Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Conceptualizing Periodicity Through Sound: Analysing Pre-service Teachers' Conceptualized Mathematical Work

Alejandro Cabrera-Baquedano

alejandro.cabrera@uda.cl

Universidad de Atacama

Elizabeth Montoya-Delgadillo

Pontificia Universidad Católica de Valparaíso

Laurent Vivier

Université Paris Cité

Fabrice Vandebrouck

Université Paris Cité

Research Article

Keywords: Conceptualization, Mathematical Working Space, Activity Theory, Periodicity, Fourier Series

Posted Date: November 26th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11131247/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Conceptualizing Periodicity Through Sound: Analysing Pre-service Teachers' Conceptualized Mathematical Work

Alejandro Cabrera Baquedano[1] – Elizabeth Montoya[2] – Laurent Vivier[3] – Fabrice Vandebrouck[3]

Abstract

Periodicity is fundamental for describing recurrent phenomena in mathematics and the sciences, yet its teaching is commonly approached through trigonometric functions and their geometric relation to angles and radians. This study explores sound as an alternative context for conceptualizing periodicity and further develops the notion of conceptualized mathematical work, developed through the networking of Mathematical Working Space and Activity Theory as an analytical lens for studying this process. Framed by Didactic Engineering, the study adopts a qualitative single-case design with embedded units of analysis involving 13 pre-service mathematics teachers organized into six groups. A technology-mediated sequence progressively connected pure sounds, function addition, trigonometric polynomials, and Fourier-based approximations. Findings from the focal embedded case of G6 show that conceptualization developed through the progressive reorganization of mathematical work across the sequence involving coordinated semiotic, instrumental, and discursive processes, together with teacher and peer mediations. The study shows how sound-based tasks can provide an alternative entry into trigonometric and periodic functions while making conceptualization analytically accessible as a longitudinal process.

Keywords

Conceptualization – Mathematical Working Space – Activity Theory – Periodicity – Fourier Series

1. Introduction

Periodicity is a fundamental property for describing recurrent phenomena in mathematics and the sciences, including oscillatory phenomena such as pendulums, mass–spring systems, and sound waves. In mathematics education, however, periodicity is commonly introduced through trigonometric functions, particularly sine and cosine, whose teaching is traditionally rooted in geometric approaches involving angles and the unit circle (Loeng et al., 2023). This route presents well-documented difficulties related to the coordination of angles and radians, the distinction between trigonometric ratios and functions, and the articulation of algebraic and graphical representations (Demir & Heck, 2013; Loeng, 2019; Moore, 2013; Tanguay, 2010; Tuna, 2013; Winsløw, 2016). Research also reports a predominance of algebraic work on trigonometric functions, which may obscure periodicity as a mathematical property (Buendía, 2006; Kepceoğlu & Yavuz, 2016).

Alejandro Cabrera Baquedano (*corresponding author*) ; alejandro.cabrera@uda.cl; https://orcid.org/0000-0002-1352-452X
Elizabeth Montoya Delgadillo; elizabeth.montoya@pucv.cl; https://orcid.org/0000-0002-1200-6261
Laurent Vivier; Laurent.vivier@univ-paris-diderot.fr; https://orcid.org/0000-0002-9203-8756
Fabrice Vandebrouck; vandebro@u-paris.fr; https://orcid.org/0000-0001-8941-6384

1.- Universidad de Atacama, Copiapó, Chile.
2.- Pontificia Universidad Católica de Valparaíso, Chile.
3.- Université Paris Cité, Paris, France.

These difficulties suggest the value of approaching periodicity through phenomena in which the variables are not initially associated with angles. Sound provides such a context: time and amplitude make periodic behavior accessible through auditory, graphical, numerical, and algebraic representations. Moreover, the progressive study of pure and complex sounds allows trigonometric functions to be interpreted as models of periodic phenomena and provides a path toward trigonometric polynomials and Fourier-based approximations.

Based on this perspective, we designed a technology-mediated didactic sequence using Audacity to visualize sound waves and obtain data from real-world situations, Excel to organize the numerical data from Audacity, and GeoGebra for the mathematical processing of the data, to study how pre-service mathematics teachers progressively construct mathematical meaning around periodicity. Rather than considering conceptualization as an outcome observable in an isolated task, we analyse it through the evolution of students' mathematical work across the sequence. For this purpose, Mathematical Working Space (MWS) and Activity Theory (AT) are networked to introduce the notion of *conceptualized mathematical work*, which enables us to examine the semiotic, instrumental, and discursive processes, mediations, and reorganizations involved in conceptualization.

Accordingly, the study addresses the following research question: How can the conceptualization of periodicity be understood through the analysis of pre-service mathematics teachers' conceptualized mathematical work in a sound-based didactic sequence?

To address this question, the article first develops the theoretical contribution of conceptualized mathematical work through the networking of MWS and AT, with particular attention to mediation, internalization, and the longitudinal reorganization of mathematical work. It then describes the sound-based didactic sequence and the analytical framework used to analyze its implementation, tracing the development of mathematical work across four didactic situations and examining how this trajectory makes the conceptualization of periodicity analytically visible.

2. Theoretical Framework

To study conceptualization, it is necessary to adopt theoretical lenses that make it analytically observable in relation to semiotic, discursive and instrumental process. MWS allows us to describe the mathematical work in these terms, while AT explains how this work develops through mediations, teacher interventions, and the Zone of Proximal Development (ZPD) across a didactic sequence. Given that both frameworks attempt to epistemological and cognitive domains, their networking is therefore used to approach conceptualization as a process inferable from conceptualized mathematical work.

2.1. Mathematical Working Space

The theory of MWS aims to understand, from a didactic point of view, what is at stake around mathematical work in an educational context (Kuzniak et al., 2016; 2022). MWS was developed to account for the ways in which mathematical work emerges from the interaction between epistemological and cognitive dimensions of mathematical work in educational settings.

Tasks play a central role in MWS because the mathematical work developed depends on the problems individual attempt to solve and on the learning situations designed by teachers

or researchers. Their expected resolution is framed by the institutional expectations represented in the reference MWS. (Kuzniak & Richard, 2014)

A central feature of MWS is the organization of mathematical work through the articulation of two complementary aspect – the epistemological and the cognitive- with three fundamental dimensions or geneses: semiotic, instrumental, and discursive (see Fig. 1). Each genesis connects an epistemological component with a cognitive process. Semiotic genesis links the representamen (R) with visualization (V); instrumental genesis links artifacts (A) with construction (C); and discursive genesis links the theoretical referential (T), composed of definitions, properties, and theorems, with proving (P). The six elements represented in the MWS diagram can therefore be understood as resulting from the crossing of these two aspects with the three geneses (Kuzniak et al., 2022).

Mathematical work develops through the activation and coordination of these geneses. When two geneses are jointly involved in the resolution or design of a task and their respective contributions cannot be meaningfully analyzed separately, their coordination is described through a vertical plane.

The [Sem-Ins] plane characterizes the coordination of semiotic and instrumental geneses, for instance when construction or artifact use is supported or constrained by signs and representations. The [Ins-Dis] plane concerns the coordination of instrumental and discursive geneses, such as when tool use is supported by elements of the theoretical referential or when justifications is grounded in artifact use. The [Sem-Dis] plane refers to the coordination of semiotic and discursive geneses, for example when discourse and visualization are articulated through justifications supported by signs or through the discursive reorganization of representations (Kuzniak et al., 2022, p. 16)

Within MWS, mathematical work is considered complete when its circulation is ensured across the different dimensions and components of the MWS diagram, avoiding confinement to a single genesis, representation, artifact, or form of reasoning (Kuzniak et al., 2022).

