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Research Article

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Posted Date: October 13th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11107057/v1>

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Additional Declarations: No competing interests reported.

Transient Probabilities and Switching Levels in a Vacation Queue with Catastrophes

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Abstract

We study a slotted (discrete-time) single-server queue in which customers accumulate during vacations and catastrophes empty the system. We consider two questions: the level and regime at a prescribed time, and the queue level at vacation completion in a service-return cycle conditioned to have a prescribed large length. A two-color Motzkin representation, with ordinary stays distinguished from the horizontal regime switch, yields a first-return decomposition for the transient state probabilities and the stationary law. In a common-rate family, the fixed-state probabilities and the probability of the whole vacation regime need not have the same leading relaxation factor; an explicit example exhibits this separation. For weighted service-return cycles, marking the switch level gives a composition scheme. Known composition-scheme limit theorems then give bounded, square-root and linear scales, with the parameters written in the original queue weights. At criticality, a specialization admits an exact finite-length mean in terms of central trinomial coefficients, together with its constant and inverse-square-root corrections. Uniformization recovers the continuous-time transient probabilities from the same path coefficients.

Keywords: Vacation queues; Motzkin paths; catastrophes; transient probabilities; switch level; singularity analysis.

MSC 2020: 60K25 (primary); 05A15, 05A16, 05A19 (secondary).

1 Introduction

In a single-server queue, customers arrive, wait if necessary, and leave after service. The classical continuous-time $M/M/1$ model has Poisson arrivals and exponential service times; the two letters “M” denote memoryless interarrival and service times, and “1” denotes one server. During a server vacation, customers continue to arrive but receive no service. Their number when the server returns determines the initial level of the next working period. We consider two questions. At a prescribed time, what is the joint distribution of the queue level and the server regime, and can a fixed state relax at a different rate from the entire vacation event? For a long cycle ending by service of the last customer, how many customers were present when the vacation ended?

We use a slotted (discrete-time) model with a vacation regime and a working regime. Each slot contains one specified transition. An arrival adds one customer, a working service completion removes one, and a stay leaves the state unchanged. The server enters vacation when the system becomes empty and can return to work only at a positive level. A catastrophe in either regime removes all customers and returns the process to the empty vacation state. Thus a catastrophe supplies a second return-to-zero mechanism, distinct from service of the last customer.

The trajectory is represented by a two-color lattice path: the horizontal coordinate counts slots and the vertical coordinate counts customers. Blue denotes vacation and red denotes work. Arrivals and services are up- and down-steps. An ordinary stay is a horizontal Motzkin step, while the end of a vacation is a second horizontal step that changes color without changing level. Catastrophes jump directly to zero. With transition probabilities as weights, the sum of path weights over a fixed endpoint is the corresponding state probability. With general nonnegative weights, the same formulas are counting identities. Deleting ordinary stays leaves the effective event sequence, or event skeleton; its order does not determine elapsed time. Uniformization supplies a continuous-time interpretation through a Poisson clock (Gross and Miller, 1984), and is different from sampling a continuous-time queue at fixed calendar intervals.

Time-dependent vacation queues have been studied by Keilson and Servi (1987), Takagi (1990) and, for a related working-vacation model, Sudhesh et al. (2017). Lattice-path methods for queueing quantities include the quasi-birth-and-death (QBD) calculations of van Leeuwen et al. (2009) and the stationary and transient-transform analysis of Liu and Fralix (2019). Our preceding manuscript, Kleiner and Perry, *Discrete-Time M/M/1-Type Queues with Vacations and Disasters* (submitted manuscript, 2026; hereafter Paper I), develops the two-color event-clock model and gives the disaster-free Motzkin generating functions. The present paper retains its transition rules and notation but studies two different observation problems: a fixed-time state law with catastrophes, and the level at which a vacation ends inside a long service-return cycle.

For the fixed-time problem, a trajectory is cut at returns to blue zero. Completed returns form a renewal sequence and the final piece is an unfinished positive prefix. This decomposition yields the state-probability generating functions and, by occupation within one return cycle, the stationary distribution. In a family with equal arrival and catastrophe probabilities in the two regimes, the resulting factorization also separates local and aggregate relaxation. In particular, we give parameters for which the empty-state error has exponential factor $(30/32)^n$ while the probability of the whole vacation regime has factor $(31/32)^n$.

The second problem uses a different conditioning. A service-return cycle starts with an arrival to an empty system and ends at the first return by service; its length includes the initial arrival, the regime switch and the final service. Among cycles of a prescribed length n , chosen proportionally to their weights, let L_n be the level at the switch. Marking L_n produces a composition scheme. The three scaling regimes follow from the composition-scheme results of Banderier et al. (2024a) and Banderier et al. (2024b); the contribution here is the translation to the queue weights and the resulting vacation-ending statistic. A critical specialization further gives an exact mean for every admissible length and its first two asymptotic corrections. A first-down-step correspondence provides a direct Motzkin interpretation of the same statistic.

The analysis uses first-passage roots, regeneration, kernel cancellation and singularity analysis. Sections 2–4 develop the path decomposition, fixed-slot probabilities and stationary law. Sections 5–6 treat elapsed time and relaxation. Section 7 studies the switch level in long service-return cycles. The appendices collect coefficient formulas, secondary performance quantities, longer limit proofs and continuous-time comparisons.

2 Paths, probabilities and the boundary at zero

The path level is the number of customers and its color is the server regime. We specify the boundary transitions as well as those at positive levels: service of the last customer and a catastrophe both empty the system, but they are distinct events.

A path starts at blue level zero. Its level is a nonnegative integer. Blue represents the

vacation regime, and red represents the working regime. When one formula applies to either regime, $c \in \{b, r\}$ denotes its color. There is no red level zero. At a positive blue level i , the possible steps are an up-step to blue $i + 1$, a stay at blue i , a switch to red i , and a catastrophe to blue zero. Their weights are p_b, h_b, γ, ξ_b , respectively. At a positive red level the up-step, stay, service step and catastrophe have weights p_r, h_r, q_r, ξ_r .

A service step from red $i > 1$ ends at red $i - 1$. A service step from red one ends at blue zero. Every catastrophe, regardless of its source regime, ends at blue zero. In particular, service and catastrophe from red one have the same endpoint but remain different marked steps. From blue zero an arrival to blue one has weight p_b^0 and a stay has weight h_b^0 . There is no switch and no catastrophe at zero.

A step takes one slot. A catastrophe can cross several levels in that slot and is distinct from an ordinary Motzkin down-step. We write H_b, H_r for the ordinary positive-level stays and S for the switch when naming step types. Their respective weights are h_b, h_r, γ .

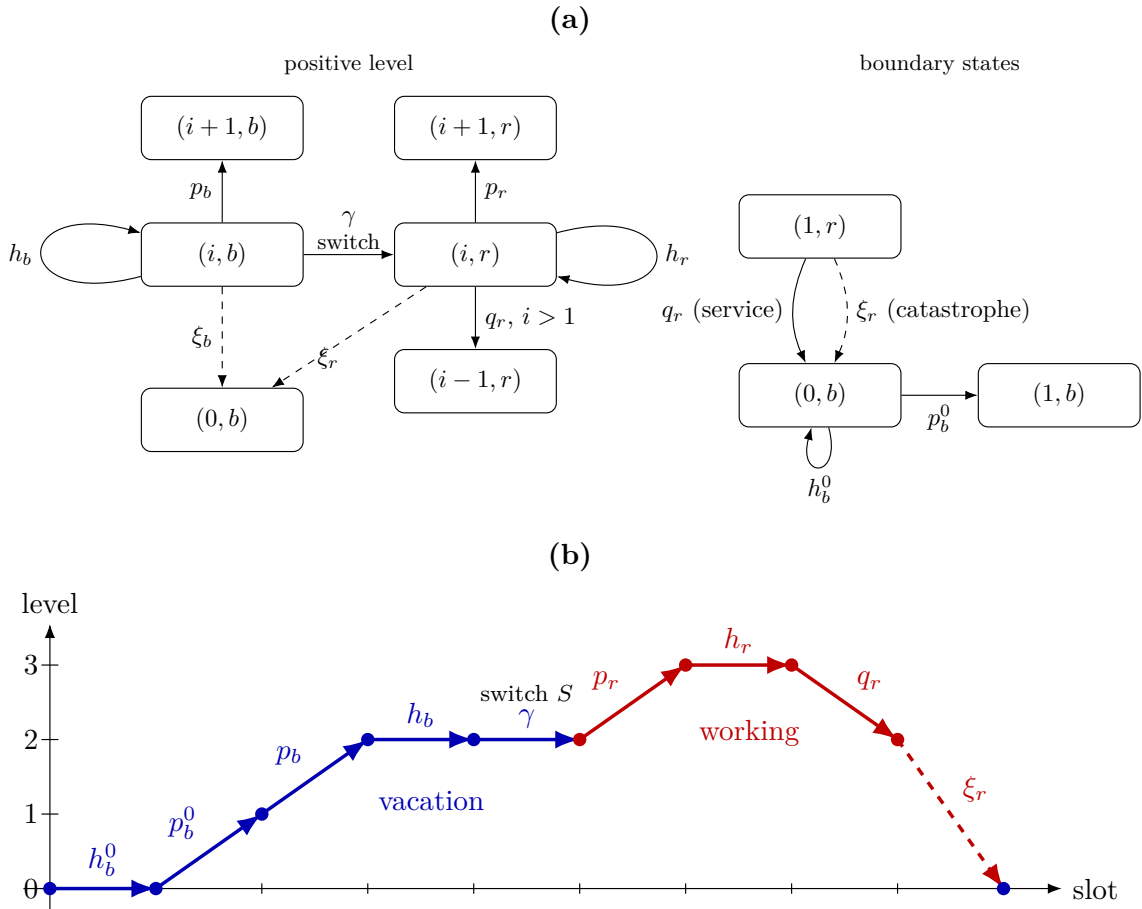


Fig. 1 (a) Allowed transitions at positive levels and at the boundary. (b) A representative weighted two-color path. Blue denotes vacation and red denotes work. Ordinary stays preserve both level and regime, whereas the switch S preserves the level but changes regime. A catastrophe returns directly to blue level zero. The labels on the sample trajectory are the corresponding step weights.

