Supplementary Information

Identifiability of Planar Cauchy Elasticity from Phase-Speed Data without Major Symmetry

Feng Yang

School of Mechanics and Safety Engineering, Zhengzhou University,
No. 100 Science Avenue, Zhengzhou 450001, China
Corresponding author: yf@zzu.edu.cn

Journal: Acta Mechanica

This Supplementary Information provides the algebraic details and numerical-validation procedures supporting the main manuscript. It uses the same notation. The density-normalized engineering constitutive vector is

$$\boldsymbol{c} = (C_{11}, C_{22}, C_{12}, C_{16}, C_{26}, C_{66}, R^0, K_a^0, K_b^0)^T.$$

The major-symmetric and major-antisymmetric coefficient subvectors are

$$\boldsymbol{c}_{\mathrm{e}} = (C_{11}, C_{22}, C_{12}, C_{16}, C_{26}, C_{66})^T, \qquad \boldsymbol{c}_{\mathrm{o}} = (R^0, K_a^0, K_b^0)^T.$$

Propagation directions are parameterized by θ modulo π, with $\hat{n}$ and $-\hat{n}$ representing the same measured direction. For $\hat{n} = (\cos\theta, \sin\theta)$, the density-normalized acoustic tensor is

$$Q_{ik}(\hat{n}) = \mathbb{G}_{ijkl} n_j n_l,$$

where $\mathbb{G} = \varrho^{-1}\mathbb{C}$ is the density-normalized fourth-order stiffness tensor.

1 Harmonic representation of the acoustic tensor

For the four scalar acoustic combinations

$$T = Q_{11} + Q_{22}, \qquad D = Q_{11} - Q_{22}, \qquad P = Q_{12} + Q_{21}, \qquad A = \frac{Q_{12} - Q_{21}}{2},$$

each scalar quantity has the form

$$f(\theta) = f_0 + f_c \cos 2\theta + f_s \sin 2\theta. \tag{S1.1}$$

Direct contraction of the planar fourth-order tensor gives

$$T_0 = \frac{C_{11} + C_{22}}{2} + C_{66}, \qquad T_c = \frac{C_{11} - C_{22}}{2}, \qquad T_s = C_{16} + C_{26}, \tag{S1.2}$$

$$D_0 = \frac{C_{11} - C_{22}}{2}, \qquad D_c = \frac{C_{11} + C_{22}}{2} - C_{66}, \qquad D_s = C_{16} - C_{26}, \tag{S1.3}$$

$$P_0 = C_{16} + C_{26}, \qquad P_c = C_{16} - C_{26}, \qquad P_s = C_{12} + C_{66}, \tag{S1.4}$$

$$A_0 = \frac{K_a^0 + K_b^0}{2}, \qquad A_c = \frac{K_a^0 - K_b^0}{2}, \qquad A_s = \frac{R^0}{2}. \tag{S1.5}$$

The twelve Fourier coefficients satisfy three constitutive consistency relations,

$$T_c = D_0, \qquad T_s = P_0, \qquad D_s = P_c. \tag{S1.6}$$

Conversely, any twelve coefficients satisfying these three relations determine the nine constitutive coefficients uniquely through the inverse formulas in the main manuscript. These relations therefore characterize the image of the local homogeneous planar Cauchy-elastic law within the twelve-dimensional harmonic coefficient space.

2 Phase-speed invariants and generic rank eight

Within the regime of positive real acoustic eigenvalues considered in the main manuscript, the two phase-speed branches determine the trace and determinant of $Q(\theta)$. In the notation above,

$$\det Q(\theta) = \frac{T(\theta)^2 - D(\theta)^2 - P(\theta)^2}{4} + A(\theta)^2. \tag{S2.1}$$

If $f(\theta) = f_0 + f_c \cos 2\theta + f_s \sin 2\theta$, then

$$\begin{aligned} f(\theta)^2 = {} & q_0(f) + q_c(f)\cos 2\theta + q_s(f)\sin 2\theta \\ & + q_{4c}(f)\cos 4\theta + q_{4s}(f)\sin 4\theta, \end{aligned} \tag{S2.2}$$

with

$$q_0(f) = f_0^2 + \frac{f_c^2 + f_s^2}{2}, \qquad q_c(f) = 2f_0 f_c, \qquad q_s(f) = 2f_0 f_s, \tag{S2.3}$$

$$q_{4c}(f) = \frac{f_c^2 - f_s^2}{2}, \qquad q_{4s}(f) = f_c f_s. \tag{S2.4}$$

Hence the determinant contains only five Fourier coefficients,

$$d_0 = \frac{1}{4}\big[q_0(T) - q_0(D) - q_0(P)\big] + q_0(A), \tag{S2.5}$$

$$d_{2c} = \frac{1}{4}\big[q_c(T) - q_c(D) - q_c(P)\big] + q_c(A), \qquad d_{2s} = \frac{1}{4}\big[q_s(T) - q_s(D) - q_s(P)\big] + q_s(A), \tag{S2.6}$$

$$d_{4c} = \frac{1}{4}\big[q_{4c}(T) - q_{4c}(D) - q_{4c}(P)\big] + q_{4c}(A), \qquad d_{4s} = \frac{1}{4}\big[q_{4s}(T) - q_{4s}(D) - q_{4s}(P)\big] + q_{4s}(A). \tag{S2.7}$$

