_1(x) E_{\alpha, \alpha}(i(a_1 + k)t^\alpha) + \mathbf{N}_2(x) E_{\alpha, \alpha}(i(a_1 - k)t^\alpha) \right] \begin{pmatrix} C_0 \\ C_h \end{pmatrix}, \quad (5.10)$$

and or in the aggregate form

$$\mathbf{E}(x, t) = \overline{\mathbf{N}}(x, t) \begin{pmatrix} C_0 \\ C_h \end{pmatrix} \quad (5.11)$$

where

$$\overline{\mathbf{N}}(x, t) = \frac{1}{2} \begin{pmatrix} (E^+ + E^-) - \frac{a}{2k}(E^+ - E^-) & \frac{\sigma}{k} e^{iax}(E^+ - E^-) \\ \frac{\sigma}{k} e^{-iax}(E^+ - E^-) & (E^+ + E^-) - \frac{a}{2k}(E^+ - E^-) \end{pmatrix}, \quad (5.12)$$

$$E^+ = \exp_{\alpha, 0}(i(a_1 + k), t), \quad E^- = \exp_{\alpha, 0}(i(a_1 - k), t).$$

5.3 The two particular cases of the RL Cauchy problem solution

In the special case when

$$\lim_{t \rightarrow 0} D_{0t}^{\alpha-1} E_0(x, t) = 1, \quad \lim_{t \rightarrow 0} D_{0t}^{\alpha-1} E_h(x, t) = 1,$$

formula (5.10) takes the form

$$\mathbf{E}(x, t) = \frac{t^{\alpha-1}}{2} \begin{pmatrix} 1 - \frac{a}{2k} + \frac{\sigma}{k} e^{iax} & 1 + \frac{a}{2k} - \frac{\sigma}{k} e^{iax} \\ 1 - \frac{a}{2k} + \frac{\sigma}{k} e^{-iax} & 1 + \frac{a}{2k} - \frac{\sigma}{k} e^{-iax} \end{pmatrix} \begin{pmatrix} E_{\alpha, \alpha}(i(a_1 + k)t^\alpha) \\ E_{\alpha, \alpha}(i(a_1 - k)t^\alpha) \end{pmatrix}. \quad (5.13)$$

Accordingly, for the following conditions as

$$\lim_{t \rightarrow 0} D_{0t}^{\alpha-1} E_0(x, t) = 1, \quad \lim_{t \rightarrow 0} D_{0t}^{\alpha-1} E_h(x, t) = 0,$$

from Eq. (5.10), one has obtained

$$\mathbf{E}(x, t) = \frac{t^{\alpha-1}}{2} \begin{pmatrix} 1 - \frac{a}{2k} & 1 + \frac{a}{2k} \\ \frac{\sigma}{k} e^{-iax} & -\frac{\sigma}{k} e^{-iax} \end{pmatrix} \begin{pmatrix} E_{\alpha, \alpha}(i(a_1 + k)t^\alpha) \\ E_{\alpha, \alpha}(i(a_1 - k)t^\alpha) \end{pmatrix}. \quad (5.14)$$

It is interesting that in the case of FOP $\alpha = 1$ from (5.12) one can see that the matrix $\overline{\mathbf{N}}(x, t)$ takes the form

$$\overline{\mathbf{N}}(x, t) = \mathbf{N}(x, t) = e^{ia_1 t} \begin{pmatrix} \cos kt - \frac{a}{2k} i \sin kt & \frac{\sigma}{k} e^{iax} i \sin kt \\ \frac{\sigma}{k} e^{-iax} i \sin kt & \cos kt - \frac{a}{2k} i \sin kt \end{pmatrix}.$$

If in addition $a = 0$ the vector $\mathbf{E}(x, t)$ is equal to

$$\mathbf{E}(x, t) = e^{i\gamma t} \begin{pmatrix} \cos kt & i \sin kt \\ i \sin kt & \cos kt \end{pmatrix} \begin{pmatrix} C_0 \\ C_h \end{pmatrix} = e^{i\gamma t} \begin{pmatrix} C_0 & C_h \\ C_h & C_0 \end{pmatrix} \begin{pmatrix} \cos kt \\ i \sin kt \end{pmatrix}. \quad (5.15)$$