Initially the weights are arbitrary nonnegative numbers. A path has the product of its step weights. For the probability interpretation we require

$$p_b + h_b + \gamma + \xi_b = 1, \quad p_r + h_r + q_r + \xi_r = 1, \quad p_b^0 + h_b^0 = 1. \quad (2.1)$$

Unit weights will be used later for enumeration. In that specialization the quantities are counting weights rather than transition probabilities, because (2.1) is no longer imposed.

Let $b_{n,i}$ and $r_{n,i}$ be the total weights of paths at blue and red level i after n steps. The indices are $n \geq 0$, $i \geq 0$ for blue and $i \geq 1$ for red. Under (2.1) these are the corresponding state probabilities, starting from blue zero. Thus $b_{0,0} = 1$, and all other coefficients at time zero vanish. For recurrence notation only, set $r_{n,0} = 0$.

Consider a path which ends at blue level $i > 1$. Its last step is an up-step from blue $i - 1$ or a stay at blue i . A switch cannot end in blue, and a catastrophe cannot end at a positive level. Hence

$$\begin{aligned} b_{n+1,i} &= p_b b_{n,i-1} + h_b b_{n,i}, & i > 1, \\ b_{n+1,1} &= p_b^0 b_{n,0} + h_b b_{n,1}. \end{aligned} \quad (2.2)$$

Similarly,

$$r_{n+1,i} = \gamma b_{n,i} + p_r r_{n,i-1} + h_r r_{n,i} + q_r r_{n,i+1}, \quad i \geq 1. \quad (2.3)$$

At zero the incoming paths are a stay, a service completion from red one, and catastrophes from all positive levels:

$$b_{n+1,0} = h_b^0 b_{n,0} + q_r r_{n,1} + \xi_b \sum_{j \geq 1} b_{n,j} + \xi_r \sum_{j \geq 1} r_{n,j}. \quad (2.4)$$

For each fixed n all sums are finite, since no path reaches a level larger than n .

Define the generating functions

$$f_i^b(z) = \sum_{n \geq 0} b_{n,i} z^n, \quad f_i^r(z) = \sum_{n \geq 0} r_{n,i} z^n. \quad (2.5)$$

The variable z marks the number of slots and x marks the final height. We write $[z^n]F(z)$ for the coefficient of z^n in $F(z)$, and $[z^n x^i]F(z, x)$ for the corresponding coefficient of a bivariate series.

3 First passage, regeneration and fixed-slot probabilities

We now address the fixed-slot question. Every trajectory can be cut uniquely at its visits to blue level zero into completed returns followed by a final unfinished positive part, which may be empty. The full state law will follow once these two pieces are known.

The working segment introduces one elementary object that will recur throughout the calculation: a catastrophe-free downward passage through one red level. We derive its generating function first, then construct the unfinished blue and red prefixes, and finally concatenate completed returns to obtain the fixed-slot probabilities.

3.1 A red first-passage series

We begin with the elementary building block for the working regime. Start at red level one and stop upon reaching zero. Count only paths that end by service and contain no catastrophe. Their generating function is denoted by $x_1(z)$. This notation agrees with the small root used in the preceding event-clock analysis. Starting at red level m , such a path splits at its first successive visits to $m - 1, m - 2, \dots, 0$. Its generating function is therefore $x_1(z)^m$.

Lemma 3.1. *The unique formal power series with zero constant term satisfying*

$$x_1(z) = z\{q_r + h_r x_1(z) + p_r x_1(z)^2\} \quad (3.1)$$

is the red service-first-passage series. For $p_r > 0$,

$$x_1(z) = \frac{1 - h_r z - \sqrt{(1 - h_r z)^2 - 4p_r q_r z^2}}{2p_r z}, \quad (3.2)$$

where the square root has constant term one. For $p_r = 0$, the expression is interpreted as $q_r z / (1 - h_r z)$.

Proof. The first step is a service step, a stay, or an up-step. A stay is followed by the same first-passage problem. After an up-step there are two successive downward first-passage segments. This gives (3.1). Its coefficient of degree n is determined by lower degrees, which proves uniqueness. Solving the quadratic and taking the branch with zero constant term gives (3.2). $\square$

Under (2.1), when $\xi_r > 0$, $x_1(1)^m$ is the probability of a service return before a catastrophe; the complementary absorption mechanism is a red catastrophe.

The first-passage coefficients have a finite ballot-number expression; it is given in Appendix A, equation (A.1).

3.2 The last unfinished part of a path

An unfinished positive prefix starts with the arrival from blue zero and has not yet returned to zero. Let $g_i^b(z)$ and $g_i^r(z)$ count such prefixes ending at level $i \geq 1$. Neither the terminal zero of a completed cycle nor any preceding completed cycle is included. Put

$$G^b(z, x) = \sum_{i \geq 1} g_i^b(z) x^i, \quad G^r(z, x) = \sum_{i \geq 1} g_i^r(z) x^i. \quad (3.3)$$

The empty prefix at zero will be represented separately by $g_0^b(z) = 1$.

Define the following auxiliary series:

$$\Delta_b(z) = 1 - z(h_b + p_b x_1(z)), \quad a(z) = \frac{p_b z}{1 - h_b z}, \quad t(z) = \frac{p_r x_1(z)}{q_r}, \quad (3.4)$$

where formulas involving q_r assume $q_r > 0$. Also put

$$S_i(z) = \sum_{j=0}^{i-1} a(z)^j t(z)^{i-1-j}, \quad i \geq 1. \quad (3.5)$$

The finite sum is valid when $a(z) = t(z)$ and avoids division by their difference.

Proposition 3.2. *For $p_r, q_r > 0$ the unfinished prefix functions are*

$$G^b(z, x) = \frac{p_b^0 x z}{1 - z(h_b + p_b x)}, \quad (3.6)$$

$$g_i^b(z) = \frac{p_b^0 z (p_b z)^{i-1}}{(1 - h_b z)^i}, \quad (3.7)$$

$$g_1^r(z) = \frac{\gamma p_b^0 z x_1(z)}{q_r \Delta_b(z)}, \quad g_i^r(z) = g_1^r(z) S_i(z), \quad (3.8)$$

$$G^r(z, x) = \frac{z \{ \gamma G^b(z, x) - q_r g_1^r(z) \}}{1 - z(h_r + p_r x + q_r/x)}. \quad (3.9)$$

These identities count catastrophe-free prefixes, using the original weights without renormalization.

Proof. A blue prefix consists of its first arrival followed by any word of blue up-steps and stays. Summing this word gives (3.6); extracting the height coefficient gives (3.7). In red, the incoming steps give

$$(1 - h_r z)g_i^r = \gamma z g_i^b + p_r z g_{i-1}^r + q_r z g_{i+1}^r, \quad g_0^r = 0.$$

Multiplying by x^i and summing gives (3.9). Its denominator vanishes at $x = x_1(z)$. Formal substitution is valid because $x_1(0) = 0$ and each time coefficient has finite height support. This is the usual kernel-cancellation step; compare the kernel-method treatment of catastrophe paths in (Banderier and Wallner, 2017). The numerator must vanish there, so $q_r g_1^r = \gamma G^b(z, x_1)$, which gives the first formula in (3.8). Substitution of the finite sum (3.5) into the recurrence verifies the remaining formula. Uniqueness also follows coefficient by coefficient from the initial empty configuration. $\square$

Thus the kernel equations have a path interpretation. The blue part builds the level at the switch, and the red part evolves until it either ends in the prescribed state or reaches zero.

3.3 A slot first return

Start at blue zero and let τ be the first strictly positive slot at which the path is again at blue zero. A stay at zero has $\tau = 1$. Define $B_M(z)$ to count the first-return paths whose first step is an arrival and whose last step is service. Define $J_b(z)$ and $J_r(z)$ to count those ending in a blue or red catastrophe. A completed positive path has just one terminating event. Therefore

$$B_M(z) = q_r z g_1^r(z) = \frac{\gamma p_b^0 z^2 x_1(z)}{\Delta_b(z)}, \quad (3.10)$$

$$J_b(z) = \xi_b z G^b(z, 1), \quad J_r(z) = \xi_r z G^r(z, 1), \quad (3.11)$$

$$B_D(z) = B_M(z) + J_b(z) + J_r(z), \quad (3.12)$$

$$C_M(z) = h_b^0 z + B_M(z), \quad C_D(z) = h_b^0 z + B_D(z). \quad (3.13)$$

The subscript M in B_M indicates the Motzkin service-return series. Under (2.1), B_M is the generating function for a first return that begins with an arrival and ends by service; its total mass may be less than one. The full first-return generating function is C_D .

For the complete paths, let

$$F_D^b(z, x) = \sum_{i \geq 0} f_i^b(z) x^i, \quad F_D^r(z, x) = \sum_{i \geq 1} f_i^r(z) x^i.$$

Thus z records the full number of slots, including earlier completed returns, and x records the final level.

