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Dust-ion-acoustic Flat-top Solitons and Supersolitons in dusty plasma with two superthermal electrons: Application to the Earth's Ionosphere

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Abstract The nonlinear propagation of dust-ion-acoustic (DIA) structures is investigated in a homogeneous, unmagnetized dusty plasma containing positive cold ions, negatively charged cold dust grains, and cold and hot superthermal electrons. The Sagdeev pseudopotential approach is employed to investigate the existence and characteristics of nonlinear electrostatic structures, including compressive and rarefactive solitary waves, double layers, flat-top solitons, and supersolitons. The negatively charged dust significantly modifies the equilibrium charge balance and nonlinear response, allowing the existence of both positive- and negative-potential supersolitons, whereas the corresponding dust-free system supports only positive-potential supersolitons in the plasma. The transition from ordinary solitons to flat-top solitons and supersolitons is associated with changes in the Sagdeev potential and the emergence of complex, wiggly-like electric-field profiles. The propagation domains are strongly affected by electron superthermality and the relative densities of the cold and hot electron populations. Increasing the spectral index generally shifts the admissible Mach-number limits toward higher values and, as the electron distributions approach the Maxwellian limit, modifies and broadens the corresponding propagation domains. The dust-free limit recovers previously reported results, confirming the consistency of the model. Furthermore, the coexistence of compressive double layers with rarefactive solitons is identified, with their distinct phase-space trajectories reflecting the difference between homoclinic and separatrix-connected nonlinear states. The results demonstrate that negatively charged dust and non-Maxwellian electrons play a crucial role in determining the polarity and nonlinear characteristics of DIA excitations in the Earth's ionosphere dusty plasmas.

Keywords: Sagdeev pseudopotential, Supersolitons, Flat-top soliton, double layers, Superthermal electrons, dusty plasmas.

Introduction

The study of nonlinear waves in dusty plasmas has attracted considerable attention in recent years. Dusty plasmas consist of micron-sized charged dust grains together with electrons and ions and are commonly found in both space environments (e.g., Saturn's rings and cometary tails) and laboratory plasmas¹⁻². In most dusty plasma models, dust grains acquire a negative charge by collecting electrons. However, under certain conditions, smaller dust grains may become positively charged through photoionization or secondary electron emission processes³⁻⁵. Owing to their large mass and consequently small charge-to-mass ratio, the dynamics of dust grains dominate the low-frequency regime below the ion-acoustic frequency, leading to the emergence of a new wave mode known as the dust acoustic waves (DAWs). This mode was first theoretically predicted by Rao et al.⁶ and later experimentally confirmed by Barkan et al.⁷. In DAWs, the dust grains provide the inertia, whereas the restoring force is supplied by the pressure of the mobile electrons and/or ions.

Small-amplitude dust acoustic solitary waves are commonly investigated using the reductive perturbation method, which leads to the nonlinear Korteweg-de Vries (K-dV) equation⁸. However, this approach is valid only for weakly nonlinear and weakly dispersive waves. Consequently, although the reductive perturbation method is mathematically elegant, it cannot accurately describe strongly super-acoustic or large-amplitude solitary waves. The Sagdeev pseudopotential technique⁹ is another fundamental approach for investigating nonlinear plasma waves. Unlike perturbative methods, it provides an exact (non-perturbative) reduction of the plasma fluid equations into a pseudo-energy balance equation. The existence, polarity, and characteristics of nonlinear structures are determined directly by the topology of the Sagdeev pseudopotential.

In addition to large-amplitude solitary waves, several other nonlinear structures, including double layers (DLs), supersolitons, and flat-top solitons, have been identified over the years. Double layers were first reported as a distinct plasma phenomenon by Temerin et al.¹⁰ and Hasegawa and Sato¹¹. A double layer is a localized, collisionless plasma structure consisting of two adjacent regions of opposite space charge separated by a narrow transition layer. This configuration produces a strong localized electric field and a sharp potential drop across the

layer, while the electric field outside the structure remains relatively weak. Although quasi-neutrality is locally violated within the double layer, it is preserved over larger spatial scales. The existence of solitary waves and double layers in various plasma systems has been extensively investigated using the Sagdeev pseudopotential method¹²⁻¹⁵. In recent years, another class of nonlinear structures, known as supersolitons, has attracted considerable attention. Although supersolitons were present in several earlier theoretical investigations¹⁶⁻²⁰, they were not initially recognized or interpreted as a distinct type of nonlinear excitation.

The concept of supersolitons in multispecies plasmas was first introduced by Dubinov and Kolotkov²¹. Supersolitons are a special class of large-amplitude solitary waves characterized by the presence of internal substructures enclosed within the main solitary profile. They investigated these nonlinear structures in a complex plasma consisting of electrons, positrons, positive and negative ions, and immobile charged dust grains. Their analysis showed that the Sagdeev pseudopotential exhibits a novel topology with three local minima and two local maxima, where the second maximum is lower than the first. Such a pseudopotential configuration constitutes the necessary condition for the existence of supersolitons. Subsequently, Dubinov and Sazonkin²² derived an exact analytical solution for nonlinear ion-acoustic waves in a dusty electron-ion plasma by solving the continuity, momentum, and Poisson equations under the assumptions of negatively charged dust grains and Boltzmann-distributed electrons. Their solution revealed three distinct families of nonlinear waves: ordinary nonlinear periodic waves, compression and rarefaction solitary waves, and super nonlinear waves whose phase trajectories enclose one or more separatrices in the phase space. It has also been established that the existence of supersolitons in the Sagdeev pseudopotential framework requires a plasma composed of at least three distinct particle species²³.

The first numerical simulation of supersolitons in an electron-ion plasma having cold ions with two electron species cold and hot obeying kappa distributions as reported by Kakad et al.²⁴. Maharaj et al.²⁵ investigated nonlinear dust acoustic structures, with particular emphasis on supersolitons, in a dusty plasma composed of cold negatively charged dust, adiabatic positively charged dust, Boltzmann electrons, and nonthermal ions described by the Cairns distribution. Their study focused on identifying the conditions for the existence of supersolitons and analyzing their nonlinear properties. Unlike ordinary solitary waves, supersolitons are

distinguished by the presence of subsidiary potential wells in the Sagdeev pseudopotential, which give rise to characteristic oscillatory "wiggles" in the corresponding electric field profile. Owing to these unique properties, supersolitons have become an important topic in the study of nonlinear plasma waves. Verheest et al.²⁶ examined dust-ion-acoustic (DIA) supersolitons in a plasma consisting of immobile negatively charged dust grains, cold protons, and nonthermal Cairns-distributed electrons using the Sagdeev pseudopotential method. They demonstrated that the existence of supersolitons is restricted to a finite range of Mach numbers and amplitudes, requires at least three plasma species, and is strongly influenced by both the nonthermal electron population and the dust concentration. El Wakil et al.²⁷ investigated large-amplitude dust acoustic supersolitons in a four-component dusty plasma composed of positively and negatively charged dust grains, together with nonextensive electrons and ions. They reported that the supersoliton existence region expands with increasing nonextensivity (corresponding to stronger superthermal tails) and contracts as the dust temperature ratio increases. More recently, the arbitrary-amplitude dynamics of dust-ion-acoustic waves in an unmagnetized multicomponent plasma containing an ion beam, positive and negative ions, stationary negatively charged dust grains, and electrons obeying a hybrid nonthermal-Tsallis distribution were investigated²⁸. The study demonstrated that the presence of heavy, immobile dust grains plays a crucial role in the formation of supersolitons, as these grains act solely as a stationary background charge that maintains overall charge neutrality without participating dynamically in the wave motion.

Several investigations^{21-23, 29-33} have demonstrated that the formation of supersolitons strongly depends on the plasma composition and the velocity distribution of the charged particles. In an unmagnetized dusty plasma consisting of immobile dust grains, adiabatic ions, isothermal positrons, and Cairns nonthermal electrons, it has been shown that, in the absence of positrons, the plasma supports rarefactive double layers and rarefactive supersolitons propagating at the acoustic speed. Furthermore, as the electron population becomes more nonthermal (i.e., the nonthermal parameter decreases), the supersolitons eventually disappear and the nonlinear structures transform back into ordinary rarefactive solitons²⁹. The influence of magnetic fields and additional plasma species has also been explored. In a collisionless magnetized plasma composed of immobile negatively charged dust grains, warm adiabatic ions, non-isothermal positrons, and two electron populations (one isothermal and the other following the Cairns

nonthermal distribution), the large number of independent plasma parameters considerably enriches the nonlinear dynamics. Analysis based on the Sagdeev pseudopotential has revealed the existence of supersolitons that are absent in simpler magnetized four-component plasma models³⁰. It has also been emphasized that most theoretical studies on supersolitons rely on simplified plasma descriptions in which electron inertia is neglected. Consequently, the existence and properties of supersolitons in more realistic plasma models that retain the inertia of all charged species remain an open research problem and require further investigation³¹.

Dust-acoustic supersolitons have also been reported in a three-component dusty plasma consisting of cold negatively charged dust grains and two populations of positive ions obeying kappa-distributions, with electrons completely absent³². The Sagdeev pseudopotential analysis showed that the formation of these structures is governed by a delicate interplay between the ion temperature ratio and the relative concentration of the cool ion population. Supersolitons appear only when the pseudopotential develops multiple potential wells, producing the characteristic oscillatory ("wiggle") electric-field profile. Even slight variations in the plasma parameters remove these additional wells, causing the supersolitons to evolve into conventional dust-acoustic solitary waves. The existence of double layers and supersolitons has also been investigated in cometary dusty plasmas using a six-component plasma model consisting of cold oxygen ions, hot hydrogen ions, negatively charged dust grains, hot solar electrons, and cold cometary electrons³³. Such studies indicate that complex plasma environments with multiple charged species provide favorable conditions for the emergence of rich nonlinear wave structures.

In many space and astrophysical plasmas, including cometary environments^{34, 35}, the Martian ionosphere^{36, 38, 44}, and Saturn's magnetosphere³⁷, particle velocity distributions frequently exhibit energetic high-energy tails produced by solar-wind interactions, particle acceleration, and other nonequilibrium processes. Under these conditions, the Maxwellian distribution is often inadequate for accurately describing the electron population. Instead, the kappa-distribution provides a more realistic representation by accounting for the enhanced number of superthermal particles through the spectral index (k), approaching the Maxwellian distribution only in the limit $k \rightarrow \infty$. Incorporating the kappa-distribution into plasma models significantly improves the description of pressure balance and charge neutrality, leading to more reliable predictions of

Mach number limits, double-layer formation, and the characteristics of solitary waves and supersolitons observed in space plasmas.