In the case when the FOP $\alpha \rightarrow 1$, the solution (5.14) has reduced to

$$\mathbf{E}(x, t) = e^{ia_1 t} \begin{pmatrix} \cos kt - \frac{a}{2k} \sin kt \\ \frac{\sigma}{k} e^{-iax} \sin kt \end{pmatrix}. \quad (5.16)$$

Finally, putting on the parameters $\gamma = 0$ and $a = 0$, the solution (5.16) goes to 'pendulum solutions' of the canonic TT equations regarding the Bloch wave-field amplitudes in the perfect crystal (cf. [5])

$$E_0 = \cos \sigma t, \quad E_h = i \sin \sigma t.$$

In the case when $a = \gamma = 0$ from (5.12) taking into account formulas (5.6) and (5.7) one obtains the following 'quasi pendulum solutions': for the RL Cauchy nonlocal problem

$$\mathbf{E}(x, t) = \begin{pmatrix} C_0 & C_h \\ C_h & C_0 \end{pmatrix} \begin{pmatrix} \cos_{\alpha,0}(k, t) \\ i \sin_{\alpha,0}(k, t) \end{pmatrix}, \quad (5.17)$$

and for GC Cauchy problem

$$\mathbf{E}(x, t) = \begin{pmatrix} C_0 & C_h \\ C_h & C_0 \end{pmatrix} \begin{pmatrix} \cos_{\alpha,1-\alpha}(k, t) \\ i \sin_{\alpha,1-\alpha}(k, t) \end{pmatrix} \quad (5.18)$$

respectively.

Below, we present graphs of quasi-periodic solutions to the Cauchy problems for $k = \sigma = 1$ and various values of the FOP α : a) for the model with Riemann–Liouville derivatives; b) for the model with Gerasimov–Caputo derivatives.

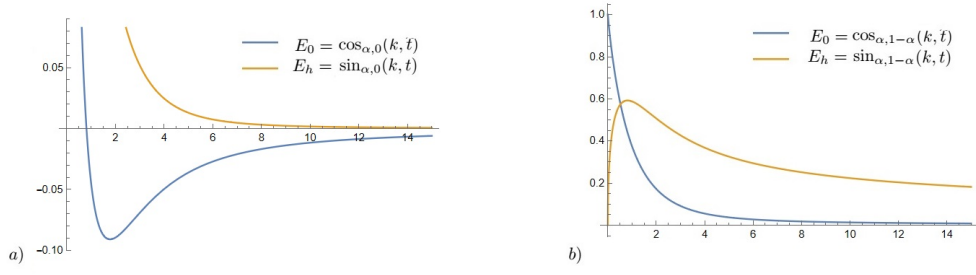


Fig. 1 Graphs of the quasi-periodic solutions for $\alpha = 0.46$: a) for the RL approach; b) for the GC approach.

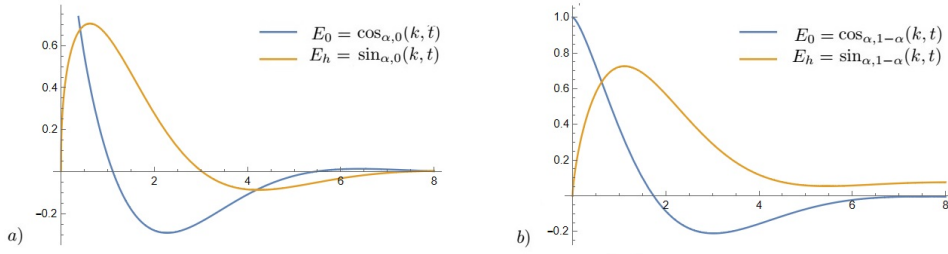


Fig. 2 Graphs of the quasi-periodic solutions for $\alpha = 0.7$: a) for the RL approach; b) for the GC approach.