The complete angular phase-speed field factors through the eight-component polynomial map

$$\mathcal{S}(\boldsymbol{c}) = (T_0, T_c, T_s, d_0, d_{2c}, d_{2s}, d_{4c}, d_{4s})^T \in \mathbb{R}^8. \tag{S2.8}$$

Five distinct propagation directions modulo π determine these eight invariants exactly in the noise-free model. To see this, write $z = e^{2i\theta}$. A trigonometric polynomial with harmonics $0, 2\theta, 4\theta$ becomes a Laurent polynomial of degree two in z, and five distinct directions modulo π give five distinct values of z. Hence its five coefficients are uniquely determined by five samples. The trace requires only its three 0 and 2θ coefficients and is therefore overdetermined by the same five directions. Thus the ten phase-speed values from five directions contain eight independent phase-speed invariants and two exact redundancy relations.

To show that the eight-dimensional phase-speed bound is attained generically, consider

$$\boldsymbol{c}_\star = (4, 5, 1, 1/5, -1/5, 2, 1/5, 1/10, -1/10)^T.$$

For the Jacobian $D\mathcal{S}(\boldsymbol{c}_\star)$, the 8×8 minor obtained by deleting the C_{11} column is

$$\det J_{\widehat{C_{11}}} = \frac{13}{80000} \neq 0. \tag{S2.9}$$

Because this minor is a polynomial in the constitutive coefficients and is not identically zero, the Jacobian has rank eight on an open dense subset of the nine-parameter space. Dimension alone already prevents a locally one-to-one map from nine constitutive coefficients to eight phase-speed invariants. The nonzero minor supplies the complementary positive statement: the generic rank reaches the maximum eight, so the local non-uniqueness is exactly one dimensional rather than larger. Wherever the eigenvalues remain simple and positive, these regular level sets correspond to identical phase-speed data. Degenerate parameter sets may have higher-dimensional level sets.

The phase-speed map also has an exact sign-reversal symmetry. Reversing the three major-antisymmetric coefficients gives

$$Q(\theta; \boldsymbol{c}_\mathrm{e}, -\boldsymbol{c}_\mathrm{o}) = Q(\theta; \boldsymbol{c}_\mathrm{e}, \boldsymbol{c}_\mathrm{o})^T, \tag{S2.10}$$

which preserves the characteristic polynomial at every θ.

At $\boldsymbol{c}_\star$, a tangent vector to the regular constant-phase-speed level set is

$$\boldsymbol{v}_\star = (-1,-1,-3,-1,1,1,-34,-12,17)^T, \qquad D\mathcal{S}(\boldsymbol{c}_\star)\boldsymbol{v}_\star = 0, \tag{S2.11}$$

where the integer scaling is arbitrary. This explicit null vector is only a tangent representation; the constant-rank theorem supplies the corresponding local nonlinear level set. The curve in Fig. 1(a) of the main manuscript is obtained by pseudo-arclength continuation of the eight equations $\mathcal{S}(\boldsymbol{c}) = \mathcal{S}(\boldsymbol{c}_\star)$ from this point. Over the displayed segment, the maximum phase-speed-invariant residual is below 10^{-8}, the acoustic eigenvalues remain simple and positive on the verification grid, and the invariant Frobenius norm of the odd part remains strictly positive. The curve therefore gives an explicit local realization of the regular constant-phase-speed family without implying a global classification of the full level set.

3 Weak odd coupling and phase-speed data generated by major-symmetric materials

Let $\mathcal{E} = \{\boldsymbol{c} : \boldsymbol{c}_\mathrm{o} = 0\}$ be the six-dimensional major-symmetric coefficient subspace and let $\mathcal{S}_\mathrm{e} = \mathcal{S}|_{\mathcal{E}}$. At $\boldsymbol{c}_\mathrm{e} = (4,5,1,1/5,-1/5,2)^T$, the 6×6 Jacobian minor formed from $(T_0, T_c, T_s, d_0, d_{2c}, d_{2s})$ is