Theorem 3.3. *Every path is a unique sequence of completed first returns followed by an unfinished prefix. Consequently, with $U_D(z) = 1/(1 - C_D(z))$,*

$$f_0^b(z) = U_D(z), \quad f_i^b(z) = U_D(z) g_i^b(z), \quad f_i^r(z) = U_D(z) g_i^r(z), \quad i \geq 1. \quad (3.14)$$

Equivalently,

$$F_D^b(z, x) = \frac{1 + G^b(z, x)}{1 - C_D(z)}, \quad F_D^r(z, x) = \frac{G^r(z, x)}{1 - C_D(z)}. \quad (3.15)$$

Proof. Cut a path at every visit to zero, treating a stay there as a one-step return. The final piece either is empty or remains strictly positive after its first arrival. These cuts are forced by the path, so the decomposition is unique. Weights and lengths multiply under concatenation. Summing the geometric series of completed pieces proves the formulas. $\square$

This is the first-return renewal decomposition applied to the present path model, in the same regenerative spirit as the preceding event-clock analysis. All possible return causes use the same prefix functions. When $\xi_b = \xi_r = 0$, equations (3.14)–(3.15) are algebraically equivalent to the disaster-free Motzkin generating functions in Section 5 of Paper I; here the first-return form is retained because it separates the possible return causes before they are recombined.

If u marks the number of catastrophes, replace the denominator in (3.15) by

$$1 - C_M(z) - u\{J_b(z) + J_r(z)\}.$$

Extracting $[u^k z^n x^i]$ gives a joint weight, or a joint probability under (2.1). A conditional probability given k catastrophes requires division by the probability of that event.

For general weights, define

$$\mathcal{H}(z) = 1 + G^b(z, 1) + G^r(z, 1). \quad (3.16)$$

Here $\mathcal{H}$ denotes the survivor-prefix series. Under (2.1), its coefficient of z^n is $\mathbb{P}(\tau > n)$. Summing the survival probabilities gives

$$1 - C_D(z) = (1 - z)\mathcal{H}(z). \quad (3.17)$$

The identity is valid for $|z| < 1$ even when the return is defective, meaning that it fails to occur with positive probability. If $\mathbb{E}\tau < \infty$, then $\mathcal{H}(1) = \mathbb{E}\tau$.

3.4 Which cycle duration is being measured?

There are three useful durations.

First, τ is the slot first-return time just defined. Second, the positive part starts at blue one immediately after the arrival to an empty system and stops at its first return to zero. Third, an idle-plus-positive cycle groups all consecutive stays at zero, includes the departure arrival and then includes the positive part. Their series are, respectively,

$$C_D(z), \quad \frac{B_D(z)}{p_b^0 z}, \quad \hat{C}_D(z) = \frac{B_D(z)}{1 - h_b^0 z}. \quad (3.18)$$

The second formula assumes $p_b^0 > 0$ and removes the forced departure factor. A slot first return conditioned to depart immediately has series $B_D(z)/p_b^0$ and is one slot longer than the positive part.

There is another grouping in the earlier continuous-time literature: a whole vacation lasts from an empty state until a successful switch to red and may contain several blue catastrophes and repeated empty intervals. This longer grouping is used in the comparison in Appendix D.

3.5 A finite calculation for any slot T

Let $c_n = [z^n]C_D(z)$ and $u_n = [z^n]U_D(z)$. The exact renewal recursion is

$$u_0 = 1, \quad u_n = \sum_{k=1}^n c_k u_{n-k}, \quad n \geq 1. \quad (3.19)$$

Together with Proposition 3.2, this gives $b_{T,i}$ and $r_{T,i}$ using finite sums only. In particular,

$$b_{T,i} = \sum_{m=0}^{T-i} u_m p_b^0 p_b^{i-1} \binom{T-m-1}{i-1} h_b^{T-m-i}, \quad i \geq 1, \quad (3.20)$$

and $b_{T,0} = u_T$. For red,

$$r_{T,i} = \sum_{m=0}^T u_m [z^{T-m}] \{g_1^r(z) S_i(z)\}. \quad (3.21)$$

Equations (A.1), (3.8) and finite series multiplication compute every term in (3.21). This gives an exact finite coefficient representation in T and i . Direct state recurrences (2.2)–(2.4) provide an independent check.

4 Stationarity, decomposition and level crossing

The fixed-slot formulas describe the queue at a finite horizon. For long-run behavior, returns to blue zero provide a regenerative structure. During one first-return cycle the path spends a certain number of slots at each blue and red level; when the mean cycle length is finite, the stationary probability of a state is its expected occupation in one cycle divided by the expected cycle length. We first obtain this stationary law. The same occupation formulas then give a conditional decomposition and level-crossing balances. The stationary probabilities will also be the reference values in the relaxation analysis of Section 6.

4.1 Existence and the stationary distribution

For the nondegenerate statements in this section assume (2.1), $p_b^0, p_b, p_r, q_r, \gamma > 0$, and either $\xi_r > 0$, or $\xi_r = 0$ and $p_r < q_r$. Put $\theta_b = \gamma + \xi_b > 0$. The number of positive-blue slots until switch or catastrophe has a geometric tail with success probability θ_b . Red occupation is bounded by a geometric catastrophe time if $\xi_r > 0$. Without red catastrophes its mean from level m is $m/(q_r - p_r)$. The level passed to red has a geometric tail. Thus the first-return time τ defined in Section 3.3 satisfies $\mathbb{E}\tau < \infty$ under these hypotheses.

For $\xi_r = 0, p_r = q_r$ the red return is almost sure but has infinite mean; with red accessible this is the null-recurrent boundary. For $\xi_r = 0, p_r > q_r$ there is a positive probability of never returning after a switch.

Define the stationary ratios and roots

$$a = \frac{p_b}{1 - h_b}, \quad t_{\pm} = \frac{1 - h_r \pm \sqrt{(1 - h_r)^2 - 4p_r q_r}}{2q_r}. \quad (4.1)$$

The smaller root is t_- . Here $0 < a < 1$, $0 < t_- < 1$, and $t_+ \geq 1$, with $t_+ = 1$ exactly when $\xi_r = 0$ in the stable case. Also $a = a(1)$ and $t_- = t(1)$. Put

$$K = \frac{\gamma p_b^0}{(1 - h_b)q_r(t_+ - a)}, \quad m_0 = 1 + \frac{p_b^0}{\theta_b} + \frac{K}{(1 - a)(1 - t_-)}. \quad (4.2)$$

Theorem 4.1. *Assume (2.1), $p_b^0, p_b, p_r, q_r, \gamma > 0$, and either $\xi_r > 0$ or $\xi_r = 0$ with $p_r < q_r$. With the roots and constants in (4.1)–(4.2), the stationary probabilities are*

$$\pi_0^b = \frac{1}{m_0}, \quad (4.3)$$

$$\pi_i^b = \pi_0^b \frac{p_b^0}{1 - h_b} a^{i-1}, \quad i \geq 1, \quad (4.4)$$

$$\pi_i^r = \pi_0^b K \sum_{j=0}^{i-1} a^j t_-^{i-1-j}, \quad i \geq 1. \quad (4.5)$$

Under the stated hypotheses the chain is irreducible and aperiodic, and $b_{T,i} \rightarrow \pi_i^b$, $r_{T,i} \rightarrow \pi_i^r$ as $T \rightarrow \infty$.

Proof. The expected number of visits to positive state i before the first return is $g_i^c(1)$. The initial visit at zero contributes one. Evaluating Proposition 3.2 at one gives

$$g_i^b(1) = \frac{p_b^0}{1 - h_b} a^{i-1}, \quad g_i^r(1) = K \sum_{j=0}^{i-1} a^j t_-^{i-1-j}.$$

For example, $x_1(1) = 1/t_+$ and $\Delta_b(1) = (1 - h_b)(t_+ - a)/t_+$ give $g_1^r(1) = K$ directly. Summing these occupations gives $m_0 = \mathbb{E}\tau$. Dividing the expected visits by the expected cycle length proves the invariant probabilities. Positive recurrence and irreducibility give uniqueness. A return path of length three has weight $p_b^0 \gamma q_r^2 > 0$, and one of length five has weight $p_b^0 p_b \gamma q_r^2 > 0$; hence the period is one. The standard convergence theorem then gives the stated time limits. $\square$

Let N denote the level of a stationary observation. The regime probabilities and mean level are consequently

$$\pi^{b,+} = \sum_{i \geq 1} \pi_i^b = \pi_0^b p_b^0 / \theta_b, \quad \pi^r = \frac{\pi_0^b K}{(1 - a)(1 - t_-)}, \quad (4.6)$$

$$\mathbb{E}_\pi N = \frac{\pi^{b,+}}{1 - a} + \pi^r \left(\frac{1}{1 - a} + \frac{t_-}{1 - t_-} \right). \quad (4.7)$$

4.2 A conditional distributional decomposition

For $0 \leq v < 1$, $\text{Geom}_0(v)$ denotes the law on $0, 1, \dots$ with mass $(1 - v)v^j$. Normalizing (4.4) and (4.5) gives

$$N \mid \{\text{positive blue}\} \stackrel{d}{=} 1 + X_a, \quad N \mid \{\text{red}\} \stackrel{d}{=} 1 + X_a + Y_{t_-}, \quad (4.8)$$

where $X_a \sim \text{Geom}_0(a)$ and $Y_{t_-} \sim \text{Geom}_0(t_-)$ may be chosen independent. Indeed the conditional red generating function is

$$\frac{(1 - a)(1 - t_-)x}{(1 - ax)(1 - t_-x)}.$$

This is a distributional factorization of the stationary level.