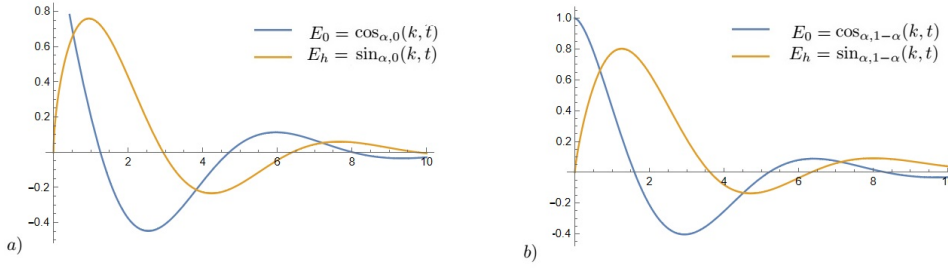


Fig. 3 Graphs of the quasi-periodic solutions for $\alpha = 0.8$: a) for the RL approach; b) for the GC approach.

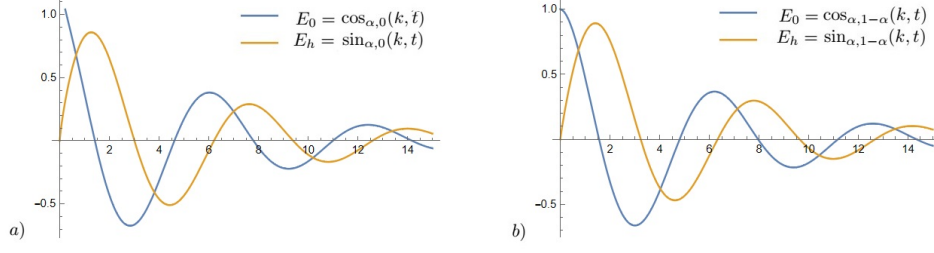


Fig. 4 Graphs of the quasi-periodic solutions for $\alpha = 0.9$: a) for the RL approach; b) for the GC approach.

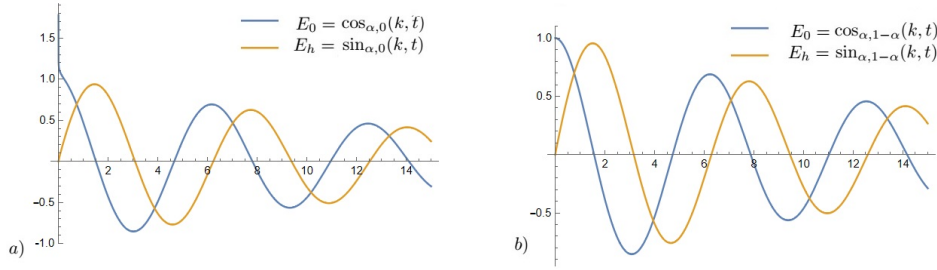


Fig. 5 Graphs of the quasi-periodic solutions for $\alpha = 0.96$: a) for the RL approach; b) for the GC approach.

It is known [13, p. 134] that, for large values of z , the following asymptotic expansion of the Mittag-Leffler-type function holds

$$E_{2\alpha,\mu}(z) = \frac{1}{2\alpha} z^{(1-\mu)/2\alpha} \exp z^{1/2\alpha} - \sum_{k=1}^n \frac{z^{-k}}{\Gamma(\mu - 2\alpha k)} + O\left(\frac{1}{|z|^{n+1}}\right), \quad |\arg z| \leq \alpha_1 \pi,$$

$$E_{2\alpha,\mu}(z) = -\sum_{k=1}^n \frac{z^{-k}}{\Gamma(\mu - \alpha k)} + O\left(\frac{1}{|z|^{n+1}}\right), \quad \alpha_1 \pi \leq |\arg z| \leq \pi,$$

where $\alpha \in (0, 1)$, and $\alpha_1 \in (\alpha, 2\alpha)$.