$$\det D\mathcal{S}_\mathrm{e}[1{:}6,:] = \frac{3}{32} \neq 0. \tag{S3.1}$$

Hence $\mathcal{S}_\mathrm{e}$ has generic rank six. Near such regular points, major-symmetric phase-speed data form a six-dimensional submanifold of the eight-dimensional phase-speed-invariant space. At points where the full map has rank eight, the inverse image of the phase-speed data generated by the major-symmetric class has local dimension $9-(8-6) = 7$, i.e. codimension two in the full constitutive space. Thus exact phase-speed equivalence with some major-symmetric material is not a generic property of an odd constitutive law. This statement also holds globally in a measure-theoretic sense. Since $\mathcal{S}_\mathrm{e}$ is polynomial on the six-dimensional semialgebraic set $\mathcal{E}$, the image $\mathcal{S}(\mathcal{E})$ is semialgebraic and has dimension at most six. On the regular set $\mathcal{R} = \{\boldsymbol{c} : \operatorname{rank} D\mathcal{S} = 8\}$, the constant-rank theorem gives local coordinates in which $\mathcal{S}$ is a projection with one-dimensional fibers, so

$$\dim\big[\mathcal{R} \cap \mathcal{S}^{-1}\big(\mathcal{S}(\mathcal{E})\big)\big] \leq 6 + 1 = 7.$$

The complement $\mathcal{R}^c$ is the common zero set of the 8×8 Jacobian minors. Because the minor $13/80000$ above shows that these minors are not all identically zero, $\mathcal{R}^c$ is a proper algebraic subset of $\mathbb{R}^9$ and has dimension at most eight. Therefore the full compatibility set $\mathcal{S}^{-1}[\mathcal{S}(\mathcal{E})]$ is contained in a semialgebraic subset of dimension at most eight and hence has nine-dimensional Lebesgue measure zero.

For $\boldsymbol{c}(\varepsilon) = (\boldsymbol{c}_\mathrm{e}, \varepsilon\boldsymbol{c}_\mathrm{o})$, the trace coefficients are independent of ε and $A(\theta;\varepsilon) = \varepsilon A(\theta;1)$. The determinant coefficients therefore satisfy the exact relation

$$\mathcal{S}(\boldsymbol{c}_\mathrm{e}, \varepsilon\boldsymbol{c}_\mathrm{o}) = \mathcal{S}(\boldsymbol{c}_\mathrm{e}, 0) + \varepsilon^2 \boldsymbol{q}(\boldsymbol{c}_\mathrm{o}). \tag{S3.2}$$

If the quadratic vector $\boldsymbol{q}$ has a nonzero component normal to the local phase-speed manifold generated by major-symmetric materials, the distance to that manifold is $\varepsilon^2 d_2 + O(\varepsilon^4)$ with $d_2 > 0$. The order of this scaling is independent of the particular smooth nondimensional norm used to measure that distance, although its coefficient is norm dependent. Special odd directions can instead have $d_2 = 0$ and may remain exactly phase-speed equivalent to major-symmetric materials.

For the dimensionless residual $\mathcal{R}_\mathrm{MS}$ defined in the main manuscript, a numerical check of the weak odd-coupling scaling used the even reference $(4,5,1,0.2,-0.2,2)$ and odd direction $(R^0, K_a^0, K_b^0) = (0.2, 0.1, -0.1)$. After minimizing over the six major-symmetric coefficients, the values at $\varepsilon = 0.02, 0.05, 0.1, 0.2, 0.4$ satisfy

$$\frac{\mathcal{R}_\mathrm{MS}}{\varepsilon^2} = (5.28196, 5.28198, 5.28205, 5.28233, 5.28345)\times 10^{-5}. \tag{S3.3}$$

The near-constant ratio is an illustrative numerical verification of the quadratic scaling, not a universal coefficient.

A numerical contrast test sampled 25 admissible major-symmetric materials and used 40 random initial guesses per material to solve the overdetermined equation $\mathcal{S}_\mathrm{e}(\boldsymbol{x}) = \mathcal{S}_\mathrm{e}(\boldsymbol{c}_\mathrm{e})$. Of the 1000 starts, 686 converged

to residual below 10^{-8}; every one returned the generating material to within 1.2×10^{-12}, and no secondary zero-residual solution was found. The remaining starts did not converge to a zero-residual solution. This is numerical evidence, not a global uniqueness proof, but it is consistent with the generic rank-six local result and with classical major-symmetric phase-speed inversion.

Two exact families built on a square-symmetric major-symmetric base are useful for locating the exceptional set. For

$$C_{11} = C_{22} = \alpha, \quad C_{12} = \gamma, \quad C_{66} = \kappa, \quad C_{16} = C_{26} = 0,$$

the major-symmetric tensor belongs to the planar square-symmetric D_4 class. A pure R^0 odd coupling breaks the quarter-turn symmetry but retains the two coordinate-axis mirrors and the half-turn, giving a D_2 family. A pure pseudoscalar coupling $K^0_a = K^0_b$ retains C_4 rotational symmetry but breaks the mirrors. For the common square-symmetric base one has $T = \alpha + \kappa$, $D = (\alpha - \kappa)\cos 2\theta$, and $P = (\gamma + \kappa)\sin 2\theta$. With a pure R^0 odd coupling, $A = (R^0/2)\sin 2\theta$. Defining

$$\gamma' = \operatorname{sgn}(\gamma + \kappa)\sqrt{(\gamma + \kappa)^2 - (R^0)^2} - \kappa \tag{S3.4}$$

when $|R^0| < |\gamma + \kappa|$ gives $P'(\theta)^2 = P(\theta)^2 - 4A(\theta)^2$ and therefore identical trace and determinant for every θ.