The level at a successful blue-to-red switch has mass $(1 - a)a^{i-1}$. To see this, remove blue stays. Before the first terminating event, the probability of an up-step is a , and that of a switch is $\gamma/(1 - h_b)$. The unnormalized switching-level mass is therefore $\gamma a^{i-1}/(1 - h_b)$; summation and conditioning give the claim. Thus the first factor in (4.8) has the switching-level distribution.

At $a = t_-$ the red conditional law has mass $i(1 - a)^2 a^{i-1}$. For $a > t_-$ its spatial decay is governed by a , and for $a < t_-$ by t_- . For $\xi_r = 0$, the continuous-time specialization of (4.8) is consistent with the classical Fuhrmann–Cooper decomposition (Fuhrmann and Cooper, 1985) and follows from the vacation-queue decomposition theorem of Kleiner et al. (2023a) after taking unit batches and exponential working service. Formula (4.8) also includes a positive red catastrophe rate through the root t_- .

4.3 Crossing a level

For a finite path and an integer $k \geq 0$, let U_k be its number of crossings from level k to $k + 1$, D_k its number of service crossings from $k + 1$ to k , and I_k^C its number of catastrophes whose source level exceeds k . If the initial and final levels are i_0 and i_1 , respectively, then

$$U_k - D_k - I_k^C = \mathbf{1}_{\{i_1 > k\}} - \mathbf{1}_{\{i_0 > k\}}. \quad (4.9)$$

Each transition changes the indicator of being above k by precisely its contribution on the left. Summing proves the identity. For a completed return to zero this gives $U_k = D_k + I_k^C$. For an unfinished prefix, the endpoint indicator in (4.9) records its remaining net crossing.

Taking stationary expectations per slot gives, for $k \geq 1$,

$$p_b \pi_k^b + p_r \pi_k^r = q_r \pi_{k+1}^r + \xi_b \sum_{j \geq k+1} \pi_j^b + \xi_r \sum_{j \geq k+1} \pi_j^r. \quad (4.10)$$

At $k = 0$, the left side is $p_b^0 \pi_0^b$. No stay or switch crosses a level. Summing (4.10) over boundaries also gives the customer-flow check

$$p_b^0 \pi_0^b + p_b \pi^{b,+} + p_r \pi^r = q_r \pi^r + \xi_b \sum_{j \geq 1} j \pi_j^b + \xi_r \sum_{j \geq 1} j \pi_j^r.$$

The geometric tails justify this summation. A catastrophe contributes its entire source height to the departure count.

When the catastrophe frequency in a regime is positive, its source-height distribution is proportional to $\xi_c \pi_i^c$. Because ξ_c is constant on positive levels, conditioning on the regime cancels this factor. The severity laws are exactly the positive-blue and red conditional laws in (4.8).

The same roots a, t_-, t_+ give a finite expression for the maximum level in a return cycle. Appendix B.1 derives it and shows that its tail changes scale at the same spatial equality $a = t_-$.

5 The event clock and continuous-time probabilities

The coefficients $b_{n,i}$ and $r_{n,i}$ describe the queue after n slots. This slot count is distinct from two related descriptions of time. Deleting ordinary stays produces the effective event sequence, or event skeleton, which preserves the order of arrivals, services, switches and catastrophes but not their elapsed time. Alternatively, uniformization embeds the slotted transition matrix in continuous time and yields state probabilities at a prescribed physical time t .

We therefore proceed in two steps. First we relate a slot path to its event skeleton through geometric holding blocks. We then use uniformization to reconstruct the continuous-time state probabilities from the slot coefficients.

5.1 The event sequence and its holding times

Assume normalized probabilities and $h_b, h_r, h_b^0 < 1$. Observe the chain only at successive non-stay transitions. Ordinary stays, including stays at zero, are deleted, while switches and catastrophes are retained. Conditional on an effective transition from a positive blue state, the probabilities of up, switch and catastrophe are

$$\bar{p}_b = \frac{p_b}{1 - h_b}, \quad \bar{\gamma} = \frac{\gamma}{1 - h_b}, \quad \bar{\xi}_b = \frac{\xi_b}{1 - h_b}.$$

The red probabilities are normalized by $1 - h_r$ in the same way. At zero the next effective transition is an arrival.

If a source state has stay probability h , the number K of deleted stays before the next effective event has

$$\mathbb{P}(K = k) = (1 - h)h^k, \quad k \geq 0.$$

This follows by summing the event probability over all possible effective destinations: $h^k(1-h)$. Conditional on the effective event sequence, the holding blocks are independent. A block including its terminal effective event has generating function

$$\phi_h(z) = \frac{(1-h)z}{1-hz}. \quad (5.1)$$

Thus, for a completed event sequence, the product of (5.1) over its source states gives the random-clock contribution. An unfinished observation also contains the terminal partial holding block. For an idle-plus-positive cycle the initial completed block has factor $p_b^0 z / (1 - h_b^0 z)$, whereas the positive part started at blue one begins after that block. Conditioning on a fixed total number of slots couples the holding blocks and is handled through the coefficient formulas rather than by independent sampling.

With the normalized effective-event probabilities fixed, inserting these holding blocks preserves the heights and crossing counts of a completed cycle and changes its duration. This is the geometric holding-time construction used in Paper I.

5.2 Uniformization and the one-hour question

Let $\lambda, \mu, \nu > 0$ denote respectively the common arrival rate, red service rate and positive-level vacation-ending rate. Let $\delta \geq 0$ be a common catastrophe rate. There is no service during vacation. Let Q be the infinitesimal generator of the continuous-time chain with the boundary rules of Section 2. All its exit rates are bounded. Choose

$$R \geq \max\{\lambda + \nu + \delta, \lambda + \mu + \delta\},$$

and let $P = I + Q/R$, where I is the identity matrix. The transition probabilities of P are

$$p_b^0 = p_b = p_r = \lambda/R, \quad q_r = \mu/R, \quad \gamma = \nu/R, \quad \xi_b = \xi_r = \delta/R,$$

with the stays given by row normalization. In particular $h_b^0 = 1 - \lambda/R$. At zero, the generator has a single outgoing transition: an arrival to blue level one. Uniformization supplies the stay probability h_b^0 .

Let $p_i^b(t), p_i^r(t)$ be continuous-time state probabilities at elapsed time t , with the same initial blue zero state. The uniformization identity is

$$p_i^b(t) = e^{-Rt} \sum_{n \geq 0} \frac{(Rt)^n}{n!} b_{n,i}, \quad p_i^r(t) = e^{-Rt} \sum_{n \geq 0} \frac{(Rt)^n}{n!} r_{n,i}. \quad (5.2)$$

Indeed $e^{tQ} = e^{-Rt} \sum_{n \geq 0} (Rt)^n P^n / n!$, a convergent operator series because Q is bounded. This is the classical uniformization, or randomization, construction (Gross and Miller, 1984; Leguesdron et al., 1993). Here its coefficients are supplied by the two-regime path representation.

Truncation at $n = N$ in (5.2) has componentwise error at most $\mathbb{P}\{\text{Poisson}(Rt) > N\}$. The retained vector is a subprobability vector with exactly this missing mass. Computing P^n by the finite-support recurrence avoids an independent height cutoff for $n \leq N$.

For comparison with continuous-time transforms, let

$$\widehat{p}_i^c(s) = \int_0^\infty e^{-st} p_i^c(t) dt, \quad s > 0, \quad c \in \{b, r\}.$$

For every $n \geq 0$,

$$\int_0^\infty e^{-(s+R)t} \frac{(Rt)^n}{n!} dt = \frac{1}{s+R} \left(\frac{R}{s+R} \right)^n.$$

Termwise integration of the nonnegative Poisson sum therefore gives

$$\hat{p}_i^c(s) = \frac{1}{s+R} f_i^c\left(\frac{R}{s+R}\right). \quad (5.3)$$

These are transforms of the state probabilities. The positive-part transforms and the common-rate formulas obtained from this identity are given in Appendix D.

For example, take $\lambda = 2$, $\mu = 3$, $\nu = 1$, $\delta = 0$, in inverse hours, and start empty in vacation. Then

$$p_3^r(1) = 0.0502257394147708\dots, \quad \pi_3^r = \frac{8}{81} = 0.0987654320987654\dots$$

The Poisson sums agree for $R = 5, 8, 13$, and a direct finite-matrix exponential provides a further numerical check. Thus $p_3^r(1)$ is the joint state probability of being in the working regime with three customers. Truncation details for the numerical evaluation are given in Appendix E.

A red segment has a classical modified-Bessel representation. Let I_k denote the modified Bessel function of the first kind. For an elapsed red-segment duration $s > 0$ and positive levels m, j , let $K_{m,j}(s)$ be the probability of a red transition from m to j before absorption at zero or by a catastrophe. Its kernel is

$$K_{m,j}(s) = e^{-(\lambda+\mu+\delta)s} \left(\frac{\lambda}{\mu}\right)^{(j-m)/2} \left[I_{|j-m|}(2s\sqrt{\lambda\mu}) - I_{j+m}(2s\sqrt{\lambda\mu}) \right]. \quad (5.4)$$

Expanding the unrestricted birth–death walk gives its $I_{|j-m|}$ term; reflection of paths that visit zero gives the subtracted I_{j+m} term. The common net increment gives the factor $(\lambda/\mu)^{(j-m)/2}$, and independent catastrophe killing gives $e^{-\delta s}$. From initial blue zero, the last switch into the current red segment gives

$$p_j^r(t) = \nu \int_0^t \sum_{m \geq 1} p_m^b(u) K_{m,j}(t-u) du.$$

Without catastrophes, this convolution is equation (5.6) of Sudhesh et al. (2017) after setting their vacation service and impatience rates to zero. The additional factor $e^{-\delta s}$ in (5.4) is independent catastrophe killing. The convolution uses the blue probabilities obtained from the same forward system. Appendix E gives the numerical comparison with uniformization and a matrix exponential.