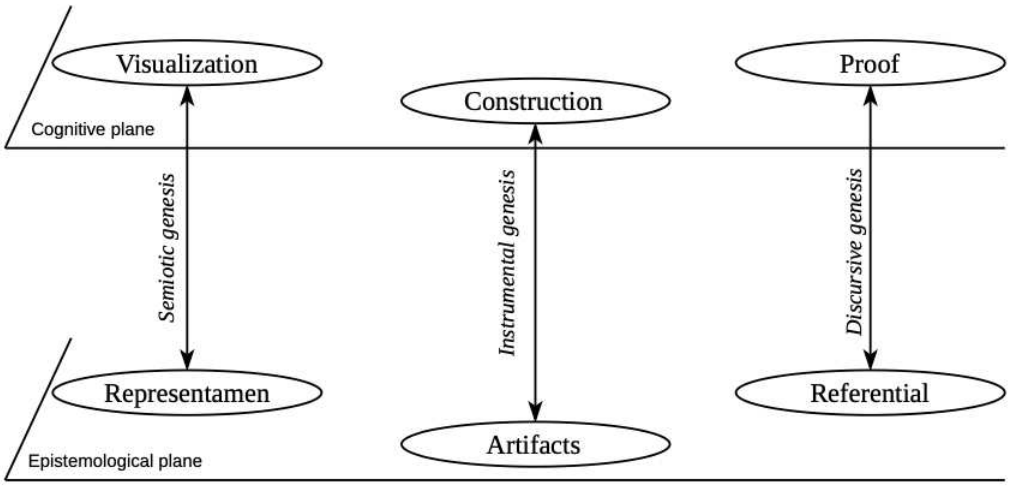


Fig. 1 MWS diagram

MWS also introduces the notion of paradigm to characterize the epistemological regime within which mathematical work is developed. Paradigms are not levels of difficulty; they refer to different forms of mathematical work, with different relations to reality, different

forms of representation, different artifacts, and different theoretical expectations. Montoya and Vivier (2016, p. 746) define the Analysis Paradigms as follows:

> A1, which enables interpretations with implicit assumptions based on geometry, arithmetic calculations or the real world.
>
> A2, where the rules of calculation are defined more or less explicitly and are applied independently of reflection on the existence and nature of the objects involved.
>
> A3, which is characterized by work involving approximation and neighborhoods, even topological; definition and properties are set theoretically that allows an "ε work" specific to this paradigm: bounds, inequality, "the negligible".

2.2. Activity Theory

While MWS offers a structured way of describing mathematical work, AT provides a developmental perspective for understanding how such work may evolve through mediation.

AT has been adapted for analyzing the teaching-learning process in the mathematics classroom, both in its collective and individual dimensions, but also according to complementary dimensions: cognitive - linked to the knowledge and tasks at stake in a session - and mediative - linked to interactions between actors during classroom proceedings (Vandebrouck, 2018). In this adaptation, the conceptualization process proposed by Vergnaud (1991) constitutes one of the bases of the definition of learning in Activity Theory.

A central contribution of this perspective lies in the distinction and relationship between external and internal activity. External activity refers to the socially organized, materially mediated, and observable actions through which students interact with tasks, tools, and others in classroom situations. Internal activity refers to the constructive reorganization of meanings and knowledge that may emerge from this external engagement. Following Vygotsky (1985) and Leont'ev (1984), these should not be understood as separate domains or linear stages, but as dialectically related dimensions of development: what first appears in the social plane may progressively be reconstructed at the internal plane through processes of internalization (Vygotsky, 1978; Bartolini Bussi & Mariotti, 2008).

This developmental movement is closely linked to the ZPD, understood as the distance between what a learner can achieve independently and what can be achieved with guidance or in collaboration with pairs. Within this zone, development is made possible through constructive mediation (Vandebrouck, 2018; Robert, 2008). In particular, signs, language, and artifacts play a central role, since they support action while also organizing thought, communication, and reflection. In mathematics classrooms, teacher mediation is therefore crucial: through guided interactions, partial institutionalizations, verbalization, and tool use, students may move beyond their current level of understanding and progressively appropriate culturally elaborated forms of mathematical thinking.

2.3. Networking: Conceptualized Mathematical Work

The notion of conceptualized mathematical work was initially proposed in the first author's doctoral research (Cabrera Baquedano, 2024) through a networking (Prediger & Bikner - Ahsbahs, 2014) between the theory of MWS and AT. The present article further develops

this construct by clarifying its longitudinal analytical scale and operationalizing it through the progressive reorganization of mathematical work across a didactic sequence.

To carry out this networking we use both theories in a coordinated approach. On the one hand, MWS provides a methodological framework for analyzing mathematical work, and AT allows us to describe the mediational and developmental processes involved in conceptualization.

Conceptualized mathematical work is understood as developing through a trajectory in which mathematical work becomes progressively more complete and coherent in its circulation, rather than requiring complete mathematical work in each individual task. Student progressively appropriates and reorganizes knowledge through mediated work within the ZPD.

Complete mathematical work and conceptualized mathematical work operate at different analytical scales. Complete mathematical work characterizes the coherence and articulation of the mathematical work developed in relation to a given task, whereas conceptualized mathematical work introduces a longitudinal and developmental dimension. Although the latter draws on the notion of complete mathematical work, it does not require students to exhibit complete mathematical work in every task. Rather, it is inferred from how, across a sequence of situations, students progressively reorganize the components of their personal MWS, establish new connections from previously available knowledge, and mobilize these connections in subsequent mathematical work. Thus, locally partial or mediated mathematical work may contribute to conceptualized mathematical work when it becomes part of a coherent trajectory of reorganization across the sequence.

Language activities play a fundamental role, since they will allow the student to verbalize the connections made in the construction of new concepts. In this sense, Vergnaud (1991) points out that "language activities in a situation and cognitive activities about language necessarily involve specific conceptualizations about the content of thought, which are therefore conditioned by the content of knowledge" (p. 85).

In the conceptualization process, the adaptation of knowledge (Robert, 2008) to new situations is fundamental to conceptualize a mathematical object through relationships between already internalized knowledge. Conceptualized mathematical work requires the student to recognize both intra- and extra-mathematical situations that can be modeled by means of the concepts developed when solving different situations.

Accordingly, conceptualized mathematical work requires, at the design level, a sequence of tasks that enables progressive knowledge mobilization and, at the student level, evidence of new connections through discourse, signs, representations, and artifact use.

2.4. Mediations, ZPD and observable traces of conceptualization

From an Activity Theory perspective, learning cannot be reduced to the completion of isolated tasks. Rather, it unfolds through the dialectical relationship between external and internal activity. In this article, activity is understood in the sense given by Activity Theory, whereas mathematical work remains the main analytical construct. Accordingly, concepts from Activity Theory are mobilized to explain the developmental mechanisms through which mathematical work evolves through constructive mediation and may become conceptually productive. Conceptualization is therefore not directly observed in activity

itself, but analytically inferred from traces left in mathematical work across the didactic sequence.

This developmental movement is closely related to the ZPD. Within this zone, development depends on mediation, particularly on the coordination of signs, artifacts, and language. Signs and language function as means of organizing communication, reflection, and verbalization, whereas artifacts support action and generate semiotic resources for interpreting and transforming mathematical work. In classroom settings, teacher mediation is therefore crucial, since it is through guided interactions, local institutionalization, verbalization, and tool use that mathematical work initially developed in shared activity may become progressively appropriated by students.

This perspective also offers a productive way of re-reading the geneses and vertical planes of the MWS. In analytical terms, the three geneses may be understood in relation to processes of internalization linked to the cognitive dimensions they organize. More specifically, they may be read through the progressive reorganization of mathematical work in terms of the appropriation and use of artifacts, the emergence, coordination, and transformation of signs and representations, as well as verbalization, explanation, justification, and validation. Through these processes, students progressively transform externally mediated work into more coherent forms of mathematical work. Rather than being considered in isolation, these processes become especially meaningful when examined through the vertical planes of the MWS, since these planes make visible the coordination of two geneses in the development of mathematical work. In this sense, the vertical planes constitute analytically significant coordinations through which the circulation and progressive reorganization of mathematical work can be studied.