Another class of nonlinear electrostatic structures, namely flat-top solitons, has also attracted considerable attention in plasma physics. These structures were first reported during the study of arbitrary-amplitude quantum dust-ion-acoustic (QDIA) waves in a one-dimensional quantum hydrodynamic model consisting of electrons, ions, and immobile dust grains. It was found that a conventional rarefactive solitary wave gradually evolved into a broad, flat-bottomed (or table-top) structure when the quantum effects became sufficiently strong and the equilibrium ion density exceeded the electron density³⁹. Subsequent studies identified these structures as flat-top solitons, also referred to as Flat-Top Solitary Waves (FTSWs). They represent an intermediate nonlinear state that appears during the transition from ordinary solitary waves to supersolitons in the Sagdeev pseudopotential framework. Unlike the smooth bell-shaped profile of a conventional soliton, a flat-top soliton exhibits a broad region where the electrostatic potential remains nearly constant, giving the wave a characteristic plateau-like shape. From the viewpoint of the Sagdeev pseudopotential, an FTSW satisfies the standard solitary-wave existence conditions at the equilibrium point while maintaining a weak but finite charge imbalance near its peak, which is responsible for the formation of the nearly flat potential core⁴⁰.

The existence of flat-top solitons is restricted to a very narrow range of propagation velocities. In particular, they are formed only when the Mach number is slightly lower than the critical value corresponding to the associated double layer or triple-root solution. The difference between these two velocities is extremely small, typically of the order of 10^{-5} or even smaller, indicating that flat-top solitons occur only within a very limited parameter region⁴¹. The evolution of conventional solitary waves into flat-top solitons has been reported in several multicomponent ion plasma models^{36, 37}. However, to the best of our knowledge, a similar transition has not yet been systematically investigated in dusty plasma systems. This gap provides additional motivation for examining the conditions under which flat-top solitons may arise in dusty plasmas and their relationship with other nonlinear structures, such as double layers and supersolitons.

The remainder of this paper is organized as follows. The basic governing equations for the ion and dust grains with two superthermal electron populations with their usual meaning is written.

In the section contains the derivation of the corresponding Sagdeev pseudopotential and energy balance equation. It is also contains the existence conditions of the various nonlinear structures supported by the dusty plasma system. In next section, we present the main results and discussion of the theoretical study. It also examines the influence of the relevant plasma parameters and highlights the limitations of the present analysis. Finally, in the last section summarizes the main findings and presents the conclusions of this study.

Basic Governing Equations

In the present investigation, we consider an unmagnetized, collisionless, and homogeneous generalized dusty plasmas system consisting of cold and hot superthermal electrons with different temperatures T_{ec} and T_{eh} , respectively, along with positive cold ions and negatively charged cold dust grains. Such plasma systems are particularly relevant to space plasma environments, where dust grains and non-Maxwellian electrons can significantly influence plasma dynamics and wave propagation. The negatively charged dust grains, which are generally much heavier than ions and electrons, carry a substantial negative charge and consequently play an important role in determining the stability and nonlinear wave characteristics of the plasma. The positive ions primarily provide the inertial response of the plasma, whereas the superthermal electrons, characterized by an enhanced high-energy tail, exhibit a modified response to the electrostatic potential. The one-dimensional wave propagation in such a dusty plasma system is governed by the following normalized basic equations:

$$\partial_t n_i + \partial_x(n_i v_i) = 0 \quad (1)$$

$$\partial_t v_i + v_i \partial_x(v_i) = -\partial_x \varphi \quad (2)$$

$$\partial_t n_d + \partial_x(n_d v_d) = 0 \quad (3)$$

$$\partial_t v_d + v_d \partial_x(v_d) = \left(\frac{z_d}{\mu_d}\right) \partial_x \varphi \quad (4)$$

The Poisson equation governing the plasma system under consideration is given by

$$\partial_{xx} \varphi = n_{ec} + n_{eh} - n_i(1 + \alpha z_d) + \alpha z_d n_d \quad (5)$$

The electron densities are assumed to follow a non-Maxwellian distribution, deviating from the standard Maxwell–Boltzmann form and being described by a kappa distribution function.

$$n_{ec} = f \left(1 - \frac{\beta \varphi}{(k_c - 3/2)} \right)^{-k_c + \frac{1}{2}} \quad (6)$$

$$n_{eh} = (1 - f) \left(1 - \frac{\varphi}{(k_h - 3/2)} \right)^{-k_h + \frac{1}{2}} \quad (7)$$

Here, the number densities and fluid velocities of the ions and negatively charged dust grains are denoted by n_i , n_d , v_i and v_d , respectively. The ions are assumed singly charged and cold. The normalized equations (1)–(5) are obtained by normalizing all the densities with their respective equilibrium densities n_{i0} , n_{d0} and n_0 , respectively. Where n_0 is the total equilibrium electron density, i.e., $n_0 = n_{ec0} + n_{eh0}$. The electrostatic potential φ is normalized by $k_B T_e / e$. The spatial coordinate is normalized by the electron Debye length λ_{de} , while the fluid velocities are normalized by the ion-acoustic speed, C_s . The time variable is normalized by the inverse ion plasma frequency ω_i^{-1} . The charge neutrality condition is given by $\frac{n_{i0}}{n_0} = 1 + \alpha z_d$, where α denotes the equilibrium dust-to-electron density ratio ($\alpha = \frac{n_{d0}}{n_0}$) and z_d denotes the dust grain charge number expressed in units of the elementary charge e . The dust grains are assumed to carry a negative charge of magnitude $Q_d = |ez_d|$. The dust-to-ion mass ratio ($\mu_d = \frac{m_d}{m_i}$) and the hot to cold electron temperature ratio $\beta = \left(\frac{T_{eh}}{T_{ec}} \right)$, are the relevant dimensionless parameters. Furthermore, the parameters $f = n_{ec0}/n_0$ and $1 - f = n_{eh0}/n_{e0}$ represent the fractional equilibrium number densities of the cold and hot electron populations, respectively. Here, $\lambda_{de} = (\epsilon_0 k T_e / n_0 e^2)^{1/2}$, $C_s = (K T_e / m_i)^{1/2}$ and $\omega_i^{-1} = (\epsilon_0 m_i / n_0 e^2)^{1/2}$.

Derivation of the Sagdeev Pseudopotential-

For the existence of localized wave structures, we employ the Sagdeev pseudopotential method by introducing a reference frame moving with the velocity of the solitary wave. In this moving frame, the space and time variables are combined into a single independent variable, generally expressed as $\eta = x - Mt$, where M is the normalized wave speed, which is referred to as the Mach number. All the plasma variables approach their respective unperturbed equilibrium values as $|\eta| \rightarrow \infty$. Under this transformation, the derivatives with respect to space and time become $\frac{\partial}{\partial x} = \frac{d}{d\eta}$ and $\frac{\partial}{\partial t} = -M \frac{d}{d\eta}$, respectively. Consequently, the governing plasma equations are

transformed into a set of ordinary differential equations describing the spatial variations of the plasma variables across the wave structure.

To obtain stationary wave solution of Eqs. (1)–(5), the ion and dust fluid equations are integrated using the appropriate boundary conditions, namely $n_1(\varphi) \rightarrow 1$, $n_2(\varphi) \rightarrow 1$, $v_i \rightarrow 0$, $v_d \rightarrow 0$, $\varphi \rightarrow 0$ and $\frac{d\varphi}{d\eta} \rightarrow 0$, as $|\eta| \rightarrow \infty$. Using these conditions, the expressions for the ion and dust number densities can be obtained as follows:

$$n_i = \frac{M}{\sqrt{(M^2 - 2\varphi)}} \quad (8)$$

$$n_d = \frac{M}{\sqrt{\left(M^2 + \frac{2Z_d\varphi}{\mu_d}\right)}} \quad (9)$$

Further, Poisson's Eq. (5) can be combined with Eqs. (6)–(9) and integrated using the appropriate boundary conditions for a localized solution. This yields an energy-like integral that can be expressed in the following form:

$$\frac{1}{2} \left(\frac{d\varphi}{d\eta} \right)^2 + S(\varphi) = 0 \quad (10)$$

The Sagdeev pseudopotential $S(\varphi)$ is then obtained in the following form

$$S(\varphi) = M(1 + z_d\alpha)(M - \sqrt{M^2 - 2\varphi}) + M\alpha\mu_d \left(M - \sqrt{M^2 + \frac{2Z_d\varphi}{\mu_d}} \right) - (1 - f) \left(\left(1 - \frac{\phi}{(k_h - \frac{3}{2})} \right)^{\left(\frac{3}{2} - k_h \right)} - 1 \right) - \frac{f}{\beta} \left(\left(1 - \frac{\phi\beta}{(k_c - \frac{3}{2})} \right)^{\left(\frac{3}{2} - k_c \right)} - 1 \right) \quad (11)$$

For the existence of localized nonlinear wave structures, the energy-balance equation requires the Sagdeev pseudopotential $S(\varphi)$ to be negative over the corresponding range of the electrostatic potential φ . Therefore, the region in which $S(\varphi) < 0$ is examined to determine the admissible domain for localized solutions. Here it can be noticed that the Sagdeev potential satisfy the basic requirement of the nonlinear structures $S(\varphi = 0) = S'(\varphi = 0) = 0$. Here the prime denotes the first derivative of the Sagdeev potential with respect to electrostatic potential φ . The existence of solitary waves can be established by satisfying certain conditions associated

with the Sagdeev pseudopotential $S(\varphi)$. For a rarefactive solitary wave, the electrostatic potential satisfies $\varphi_m < 0$ at the maximum amplitude, where φ_m denotes the amplitude of the solitary wave. In this case, the Sagdeev pseudopotential must satisfy $S(\varphi_m) = 0$, with $S(\varphi) < 0$ for $\varphi_m < \varphi < 0$. Similarly, for a compressive solitary wave, $\varphi_m > 0$, and the corresponding conditions are $S(\varphi_m) = 0$ and $S(\varphi) < 0$ for $0 < \varphi < \varphi_m$. The condition $S''(\varphi) < 0$ ensures that the equilibrium point at $\varphi = 0$ is unstable and permits the existence of solitary-wave solutions³⁶⁻³⁸.

The existence of nonlinear localized structures such as double layers (DLs), supersolitons, and flat-top solitary waves (FTSWs) requires additional criteria beyond the standard conditions for conventional solitary waves discussed above. These criteria are determined by the detailed topology of the Sagdeev pseudopotential and its derivatives. For a double layers (DLs), the pseudopotential must possess a second-order unstable equilibrium at a finite potential $\varphi = \varphi_D$. Thus, in addition to the usual equilibrium condition at $\varphi = 0$, the DL amplitude must satisfy $S(\varphi_D) = S'(\varphi_D) = 0$ and $S''(\varphi) < 0$ ^{38, 43}. These conditions indicate that the pseudopotential returns to zero at a finite potential while maintaining an unstable equilibrium there. The corresponding electric potential therefore approaches two different asymptotic states, which is the characteristic feature of a double layer.