Joint occupation times in blue and red are considered in Appendix B.2. In that calculation, a slot belongs to its source regime; in particular, the switch slot counts as blue time. This convention is also used for the continuous-time comparison in Appendix D.

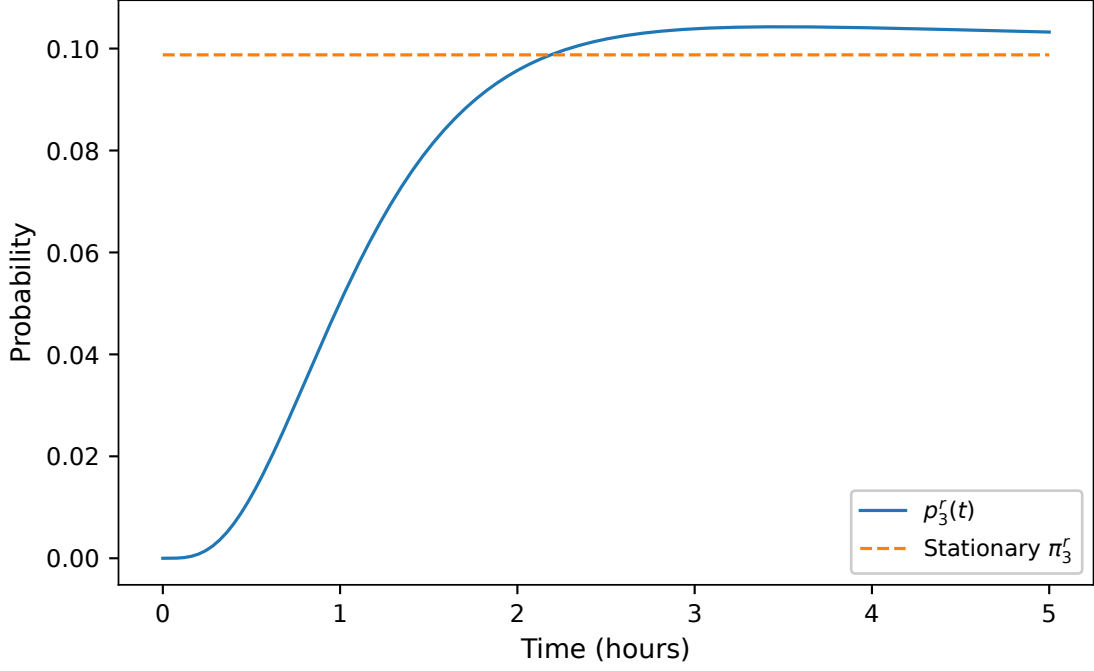


Fig. 2 The probability $p_3^r(t)$ for $\lambda = 2$, $\mu = 3$, $\nu = 1$, $\delta = 0$, starting from blue level zero. Time is measured in hours. The continuous curve is obtained by Poisson summation, and the horizontal line is the stationary probability. The transient probability rises above its stationary value and then approaches it from above.

6 Fixed-state relaxation and the vacation event

How quickly does the probability of an empty system approach its stationary value, and does the same rate describe the probability that the server is on vacation? The first event is a single state; the second includes every blue level. We calculate both from the functions of Section 3. Equal arrival and catastrophe probabilities across the two regimes allow the relevant cancellations to be written explicitly.

6.1 A specified common-rate family

Assume normalized transition probabilities in the common-rate family

$$\begin{aligned} p_b^0 = p_b = p_r = p > 0, & \quad \xi_b = \xi_r = \xi \geq 0, \\ h_b = 1 - p - \gamma - \xi, & \quad h_r = 1 - p - q_r - \xi. \end{aligned} \quad (6.1)$$

with $q_r, \gamma > 0$, $h_b \geq 0$, $h_r > 0$, $h_b^0 = 1 - p$. Assume $\xi > 0$ or, if $\xi = 0$, $p < q_r$. These are the hypotheses of Theorem 4.1; the stationary probabilities used below are therefore well defined.

Theorem 6.1. *For the family (6.1),*

$$f_0^b(z) = \frac{\Delta_b(z)(1 - t(z))}{(1 - z)(1 - h_b z)}, \quad (6.2)$$

$$f_i^b(z) = a(z)^i f_0^b(z), \quad i \geq 1, \quad (6.3)$$

$$f_i^r(z) = \frac{\gamma p z x_1(z)(1 - t(z))}{q_r(1 - z)(1 - h_b z)} S_i(z), \quad i \geq 1. \quad (6.4)$$

Proof. Define $D_s(z) = 1 - (1 - \xi)z$, $D_b(z) = 1 - (h_b + p)z = D_s(z) + \gamma z$ and $E(z) = 1 - z(h_r + q_r + px_1(z))$. The red kernel equation implies $D_s = E(1 - t)$, and $\Delta_b = E + \gamma z$. The prefix formulas give

$$G^b(z, 1) = \frac{pz}{D_b(z)}, \quad G^r(z, 1) = \frac{\gamma pz^2(1 - x_1(z))(1 - h_b z)}{D_b(z)\Delta_b(z)D_s(z)}.$$

On summing the two prefixes and the empty prefix, cancellation gives

$$\mathcal{H}(z) = \frac{(1 - h_b z)E(z)}{\Delta_b(z)D_s(z)} = \frac{1 - h_b z}{\Delta_b(z)(1 - t(z))}.$$

Equation (3.17) now gives (6.2). Multiplication by the fixed-height prefixes proves the other two formulas. All cancellations are formal-series identities, continued through removable singularities. $\square$

The factor Δ_b now appears in the numerator. This cancellation leaves the holding factor $1 - h_b z$ and the red branch point to govern fixed-state relaxation.

6.2 The polynomial factors at a fixed level

Let $K_r = h_r + 2\sqrt{pq_r}$. If $h_b \geq K_r$, put $z_b = 1/h_b$ and

$$A_0 = \frac{\Delta_b(z_b)(1 - t(z_b))}{1 - z_b}.$$

Under (6.1), this constant is nonzero. At z_b we have $\Delta_b(z_b) = -pz_b x_1(z_b)$ and $g_1^r(z_b) = -\gamma/q_r$. For every fixed $i \geq 1$,

$$b_{n,0} - \pi_0^b \sim A_0 h_b^n, \quad (6.5)$$

$$b_{n,i} - \pi_i^b \sim \frac{A_0 (pz_b)^i}{i!} n^i h_b^n, \quad (6.6)$$

$$r_{n,i} - \pi_i^r \sim -\frac{\gamma}{q_r} \frac{A_0 (pz_b)^{i-1}}{(i-1)!} n^{i-1} h_b^n. \quad (6.7)$$

If $h_b > K_r$, then $z_b = 1/h_b$ lies strictly before the red branch point. The function $a(z)$ has a simple pole at z_b , and the largest pole in S_i is its term $a(z)^{i-1}$. Thus the pole orders of f_0^b, f_i^b, f_i^r are $1, i+1, i$. A factor $(1 - z/z_b)^{-j}$ has coefficient $\binom{n+j-1}{j-1} z_b^{-n}$, whose polynomial part is $n^{j-1}/(j-1)!$. Applying this rule to the three pole orders gives the powers of n and the factorials in (6.5)–(6.7). Subtracting the stationary term $\pi_i^c/(1 - z)$ does not affect a principal part at $z_b > 1$.

If $h_b = K_r$, the same point is also the red branch point. The functions have Puiseux expansions in integer and half-integer powers of $1 - z/z_b$. In each numerator its value at z_b gives the same leading term as above; the square-root correction reduces the coefficient order by a factor $n^{-1/2}$. The nonzero leading constants therefore give the same equivalents at the boundary.

If $h_b < K_r$, center each fixed-state function by subtracting its stationary pole. It then has a square-root expansion at $1/K_r$. If the square-root coefficient for the specified state is nonzero, its deviation is equivalent to a signed constant times $K_r^n n^{-3/2}$. When the square-root amplitude vanishes, the next singular term determines the decay. For $h_r = 0$, both red branch points contribute to the corresponding coefficient calculation.

For common continuous-time rates $\lambda, \mu, \nu, \delta$, comparison of h_b and K_r gives the local boundary

$$\nu_{\text{loc}} = \mu - 2\sqrt{\lambda\mu}. \quad (6.8)$$

The label “loc” refers to a fixed level. A nonpositive value in (6.8) means that the corresponding positive-rate local blue-pole region is absent.

6.3 Different rates for a state and the vacation regime

Summing (6.3) over all blue levels gives a different denominator:

$$F_D^b(z, 1) = \frac{\Delta_b(z)(1 - t(z))}{(1 - z)\{1 - (h_b + p)z\}}. \quad (6.9)$$

Define $B_n = \sum_{i \geq 0} b_{n,i}$ and $\pi^b = \sum_{i \geq 0} \pi_i^b$. If $h_b + p > K_r$ and the numerator at $z_A = 1/(h_b + p)$ is nonzero, then

$$B_n - \pi^b \sim \frac{\Delta_b(z_A)(1 - t(z_A))}{1 - z_A} (h_b + p)^n.$$

For example, take $p = 1/32$, $q_r = 25/32$, $\gamma = 1/32$ and $\xi = 0$. Then $h_b = 30/32$, $h_r = 6/32$, $K_r = 16/32$. The fixed-zero error has exponential factor $(30/32)^n$, while the blue-regime error has factor $(31/32)^n$, with nonzero constants in both cases.