Paradigms add a further epistemological dimension to this analysis. Unlike signs, artifacts, and language, paradigms are not mediators of activity but epistemological orientations that characterize the form and validity of mathematical work. They make it possible to identify the epistemological orientation of the work developed by students and to examine whether this work evolves toward more theoretically elaborated forms of mathematical analysis. For this reason, paradigmatic shifts are especially relevant in the present study. The progression from work mainly developed within A1 and A2 paradigms toward forms of work associated with an A3 paradigm is interpreted not simply as a change in performance, but as an indicator that students' mathematical work is becoming more coordinated, more explicit in its mathematical relations, and more conceptually productive across the sequence.

Building on the notion of conceptualized mathematical work introduced in the previous section, conceptualization is approached here not as a generic label for learning, but as an analytically inferable process. More specifically, conceptualization becomes observable through traces left in students' mathematical work as, across a didactic sequence, they progressively coordinate instrumental, semiotic, and discursive processes together with the mediations unfolding within their ZPD, activate and articulate different vertical planes of the MWS, and establish new connections through the progressive reorganization of the epistemological components of their personal MWS.

In conclusion, the networking of MWS and AT does more than juxtapose two descriptions of learning: it makes conceptualization analytically accessible by relating the evolving form of mathematical work to the mediational and developmental processes through which that work is progressively reorganized.

3. Methodology

3.1. Method and context

This study was framed by Didactic Engineering (Artigue, 1988), which guided the design, implementation, and analysis of the teaching sequence. In line with the research objective, the study adopted a qualitative approach (Bikner-Ashbahs et al., 2015) aimed at understanding how pre-service mathematics teachers conceptualized periodicity while engaging in a sound-based task sequence. Within the experimentation phase of Didactic Engineering, the implementation was examined as a single-case study (Yin, 2009) with embedded units of analysis. Following the analysis of the six embedded units, G6 was selected as the focal case because its work provided the clearest longitudinal evidence of connections established and subsequently mobilized across the four didactic situations. The selection was therefore analytical rather than representative, allowing the proposed construct to be examined through a sustained trajectory of mathematical work.

The implementation involved 13 pre-service mathematics teachers enrolled in the fifth semester (of a total of 9) of a teacher education programme at a Chilean university. The participants were organized into six working groups: one group of three students and five groups of two students (labelled G1–G6). The teaching sequence consisted of four didactic situations implemented across six lessons of 75 minutes each.

The design situation incorporated the use of three softwares: Geogebra, Excel, and Audacity. Audacity is an audio recording and playback program whose interface displays the waveform of each sound. This software allows users to create mixes, export audio data, among other functions. Due to the fact that the pre-service mathematics teachers were unfamiliar with this software, they were provided with a tutorial covering its main features.

The empirical material analyzed in this article was collected as part of the first author's broader doctoral research (Cabrera Baquedano, 2024). The present article focuses specifically on the Chilean implementation and re-examines the focal case of G6 through the refined longitudinal formulation of conceptualized mathematical work developed here.

3.2 Data Analysis

The researcher adopted a participant-observer role, as he also acted as the instructor during the implementation. The data corpus included students' written productions, digital files produced in GeoGebra and Excel, screen recordings of each group's work while solving the tasks, transcriptions of sessions and video recordings of each sessions. For each session, written productions, students' work on the computer, screen recordings, and group dialogues were analyzed in an integrated way. From this perspective, the longitudinal analysis of the four didactic situations made it possible to examine how conceptualized mathematical work developed across the sequence.

The analysis proceeded in two stages. First, the mathematical work of the six groups was examined by confronting the a priori analysis with the work actually developed in each didactic situation; the trajectories were then compared longitudinally across the sequence, after which G6 was examined in depth because it provided the clearest evidence of connections established and subsequently mobilized across situations.

The analysis focused on five main elements:

- **Mediations:** teacher devolutions, institutionalizations and peer exchanges that led groups to modify strategies or deepen their arguments.
- **Geneses and vertical planes:** identified through the theoretical tools of MWS in relation to students' semiotic, instrumental, and discursive work.
- **Circulation of mathematical work:** analyzed through the chaining or combining of geneses and vertical planes within and across tasks.
- **Paradigms of analysis (A1, A2, A3):** used to characterize the epistemological form of the work developed by each group.
- **Conceptualized mathematical work:** examined longitudinally through the reinterpretation of trigonometric and periodic functions in relation to sound waves, the understanding of function addition as sound mixing, the interpretation of trigonometric polynomials and Fourier series, and the progressive reorganization of the epistemological components of the groups' personal MWS.

To strengthen the interpretation of the mathematical work, data were triangulated across students' different sources in the corpus. Rather than treating these sources independently, evidence was contrasted across them to reconstruct the mathematical work developed by each group and to support the identification of geneses, vertical planes, mediations, and changes across the sequence.

3.3. Didactic sequence

The didactic sequence consisted of four didactic situations designed to progressively support the conceptualization of periodicity through sound. Each task focused on a different stage in the transition from the recognition of pure sounds to the description of complex sounds through trigonometric polynomials and Fourier-based approximations.

The first situation was developed over the first two sessions, the second during the third session, the third during the fourth session, and the fourth over the final two sessions. (see *Fig. 2*)

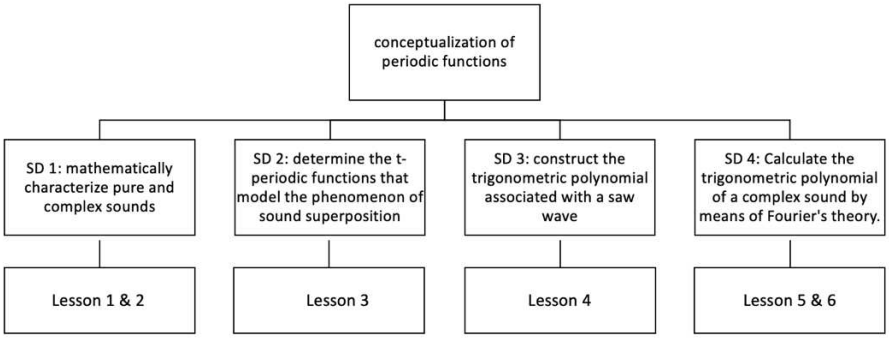


Fig. 2 Objective of each didactic situations and corresponding lessons

Didactic Situation 1

In the first situation, the objective was to recognize that pure sounds can be described by trigonometric functions. Two tasks were designed. In Task 1.1, pre-service mathematics teachers were asked to characterize the sounds contained in a folder of 12 audio files using Audacity. They were expected to classify the sounds in terms of their nature, the shape of the waveform displayed in the software, or the vibrating body producing the sound.

In Task 1.2, they were asked to determine the function associated with each sound. To do so, they exported data from Audacity to Excel and then used GeoGebra's two-variable

regression tool to obtain a graphical representation and an analytic approximation of each waveform. Through this task, students were expected to recognize that pure sounds could be approximated by trigonometric functions.

Didactic Situation 2

This situation aimed to determine periodic functions modelling sound superposition. Participants mixed pure sounds of different frequencies in Audacity and were asked to determine the function and frequency of the resulting sound. Through successive modifications of the component frequencies, they were expected to interpret sound superposition as addition of sinusoidal functions and to recognize that, for the commensurate frequencies considered in the tasks, the fundamental frequency of the resulting trigonometric polynomial corresponds to the greatest common divisor of its component frequencies.