Supersolitons represent a more complicated phase-space structure than ordinary solitary waves. Their identification requires the Sagdeev pseudopotential to possess an additional internal structure, usually manifested as a secondary negative well or a pronounced wiggle within the main potential well. In addition to satisfying the basic solitary-wave conditions, the pseudopotential must therefore exhibit this additional internal extremum. The associated phase-space trajectory contains additional critical points and consequently encloses the structures associated with the conventional solitary-wave and double-layer regimes. A special class of localized structures, known as flat-top solitary waves (FTSWs), can occur in the parameter regime connecting double layers and supersolitons. The term *flat-top* arises from the characteristic broad and nearly constant-amplitude region observed in the electrostatic potential profile. Unlike an ideal double layer, the pseudopotential does not exactly satisfy the double-layer conditions at the corresponding finite potential. Instead, it develops an extended, nearly flat region around the turning point. This behavior can be characterized by $S(\varphi = \varphi_{FT}) = 0, S'(\varphi =$

$\varphi_{FT}) = \epsilon, S''(\varphi = \varphi_{FT}) = \delta$, where δ & $\epsilon \neq 0$. Where φ_{FT} denotes the potential associated with the flat region^{36, 37}. Thus, $S(\varphi_{FT})$ and $S'(\varphi_{FT})$ are small but finite rather than exactly zero. Consequently, the potential remains nearly constant over an appreciable spatial interval, producing the characteristic flat-top profile.

Finally, the inequality given by $S''(\varphi) \leq 0$, together with the equilibrium requirements of the Sagdeev pseudopotential, determines the lower bound of the Mach number for the formation of nonlinear structures. Accordingly, the minimum allowable Mach number can be expressed as $M > M_0$. This threshold defines the lower boundary of the parameter regime in which localized nonlinear structures can exist and obtained as follows.

$$M_0^2 \geq \frac{(1+\alpha z_d)\mu_d + \alpha z_d^2}{\mu_d \left[f\beta \left(\frac{2k_c-1}{2k_c-3} \right) + (1-f) \left(\frac{2k_h-1}{2k_h-3} \right) \right]} \quad (12)$$

From Eq. (12), the lower limit of the Mach number, M_0 , is obtained, which corresponds to the acoustic speed of dust-ion-acoustic wave propagation. In the limit of $\alpha = 0$, the considered dusty plasma system reduces to a dust-free plasma, and Eq. (12) reduces to $M_0^2 \geq \left[\frac{f\beta(2k_c-1)}{(2k_c-3)} + \frac{(1-f)(2k_h-1)}{(2k_h-3)} \right]^{-1}$, which is identical to the expression obtained in Ref. (42) for a plasma containing two-temperature electrons. Furthermore, when the cold-electron contribution vanishes, i.e., $f = 0$, the lower Mach-number limit reduces to $M_0^2 \geq \frac{(2k_h-3)}{(2k_h-1)}$, corresponding to an electron-ion plasma with superthermal electrons. In the further limit $k \rightarrow \infty$, the superthermal electron distribution approaches the Maxwellian distribution, and the corresponding Mach-number expression becomes $M_0 \geq 1$. Consequently, the plasma model reduces to the familiar electron-ion plasma with Boltzmann electrons. These limiting cases demonstrate the consistency of the present formulation with previously established plasma models. In the present study, both the ion and dust fluids are assumed to propagate at speeds exceeding the acoustic speed. Accordingly, only the positive roots of Eq. (12) are physically relevant and are considered in the subsequent analysis.

The upper limit of the Mach number is determined by requiring the ion and dust densities, given by Eqs. (8) and (9), respectively, to remain real and physically admissible. These reality

conditions impose critical values on the electrostatic potential for the ion and dust species, which can be expressed as

$$\varphi_{ic} = \frac{M^2}{2} \quad (13)$$

$$\varphi_{dc} = -\frac{M^2 \mu_d}{2z_d} \quad (14)$$

Where φ_{ic} and φ_{dc} denote the critical electrostatic potential values associated with the ion and dust density constraints, respectively. Substituting these critical potential values into the Sagdeev pseudopotential Eq. (11) yields two corresponding upper limits for the Mach number. These are denoted by M_{ur} and M_{uc} , representing the upper Mach-number limits for the rarefactive and compressive structures, respectively. Since the critical potential associated with the dust density occurs on the negative-potential side, it determines the upper Mach-number limit for rarefactive solitary structures, M_{ur} . In contrast, the critical potential associated with the ion density occurs on the positive-potential side and therefore determines the upper Mach-number limit for compressive solitary structures, M_{uc} . Hence, the allowable Mach-number ranges for the two types of nonlinear structures are constrained by the corresponding reality conditions of the dust and ion densities. Therefore these two constraints equation is given as follows.

$$\begin{aligned} \frac{f}{\beta} \left[\left(1 - \frac{\beta M_{uc}^2}{(2k_c - 3)} \right)^{-k_c + 3/2} - 1 \right] + (1 - f) \left[\left(1 - \frac{M_{uc}^2}{(2k_h - 3)} \right)^{-k_h + 3/2} - 1 \right] = M_{uc}^2 \left[(1 + \alpha z_d) + \right. \\ \left. \alpha \mu_d \left(1 - \left(1 + Z_d / \mu_d \right)^{1/2} \right) \right] \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{f}{\beta} \left[\left(1 + \frac{\beta M_{ur}^2 \mu_d}{z_d (2k_c - 3)} \right)^{-k_c + 3/2} - 1 \right] + (1 - f) \left[\left(1 + \frac{M_{ur}^2 \mu_d}{z_d (2k_h - 3)} \right)^{-k_h + 3/2} - 1 \right] = M_{ur}^2 \left[\alpha \mu_d + \right. \\ \left. (1 + \alpha z_d) \left(1 - \left(1 + \mu_d / Z_d \right)^{1/2} \right) \right] \end{aligned} \quad (16)$$

Thus, the permissible Mach number ranges for the rarefactive and compressive nonlinear structures are $M_0 < M < M_{ur}$ and $M_0 < M < M_{uc}$, respectively. Here, M_0 denotes the lower

Mach-number limit (i.e., the acoustic speed), while M_{ur} and M_{uc} represent the upper Mach-number limits for the rarefactive and compressive solitary-waves, respectively.

Parametric Analysis

In the prescribed plasma model, we have considered a homogeneous, unmagnetized plasma consisting of positive cold ions, negatively charged cold dust grains, and two populations of superthermal electrons, namely cold and hot electrons. Various nonlinear structures, including solitons, double layers, flat-top solitons, and supersolitons of both compressive and rarefactive types, are investigated using the Sagdeev pseudopotential approach. As discussed earlier by Ref. (42), ion-acoustic compressive solitons and supersolitons can exist in a plasma containing two-temperature superthermal electrons. In the present study, negatively charged cold dust grains are additionally included, and we demonstrate that the dusty plasma system can support both compressive and rarefactive supersolitons as well as solitons. Several authors^{26, 32, 33, 37} have reported supersolitons and double layers of either compressive or rarefactive polarity in different plasma models. To the best of our knowledge, the existence of both compressive and rarefactive supersolitons, flat-top solitons, within the same dusty plasma system has not been reported previously and therefore represents a novel finding.

In addition to identifying these nonlinear structures, we have systematically examined the variation of their amplitudes, the transitions between different nonlinear modes, and the corresponding changes in the electrostatic potential, electric field, and phase-space trajectories with variations in the plasma parameters. The existence of supersolitons is established from the corresponding Sagdeev pseudopotential, $S(\varphi)$, as a function of the electrostatic potential φ . For a physically admissible solitary structure, the Sagdeev potential satisfies $S(0) = S'(0) = 0$ and $S''(0) < 0$, while the potential must remain in the appropriate allowed region until it reaches a finite-amplitude root, where $S(\varphi_m) = 0$. In the supersoliton regime, the Sagdeev pseudopotential possesses additional extrema inside the principal potential well. These additional extrema produce a characteristic substructure in the pseudopotential and distinguish a supersoliton from an ordinary solitary wave^{21, 40, 42}. Thus, the occurrence of a supersoliton is associated with a sequence of additional maxima and minima of $S(\varphi)$ between the equilibrium point $\varphi = 0$ and the finite-amplitude root φ_m , rather than merely with the presence of multiple roots of $S(\varphi)$. The

corresponding electric field^{40, 41}, $E = -\frac{d\phi}{d\eta}$, consequently exhibits a more complicated spatial structure than that of an ordinary soliton. Numerically, Eq. (12) shows that the curvature of the Sagdeev pseudopotential near the equilibrium point imposes a specific constraint on the admissible nonlinear structures. In the present plasma model, finite-amplitude positive and negative roots can occur for appropriate parameter ranges, corresponding respectively to compressive and rarefactive nonlinear structures. We find that both positive and negative finite-amplitude roots can occur beyond the corresponding compressive and rarefactive double-layer regimes. This demonstrates that the present dusty plasma system can support both positive- and negative-polarity dust-ion-acoustic supersolitons, in addition to the corresponding double layers.

The present results differs from the model considered in Ref. (42), where the parameter constraints permit only a positive-potential root in the relevant parameter regime, and consequently only positive-polarity supersolitons were obtained. The inclusion of negatively charged dust grains in the present plasma model modifies the charge balance and the nonlinear Sagdeev potential sufficiently to permit both polarities of supersolitons. Therefore, the existence of both compressive and rarefactive supersolitons is a distinctive feature of the present dusty plasma model. In contrast, earlier studies demonstrated either compressive supersolitons^{36, 37, 42} or rarefactive supersolitons^{21, 38, 43} in their respective plasma systems, which differs from the present case.