Let P_n denote the complete state law at slot n . Total variation between two laws is the supremum of their differences on events. Applying this definition to the event “blue regime” gives

$$\|P_n - \pi\|_{\text{TV}} \geq |B_n - \pi^b|.$$

This lower bound separates the fixed-zero rate from convergence of the complete distribution. Determining an exact total-variation rate requires uniform control over the level index.

Figure 3 compares these errors. The slower blue-event term arises after summing over an unbounded set of levels, even though every fixed-level probability is given by the same renewal construction.

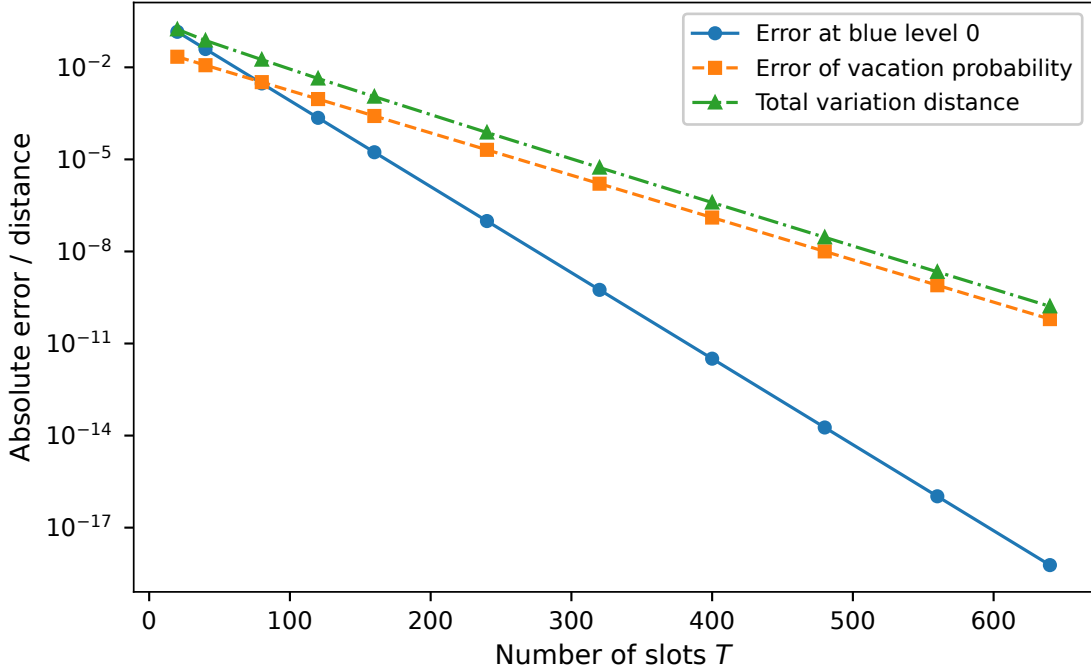


Fig. 3 Local and aggregate relaxation for $p = \gamma = 1/32$, $q_r = 25/32$ and $\xi = 0$, starting from blue zero. The fixed-zero error has exponential factor $(30/32)^n$, whereas the blue-regime error has factor $(31/32)^n$. Total variation is shown numerically; the blue-event error gives its lower bound. The small fixed-zero differences are computed by exact integer subtraction.

7 Service cycles and the level at the switch

We now condition on a cycle length rather than observe the queue at a chosen time. A service-return cycle begins with an arrival to an empty system and ends at its first return by service. Its length includes the initial arrival, the switch and the final service, and is one slot longer than the positive part of Section 3.4. We ask how many customers are present at the switch in a long such cycle. The denominator of the conditional law counts all service-return cycles of the specified length; we calculate its asymptotics first and then mark the switch level.

This section counts service-return cycles with nonnegative weights. No catastrophe occurs in a counted cycle, so the catastrophe weights do not enter these coefficients. Assume $p_b, p_b^0, p_r, q_r, \gamma > 0$, $h_b \geq 0$ and $h_r > 0$. The radical in (3.2) has its first positive branch point at

$$z_1 = \frac{1}{h_r + 2\sqrt{p_r q_r}},$$

which is the radius of convergence of $x_1(z)$. At this point $x_1(z_1) = x_* = \sqrt{q_r/p_r}$. For short formulas put $\kappa = \sqrt{p_r q_r}$ and $\chi = \sqrt{\kappa/z_1}/p_r$. Then

$$x_1(z) = x_* - \chi\sqrt{1 - z/z_1} + O(1 - z/z_1).$$

Let $\Delta_1 = \Delta_b(z_1)$, where Δ_b was defined in (3.4). The competition is between the zero of Δ_b and the red kernel branch point.

Theorem 7.1. *Under these hypotheses, the service-cycle coefficients have the following equivalents. If $\Delta_1 > 0$,*

$$[z^n]B_M(z) \sim \frac{\gamma p_b^0 z_1^2 \chi (1 - h_b z_1)}{2\sqrt{\pi} \Delta_1^2} z_1^{-n} n^{-3/2}. \quad (7.1)$$

If $\Delta_1 = 0$,

$$[z^n]B_M(z) \sim \frac{\gamma p_b^0 z_1 x_*}{p_b \chi \sqrt{\pi}} z_1^{-n} n^{-1/2}. \quad (7.2)$$

If $\Delta_1 < 0$, there is a unique positive zero $z_2 < z_1$ of Δ_b and

$$[z^n]B_M(z) \sim \frac{\gamma p_b^0 z_2^2 x_1(z_2)}{1 + p_b z_2^2 x_1'(z_2)} z_2^{-n}. \quad (7.3)$$

Proof. The function $z(h_b + p_b x_1(z))$ is strictly increasing for positive $z < z_1$. This proves existence and uniqueness of the positive pole when $\Delta_1 < 0$. Positivity and the consecutive nonzero powers supplied by $h_r > 0$ exclude another pole of equal modulus. At that zero, $-z_2 \Delta_b'(z_2) = 1 + p_b z_2^2 x_1'(z_2)$, giving (7.3). When $\Delta_1 > 0$, regard B_M as a function of x_1 at $z = z_1$. Its derivative with respect to x_1 is $\gamma p_b^0 z_1^2 (1 - h_b z_1)/\Delta_1^2$. Multiplication by the square-root coefficient $-\chi$ of x_1 gives the coefficient $-\gamma p_b^0 z_1^2 \chi (1 - h_b z_1)/\Delta_1^2$ in B_M . If $\Delta_1 = 0$, the constant denominator term vanishes. Its leading term is $z_1 p_b \chi \sqrt{1 - z/z_1}$, while the numerator tends to $\gamma p_b^0 z_1^2 x_*$. Division gives the inverse-square-root term. Applying the coefficient rules for exponents $1/2$ and $-1/2$ proves (7.1) and (7.2); see Flajolet and Sedgewick (2009). $\square$

Thus the critical equality is

$$h_b + p_b \sqrt{q_r/p_r} = h_r + 2\sqrt{p_r q_r}. \quad (7.4)$$

If $h_r = 0$, the two red branch points have equal modulus. Their coefficient contributions must then be combined. When both positive-level stay weights vanish, exact formulas reduce to the preceding event-clock model and its parity-sensitive asymptotics.

7.1 A first-down-step correspondence

An ordinary Motzkin meander starts at zero, uses up, down and horizontal steps, stays nonnegative, and may end at any level. An excursion is such a path ending at zero. The following correspondence explains how one switch records the first down-step of an ordinary path.

Define the ordinary Motzkin excursion series by

$$M(z) = 1 + zM(z) + z^2M(z)^2, \quad M(0) = 1. \quad (7.5)$$

At unit red weights, the kernel root is $x_1(z) = zM(z)$.

Proposition 7.2. *Nonempty two-color unfinished positive prefixes correspond bijectively to nonnegative ordinary Motzkin meanders starting with an up-step, preserving length. Service-return cycles of length n correspond bijectively to ordinary Motzkin excursions of length $n - 1$ starting with an up-step.*

Proof. Before a switch there are no down-steps. Replace the switch S by a down-step D and lower every subsequent red vertex by one. The red part was strictly positive, so the resulting path is nonnegative. The replacement D is the first down-step. If no switch occurred, leave the up/stay prefix unchanged. Conversely, locate the first down-step of the meander, replace it by S , raise the following segment by one, and color the two segments blue and red. A meander with no down-step is all blue. These operations are inverses.

For a service-return cycle remove its final red service step first, and apply the same map. The remaining endpoint was red one, so the image ends at zero. Conversely the excursion map gives a prefix ending at red one; appending a service step returns it to blue zero. The length changes by one. This proves both assertions. $\square$

The source height of the first down-step is exactly the queue level at vacation completion. It counts all preceding up-steps, including those separated by stays. For example, the ordinary excursion $UH UDD$ has initial consecutive rise length one but first-down-step source height two. However, if all horizontal stays are deleted, the source height of the first down-step becomes the initial-rise length of the resulting Dyck path. Thus the coefficient refinement in Appendix A can also be obtained from the classical initial-rise count followed by insertion of horizontal steps; compare (Sapounakis and Tsikouras, 2004). The correspondence above is used because it keeps the two-regime weights visible.