Didactic Situation 3

The third situation aimed to construct a trigonometric polynomial associated with a sawtooth wave and to introduce the main idea underlying Fourier's theorem. Using a GeoGebra applet designed for the sequence, pre-service mathematics teachers first explored the additive synthesis of a fundamental frequency of 220 Hz and its harmonics, modifying their amplitudes and examining the resulting graphical and auditory changes. They were then asked to determine the trigonometric polynomial modelling a 220 Hz sawtooth wave, using Audacity to construct and visualize the sound. Through this work, students were expected to identify the regularity in the harmonic amplitudes and express the resulting finite trigonometric series in summation form. The teacher subsequently institutionalized the idea that progressively adding harmonics modifies the waveform while preserving periodicity, providing a basis for interpreting the sawtooth wave as an approximation by a trigonometric polynomial and for introducing Fourier's theorem.

Didactic Situation 4

The fourth situation, implemented over two sessions, aimed to determine a trigonometric polynomial modelling a complex sound. Pre-service mathematics teachers selected either a flute or violin sound and analysed its waveform in Audacity to determine its period and frequency. The sound data were then exported to Excel, where Fourier coefficients were approximated through Riemann sums, and subsequently used in GeoGebra to construct and graph the corresponding trigonometric polynomial. Students were expected to compare the resulting model with the original waveform and refine it by calculating additional coefficients. This final situation required them to mobilize the technological and mathematical resources developed in previous sessions and to use Fourier theory as a means of approximating and validating a real periodic phenomenon.

4. Findings

The broader analysis revealed differentiated trajectories across the six groups. This section focuses on G6 as the focal embedded case because it exhibited the most coherent development of conceptualized mathematical work across the four didactic situations.

4.1 Didactic Situation 1:

To begin the first lesson, students were given a folder containing twelve sounds and were asked to classify them. This first task had an exploratory character: the groups were expected to attend to pitch, timbre, loudness, and duration. G6 initially organized the sounds according to perceptual criteria, especially sonority and duration (see *Fig. 3*), and even grouped some sounds according to the pleasure they felt while listening to them.

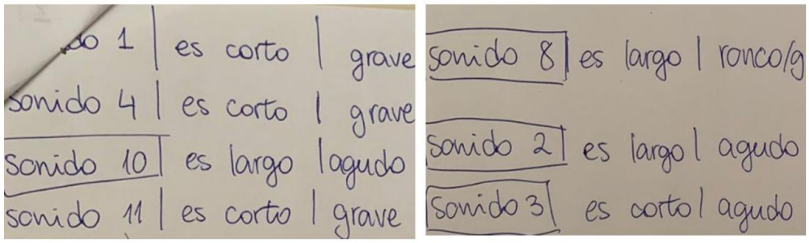


Fig. 3 Analysis developed by G6, in which auditory perception plays a central role. [**English translation:** Sound 1 is short and low-pitched; Sound 4 is short and low-pitched; Sound 10 is long and high-pitched; Sound 11 is short and low-pitched; Sound 8 is long and hoarse; Sound 2 is long and high-pitched; Sound 3 is short and high-pitched"]

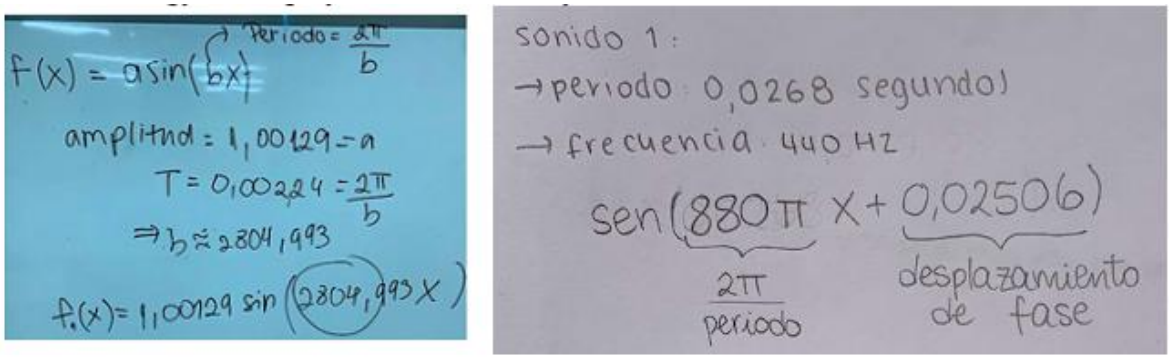
Although Sounds 1, 2, and 3 had the same frequency, G6 classified some of them as high-pitched and others as low-pitched on the basis of timbre rather than on physical criteria. In this first moment, the group's work remained close to auditory perception and did not yet establish the mathematical or physical distinctions anticipated in the a priori analysis. Instrumentalization of Audacity allowed the group to begin recognizing relevant features of the sounds through the interface, especially waveform shape, intensity, and duration, as shown in *Fig. 4*.



Fig. 4 Audacity interface of intensity and waveform

At the beginning of Lesson 2, the professor reformulated the work of the previous session by introducing a second task: to determine mathematically the function that models each sound. The pre-service teachers exported the sound data from Audacity in a table with time and amplitude as variables. According to the a priori analysis, G6 did not initially succeed in constructing the function describing each sound. A decisive mediation then occurred when a member of G3 presented their strategy to the class (see *Fig. 5a*). From the values visualized in Audacity, G3 determined amplitude and period, converted period into frequency, and proposed a sinusoidal model.

G6 adapted this strategy to its own work, refining the measurement of the period according to the interval selected in Audacity and incorporating phase information from the interface. This led the group to construct a sinusoidal model for Sound 1 that was much closer to the one expected in the a priori analysis, as shown in *Fig. 5b*.



(a) **(b)**

Fig. 5 Trigonometric functions constructed during DS1: (a) function generated by G3; (b) function subsequently obtained by G6 after adapting G3's strategy

From the perspective of mathematical work, this stage shows how Audacity supported a [Sem-Ins] circulation through which G6 identified period, frequency, amplitude, and waveform. In analytical terms, the [Sem–Dis] plane activation is observable when these visual elements were reorganized discursively to determine the trigonometric function describing the sound. In addition, the group moved from a perceptual entry into the phenomenon toward an initial mathematization of a pure sound. Even so, the mathematical work developed at this stage remained associated with the A1 paradigm, since G6 still relied on particular values such as sound period and frequency and only reached the expected model through mediation and the sharing phase. Stage 1 therefore marks the beginning of the conceptualization process: trigonometric functions start to emerge not as decontextualized expressions, but as function that describe a pure sound. *Fig. 6* (see below) follows the organizational logic of MWS to represent Audacity's instrumentalization and visualization, without constituting an MWS diagram itself.