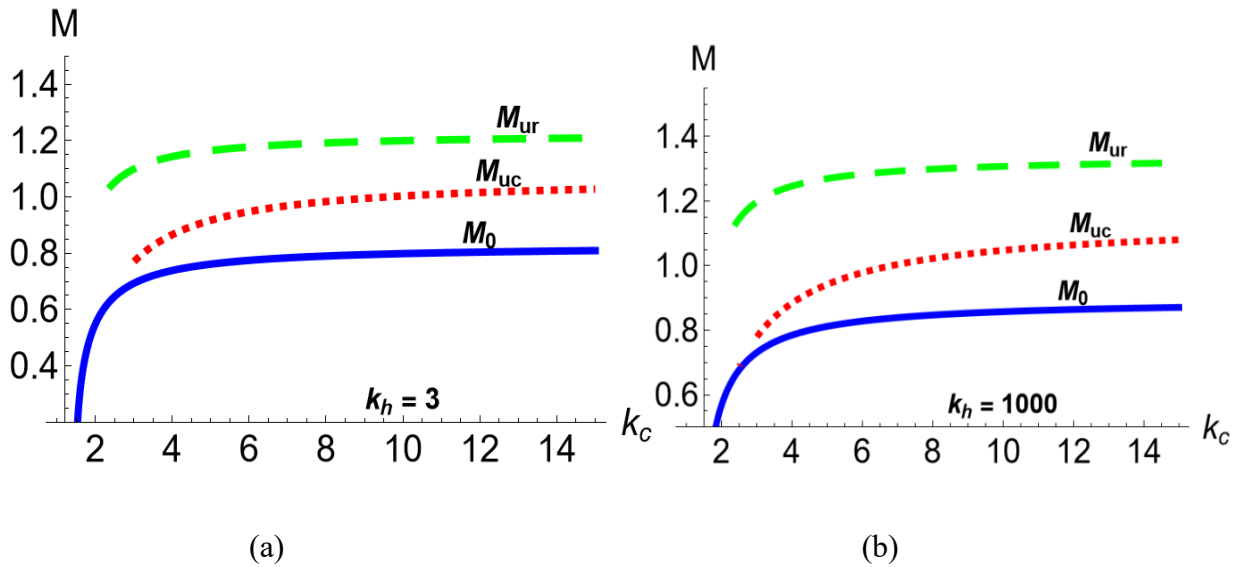


Fig. 1 (color online). Variation the Mach number, M , with the spectral index for cold electrons, k_c , for the plasma parameters $f = 0.5$, $\alpha = 0.1 \times 10^{-3}$, $\mu_d = 0.1 \times 10^4$, $z_d = 2 \times 10^3$ and $\beta = 3$. (a) For $k_h = 3$ (superthermal electrons); (b) For $k_h = 1000$ (Maxwellian electrons).

To illustrate the existence regions of both positive- and negative polarities nonlinear structures, Figure 1(a–b) illustrates the variation of the allowed Mach-number regions with the parameter k_c , where k_c corresponds to the superthermal cold-electron populations. In Fig. 1(a), the green dashed curve represents the upper Mach-number limit for the rarefactive structures, M_{ur} , whereas the red dotted curve represents the upper Mach-number limit for the compressive structures, M_{uc} . The blue solid curve represents the lower Mach-number limit, M_0 . These curves demonstrate the dependence of the Mach-number limits on the degree of superthermality of the cold-electron population through the parameter k_c , while all other plasma parameters are kept fixed at $\mu_d = 0.1 \times 10^4$, $z_d = 2 \times 10^3$, $\alpha = 0.1 \times 10^{-3}$, $f = 0.5$ and $\beta = 3$. The spectral index for hot electron populations is taken $k_h = 3$ for Fig. 1(a), which means that the hot electrons have a strongly non-Maxwellian distribution with an enhanced population of energetic electrons. It is observed that the Mach-number limits increase with increasing k_c and gradually approach saturation at larger values of k_c , as shown in Fig. 1(a). This behavior indicates that, as the cold-electron population becomes less superthermal and approaches the Maxwellian limit, the characteristic Mach-number limits become progressively less sensitive to further increases in k_c . It is also found that, for a fixed k_c , the upper Mach-number limit for the rarefactive structures, M_{ur} , is higher than that for the compressive structures, M_{uc} , indicating a relatively wider Mach-number range for the existence of rarefactive structures.

Figure 1(b) shows again the variation of the Mach-number with the superthermal parameter k_c for $k_h = 1000$ (Maxwellian electron populations), while the other plasma parameters are kept the same as those used in Fig. 1(a), except for the specified value of k_h . Here it is noticed that, as k_h increases, the lower and upper Mach-number limits shift toward higher Mach number regimes, while the corresponding permissible Mach-number intervals become wider. Thus, the nonlinear structures require progressively higher Mach numbers as the hot-electron population becomes less superthermal and approaches the Maxwellian limit. This indicates that the degree of superthermality of the hot-electron population plays an important role in determining the range of Mach numbers over which compressive and rarefactive nonlinear structures can exist.

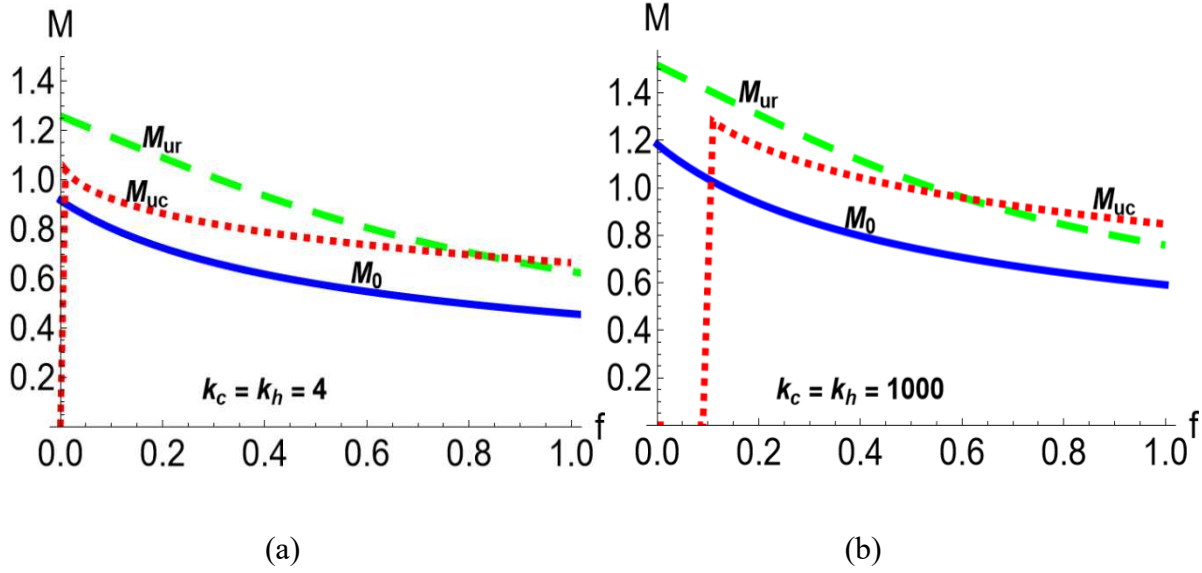
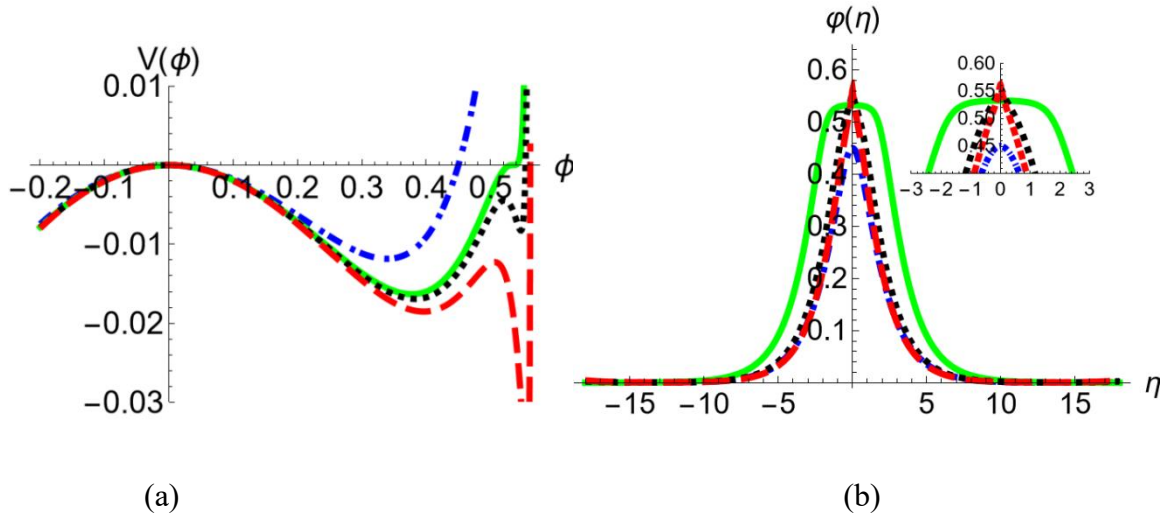


Fig. 2(color online). Variation the Mach number, M , with the density ratio of cold to total electrons, f , for the plasma parameters $\alpha = 0.1 \times 10^{-3}$, $\mu_d = 0.1 \times 10^4$, $z_d = 2 \times 10^3$ and $\beta = 3$. (a) For superthermal hot electrons ($k_c = k_h = 4$); (b) For Maxwellian hot electrons ($k_c = k_h = 1000$).

Figures 2(a) and 2(b) illustrate the variation of the Mach-number with the cold-electron density, f , for superthermal and Maxwellian electron populations, respectively. In Fig. 2(a), the spectral indices of both cold and hot superthermal electron populations are $k_c = k_h = 4$, respectively, while all other plasma parameters are kept fixed at the values specified in the figure. It is observed that, with increasing cold-electron density, the permissible Mach-number range decreases. This indicates that an increasing contribution of the cold-electron population restricts the parameter space over which nonlinear structures can exist. This behavior can be understood in terms of the modified electron density response to the electrostatic potential resulting from the changing relative populations of the cold and hot electrons. Since the cold-electron fraction directly affects the total electron response, its increase changes the balance between the electron restoring force and the ion inertial response. For a superthermal electron population, the spectral index further controls the strength of the suprathermal tail and, consequently, the nonlinear response of the electrons to the electrostatic potential. In the present parameter regime, the combined effect of increasing cold-electron density and the associated change in the electron response modifies the curvature and higher-order properties of the Sagdeev pseudopotential. As a result, the conditions required for the formation of an admissible negative Sagdeev potential

well are satisfied over a narrower range of Mach numbers, leading to a reduction in the parameter space available for localized nonlinear solitary structures.

Figure 2(b) further illustrates the limiting behavior as both the cold- and hot-electron populations progressively approach their Maxwellian limits, i.e., as their respective spectral indices increase. In this limit, the suprathermal tails become progressively weaker, and the electron density response approaches that of a Maxwellian population. The corresponding Mach-number interval expands, indicating that the nonlinear structures can exist over a broader range of propagation speeds. These results demonstrate that the admissible Mach-number domain is sensitive not only to the relative densities of the cold- and hot-electron populations but also to the characteristics of their velocity distributions. Overall, the results indicate that the electron distribution functions play a crucial role in determining the existence domain of nonlinear electrostatic structures. In the present parameter regime, strongly superthermal cold- and hot-electron populations tend to narrow the permissible Mach-number range, whereas progressively weaker superthermality broadens the allowed Mach-number domain and permits nonlinear structures over a wider range of Mach numbers.