For a pure pseudoscalar coupling $K^0_a = K^0_b = K^0$, let $d = \alpha - \kappa$ and $p = \gamma + \kappa$. If $|d|, |p| > 2|K^0|$, set

$$a = \operatorname{sgn}(d)\sqrt{d^2 - 4(K^0)^2}, \qquad b = \operatorname{sgn}(p)\sqrt{p^2 - 4(K^0)^2}, \tag{S3.5}$$

and

$$\alpha' = \frac{\alpha + \kappa + a}{2}, \qquad \kappa' = \frac{\alpha + \kappa - a}{2}, \qquad \gamma' = b - \kappa'. \tag{S3.6}$$

Then $T' = T$ and $D'^2 + P'^2 = D^2 + P^2 - 4(K^0)^2$, so the odd and major-symmetric materials are identical in phase speed in every direction. The $SO(2)$ family in the next section is the subcase $d = p = B$. These examples do not constitute an exhaustive classification of odd constitutive laws whose phase-speed data are compatible with major-symmetric materials.

4 Phase-speed ambiguity in the $SO(2)$-isotropic subclass

Here $SO(2)$ denotes the group of proper in-plane rotations. Let B and μ be the density-normalized two-dimensional bulk and shear moduli and let K^0 be the density-normalized odd modulus. For this $SO(2)$-invariant law,

$$Q = \begin{pmatrix} \mu & K^0 \\ -K^0 & \mu + B \end{pmatrix} \tag{S4.1}$$

in the ordered transverse–longitudinal basis. The phase-speed invariants are

$$\tau = \operatorname{tr} Q = 2\mu + B, \qquad \Pi = \det Q = \mu(\mu + B) + (K^0)^2. \tag{S4.2}$$

Suppose $|B| \geq 2|K^0|$ and define

$$B' = \operatorname{sgn}(B)\sqrt{B^2 - 4(K^0)^2}, \qquad \mu' = \mu + \frac{B - B'}{2}, \qquad K^{0\prime} = 0, \tag{S4.3}$$

where sgn denotes the sign function. Let $\delta = (B - B')/2$, so $B' = B - 2\delta$. Then

$$2\mu' + B' = 2(\mu + \delta) + B - 2\delta = 2\mu + B,$$

and

$$\begin{aligned} \mu'(\mu' + B') &= (\mu + \delta)(\mu + B - \delta) && \text{(S4.4)} \\ &= \mu(\mu + B) + \delta(B - \delta) && \text{(S4.5)} \\ &= \mu(\mu + B) + \frac{B^2 - B'^2}{4} && \text{(S4.6)} \\ &= \mu(\mu + B) + (K^0)^2. && \text{(S4.7)} \end{aligned}$$

The major-symmetric medium $(\mu', B', 0)$ has the same trace and determinant as (μ, B, K^0) and therefore the same acoustic eigenvalues in every direction. When the common eigenvalues are positive, the two media therefore have the same phase-speed branches. If in addition $B > 0$, then $B' = \Delta_\lambda > 0$ and $\mu' = \lambda_- > 0$, so the major-symmetric counterpart is a positive-definite classical isotropic elastic solid. Strict inequality $|B| > 2|K^0|$ gives two distinct real acoustic eigenvalues; equality is the coalescence boundary. Distinguishing the odd medium from the major-symmetric material with identical phase-speed branches then requires polarization information.

5 Explicit null spaces for two propagation directions

For two distinct propagation directions θ_1 and θ_2, define

$$g(\theta) = \cos(2\theta - \theta_1 - \theta_2) - \cos(\theta_1 - \theta_2) = g_0 + g_c \cos 2\theta + g_s \sin 2\theta, \tag{S5.1}$$

where

$$g_0 = -\cos(\theta_1 - \theta_2), \qquad g_c = \cos(\theta_1 + \theta_2), \qquad g_s = \sin(\theta_1 + \theta_2). \tag{S5.2}$$

By construction, $g(\theta_1) = g(\theta_2) = 0$.

Major-antisymmetric null-space direction

Set $A = b\, g$. The corresponding constitutive perturbation is

$$\delta K^0_a = b(g_0 + g_c), \qquad \delta K^0_b = b(g_0 - g_c), \qquad \delta R^0 = 2b g_s. \tag{S5.3}$$

It produces zero skew-symmetric acoustic response in both sampled directions.