The correspondence acts on each unfinished prefix or service-return piece separately. Its weighted version uses the same division at the first down-step: use blue weights before the first D , give that first D weight γ , and use red weights after it. The initial up-step has weight p_b^0 . In the cycle correspondence the removed final step contributes the additional factor $q_r z$.

7.2 Marking the level of the switch

Let $B_m(z)$ count service-return cycles whose switch occurs at level $m \geq 1$. The blue segment has its initial arrival, $m - 1$ subsequent arrivals, and arbitrary stays before its switch. The red segment has series $x_1(z)^m$. Thus, for general weights,

$$B_m(z) = \frac{\gamma p_b^0 z^2 (p_b z)^{m-1} x_1(z)^m}{(1 - h_b z)^m}. \quad (7.6)$$

At unit weights this is $z^{2m+1}M(z)^m/(1-z)^m$. Under the bijection, m is the source height of the first down-step of the ordinary excursion.

7.3 The weighted switch level and three limiting regimes

Continue with the general weights of this section. Assume the hypotheses of Theorem 7.1: all up, service, switch and initial-arrival weights are positive, $h_b \geq 0$ and $h_r > 0$. Only service-return cycles are counted in this subsection. Each such cycle contains one switch and no catastrophe. For each length $n \geq 3$, choose a cycle with probability proportional to its weight, and let L_n be the level at which the switch occurs. In particular,

$$\mathbb{P}(L_n = m) = \frac{[z^n]B_m(z)}{[z^n]B_M(z)}, \quad m \geq 1. \quad (7.7)$$

The denominator is positive for every $n \geq 3$: a level-one cycle can have any number of red stays, since $h_r > 0$. Thus (7.7) defines a probability measure on the finite set of service-return paths of length n using their combinatorial weights. Under (2.1), this is the law of the switch level conditional on departure in the first slot and on the first return to zero occurring by service at slot n .

The blue segment before the switch contains its initial arrival, $m - 1$ additional arrivals and stays. Following the switch, the red part must make m successive downward passages. Thus (7.6) may be written in terms of one block series,

$$Y(z) = \frac{p_b z x_1(z)}{1 - h_b z}, \quad B_m(z) = \frac{\gamma p_b^0}{p_b} z Y(z)^m. \quad (7.8)$$

If w marks switch level, the bivariate generating function is

$$\mathcal{B}(z, w) = \sum_{m \geq 1} w^m B_m(z) = \frac{\gamma p_b^0}{p_b} \frac{z w Y(z)}{1 - w Y(z)}. \quad (7.9)$$

In particular, $\mathcal{B}(z, 1) = B_M(z)$, and the common factor $\gamma p_b^0/p_b$ cancels from the conditional law (7.7).

Recall z_1, x_*, χ, Δ_1 defined above. If $\Delta_1 \geq 0$, then $h_b z_1 < 1$. Define

$$y_* = Y(z_1), \quad \beta_* = \frac{p_b z_1 \chi}{1 - h_b z_1} > 0.$$

The expansion at the red branch point is

$$Y(z) = y_* - \beta_* \sqrt{1 - z/z_1} + O(1 - z/z_1). \quad (7.10)$$

When $\Delta_1 = 0$, $y_* = 1$ and $\beta_* = \chi/x_* = 1/\sqrt{\kappa z_1}$.

Theorem 7.3 (Switch level in a long service-return cycle). *For the weighted law (7.7), the three cases of Theorem 7.1 have the following consequences.*

(i) If $\Delta_1 > 0$, then $0 < y_* < 1$ and, for every $m \geq 1$,

$$\mathbb{P}(L_n = m) \longrightarrow m(1 - y_*)^2 y_*^{m-1}. \quad (7.11)$$

All moments converge. In particular,

$$\mathbb{E}L_n \longrightarrow \frac{1 + y_*}{1 - y_*}, \quad \text{Var}(L_n) \longrightarrow \frac{2y_*}{(1 - y_*)^2}. \quad (7.12)$$

(ii) If $\Delta_1 = 0$, then $L_n/\sqrt{n}$ converges in distribution and in all moments to a nonnegative random variable Z with density

$$f_Z(s) = \frac{\beta_*^2 s}{2} \exp\left(-\frac{\beta_*^2 s^2}{4}\right), \quad s \geq 0. \quad (7.13)$$

This is the Rayleigh distribution with scale $\sqrt{2}/\beta_*$. Consequently,

$$\mathbb{E}L_n \sim \frac{\sqrt{\pi}}{\beta_*} \sqrt{n}, \quad \text{Var}(L_n) \sim \frac{4 - \pi}{\beta_*^2} n. \quad (7.14)$$

(iii) If $\Delta_1 < 0$, let z_2 be the pole of Theorem 7.1, equivalently $Y(z_2) = 1$. Define

$$m_B = z_2 Y'(z_2), \quad v_B = z_2^2 Y''(z_2) + m_B - m_B^2, \quad \alpha_L = \frac{1}{m_B}, \quad \sigma_L^2 = \frac{v_B}{m_B^3}. \quad (7.15)$$

Here $\mathcal{N}(0, 1)$ denotes the normal law with mean zero and variance one. Then $v_B > 0$ and

$$\frac{L_n - \alpha_L n}{\sqrt{\sigma_L^2 n}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \mathbb{E}L_n = \alpha_L n + O(1), \quad \text{Var}(L_n) = \sigma_L^2 n + O(1). \quad (7.16)$$

The proof is given in Appendix C.1. It uses the square-root expansion in the first two cases and the simple pole of $1 - wY(z)$ in the third. In particular, it supplies the tail control required for the moment assertions.

The limit mechanisms specialize Theorem 6 of Banderier et al. (2024a). In their terminology, the outer function is $v/(1 - v)$, the inner function is $zx_1(z)/(1 - h_b z)$, and the Gibbs weight is p_b . When $h_b z_1 < 1$, the critical weight is $(1 - h_b z_1)/(z_1 x_*)$, equivalently $\Delta_1 = 0$. If $h_b z_1 \geq 1$, the model is on the pole side and the direct pole proof applies. The critical case is also covered by the composition-scheme analysis of Banderier et al. (2024b). Here the marked parameter is the vacation-ending level.

The equality $\Delta_1 = 0$ is shared by the length coefficients and the conditional switch level. It differs from the stationary spatial equality $a = t_-$ in Section 4, because conditioning on a long service cycle is not stationary sampling.

7.4 Examples at fixed counting weights

At unit weights, (7.6) gives $Y(z) = z^2 M(z)/(1 - z)$ and $y_* = 1/2$. The first case of Theorem 7.3 therefore gives

$$\mathbb{P}(L_n = m) \longrightarrow \frac{m}{2^{m+1}}, \quad \mathbb{E}L_n \longrightarrow 3, \quad \text{Var}(L_n) \longrightarrow 4. \quad (7.17)$$

The exponential-moment argument in the theorem justifies both moment limits. Equivalently, the limiting level has the law of $1 + G_1 + G_2$, where G_1, G_2 are independent $\text{Geom}_0(1/2)$ variables. This is a shifted negative-binomial law. Thus the switch level remains of order one as the cycle length increases.

For an example displaying all three scales, use counting weights and leave all service-cycle weights at one except p_b . Then $\Delta_1 = (2 - p_b)/3$, and $p_b = 1, 2, 3$ give the subcritical, critical and supercritical cases, respectively. At $p_b = 2$, $\beta_* = \sqrt{3}$, so

$$\mathbb{E}L_n \sim \sqrt{\pi n/3}, \quad \mathbb{P}(L_n/\sqrt{n} \leq s) \longrightarrow 1 - e^{-3s^2/4} \quad (s \geq 0).$$

At $p_b = 3$, $z_2 = \sqrt{2}/(3 + \sqrt{2})$ and (7.15) gives $\alpha_L \approx 0.1699056897$ and $\sigma_L^2 \approx 0.2684670136$.

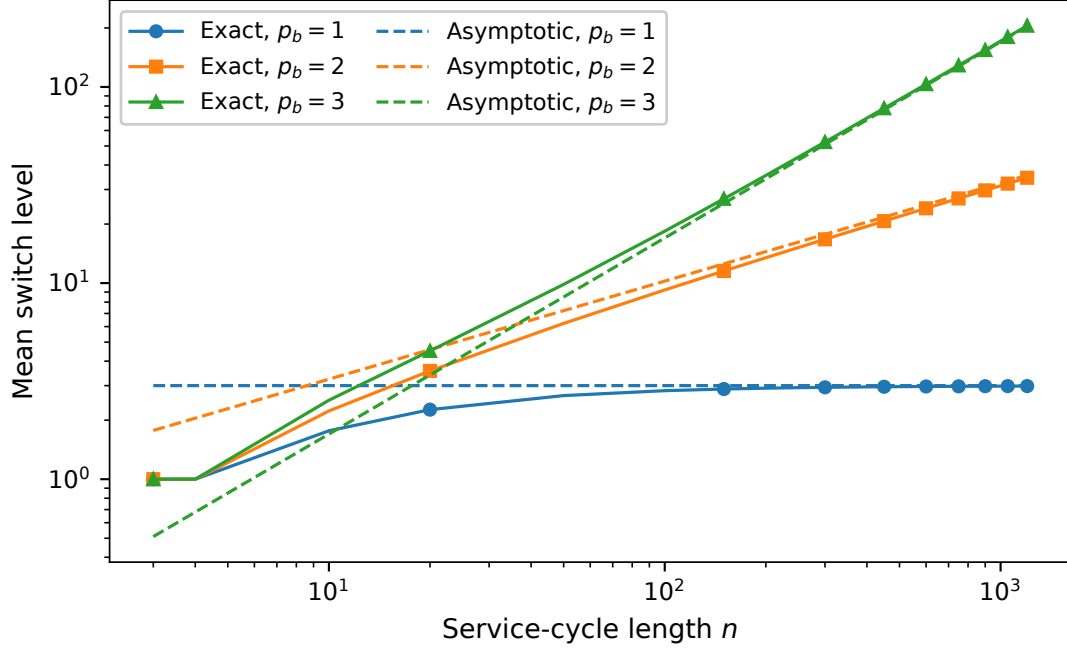


Fig. 4 Mean switch level under weighted sampling of service cycles of length n . Only the blue up-weight changes: $p_b = 1, 2, 3$; all other service-cycle weights equal one. Solid lines are exact coefficient ratios. Dashed curves give the leading asymptotic means in Theorem 7.3.