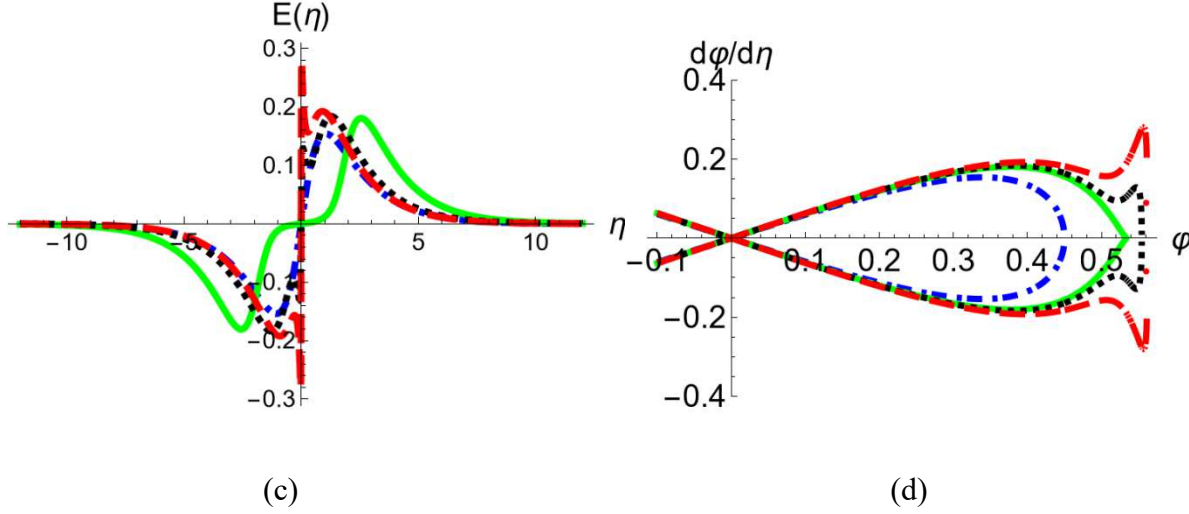


Fig. 3(color online). Plots of (a) the Sagdeev potential, (b) the electrostatic potential, (c) electric fields, and (d) phase space trajectory for compressive soliton, flat top soliton and supersolitons. The plasma parameters are kept $f = 0.006$, $\alpha = 0.3 \times 10^{-4}$, $\mu_d = 0.2 \times 10^4$, $z_d = 1020$, $\beta = 3.275$ and $k_c = 3.5$ and $k_h = 4.68$. The dash-dotted blue, solid green, dotted black, and dashed red curves correspond to $M = 1.04$, $M = 1.054375$, $M = 1.056$, and $M = 1.06$, respectively.

Figures 3(a-d) demonstrates the progressive transition from an ordinary soliton to a flattop soliton and, subsequently, to a supersoliton in the present plasma system. Figure 3(a) shows the Sagdeev pseudopotential $S(\phi)$ for compressive nonlinear structures obtained by varying the plasma parameter M , while all other parameters are kept fixed at the values indicated in the figure. The blue dash-dotted, solid green, black dotted and red dashed curves correspond to the ordinary soliton, flattop soliton, and supersolitons, respectively. The transition between these nonlinear structures is governed by the modification of the Sagdeev pseudopotential as the Mach number are varied by keeping fixed all other plasma parameters. For the ordinary soliton, the Sagdeev potential possesses a conventional single potential well around the equilibrium state $\phi = 0$, with $S(\phi) = 0$, and $S'(\phi) = 0$ at the Mach number $M = 1.04$. The existence of such a negative pseudopotential well provides the necessary condition for a localized solitary disturbance. As the Mach number is increased, the shape of the pseudopotential becomes progressively flatter over a finite range of ϕ . This produces a broad region in the corresponding solitary-wave profile, leading to the formation of a flattop soliton as indicated by the green solid line curve (at $M = 1.054375$). The flattop soliton is particularly sensitive to the plasma parameters because it occurs in a relatively narrow region of parameter space at $\phi = 0.5326$,

where the nonlinear electron response and the inertial response of the positive ions produce an extended region of weak effective restoring force. In this regime, the Sagdeev potential varies only slowly with φ , so that the pseudo-particle associated with the Sagdeev formalism spends a comparatively long distance near the upper part of the potential structure. Consequently, the electrostatic potential remains nearly constant over an extended spatial interval, giving rise to the characteristic flat-top profile.

With a further increase in the relevant parameter, $M(= 1.056, 1.06)$, the Sagdeev pseudopotential develops additional curvature/inflection structure within the admissible potential domain. The corresponding pseudo-particle trajectory is then qualitatively different from that of an ordinary solitary wave, giving rise to the supersoliton. Thus, the supersoliton is not merely a larger-amplitude soliton; rather, it represents a distinct nonlinear state associated with a more complicated Sagdeev-potential topology. The appearance of the supersoliton therefore reflects a delicate balance between the nonlinear electron density response, ion inertia, and the electrostatic restoring force. It is important to note that the occurrence of a supersoliton within a continuous parameter scan does not imply that every intermediate parameter value supports a supersoliton. Instead, the supersoliton generally occupies a restricted region of the permissible parameter space, and its formation is preceded by the flattop or conventional solitary-wave regime. Therefore, the present results demonstrate that the supersoliton domain is embedded within the broader region supporting solitary structures. In this sense, the observation of a supersoliton is accompanied by the possible existence of other solitary modes in neighboring regions of the same plasma parameter space.

Figure 3(b) presents the corresponding electrostatic potential profiles obtained from the energy balance equation. The ordinary soliton exhibits the usual localized potential hump characteristic of a compressive solitary structure. In contrast, the green solid curve corresponding to the flattop soliton^{36, 37} displays an extended nearly constant-amplitude region near its maximum. This flat region is the direct spatial manifestation of the shallow portion of the Sagdeev pseudopotential. The supersoliton profile, represented by the black dotted and red dashed curves (with increased amplitude), exhibits a more complicated localized structure, reflecting the additional nonlinear features present in its Sagdeev potential. The associated electric-field profiles are shown in Fig.

3(c), where the electric field is obtained from $E = -\frac{d\phi}{d\eta}$. The ordinary soliton produces a relatively simple bipolar electric-field profile, whereas the flattop soliton exhibits a broadened field structure because the electric field becomes small over the nearly constant-potential region. The most distinctive feature is observed for the supersoliton, whose electric-field profile exhibits the characteristic wiggly-like structure, as shown by the black dotted and red dashed curves, consistent with earlier observations^{36, 37, 42}. These additional oscillatory features are a direct consequence of the more complicated spatial variation of the supersoliton potential and provide an important diagnostic for distinguishing supersolitons from conventional solitary waves.

Figure 3(d) shows the corresponding phase portraits in the $(\phi, \frac{d\phi}{d\eta})$ phase plane. The solitary solutions appear as closed trajectories surrounding the equilibrium point, with the homoclinic orbit connecting the saddle point at the unperturbed state⁴⁰. The deformation of the phase trajectory from the ordinary soliton to the flattop state reflects the change in the nonlinear restoring force. For the supersoliton, the phase-space trajectory acquires a more intricate structure, consistent with the additional features of the Sagdeev pseudopotential and the associated wiggly-shaped electric field. Overall, Figs. 3 (a-d) demonstrates that the progression from soliton to flattop soliton and finally to supersoliton highlights the strong sensitivity of nonlinear wave morphology to the Mach number. In particular, the narrow parameter interval associated with the flattop state emphasizes its transitional character, while the emergence of the supersoliton indicates a qualitative modification of the underlying Sagdeev-pseudopotential topology.

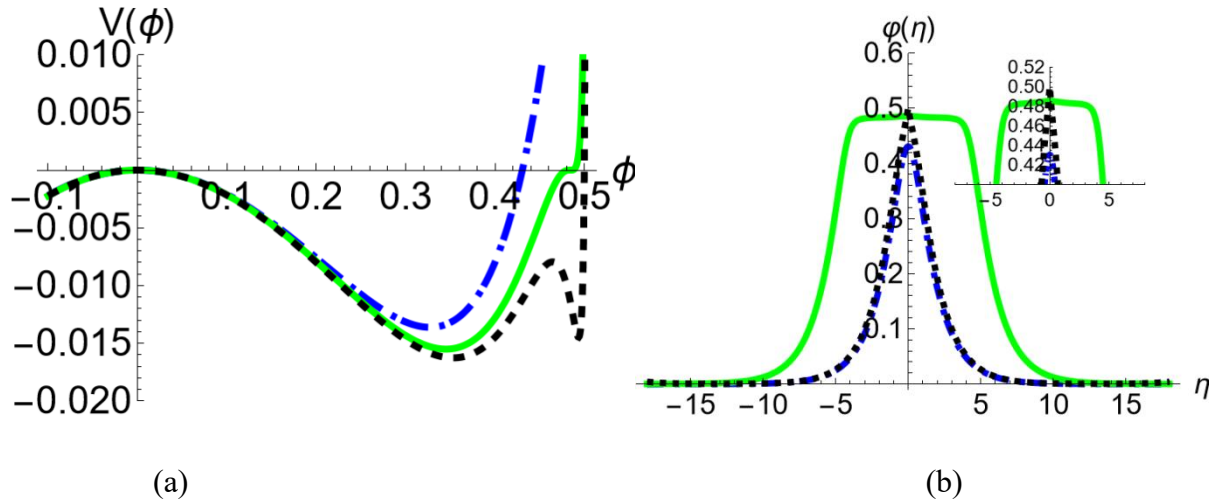


Fig. 4(color online). Plots of (a) the Sagdeev potential, (b) the electrostatic potential for soliton, flat top soliton and supersoliton structures in the dust-free plasma $\alpha = 0$. The plasma parameters are fixed at $M = 1.001$, $\beta = 3.4874$, $k_c = 3.4008$ and $k_h = 3.89$. The dash-dotted blue, solid green, and dotted black curves correspond to $f = 0.004$, $f = 0.00574$, and $f = 0.064$, respectively.

Figure 4 illustrates the dust-free limit of the present plasma model, corresponding to $\alpha = 0$, with cold and hot superthermal electron populations. This limiting case has previously been investigated in Ref. (42), where the existence of compressive solitons and supersolitons was demonstrated. In the present model, the results obtained in Ref. (42) are recovered in the appropriate dust-free limit, $\alpha = 0$, thereby providing a consistency check for the present theoretical formulation. In this limit, the dust-related parameters μ_d and z_d do not influence the existence or characteristics of the nonlinear structures because the dust contribution to Poisson's equation vanishes. Interestingly, the present analysis further demonstrates, as a novel result, that the dust-free plasma can also support flat-top solitary structures in the positive-potential regime. Figure 4(a) shows the corresponding Sagdeev pseudopotential for different values of f (cold-electron density). The dash-dotted blue curve represents an ordinary compressive solitary wave for $f = 0.004$ at the speed of the wave $M = 1.05$. With an increase in cold-electron density, the Sagdeev potential develops a broader region near its zero, resulting in a flat-top solitary structure (at $f = 0.00574$), represented by the solid green curve. For a further increase in this cold electron density at $f = 0.006$, the pseudopotential acquires the characteristic structure required for the formation of a supersoliton, represented by the dotted black curve. Thus, the sequence from an ordinary soliton to a flat-top soliton and subsequently to a supersoliton is obtained through a systematic variation of the cold superthermal electron population.