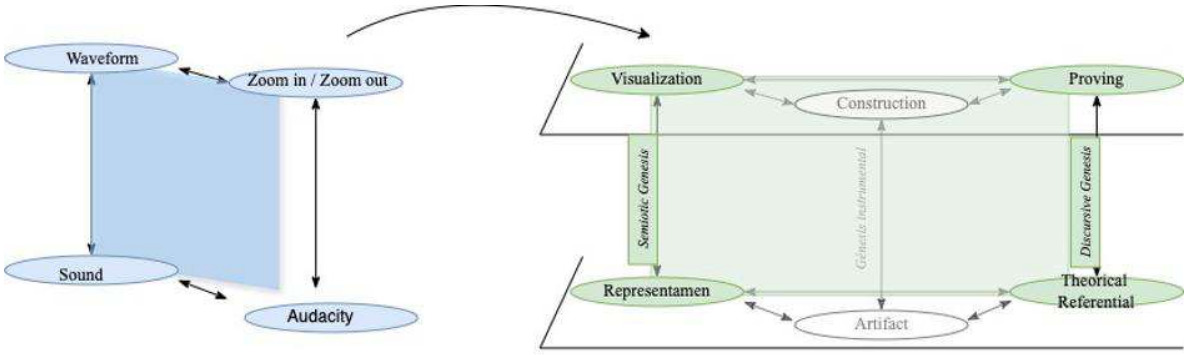


Fig. 6 Path of mathematical work developed by G6 in the first didactic situation

4.2 Didactic Situation 2:

At the beginning of the second stage, the professor asked the pre-service teachers to construct two pure sounds in Audacity and then mix them. The first task required them to build a 220 Hz sound with amplitude 1 and a 330 Hz sound with amplitude 0.8, and to determine the function describing the resulting sound. In G6, a first important shift occurred when one of the pre service teachers related the phenomenon of sound cancellation in headphones to the superposition of waves and proposed that the sound mix should be described as the sum of functions. The group then graphed the two sinusoidal functions in GeoGebra, added them, and compared the resulting graph with the mix produced in Audacity (see *Fig. 7*). In this sense, the sound phenomenon was no longer treated only perceptually, but was reinterpreted as a mathematical process of function addition. G6 also calculated the period of the resulting graph in GeoGebra and used it to determine the frequency of the mixture. At this stage, the instrumentalization of Audacity and GeoGebra supported the transition from the construction of the sound to the construction of its mathematical function.

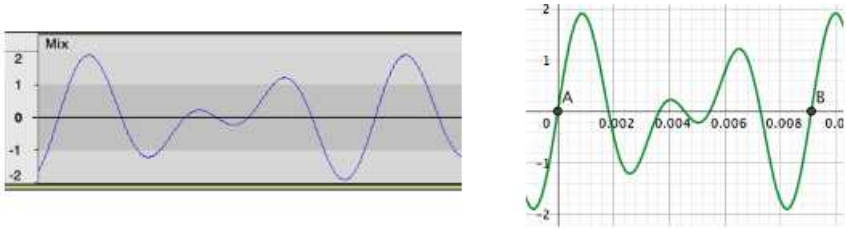


Fig. 7 Visualization of the sound mix in Audacity and its graph in Geogebra. A and B allow G6 to determine the period of the sound and the function

A second task then asked students to replace the 330 Hz sound with a 440 Hz sound and again determine the function and its frequency. G6 repeated the previous strategy and correctly represented the new sound as the sum of two sinusoidal functions: $f(x) = \sin(2\pi * 220x) + 0.8\sin(2\pi * 440x)$. However, the group initially formulated an incorrect hypothesis: they interpreted the frequency of the trigonometric polynomial as the difference between the frequencies of the pure sounds. This hypothesis seemed to fit the first two examples and was therefore provisionally stabilized by the group. The third task, which introduced additional sounds of 660 Hz and 880 Hz, made this interpretation insufficient. At that point, teacher mediation became decisive. Through a sequence of questions, the professor redirected the pre-service teachers' attention from numerical difference to multiplicative structure, prompting them to reinterpret the frequencies as multiples of 220. This mediation enabled G6 to recognize that the frequency of the trigonometric polynomial was not given by subtraction, but by the greatest common divisor of the component frequencies, as shown in *Fig. 8*.

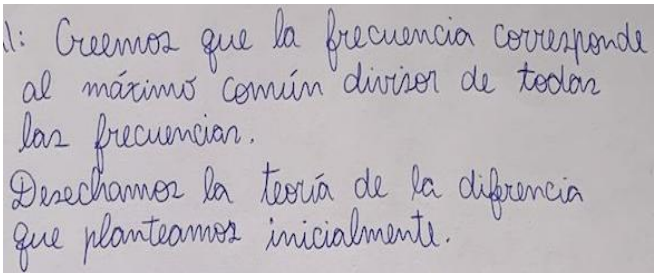


Fig. 8 G6 acknowledges that the frequency is derived from the GCD and not from the difference between the frequencies. [**English translation:** We believe that the frequency corresponds to the greatest common divisor of all the frequencies. We discard the difference hypothesis that we initially proposed]

From the perspective of mathematical work, Stage 2 shows substantial progress in G6. The group not only identified the mixture of sounds as a sum of functions thanks to their theoretical framework, in which they activated the [Sem-Ins] plane to coordinate the visualization in GeoGebra with Audacity, but also generalized the rule governing the frequency of the resulting trigonometric polynomial by activating the [Ins-Dis] plane. In analytical terms, this stage highlights an articulation between instrumental work in GeoGebra and discursive work around emerging numerical regularities, along with a [Sem–Dis] activation when the group, with the support of mediation, reorganized signs and arguments to establish the role of the greatest common divisor. For this reason, the mathematical work developed by G6 in this stage can be characterized as belonging to the A2 paradigm: the group correctly determined the functions modeling the sound mixtures and, through a numerical analysis of frequency values, identified a regularity that allowed a more formal interpretation of the trigonometric polynomial. Stage 2 therefore marks a decisive step in the conceptualization process, as function addition and polynomial frequency become mathematically meaningful in relation to sound superposition. *Fig. 9* shows the reorganization of mathematical work in Stage 2.

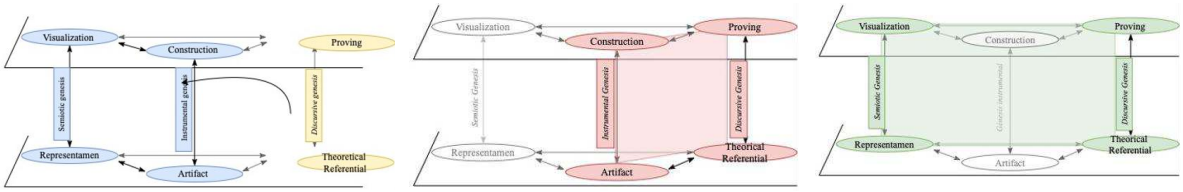


Fig. 9 Path of different genesis and planes activated by G6 in the didactic situation 2

4.3 Didactic Situation 3:

Didactic situation 3 began by recalling the knowledge institutionalized in the previous lesson and asking students to construct the trigonometric polynomial associated with a fundamental frequency of 220 Hz and its first harmonics. In this first task, G6 explicitly represented the polynomial not only as a sum of sinusoidal functions, but also in terms of the harmonic position of each component.

The group correctly identified the first harmonics of the fundamental frequency and determined the sinusoidal function corresponding to each of them, treating amplitude as a parameter. A second task then required the group to represent a complex sound obtained by mixing inharmonic frequencies. G6 distinguished this case from the previous one by noting that, unlike the harmonic case, the new sound did not arise from a fundamental frequency and its multiples. They justified this relationship using GCD (see *Fig. 10*)

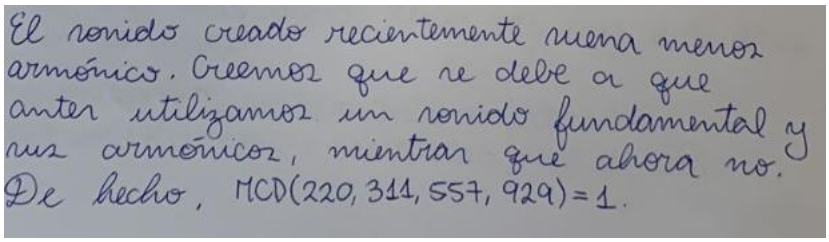


Fig. 10 G6's justification for distinguishing the new sound from the previous harmonic construction. [**English translation:** The newly created sound sounds less harmonic. We believe this is because previously we used a fundamental tone and its harmonics, whereas now we do not. In fact, GCD (220, 314, 557, 929) = 1]

Even so, the group correctly determined the trigonometric polynomial describing the sound, first constructing and sonifying it in Audacity and then graphing the corresponding polynomial in GeoGebra. At this stage, sound construction, sonification, and graphical visualization were already coordinated as part of the group's mathematical work.