Major-symmetric null-space direction

Set

$$T = a g_0 g, \qquad D = a g_c g, \qquad P = a g_s g.$$

The consistency relations are then satisfied identically. Substitution into the constitutive inverse gives the explicit null vector

$$\delta C_{11} = \frac{a}{2}(g_0 + g_c)^2, \qquad \delta C_{22} = \frac{a}{2}(g_0 - g_c)^2, \tag{S5.4}$$

$$\delta C_{66} = \frac{a}{2}(g_0^2 - g_c^2), \qquad \delta C_{12} = a\left[g_s^2 - \frac{g_0^2 - g_c^2}{2}\right], \tag{S5.5}$$

$$\delta C_{16} = \frac{a}{2} g_s (g_0 + g_c), \qquad \delta C_{26} = \frac{a}{2} g_s (g_0 - g_c). \tag{S5.6}$$

The major-symmetric six-parameter two-direction operator has rank five and nullity one, whereas the full nine-parameter operator has rank seven and nullity two.

For $\theta_1 = 0$ and $\theta_2 = \pi/2$, one has $(g_0, g_c, g_s) = (0, 0, 1)$. The major-symmetric null vector reduces to a pure C_{12} perturbation, and the major-antisymmetric null vector reduces to a pure R^0 perturbation. This case gives the most direct illustration of the two unidentified constitutive coefficients.

6 Three-direction interpolation and the equal-spacing design

For three propagation directions, the scalar Fourier coefficients satisfy

$$\begin{pmatrix} f(\theta_1) \\ f(\theta_2) \\ f(\theta_3) \end{pmatrix} = M \begin{pmatrix} f_0 \\ f_c \\ f_s \end{pmatrix}, \qquad M_{j\cdot} = (1, \cos 2\theta_j, \sin 2\theta_j), \tag{S6.1}$$

with

$$\det M = 4 \sin(\theta_1 - \theta_2) \sin(\theta_2 - \theta_3) \sin(\theta_3 - \theta_1). \tag{S6.2}$$

Any three distinct propagation directions determine (f_0, f_c, f_s) uniquely.

For the canonical equal-spacing design $(0, \pi/3, 2\pi/3)$,

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1/2 & \sqrt{3}/2 \\ 1 & -1/2 & -\sqrt{3}/2 \end{pmatrix}, \tag{S6.3}$$

and the inverse is particularly simple:

$$f_0 = \frac{f(0) + f(\pi/3) + f(2\pi/3)}{3}, \tag{S6.4}$$

$$f_c = \frac{2f(0) - f(\pi/3) - f(2\pi/3)}{3}, \tag{S6.5}$$

$$f_s = \frac{f(\pi/3) - f(2\pi/3)}{\sqrt{3}}. \tag{S6.6}$$

Moreover, with $\|\cdot\|_2$ denoting the spectral norm and κ_2 its associated matrix condition number,

$$M^T M = \mathrm{diag}(3, 3/2, 3/2), \qquad \kappa_2(M) = \sqrt{2}, \qquad \|M^{-1}\|_2 = \sqrt{2/3}. \tag{S6.7}$$

Equality in the scalar interpolation optimum requires

$$\sum_{j=1}^{3} e^{2i\theta_j} = 0. \tag{S6.8}$$

The three unit phasors $e^{2i\theta_j}$ therefore form an equilateral triple, so the propagation directions differ by $\pi/3$, up to common rotation, permutation, and the identification $\theta \sim \theta + \pi$.

7 Full constitutive-to-acoustic operator under invariant norms

Let $\Theta = (\theta_1, \theta_2, \theta_3)$ and let $\mathcal{L}_\Theta$ map the constitutive vector to the stacked entries of the three acoustic tensors. The fourth-order tensor Frobenius norm, denoted by $\|\cdot\|_F$, is

$$\|\mathbb{G}\|_F^2 = \boldsymbol{c}^T W \boldsymbol{c}, \qquad W = \mathrm{diag}(1, 1, 2, 4, 4, 4, 2, 4, 4). \tag{S7.1}$$

The Euclidean representation of the invariantly normalized operator is

$$\widetilde{\mathcal{L}}_\Theta = \mathcal{L}_\Theta W^{-1/2}. \tag{S7.2}$$

Every design contains the isotropic major-symmetric subspace parameterized by (B, μ), on which

$$\|\mathbb{G}\|_F^2 = 4B^2 + 8\mu^2, \qquad \sum_{j=1}^{3} \|Q(\theta_j)\|_F^2 = 3(B^2 + 2B\mu + 2\mu^2). \tag{S7.3}$$

The input and output quadratic-form matrices on this subspace are

$$H_{\mathrm{in}} = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}, \qquad H_{\mathrm{out}} = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}. \tag{S7.4}$$

The generalized squared singular values are the eigenvalues of $H_{\mathrm{in}}^{-1/2} H_{\mathrm{out}} H_{\mathrm{in}}^{-1/2}$,

$$\nu_\pm = \frac{3}{4} \pm \frac{3\sqrt{2}}{8}. \tag{S7.5}$$

Because this subspace is present for every directional design, the full operator satisfies, with $\sigma_{\min}$ and $\sigma_{\max}$ denoting the smallest and largest singular values,

$$\sigma_{\min}^2(\widetilde{\mathcal{L}}_\Theta) \le \nu_-, \qquad \sigma_{\max}^2(\widetilde{\mathcal{L}}_\Theta) \ge \nu_+, \tag{S7.6}$$

and hence

$$\kappa_2(\widetilde{\mathcal{L}}_\Theta) \geq \sqrt{\nu_+/\nu_-} = 1+\sqrt{2}. \tag{S7.7}$$