The corresponding electrostatic potential profiles are shown in Fig. 4(b). As the density of the cold superthermal electrons increases, the amplitude of the positive-potential nonlinear structures increases, while their spatial profiles also evolve from conventional solitary waves to flat-top structures and finally to supersolitons. This behavior is consistent with the modification of the Sagdeev pseudopotential shown in Fig. 4(a), since the existence and morphology of solitary structures are directly determined by the shape of the pseudopotential in the accessible potential region. It is also found numerically that, for $f > f_c = 0.0072$, no supersoliton solution exists for the selected plasma parameters shown in Fig. 4. This indicates that supersoliton formation occurs

only within a restricted parameter range. Outside this range, the Sagdeev potential does not satisfy the necessary conditions for a supersoliton, and consequently the corresponding supersoliton solution disappears.

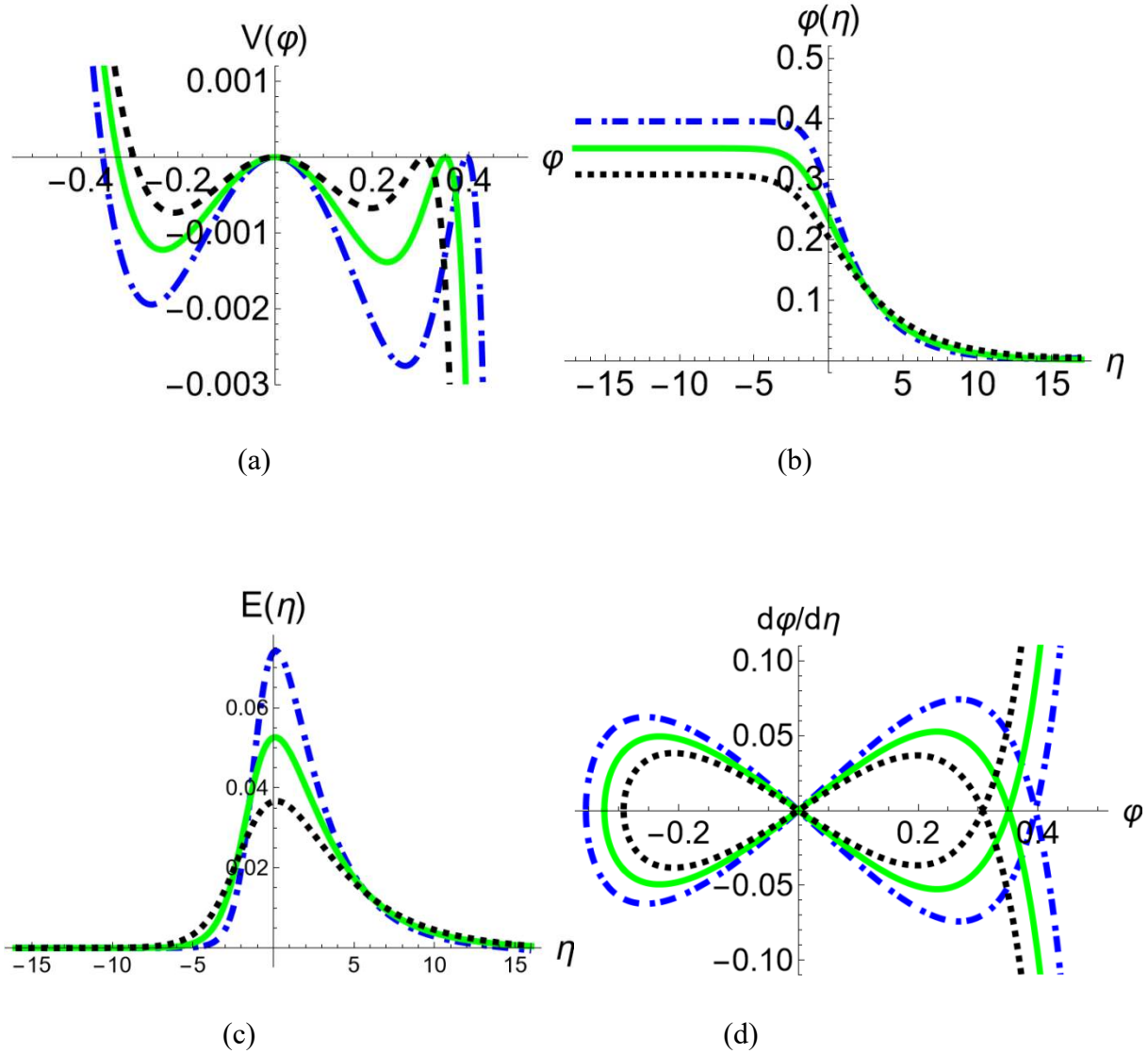


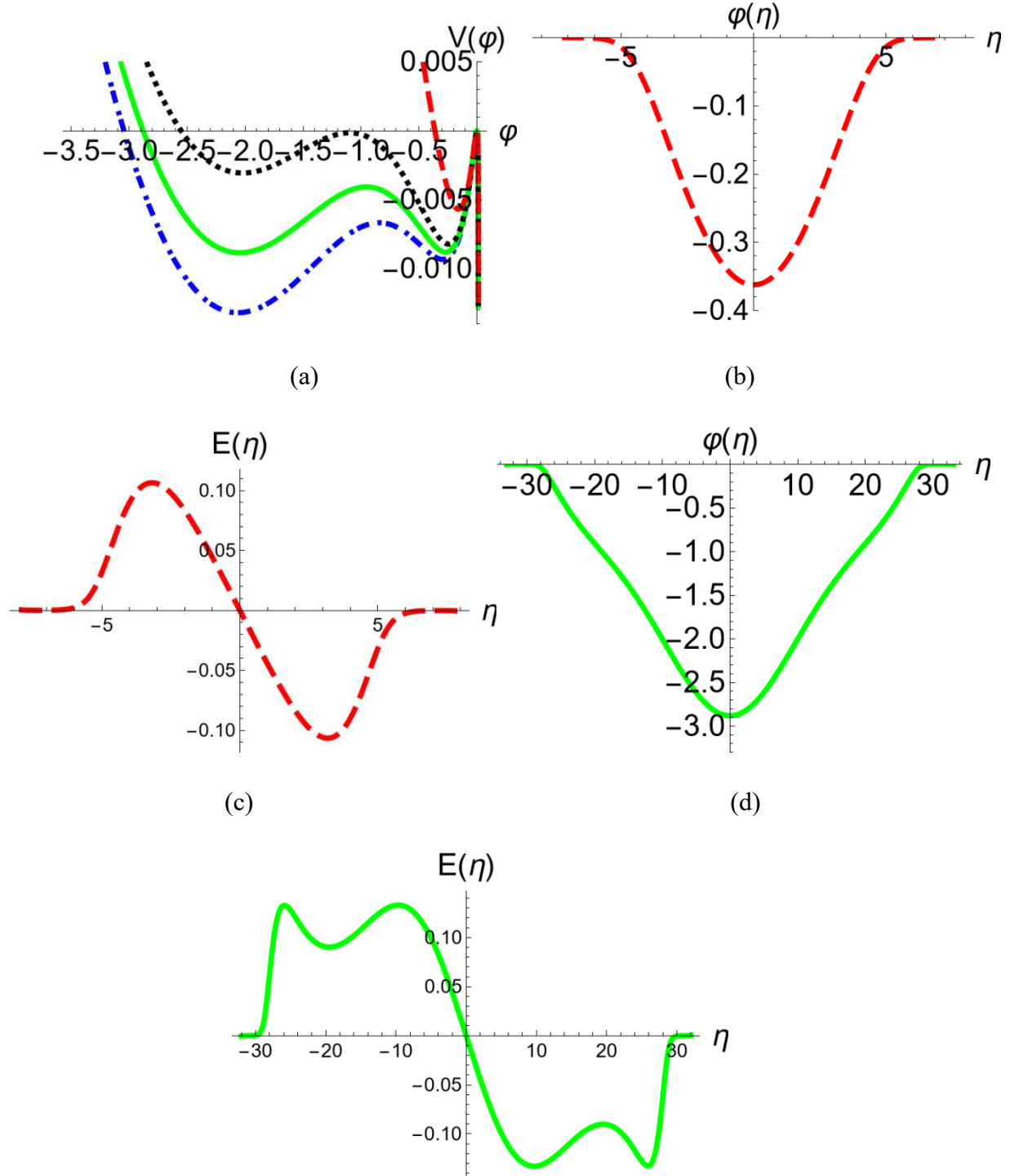
Fig. 5(color online). Plots of (a) the Sagdeev potential, (b) the electrostatic potential, (c) electric fields, and (d) phase space trajectory showing the coexistence of solitons and DLs. The plasma parameters are kept at $M = 1.02$, $\alpha = 0.2 \times 10^{-3}$, $\mu_d = 0.1 \times 10^4$, $z_d = 10^3$, $\beta = 3.3$, and $f = 0.01$. The dotted black, solid green, and blue dash-dotted curves correspond to $(k_c, k_h) = (2.9992, 4.1892)$, $(3.1206, 3.9967)$ and $(3.2359, 3.7919)$, respectively.

Fig. 5 depicts the parametric regime in which a compressive double layer and a rarefactive solitary structure are supported simultaneously for considered dusty plasmas. To examine the

role of the two electron populations, three pairs of spectral indices, k_c and k_h , are considered. The dotted black, solid green, and blue dash-dotted curves in Fig. 5(a) correspond to (2.999, 4.1892), (3.121, 3.997), and (3.236, 3.792), respectively, while the remaining plasma parameters are fixed at $\alpha = 0.2 \times 10^{-3}$, $\mu_d = 0.1 \times 10^4$, $z_d = 10^3$ and $\beta = 3.3$ for $M = 1.02$. A clear modification of the Sagdeev pseudopotential occurs as k_c is increased and k_h is decreased. In particular, the potential well associated with the nonlinear structure becomes deeper, indicating a stronger nonlinear response of the plasma. It is found that an increase in the cold-electron contribution enhances the amplitude of the solitons and DLs, whereas an increase in the hot-electron contribution reduces the amplitudes of both solitons and DLs. This parameter variation simultaneously makes the hot-electron population more strongly superthermal and moves the cold-electron population toward the Maxwellian regime. The resulting change in the electron charge-density response alters the balance required for the formation of nonlinear electrostatic structures. Consequently, the potential jump of the compressive double layer becomes progressively larger. The important point is that the nonlinear response is governed not by either electron population independently, but by their combined contribution to the total charge density.