The most significant development of this stage emerged in the task of determining the trigonometric polynomial associated with a sawtooth wave of frequency 220 Hz. G6 first constructed the waveform in Audacity, using the software to hear the sound and visualize its shape. They then used the GeoGebra applet (*Fig. 11*) introduced by the professor to vary the amplitudes of successive harmonics and to compare the resulting waveforms. Initially, their work remained strongly linked to sound perception: as in earlier stages, perceptual criteria prevailed over more explicit physical knowledge of sound, and acute sound continued to be associated more with timbre than with frequency.

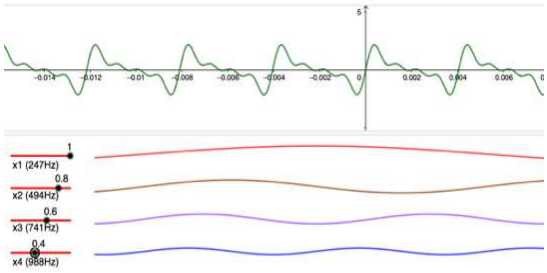


Fig. 11 Sawtooth wave constructed by G6 in GeoGebra applet

However, through teacher mediation and collective discussion involving a new collective use of the applet, G6 identified the regularity governing the amplitudes of the harmonics and used it to determine the analytical expression of the polynomial modeling the sawtooth wave. The final expression $\sum_{n=1}^{\infty} a_n \sin(nw_0x)$ with $w_0 = \frac{2\pi}{T}$, $a_n = \frac{1}{n}$, was therefore not produced autonomously from the beginning, but emerged from a process of guided adjustment, comparison, and validation.

From the perspective of mathematical work, Stage 3 marks a substantial enrichment of G6's personal MWS. To answer the initial questions from the didactic situation, pre-services teachers activate the [Sem-Ins] plane by coordinating the graphical and sonifying representations in Geogebra with the GCD of the mixed frequencies. G6 coordinated artifact use, waveform visualization, numerical regularities, and formal mathematical expressions in a way that made possible a complete circulation across its personal MWS in the last task. In particular, the GeoGebra applet enabled students to modify harmonic amplitudes while immediately observing the effect of these changes on the waveform, which supported the search for regularities and the construction of the corresponding polynomial. The analytical expression of the sawtooth wave was thus obtained through the coordination of semiotic, instrumental, and discursive processes rather than through isolated calculation alone. Even so, the work remained within the A2 paradigm: G6 was able to construct and validate the trigonometric polynomial, but only through the mediations and collective work that progressively prepared the institutionalization of Fourier's theorem. *Fig. 12* shows the reorganization of mathematical work

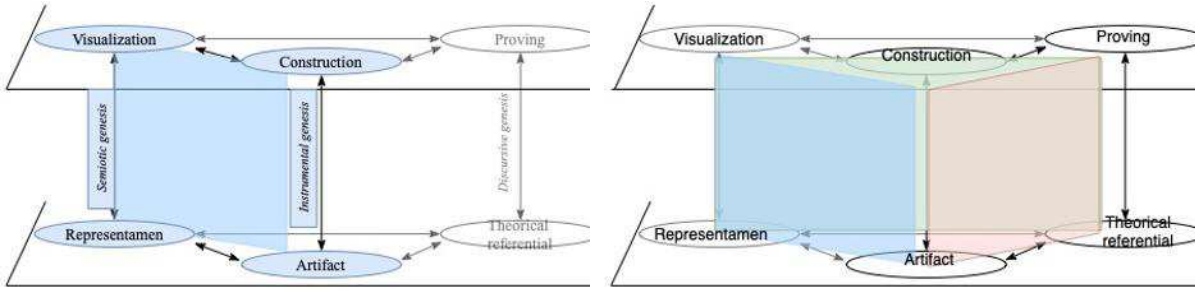


Fig. 12 Path of different genesis and planes activated by G6 in the didactic situation 3

4.4 Didactic Situation 4:

In the final stage, the professor asked the groups to choose one of two complex sounds—a flute or a violin—and determine the trigonometric polynomial that modeled it. G6 chose the violin sound and began by analyzing its waveform in Audacity. The group recognized that, unlike the idealized sounds studied in previous stages, this waveform did not repeat in a perfectly regular way, since it corresponded to a real phenomenon. Using the Audacity interface, G6 estimated the period of the sound and obtained a value that led them to a

frequency of 400 Hz (see *Fig. 13*), close to but different from the theoretical value of 440 Hz.

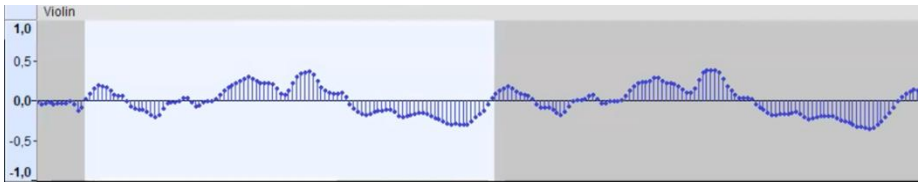


Fig. 13 Violin sound wave analyzed in Audacity using the zoom tool to determine its period

This initial phase already reveals a shift in the group's work: sound is analyzed as a phenomenon that required examination and interpretation. At the same time, the Fourier series became a symbolic artifact through which the group attempted to calculate the coefficients needed to model the sound.

A major difficulty then emerged. Like the other groups, G6 was unable to determine directly the function involved in the Fourier coefficients. At this point, teacher mediation was decisive. Rather than solving the task for the students, the professor redirected their attention to the meaning of the function to be integrated and to its relationship with the sound data exported from Audacity. Once these data were transferred to Excel, G6 organized a spreadsheet to approximate the coefficients through a Riemann-sum procedure, as shown in *Fig. 14*.

	A	B	C	D	E	F
1	tiempo	amplitud		suma superior		área total
2	0	0,09416		3,1066000E-06		1,4741000E-06
3	0,00002	0,15533		0,000005931		
4	0,00005	0,1977		3,8016E-06		
5	0,00007	0,19008		3,4628E-06		
6	0,00009	0,17314		2,6626E-06		

Fig. 14 Excel spreadsheet configured by G6 to calculate Fourier coefficients. [**English Translation:** columns label: time, amplitude, upper sum, total area]

In this process, G6 adapted previously acquired knowledge of integration to approximate the Fourier coefficients. The group distinguished its right-endpoint procedure from the upper and lower sums previously studied in calculus, although right-endpoint sums are also Riemann sums. This distinction provides evidence of how students interpreted and adapted the prior mathematical knowledge. *Fig. 15* shows the various calculations and arguments used to determine the period and the first Fourier coefficient, taking into account the restrictions associated with the artifacts in use.

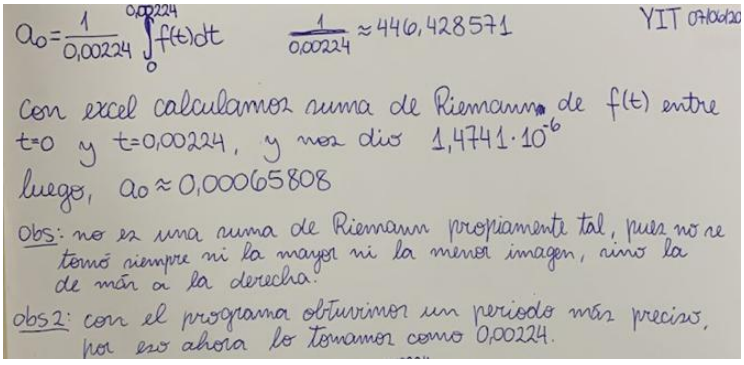


Fig. 15 G6's explanation of its processes for determining Fourier coefficients and calculating the trigonometric polynomial. [**English translation:** Using Excel, we calculated a Riemann sum of *f(t)* between *t=0* and *t=0.002241*, which gave 1.1741×10^{-6}. Then, $a_0 \approx 0.000065808$. Note: It is not strictly a Riemann sum, since we did not consistently take either the maximum or minimum function value, but rather the value at the right endpoint. Note 2: Using the software, we obtained a more precise period, so we now take it as 0.002241]

From the perspective of mathematical work, this stage makes visible an [Ins–Dis] activation, since the spreadsheet became an artifact for coefficient calculation through the coordination of software use and theoretical knowledge about integrals.