For the 60° equal-spacing design, direct singular-value decomposition yields

$$\left\{\frac{3}{4}+\frac{3\sqrt{2}}{8}, \frac{9}{8}, \frac{9}{8}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{8}, \frac{3}{8}, \frac{3}{4}-\frac{3\sqrt{2}}{8}\right\} \tag{S7.8}$$

for the nine squared singular values. The equal-spacing design attains the universal lower bound. As a numerical corroboration of the full constitutive-to-acoustic operator result, a 1° grid scan with one direction fixed by rotational invariance found the minimum $1+\sqrt{2}$ only at the $0°, 60°, 120°$ triple on that grid. This grid observation is not used as a uniqueness proof; the rigorous uniqueness statement in the manuscript concerns the scalar interpolation problem. With † denoting the Moore–Penrose pseudoinverse, define

$$C_\Theta^{\mathrm{LS}} = \|\widetilde{\mathcal{L}}_\Theta^\dagger\|_2. \tag{S7.9}$$

For equal spacing,

$$C_\star = C_\Theta^{\mathrm{LS}} = \left(\frac{3}{4}-\frac{3\sqrt{2}}{8}\right)^{-1/2} = \sqrt{\frac{8+4\sqrt{2}}{3}} \approx 2.133608. \tag{S7.10}$$

Let $\widetilde{\mathcal{R}}_\Theta$ be the normalized matrix of the explicit reconstruction obtained by applying M^{-1} separately to T, D, P, A and then using the constitutive formulas in the main manuscript. It satisfies $\widetilde{\mathcal{R}}_\Theta \widetilde{\mathcal{L}}_\Theta = I_9$, so

$$C_\Theta^{\mathrm{dir}} = \|\widetilde{\mathcal{R}}_\Theta\|_2 \geq C_\Theta^{\mathrm{LS}}. \tag{S7.11}$$

For $(0, 60°, 120°)$,

$$C_\Theta^{\mathrm{dir}} = C_\Theta^{\mathrm{LS}} = 2.133608387177\ldots, \tag{S7.12}$$

even though $\widetilde{\mathcal{R}}_\Theta \neq \widetilde{\mathcal{L}}_\Theta^\dagger$. For $(0, 15°, 45°)$ the two values are

$$C_\Theta^{\mathrm{LS}} = 12.353425826\ldots, \qquad C_\Theta^{\mathrm{dir}} = 13.124118528\ldots. \tag{S7.13}$$

If $P_\Theta = \widetilde{\mathcal{L}}_\Theta \widetilde{\mathcal{L}}_\Theta^\dagger$ is the orthogonal projector onto the constitutively admissible data subspace, then

$$\widetilde{\mathcal{R}}_\Theta P_\Theta = \widetilde{\mathcal{L}}_\Theta^\dagger. \tag{S7.14}$$

Thus projection can alter typical noise propagation even at equal spacing, where the worst-case norms happen to coincide. These numerical constants are norm dependent; the statements above use the fourth-order stiffness-tensor Frobenius norm on constitutive space and the stacked acoustic-tensor Frobenius norm on data space, as defined in the main manuscript.

Admissible equal-spacing triples

Let $\mathcal{F} \subset [0, \pi)$ be the set of propagation directions with two distinct positive real acoustic eigenvalues, and let $|\mathcal{F}^c|$ be the angular measure of its complement. For

$$N(\alpha) = \sum_{j=0}^{2} \mathbf{1}_{\mathcal{F}^c}(\alpha + j\pi/3),$$

with angles understood modulo π,

$$\frac{1}{\pi}\int_0^\pi N(\alpha)\, d\alpha = \frac{3|\mathcal{F}^c|}{\pi}. \tag{S7.15}$$

If $|\mathcal{F}^c| < \pi/3$, the mean of the nonnegative integer-valued function N is smaller than one. Hence $N(\alpha) = 0$ for at least one α, and an admissible 60° triple exists. The criterion is sufficient but not necessary.