The corresponding potential profiles of compressive DLs in Fig. 5(b) provide a direct representation of this change. With the above variation of k_c and k_h , the potential excursion across the double layer increases, confirming the strengthening of the compressive structure. The potential remains localized in the moving frame, while its two asymptotic states satisfy the double-layer conditions required by the Sagdeev formalism. The monopolar electric-field profiles shown in Fig. 5(c) further characterize the localized nature of DLs, consistent with the behavior reported earlier in Ref. (37). Because the electric field is determined by the spatial gradient of the electrostatic potential, the enhanced potential jump produces a corresponding change in the field strength. The electric field is concentrated around the transition region of the double layer, where the potential changes most rapidly. The nonlinear coexistence becomes particularly evident from the phase portraits in Fig. 5(d). The trajectories associated with the rarefactive solitary wave and the compressive double layer occupy the same admissible nonlinear parameter regime but possess distinct phase-space characteristics. The solitary structure corresponds to a closed orbit, whereas the double layer is represented by a separatrix connecting

two equilibrium points³⁷. Thus, the phase-space analysis confirms that the present dusty plasma supports two different nonlinear excitations simultaneously: a rarefactive soliton together with a compressive double layer. This coexistence results from the competition between the nonlinear responses of the cold and hot superthermal electrons and the dust-ion components.



(e)

Fig. 6(color online). Plots of (a) the Sagdeev potential, (b) soliton potential profile, (c) the electric field corresponding to the soliton, (d) supersoliton potential profile and (e) the electric field corresponding to the supersoliton. The plasma parameters are kept $M = 0.8$, $\alpha = 0.85 \times 10^{-3}$, $\mu_d = 6 \times 10^4$, $z_d = 460$, $\beta = 8$, $f = 0.135$ and $k_c = 1.6$. The dash-dotted blue, solid green, dotted black, and dashed red curves correspond to $k_h = 5.6$, $k_h = 6$, $k_h = 6.6$, and $k_h = 1000$, respectively.

As discussed above, the present plasma model admits a rich variety of nonlinear excitations, including compressive and rarefactive supersolitary waves as well as solitary structures. Figure 6 shows that rarefactive supersolitons^{21, 38, 43} occur at relatively low Mach numbers, i.e., $M < 1$, in the considered dusty plasma. This figure examines the evolution of these nonlinear structures with the spectral index of the hot-electron population, while keeping all other plasma parameters fixed. Figure 6(a) displays the Sagdeev pseudopotential together with the corresponding electrostatic potential for the selected plasma parameters, namely the dust to ions concentration ratio $\alpha = 0.85 \times 10^{-3}$, dust-to-ion mass ratio $\mu_d = 6 \times 10^4$, dust charge number $z_d = 460$, cold-electron to total-electron density ratio $f = 0.135$, and hot-to-cold electron temperature ratio $\beta = 8$, at the specified Mach number $M = 0.8$. The cold-electron spectral index is fixed at $k_c = 1.6$, whereas the hot-electron spectral index k_h is varied. The dash-dotted blue, solid green, dotted black, and dashed red curves correspond to $k_h = 5.6, 6, 6.6$ and 1000 , respectively. The first three curves represent rarefactive supersolitary solutions, while the red curve represents an ordinary rarefactive solitary wave. A systematic weakening of the nonlinear structures is obtained as k_h increases. In the framework of the present model, increasing k_h reduces the departure of the hot-electron distribution from the Maxwellian form. The associated reduction in the superthermal electron response modifies the Sagdeev potential and consequently decreases the amplitude of both the ordinary solitons and supersolitons. At the same time, the supersolitons become spatially broader. Thus, the transition toward a Maxwellian hot-electron population does not simply alter the amplitude; it also changes the spatial scale over which the nonlinear disturbance is sustained. For the chosen plasma composition, the supersoliton structure is numerically obtained within the range $k_h < 6.68$. Beyond $k_h \geq 6.68$, the additional second potential well required for a supersoliton disappears. In particular, at $k_h = 1000$, only negative-potential ordinary solitary waves remain admissible. The red curve in Fig. 6(a), drawn for $k_h =$

1000, effectively represents the Maxwellian limit. The disappearance of the supersoliton structure in this limit highlights the important role played by hot-electron superthermality in generating the more complex Sagdeev-potential topology.

The spatial characteristics of these solutions are shown separately in Fig. 6(b–e). Figure 6(b) gives the potential profile corresponding to the ordinary rarefactive soliton. Its localized potential excursion produces the conventional bipolar electric-field signature shown in Fig. 6(c). In contrast, the supersoliton potential displayed in Fig. 6(d) possesses a characteristic internal modulation. This additional structure originates from the more complicated Sagdeev potential associated with the supersoliton regime. Its electric-field counterpart, plotted in Fig. 6(e), consequently exhibits a modified bipolar profile with an internal structure that distinguishes it from the ordinary solitary wave. Moreover, the effect of the cold-electron spectral index provides another interesting aspect of the nonlinear response. When k_c is increased from 1.6 to 2.5, while the remaining parameters are kept at the values used in Fig. 6, the system enters a regime in which compressive and rarefactive ordinary solitary waves can coexist for $k_h > 5.3$. In this case, the two types of solitary excitations occur on opposite sides of the equilibrium potential, reflecting the different signs of their potential amplitudes.

An opposite trend in their amplitudes is obtained as the hot-electron spectral index is increased: the amplitude of the rarefactive soliton decreases, whereas that of the compressive soliton increases. This contrasting response demonstrates that the two nonlinear structures are controlled differently by the electron charge-density contribution. More notably, at $k_c = 2.5$ and for the range $k_h < 5.3$, the numerical solutions indicate the simultaneous occurrence of a compressive ordinary soliton and a rarefactive supersoliton. Hence, variation of the electron spectral indices can drive the system between different nonlinear regimes and can even produce the coexistence of qualitatively different solitary structures within the same plasma model.

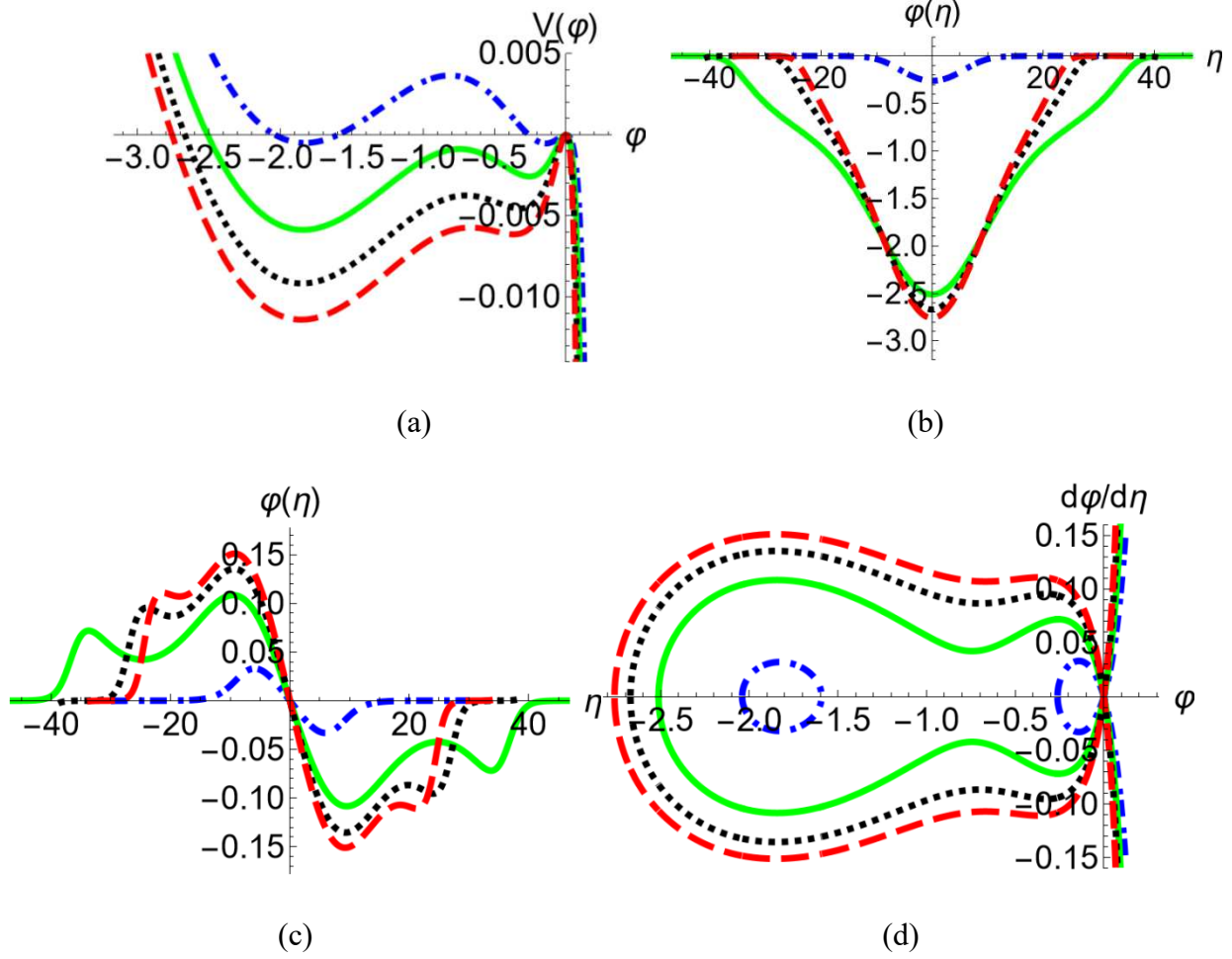


Fig. 7(color online). Plots of (a) the Sagdeev potential, (b) the electrostatic potential, (c) electric fields, and (d) phase space trajectory for rarefactive soliton and supersolitons. The plasma parameters are kept $M = 0.8$, $\alpha = 0.8 \times 10^{-3}$, $\mu_d = 6 \times 10^4$, $z_d = 0.5 \times 10^3$, $f = 0.14$, $k_c = 3$ and $k_h = 5$. The dash-dotted blue, solid green, dotted black, and dashed red curves correspond to $\beta = 6$, $\beta = 8$, $\beta = 10$, and $\beta = 12$, respectively.