After calculating the first Fourier coefficients in Excel, G6 entered them into GeoGebra, constructed the corresponding trigonometric polynomial, and compared the resulting graph with the waveform previously observed in Audacity. The group then refined the scale of the axes and validated the model visually by confronting both curves, even superimposing the Audacity graph on the GeoGebra screen (see *Fig. 16*). This validation process led the students to recognize that the polynomial already provided a good approximation, but also that the model could be improved by calculating additional coefficients.

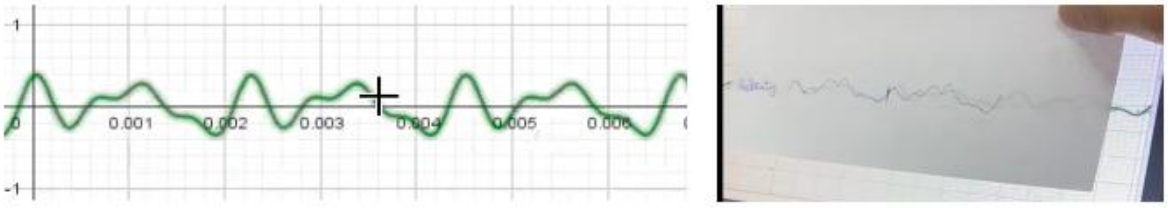


Fig. 16 A trigonometric polynomial plotted in GeoGebra by G6 and subsequently validated by superimposing it with the waveform in Audacity

In this way, Stage 4 marks the most advanced point in G6's trajectory: the group not only constructed a Fourier-based model for a complex sound, but also interpreted the quality of that model in terms of approximation and convergence. For this reason, the mathematical work remained predominantly within A2 while displaying emerging features associated with A3. It is here that the conceptualization of periodicity becomes most visible, since Fourier theory is no longer treated as a formal procedure alone, but as a mobilizable means of describing and refining the model of a real periodic phenomenon. *Fig. 17* shows the reorganization of mathematical work in the final stage.

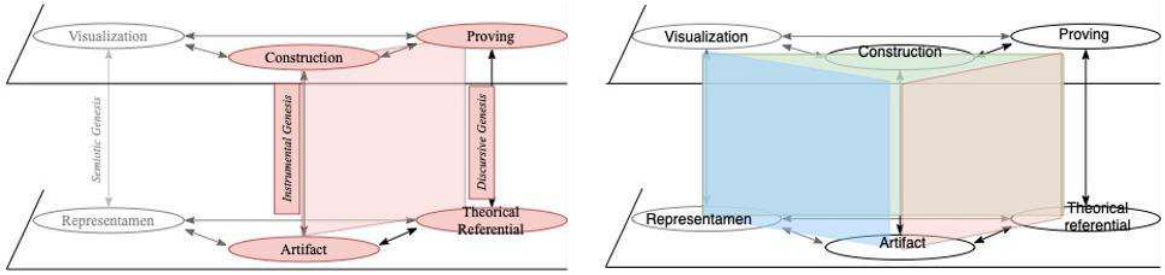


Fig. 17 Path of different genesis and planes activated by G6 in the didactic situation 4

5. Discussion

The case of G6 supports the central claim of this study: the conceptualization of periodicity is better understood through the progressive evolution of mathematical work across a

sequence than through isolated task performance. In G6, this development became visible through the reorganization of previously mediated work and through changes in the epistemological orientation of that work. The broader implementation also showed differentiated trajectories: while some groups established and later mobilized new connections from their personal MWS, others mainly broadened their understanding of notions such as frequency, period, and amplitude. These differences reinforce the value of examining conceptualization longitudinally rather than inferring it from success in a single task.

The trajectory observed in G6 also contributes to research on the teaching of trigonometric and periodic functions. Previous studies have documented difficulties associated with approaches rooted in angles, radians, and the unit circle (Loeng et al., 2023; Moore, 2013; Tanguay, 2010), as well as forms of algebraic work that may obscure periodicity as a mathematical property (Buendía, 2006). In the present study, sound provided a different entry: trigonometric functions were progressively reinterpreted through waveform, period, frequency, and amplitude; function addition acquired meaning through sound superposition; and trigonometric polynomials and Fourier-based approximations extended this work toward more complex periodic phenomena. The role of GeoGebra in coordinating graphical and algebraic representations is also consistent with research highlighting its potential for work on periodicity (Kepceoğlu & Yavuz, 2016). These results do not replace geometric approaches to trigonometry, but show how a sound-based sequence can complement them by making periodicity itself an organizing idea.

This development did not result from task sequencing alone. Audacity, GeoGebra, and Excel became progressively coordinated with signs and discourse, while teacher and peer mediations enabled students to reorganize mathematical work that they could not initially complete autonomously. This was particularly visible in the identification of harmonic amplitude regularities and in the calculation of Fourier coefficients from exported sound data. From an AT perspective, these episodes illustrate how mediation within the ZPD can support the appropriation of new forms of reasoning (Robert, 2008; Vandebrouck, 2018). From the MWS perspective, the same episodes become observable through the circulation among semiotic, instrumental, and discursive geneses and their vertical planes. The networking of the two frameworks therefore makes it possible to relate the observable form of mathematical work to the developmental processes through which that work is reorganized.

The case of G6 consequently refines the notion of conceptualized mathematical work. CMW should not be identified with the occurrence of complete mathematical work in an isolated task; rather, it refers to a trajectory in which the components of the personal MWS are progressively reorganized and connections established in one situation become available in subsequent work. In G6, this was visible in the progression from trigonometric functions as descriptions of pure sounds, to function addition and harmonic structure, and finally to Fourier-based approximation and refinement. This interpretation complements approaches that study trigonometric functions through modelling contexts (Reyes-Avendaño, 2020) while adding a longitudinal analytical lens centered on the development of mathematical work. Conceptualization thus becomes empirically accessible through the coordinated evolution of signs, artifacts, discourse, mediations, and epistemological orientations across the sequence.

6. Conclusions

This study further develops the notion of conceptualized mathematical work through the networking of MWS and AT. CMW makes it possible to approach conceptualization not as an outcome inferred from isolated task performance, but as a longitudinal process in which mathematical work is progressively reorganized through the coordination of signs, artifacts, discourse, and mediation across a didactic sequence.

The analysis of G6 showed how this process developed from perceptually grounded work on sound toward increasingly articulated mathematical work involving periodicity, function addition, trigonometric polynomials, and Fourier-based approximation. The trajectory remained predominantly within A1 and A2 forms of work, while the final situation displayed emerging features associated with A3 through approximation, refinement, and convergence. These developments were supported by the progressive coordination of Audacity, GeoGebra, and Excel with semiotic, instrumental, and discursive processes.

The study also shows that sound can provide a productive entry into periodicity and trigonometric functions without making angles and radians the initial basis for meaning-making. In this context, periodicity became progressively connected to time, frequency, amplitude, sound superposition, and approximation, allowing mathematical relations to acquire meaning through the study of a recurrent phenomenon.