8 Acoustic-tensor reconstruction from modal data and nonorthogonality

For $Q = V\Lambda V^{-1}$, where V has unit right-eigenvector columns and $\Lambda = \mathrm{diag}(\lambda_-, \lambda_+)$, differentiation gives

$$\delta Q = V\big(\delta\Lambda + [\Xi, \Lambda]\big)V^{-1} + \mathcal{O}(\eta^2), \qquad \Xi = V^{-1}\delta V. \tag{S8.1}$$

Here η is the perturbation magnitude, and $[\Xi, \Lambda] = \Xi\Lambda - \Lambda\Xi$ is the commutator. If the two unit polarization directions (right eigenvectors) have acute separation β and small angular errors $\delta\phi_-$ and $\delta\phi_+$, then

$$|\Xi_{12}| = \frac{|\delta\phi_+|}{\sin\beta}, \qquad |\Xi_{21}| = \frac{|\delta\phi_-|}{\sin\beta}. \tag{S8.2}$$

Writing $\Delta_\lambda = |\lambda_+ - \lambda_-|$, one obtains

$$\|\delta Q\|_F \le \kappa_2(V)\left[\|\delta\Lambda\|_F + \frac{\Delta_\lambda}{\sin\beta}\sqrt{(\delta\phi_-)^2 + (\delta\phi_+)^2}\right] + \mathcal{O}(\eta^2). \tag{S8.3}$$

Let $Q_s = (Q + Q^T)/2$, let $m = \mathrm{tr}\, Q_s/2$, and let $\delta_s > 0$ denote half the eigenvalue splitting of Q_s. In an oriented eigenbasis of Q_s,

$$Q = \begin{pmatrix} m - \delta_s & A \\ -A & m + \delta_s \end{pmatrix}, \qquad |A| < \delta_s. \tag{S8.4}$$

Then

$$\Delta_\lambda = 2\sqrt{\delta_s^2 - A^2}, \qquad \cos\beta = \frac{|A|}{\delta_s}, \qquad \sin\beta = \frac{\Delta_\lambda}{2\delta_s}, \tag{S8.5}$$

and, for unit-norm eigenvector columns,

$$\kappa_2(V) = \sqrt{\frac{\delta_s + |A|}{\delta_s - |A|}} = \cot\frac{\beta}{2}. \tag{S8.6}$$

The identity $\Delta_\lambda/\sin\beta = 2\delta_s$ remains finite in the coalescence limit; the singular factor is the eigenvector nonorthogonality $\kappa_2(V)$.

9 Overdetermined residuals for the Fourier form and constitutive consistency

For $N > 3$ propagation directions, let M_N contain the rows $(1, \cos 2\theta_j, \sin 2\theta_j)$ and assume that M_N has full column rank. Let I_N be the $N \times N$ identity matrix, and let $\boldsymbol{y}_f$ collect the measured values of $f \in \{T, D, P, A\}$. Define

$$\Pi_N = M_N(M_N^T M_N)^{-1} M_N^T. \tag{S9.1}$$

For each measured scalar quantity $f \in \{T, D, P, A\}$,

$$\mathcal{R}_f = \|(I_N - \Pi_N)\boldsymbol{y}_f\|_2. \tag{S9.2}$$

The fitted major-symmetric coefficients also give

$$\mathcal{R}_{\mathrm{cons}}^2 = (T_c - D_0)^2 + (T_s - P_0)^2 + (D_s - P_c)^2. \tag{S9.3}$$

The residual space has dimension

$$4(N - 3) + 3 = 4N - 9, \tag{S9.4}$$

matching the difference between the $4N$ reconstructed acoustic entries and the nine constitutive parameters. The residual $\mathcal{R}_A$ is essential for checking the skew-symmetric acoustic component: an additional 4θ term in $A(\theta)$ is orthogonal to the assumed basis for equally spaced overdetermined measurements, so it produces $\mathcal{R}_A > 0$ while leaving $\mathcal{R}_{\mathrm{cons}}$ at roundoff level.

The figure-generation script also calibrates this residual under the modal-noise model used in the synthetic reconstruction. For the first retained reference material and $N = 12$ equally spaced additional propagation directions, 5000 null-model realizations generated with pseudorandom seed 20260913 give

$$Q_{0.95}(\mathcal{R}_A) = 5.8230 \times 10^{-3}. \tag{S9.5}$$

The mean acoustic eigenvalue of this reference state is 2.25180, so the corresponding normalized threshold is 2.5859×10^{-3}. On the logarithmic injected-amplitude grid, the first amplitude for which more than 95% of 1200 realizations exceed this null threshold is

$$\frac{\eta_{4\theta}}{\overline{\lambda}} = 2.0613 \times 10^{-3}. \tag{S9.6}$$

These numbers calibrate Fig. 2(c) for the stated reference state, angular sampling and noise model; they are not proposed as material-independent thresholds.

10 Synthetic reconstruction test

The synthetic calculation starts from constitutive coefficients, computes the corresponding modal data, perturbs those data, and reconstructs the constitutive tensor. The pseudorandom generator seed is 20260913. Here $\mathcal{N}(0, \sigma^2)$ denotes a zero-mean normal distribution with variance σ^2. Each retained material is generated from

$$C_{11} = 4.5 + \mathcal{N}(0, 0.6^2), \qquad C_{22} = 4.0 + \mathcal{N}(0, 0.6^2), \tag{S10.1}$$

$$C_{12} = 1.3 + \mathcal{N}(0, 0.35^2), \qquad C_{66} = 1.4 + \mathcal{N}(0, 0.3^2), \tag{S10.2}$$

$$C_{16}, C_{26} \sim \mathcal{N}(0, 0.25^2), \qquad R^0, K_a^0, K_b^0 \sim \mathcal{N}(0, 0.12^2). \tag{S10.3}$$

A candidate is retained only if a 181-point angular scan over $[0, \pi)$ gives two real eigenvalues such that the smaller eigenvalue exceeds 0.3 and the eigenvalue separation exceeds 0.25 at every sampled angle. Sixty materials are retained. For each design and each material, 100 independent noise realizations are applied, yielding 6000 reconstructed tensors per design.