From the last relations, one can conclude that for $0 < \alpha < 0.5$, the quasi-periodic solutions decay according to a power law

$$\cos_{\alpha,0}(k,t) \sim \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}, \quad \sin_{\alpha,0}(k,t) \sim \frac{t^{-2\alpha-1}}{\Gamma(1+2\alpha)}, \quad t \rightarrow +\infty,$$

$$\cos_{\alpha,1-\alpha}(k,t) \sim \frac{t^{-2\alpha}}{\Gamma(1-2\alpha)}, \quad \sin_{\alpha,1-\alpha}(k,t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \rightarrow +\infty.$$

For $0.5 \leq \alpha < 1$ and $t \rightarrow +\infty$, the solutions exponentially decay with oscillating modulation

$$\cos_{\alpha,\beta}(k,t) \sim \frac{1}{\alpha} k^{\frac{1-\alpha-\beta}{\alpha}} \exp\left(k^{\frac{1}{\alpha}} t \cos\left(\frac{\pi}{2\alpha}\right)\right) \cos\left(k^{\frac{1}{\alpha}} t \sin\left(\frac{\pi}{2\alpha}\right) + \frac{\pi(1-\alpha-\beta)}{2\alpha}\right),$$

$$\sin_{\alpha,\beta}(k,t) \sim -\frac{1}{\alpha} k^{\frac{1-\alpha-\beta}{\alpha}} \exp\left(k^{\frac{1}{\alpha}} t \cos\left(\frac{\pi}{2\alpha}\right)\right) \cos\left(k^{\frac{1}{\alpha}} t \sin\left(\frac{\pi}{2\alpha}\right) + \frac{\pi(1-\beta)}{2\alpha}\right),$$

where the case $\beta = 0$ corresponds to the RL approach, and the case $\beta = 1 - \alpha$ to the GC approach.

6 Conclusion

In the framework of the RL derivative concept, based on the resolvent-solution of the integral Fredholm–Volterra matrix equation, a new theoretical approach of the processing the fractional X-ray diffraction crystal optics data has been elaborated. Noteworthy is the fact that the worldwide X-ray diffraction crystal optics related to the fractional order parameter (FOP) α equal to unity, has been reformulated using the fractional derivatives concept in the Riemann–Liouville sense $D_{0t}^\alpha E$ with an arbitrary FOP α , $\alpha \in (0, 1]$.

On the example of the X-ray diffraction by perfect crystal, the function $f(r) = ax$, one has found the exact analytical matrix solution of the RL-approach-Cauchy problem for an arbitrary value of the FOP α , $\alpha \in (0, 1]$. In the case of constant initial data, these solutions are expressed in terms of Mittag-Leffler-type functions. The latter have been analyzed in comparing with the corresponding productive solutions obtained in the Gerasimov – Caputo fractional calculus, respectively.

In concluding, as a viewpoint of fundamental physics as well an endeavor of revealing the non-locality of the X-ray quanta-medium atoms interaction feature, authors could suggest a new fractional integral-differential derivative of kind

$$D_{at}^{\alpha,\lambda} g(t) = \frac{1}{\Gamma(-\alpha+1)} \frac{d}{dt} \int_a^t \frac{g(s)ds}{(t-s+\lambda)^\alpha}, \quad 0 < \alpha \leq 1, \quad 0 \leq \lambda \leq 1.$$

In the case when $\lambda = 0$, the derivative $D_{at}^{\alpha,\lambda} g(t)$ goes to the RL one $D_{at}^\alpha g(t)$ exactly. Clearly, using of the above fractional integral-differential derivative $D_{at}^{\alpha,\lambda} g(t)$ is of a definite interest for solving the non-local X-ray diffraction Cauchy problem being nothing else a good topic for future job.

Compliance with ethical standards

Data Availability Statement:. This manuscript has no associated data.

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