At the same time, the differentiated trajectories observed across the six groups indicate that conceptualization cannot be attributed to task design alone. It also depends on the mathematical resources available to students, the mediations that occur during the sequence, and the extent to which connections established in one situation are subsequently mobilized and reorganized.

Overall, the study contributes both theoretically and empirically to mathematics education research. Theoretically, it proposes conceptualized mathematical work as a useful lens for studying conceptualization processes in complex mathematical domains. Empirically, it shows that sound-based, technology-mediated didactic sequences can support deep forms of mathematical work, while also revealing differentiated trajectories of conceptualization. These findings have implications for the design of teacher education experiences aimed at fostering integrative understandings of mathematical objects such as periodicity.
A further line of research would be to implement the sequence with other target populations, such as engineering students or musicians, to explore how conceptualized mathematical work emerge in interdisciplinary environments.

Statements and Declarations

Funding The doctoral studies were funded through a doctoral contract awarded by Université Paris Cité, France and a tuition scholarship from Pontificia Universidad Católica de Valparaíso, Chile.

Ethics approval The research was conducted with institutional authorization from Université Paris Cité, Paris, France and Pontificia Universidad Católica de Valparaíso, Chile. All participants were adults and provided written informed consent prior to participation.

Consent to participate All the participants were informed about the aim of the study and completed a consent form.

Data availability The data supporting the findings of this study are available from the corresponding author upon reasonable request. Access may be provided subject to the protection of participant confidentiality and the conditions established in the informed consent.

Conflict of interest Authors declare no competing interests.

Author contributions declaration
ACB: contributed to the writing of all sections of the manuscript, prepared the figures, and performed the final review of the manuscript.
EMD: contributed to the writing of Sections 1, 2.1, 3.1, 3.2, 3.3, 4, and 6.
LV contributed to the writing of Sections 1, 2.1, 3.2, 4, 5, and 6.
FV contributed to the writing of Sections 1, 2.2, 2.4, 3.3, and 4.
All authors contributed to the writing of Sections 1 and 4.
All authors contributed to the design of the didactic situation.
ACB carried out the implementation of the didactic situation.
All authors read and approved the final manuscript.

Reference

Artigue, M. (1988). Ingénierie didactique. *Recherches en Didactique des Mathématiques, 9*(3), 281–308.

Bartolini Bussi, M.G. & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. *In L. D. English (Ed.), Handbook of International Research in Mathematics Education, (2nd ed., pp. 746-783). Routledge* https://doi.org/10.4324/9780203930236.ch28

Bikner-Ashbahs, A., Knipping, C., & Presmeg, N. (Eds.). (2015). *Approaches to qualitative research in mathematics education: Examples of methodology and methods*. Springer. https://doi.org/10.1007/978-94-017-9181-6

Buendía, G. (2006). Una socioepistemología del aspecto periódico de las funciones. *Revista Latinoamericana de Investigación en Matemática Educativa*, *9*(2), 227–251.

Cabrera Baquedano, A. (2024). *Modeling periodic phenomena integrating digital technologies in superior education: A study in France and Chile* [Doctoral thesis, Université Paris Cité, in cotutelle with Pontificia Universidad Católica de Valparaíso]. HAL Thèses https://theses.hal.science/tel-05268526

Demir, Ö., & Heck, A. (2013). A new learning trajectory for trigonometric functions. In E. Faggiano & A. Montone (Eds.), *Proceedings of the 11th International Conference on Technology in Mathematics Teaching*, (pp. 119–124). University of Bari.

Kepceoğlu, I., & Yavuz, I. (2016). Teaching a concept with GeoGebra: Periodicity of trigonometric functions. *Educational Research and Reviews*, *11*(8), 573–581. https://doi.org/10.5897/ERR2016.2701

Kuzniak, A. y Richard, P. (2014). Espacios de trabajo matemático. Puntos de vista y perspectivas. *Revista Latinoamericana de Investigación en Matemática Educativa, 17*(4),5–15. https://doi.org/10.12802/relime.13.1741a

Kuzniak, A., Tanguay, D., and Elia, I. (2016). Mathematical Working Spaces in schooling: an introduction. *ZDM Mathematics Education, 48(*6), 721–737. https://doi.org/10.1007/s11858-016-0812-x

Kuzniak, A., Montoya Delgadillo, E., & Richard, P. (Eds.). (2022). *Mathematical Work in Educational Context: The perspective of the Theory of Mathematical Working Spaces.*Springer. https://doi.org/10.1007/978-3-030-90850-8

Leontiev, A. (1984). *Activité, conscience, personnalité*. Éditions du Progrès. (Original work published 1975)

Loeng, R. (2019). *Les fonctions sinus et cosinus dans le secondaire en France et au Cambodge*. [Doctoral thesis, Université de Paris]. HAL Thèses. https://theses.hal.science/tel-02983421

Loeng, R., Reyes Avendaño, C. G. & Vivier, L. (2023). La introducción de las funciones trigonométricas circulares, ¿por qué elegir un acercamiento a partir de los ángulos? In C. A. Cuevas-Vallejo, M. Martínez Reyes y J. A. Hernández-Sanchez (Coords.), *Investigaciones y experiencias en enseñanza de las ciencias y la matemática*. UAEM, 327-340. http://hdl.handle.net/20.500.11799/138309

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, *83*(2), 225–245. https://doi.org/10.1007/s10649-012-9450-6

Montoya-Delgadillo, E., & Vivier, L. (2016). Mathematical Working Spaces and paradigms as an analysis tool for the teaching and learning of analysis. *ZDM Mathematics Education, 48*(6), 739–754. https://doi.org/10.1007/s11858-016-0777-9

Prediger, S. & Bikner-Ahsbahs, A. (2014) Introduction to networking: networking strategies and their background. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of Theories as a Research Practice in Mathematics Education* (pp. 117–125). Springer. https://doi.org/10.1007/978-3-319-05389-9_8

Reyes Avendaño, C. G. (2020). Enseignement et apprentissage des fonctions numériques dans un contexte de modélisation et de travail mathématique. [PhD thesis, Université de Paris]. HAL Thèses. https://theses.hal.science/tel-03211997

Robert, A. (2008). La double approche didactique et ergonomique pour l'analyse des pratiques d'enseignants de mathématiques et une méthodologie pour analyser les activités (possibles) des élèves en classe. In F. Vandebrouck (Éd.), *La classe de mathématiques: Activité des élèves et pratiques des enseignants* (pp. 45-68). Octarès.

Tanguay, D. (2010). Degrés, radians, arcs et sinusoïdes. *Petit x*, *82*, 59–71.

Tuna, A. (2013). A Conceptual Analysis of the Knowledge of Prospective Mathematics Teachers about Degree and Radian. *World Journal of Education*, *3*(4), 1–9. https://doi.org/10.5430/wje.v3n4p1

Vandebrouck, F. (2018). Activity Theory in French didactic research. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt & B. Xu (Eds.), *Invited lectures from the 13th International Congress on Mathematical Education* (pp. 679-698). Springer. https://doi.org/10.1007/978-3-319-72170-5_38

Vergnaud, G. (1991). Langage et pensée dans l'apprentissage des mathématiques. *Revue française de pédagogie*, *96*, 79–86. https://doi.org/10.3406/rfp.1991.1350

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes.* Harvard University Press. https://doi.org/10.2307/j.ctvjf9vz4

Vygotski, L. (1985). *Pensée et langage*. (F. Sève, Trans.) Messidor/Éditions sociales. (Original work published 1934)

Winsløw, C. (2016). Angles, trigonometric functions, and university level Analysis. In E. Nardi, C. Winsløw, & T. Hausberger (Eds), *Proceedings of the First conference of*

International Network for Didactic Research in University Mathematics (pp. 163–172). Université de Montpellier.

Yin, R. K. (2009). *Case study research. Design and methods* (4th ed). SAGE.