The modal noise model uses independent Gaussian perturbations with eigenvalue standard deviation

$$\sigma_\lambda = 5 \times 10^{-4} \, \frac{|\lambda_-| + |\lambda_+|}{2} \tag{S10.4}$$

and polarization-angle standard deviation

$$\sigma_\phi = 0.05^\circ. \tag{S10.5}$$

Noisy acoustic tensors are reconstructed as $V_{\mathrm{noisy}} \Lambda_{\mathrm{noisy}} V_{\mathrm{noisy}}^{-1}$ before the three-direction constitutive inversion is applied. Relative constitutive error is measured in the invariant fourth-order tensor Frobenius norm.

Table S1: Synthetic reconstruction results.

Quantity	Equal-spacing design	Clustered design
Propagation directions	$0^\circ, 60^\circ, 120^\circ$	$0^\circ, 15^\circ, 45^\circ$
Least-squares inverse norm C_Θ^{LS}	2.1336	12.353
Closed-form inverse norm C_Θ^{dir}	2.1336	13.124
Median relative tensor error	1.22×10^{-3}	5.61×10^{-3}
Median first-order bound	2.84×10^{-3}	–
Median bound/error ratio	2.29	–

Table S1 summarizes the synthetic reconstruction results. The numerical experiment evaluates the ordering predicted by the directional conditioning and the scale of the combined local error bound; no analytical constant is fitted from these data.

11 Independent validation summary

The supplied validation script recomputes the algebra independently from the manuscript equations. Representative validation results are listed in Table S2.

Table S2: Selected independent validation results.

Check	Result
Random general three-direction reconstructions	1451 cases; maximum error 8.98×10^{-14}
Two-direction ranks (full/even/odd)	$(7, 5, 2)$
Maximum constitutive-consistency residual	1.16×10^{-13}
Maximum determinant-formula error for M	1.33×10^{-15}
Maximum explicit two-direction null-space residual	5.55×10^{-16}
Real-eigenpair acoustic-tensor reconstructions	500 cases; maximum error 5.33×10^{-15}
Complex-eigendata algebraic reconstructions	300 cases; maximum error 6.22×10^{-15}
Generic phase-speed-invariant Jacobian rank	8
Explicit Jacobian minor of the nine-parameter phase-speed map	$13/80000 = 1.625 \times 10^{-4}$
Generic Jacobian rank of the major-symmetric phase-speed map	6
Explicit Jacobian minor of the major-symmetric phase-speed map	$3/32 = 9.375 \times 10^{-2}$
$\mathcal{R}_{\mathrm{MS}}/\varepsilon^2$ for the one-parameter weak odd-coupling family defined above	5.282×10^{-5}; relative variation $< 3 \times 10^{-4}$
Constant-phase-speed tangent residual at $\boldsymbol{c}_\star$	2.62×10^{-16}
Maximum transpose-ambiguity residual	0
Maximum D_2 pure-R^0 twin invariant error	1.78×10^{-15}
Maximum C_4 pseudoscalar twin invariant error	4.44×10^{-15}
Maximum $SO(2)$ phase-speed-equivalence invariant error	2.13×10^{-14}
Equal-spacing scalar condition number	$\sqrt{2}$
Equal-spacing constitutive-to-acoustic condition number	$1 + \sqrt{2}$
1° grid minimum for the constitutive-to-acoustic operator	only $(0^\circ, 60^\circ, 120^\circ)$ with first direction fixed
Equal-spacing inverse norms $C_\Theta^{\mathrm{LS}}/C_\Theta^{\mathrm{dir}}$	2.133608387/2.133608387
Clustered inverse norms $C_\Theta^{\mathrm{LS}}/C_\Theta^{\mathrm{dir}}$	12.35343/13.12412
Maximum modal finite-difference relative remainder	3.10×10^{-6}
Maximum mirror-identity residual	3.77×10^{-15}

12 Reproducibility archive

The separate reproducibility archive contains the Python scripts used to generate all three figures and to execute the algebraic validation suite. It requires Python 3, NumPy, Matplotlib, and SciPy for the major-symmetric multistart check. From the root of the archive, run

```
python scripts/validate_acta_science.py
python scripts/validate_major_symmetric_spectral_geometry.py
python scripts/generate_all_figures.py
```

The validation scripts do not read stored numerical result files; the reported checks are recomputed from the formulas and fixed pseudorandom seeds.