We next examine the role of the temperature ratio $\beta = \frac{T_h}{T_c}$, which measures the relative thermal energy of the hot and cold electron populations, in determining the nonlinear wave structures. Figure 7 is obtained for a fixed plasma composition, with the parameters $\alpha = 0.8 \times 10^{-3}$, $\mu_d = 6 \times 10^4$, $z_d = 0.5 \times 10^3$, $f = 0.14$, $k_c = 3$ and $k_h = 5$, for the same Mach number $M = 0.8$. Only the temperature ratio β is varied so that its influence on the nonlinear solutions can be identified independently. The evolution of the Sagdeev pseudopotential with different β is displayed in Fig. 7(a). The dash-dotted blue curve represents the ordinary rarefactive soliton for $\beta = 6$, whereas the solid green, dotted black, and dashed red curves correspond to supersoliton solutions for $\beta =$

8, 10, 12 respectively. An increase in hot electrons temperature produces a pronounced change in the shape of the pseudopotential. The nonlinear potential well associated with the accessible rarefactive structures becomes progressively deeper, allowing solitary structures with larger potential excursions to develop. Consequently, both the ordinary soliton and supersoliton amplitudes increase as the hot-electron temperature becomes larger relative to the cold-electron temperature.

The change in the spatial extent of the nonlinear structures is evident from Fig. 7(b). In contrast to the amplitude, the width of the supersolitons decreases with increasing β . Thus, increasing the hot-electron temperature produces a stronger but more localized rarefactive nonlinear structures. Physically, this reflects the modification of the electron density response to the electrostatic potential when the hot population becomes increasingly energetic. The resulting balance between the electron response and the dust-ion contribution permits a larger potential excursion over a shorter spatial scale. A particularly interesting feature appears in the Sagdeev potential as β is increased. The relative depths of the two potential wells change continuously: the second well becomes increasingly pronounced, whereas the first well is progressively weakened. With further increase in β , the secondary-well structure eventually disappears and the pseudopotential evolves toward a single-well form. This change in topology is significant because the presence of the additional well is associated with the existence of the supersoliton structure. Our numerical calculations show that supersolitons are permitted only within the finite interval $7.5 < \beta < 200$. Outside this interval, the additional phase-space structure required for a supersoliton is lost and the system supports only the ordinary solitary structures.

The corresponding electric-field profiles are presented in Fig. 7(c) for solitons and supersolitons. The supersolitons possess a characteristic wiggly bipolar electric-field structure²¹ rather than the simple bipolar signature (dash-dotted blue curve) of an ordinary soliton. As β increases, the field amplitude becomes larger, consistent with the increase in the potential amplitude seen in Fig. 7(b). The internal oscillatory feature of the electric field provides a direct signature of the more intricate spatial structure associated with the supersoliton solution. The distinction between the two classes of nonlinear waves is further illustrated by the phase-space trajectories in Fig. 7(d). The dash-dotted blue trajectory forms the conventional homoclinic orbit associated with the ordinary rarefactive soliton. In contrast, the remaining trajectories exhibit additional extrema and

a more intricate orbit structure, characteristic of supersolitons. These trajectories remain homoclinic in nature, but their internal structure is considerably more complicated because the corresponding Sagdeev potential contains an additional potential feature. The resulting phase portraits are consistent with the rarefactive supersoliton and ordinary-soliton structures discussed earlier in Fig. 6(d). Overall, Fig. 7 demonstrates that β is an important control parameter for the formation and morphology of rarefactive nonlinear waves. Increasing the relative temperature of the hot electrons strengthens the solitary potential, increases the electric-field amplitude, and compresses the spatial width of the supersoliton. At the same time, the associated modification of the Sagdeev-potential topology determines the finite parameter window within which supersolitons can exist.

5. Conclusion

In this work, we have investigated the nonlinear propagation of dust-ion-acoustic structures in a homogeneous, unmagnetized dusty plasma containing positive cold ions, negatively charged cold dust grains, and two populations of superthermal electrons, namely cold and hot electrons. By employing the Sagdeev pseudopotential technique, we have systematically examined the conditions for the existence and evolution of compressive and rarefactive solitary structures, double layers, flat-top solitons, and supersolitons. The analysis demonstrates that the inclusion of negatively charged dust substantially modifies the nonlinear charge balance and, consequently, the topology of the Sagdeev pseudopotential. In particular, the dusty plasma considered here supports both positive- and negative-potential supersolitons, providing a nonlinear response that is richer than that obtained in the corresponding dust-free system.

The existence of solitary structures is determined by the admissibility conditions of the Sagdeev potential, which requires a suitable negative potential well around the equilibrium state and a finite-amplitude root accessible to the pseudo-particle trajectory. The present analysis shows that both positive and negative finite-amplitude roots can occur for appropriate ranges of the plasma parameters. Positive-potential roots correspond to compressive structures, whereas negative-potential roots correspond to rarefactive structures. More importantly, the Sagdeev pseudopotential may develop additional extrema within the potential well. These additional extrema lead to a more complicated pseudo-particle trajectory and distinguish supersolitons from conventional solitary waves. Thus, the supersoliton obtained in the present model is not simply a

higher-amplitude solitary wave; rather, it represents a distinct nonlinear state associated with a modified topology of the Sagdeev potential and a corresponding internal structure in the electric field.

The existence of both compressive and rarefactive supersolitons, together with flat-top solitons, constitutes a novel finding of the present investigation. In contrast, the corresponding model without negatively charged dust considered in earlier work allows only positive-potential supersolitons in the relevant parameter regime. The dust contribution therefore provides an additional degree of freedom in determining the nonlinear potential structure and permits the formation of supersolitons of both polarities. For the numerical analysis, the dust-to-ion density ratio, dust-to-ion mass ratio, and dust charge number are chosen as $\alpha \cong 10^{-3} - 10^{-4}$, $\mu_d \cong 10^4$, and $z_d \cong 10^3$, respectively. These parameters are consistent with Earth's ionosphere dusty plasma parameters⁴⁴. It is observed that compressive supersolitons occur within the range $M > 1$ and relatively low β values, while rarefactive supersolitons are obtained within the range $M < 1$ and are associated with relatively large β values. These results demonstrate that the dust concentration and charge state play an important role in determining the polarity and admissible Mach-number range of the supersolitons.

The parametric study of the Mach-number limits further demonstrates that the propagation speed of the nonlinear structures is strongly controlled by the electron distribution functions. Increasing the spectral index (k_c) of the cold superthermal electrons causes the lower and upper Mach-number limits to increase and gradually approach saturation, indicating a reduced sensitivity as the electron population approaches the Maxwellian limit. The upper Mach-number limit for rarefactive structures is also found to be higher than that for compressive structures for the considered parameters, providing a comparatively wider Mach-number domain for rarefactive nonlinear excitations. For Maxwellian hot electrons, increasing the relevant spectral index (k_h) shifts the admissible Mach-number domain toward higher values and broadens the corresponding propagation-speed interval. These results establish that the degree of electron superthermality plays a fundamental role in controlling the nonlinear propagation regime. The relative density (f) of the cold-electron population provides another important parameter. An increase in the cold-electron fraction generally reduces the permissible Mach-number range for nonlinear structures in the considered dusty plasmas. This behavior originates from the modification of the

total electron charge-density response and hence of the balance between the electron restoring response and the inertial response of the ions and dust. Conversely, as the electron populations become progressively less superthermal and approach the Maxwellian limit, the admissible Mach-number interval becomes broader.

The evolution from an ordinary soliton to a flat-top soliton and subsequently to a supersoliton has also been demonstrated for the positive potential. For an ordinary solitary wave, the Sagdeev pseudopotential possesses a conventional potential well, producing a localized electrostatic potential and a simple bipolar electric-field structure. With suitable variation of the parameter (M), the potential well becomes flatter over a finite range of the electrostatic potential. This produces an extended region of nearly constant potential and gives rise to a flat-top solitary structure. With a further variation of the Mach number, additional curvature and extrema develop inside the accessible potential domain, resulting in the supersoliton regime. The corresponding electric field develops an internal wiggle-like structure, while the phase-space trajectory becomes more intricate. These results establish a direct correspondence between the topology of the Sagdeev pseudopotential and the morphology of the observable nonlinear structures.

The dust-free limit provides an important consistency check of the present theoretical formulation. When the dust concentration tends to zero, i.e., $\alpha = 0$, the dust contribution to Poisson's equation vanishes, and the results reduce to the previously studied plasma containing cold and hot superthermal electrons with cold inertial ions. The previously reported compressive solitons and supersolitons are recovered in this limiting case. In addition, the present analysis indicates that flat-top positive-potential solitary structures can also occur in the dust-free limit for suitable plasma parameters. This limiting behavior confirms that the nonlinear structures obtained in the dusty system arise from a consistent extension of the dust-free plasma model rather than from an artifact of the inclusion of dust.

The analysis is also reported that the coexistence of compressive DLs and rarefactive solitons in the admissible parameter regime can occur. The effect of the cold- and hot-superthermal electrons modifies the depth and shape of the Sagdeev potential and consequently changes the potential jump across the double layer. The corresponding phase portraits provide a clear distinction between the two structures: the solitary wave is associated with a homoclinic orbit, whereas the double layer corresponds to a separatrix connecting two equilibrium states. This

coexistence highlights the interplay between the nonlinear responses of the two electron populations and the dust-ion components in establishing the accessible nonlinear states.

The spectral index (k_h) and hot-to-cold electron temperature ratio (β) are found to exert particularly strong control over rarefactive structures. It is found that with increasing the parameter k_h , weakens the departure from the Maxwellian distribution, resulting in reduced amplitudes and increased spatial widths of the rarefactive solitons and supersolitons. In the Maxwellian limit, the additional Sagdeev-potential structure necessary for supersoliton formation disappears for the selected plasma parameters, leaving only ordinary rarefactive solitary waves. In contrast, increasing β strengthens the nonlinear potential well, leading to larger amplitudes and stronger electric fields, while the supersoliton width decreases. These trends demonstrate that the electron temperature contrast and non-Maxwellian character jointly determine both the strength and spatial localization of the nonlinear structures.

The most significant outcome is the possibility of both compressive and rarefactive supersolitons in the dusty plasma system, together with the coexistence of solitons and DLs for parameter regimes. These results provide a broader theoretical framework for understanding nonlinear electrostatic excitations in dusty plasma environments where non-Maxwellian electrons and negatively charged dust grains with cold ions are simultaneously present. The identified transitions and polarity-dependent structures may be useful for interpreting complex electrostatic nonlinear phenomena in laboratory and space dusty-plasma environments.

Data Availability

All data supporting the findings of this study are available within the paper and its Supplementary Information.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CrediT authorship contribution statement

Kuldeep Singh: Writing original draft, investigation, Validation, Funding acquisition, Data curation, Rajendra Prasad Meena: Methodology, software, investigation, Sukhmander Singh: Formal analysis, review & editing, and Kishan Kumar: Writing original draft, investigation, Validation, Data curation, review & editing.

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