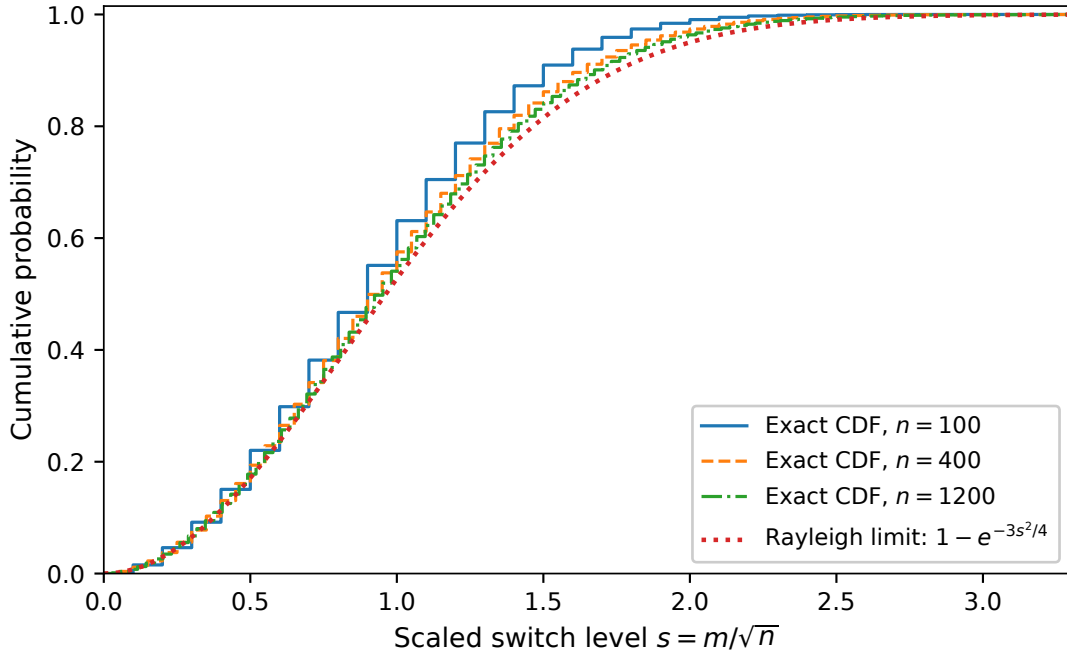


Fig. 5 Critical switch-level distribution for $p_b = 2$ and the remaining service-cycle weights equal to one. The step functions are the exact distribution functions of $L_n/\sqrt{n}$ for $n = 100, 400, 1200$. The continuous curve is the Rayleigh limit $1 - e^{-3s^2/4}$.

7.5 An exact critical mean and its finite-length correction

The critical example $p_b = 2$ permits a further calculation. Keep the other service-cycle weights equal to one. Define the central trinomial coefficients by

$$\mathsf{T}_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{j!^2(k-2j)!}, \quad k \geq 0.$$

They form sequence A002426 of the On-Line Encyclopedia of Integer Sequences (OEIS) (OEIS Foundation Inc., 2026b). Their generating function is

$$\sum_{k \geq 0} \mathsf{T}_k z^k = \frac{1}{\sqrt{1 - 2z - 3z^2}}. \quad (7.18)$$

The index k is the order of the central trinomial coefficient.

Proposition 7.4 (Critical mean at finite length). *For $p_b = 2$ and the other service-cycle weights equal to one, and for every $n \geq 3$,*

$$\mathbb{E}L_n = \frac{3^{n-2} + (-1)^{n-1}}{\mathsf{T}_{n-1} - \mathsf{T}_{n-2}} - 1. \quad (7.19)$$

As $n \rightarrow \infty$,

$$\mathbb{E}L_n = \sqrt{\frac{\pi n}{3}} - 1 - \frac{1}{16} \sqrt{\frac{\pi}{3n}} + O(n^{-3/2}). \quad (7.20)$$

The proof and the approximation-error plot are in Appendix C.2. The constant correction removes the limiting absolute error of one in the leading square-root approximation. At $n = 1200$ the exact mean is 34.44723885..., the leading term is 35.44907702..., and (7.20) without the remainder gives 34.44723071.... These values are obtained from exact weighted-path coefficients.

7.6 Unit weights and ordinary Motzkin paths

At unit weights the first-down-step correspondence explains the ordinary Motzkin sequences obtained from the two-color paths. We record the two identities here; full renewal counts with and without catastrophes are given in Appendix A.3.

Set $p_b^0 = p_b = h_b = \gamma = p_r = h_r = q_r = 1$, and set $h_b^0 = 1$ when counting full renewal paths. The catastrophe weights are specified separately. Use the Motzkin series $M(z)$ defined in (7.5). At the present weights $x_1(z) = zM(z)$. Equation (7.5) also gives $1 - z - z^2M(z) = 1/M(z)$, so

$$B_M(z) = z^3M(z)^2. \quad (7.21)$$

Its nonzero coefficients, beginning at length three, are

$$1, 2, 5, 12, 30, 76, 196, 512, 1353, \dots$$

Precisely, $[z^n]B_M(z)$ equals OEIS A002026($n - 2$) for $n \geq 3$ (OEIS Foundation Inc., 2026a). The shift matters: that OEIS entry includes an initial zero.

The total unfinished-prefix series from (3.16), without any preceding completed returns, simplifies to

$$\mathcal{H}(z) = \frac{(1-z)(1-zM(z))}{1-3z} = 1 + z + 3z^2 + 8z^3 + 22z^4 + 61z^5 + \dots \quad (7.22)$$

Here $[z^n]\mathcal{H}(z)$ is OEIS A025566($n+1$) for $n \geq 0$ (OEIS Foundation Inc., 2026d). For nonempty prefixes, Proposition 7.2 gives the meander interpretation; the empty prefix contributes the constant term.

8 Concluding remarks

For the fixed-time question, the last-return decomposition gives the probability of each level and regime. In the common-rate family, the same factorization distinguishes the relaxation of a single state from that of the whole vacation event. The explicit example in Section 6 has a slower aggregate exponential term, so summing over levels is part of the asymptotic calculation for the event being observed.

For a service-return cycle of prescribed length, the relevant quantity is the level at which service becomes available. Marking that level gives the three scales in Theorem 7.3, with parameters obtained from the cycle series. A long cycle can have a switch level that remains tight, grows on the square-root scale, or has a linear mean with normal fluctuations. In the critical specialization, Proposition 7.4 gives a finite-length mean and explains the constant error of its leading square-root approximation.

The two calculations use the same vacation and working pieces but normalize them for different observations. Additional vacation phases would require a corresponding enlargement of the regime structure; the two-regime analysis here gives the basic case.

A Finite coefficient formulas

A.1 First-passage coefficients

For $m \geq 1$ and $n \geq m$, the coefficients also have the finite expression

$$[z^n]x_1(z)^m = \sum_{u=0}^{\lfloor (n-m)/2 \rfloor} \frac{m}{2u+m} \binom{2u+m}{u} \binom{n-1}{n-2u-m} p_r^u q_r^{u+m} h_r^{n-2u-m}. \quad (\text{A.1})$$

Indeed, after removing stays the path contains u up-steps and $u+m$ down-steps. The ballot number for first reaching zero at the end is $m \binom{2u+m}{u} / (2u+m)$. Stays can be inserted before these $2u+m$ steps, but not after the final step. Distributing $n-2u-m$ stays among these slots gives the second binomial coefficient. Coefficients with $n < m$ vanish.

A.2 The unit-weight switch-level triangle

For this subsection, $M(z)$ is the Motzkin series of (7.5) and all service-cycle weights equal one. The row index is the full service-cycle length, including the initial arrival and the switch. There is a finite formula for every entry of the switch-level triangle. Let $c_{n,m} = [z^n]B_m(z)$ at unit weights. For $n \geq 2m+1$,

$$c_{n,m} = \sum_{k=m}^{\lfloor (n-1)/2 \rfloor} \frac{m}{2k-m} \binom{2k-m}{k-m} \binom{n-2}{2k-1}; \quad (\text{A.2})$$

otherwise $c_{n,m} = 0$. To see this, (7.5) gives $Y(1-Y) = z^2/(1-z)^2$. Write $\zeta = z^2/(1-z)^2$ and solve the formal equation $Y = \zeta/(1-Y)$. Lagrange inversion gives

$$[\zeta^k]Y^m = \frac{m}{2k-m} \binom{2k-m}{k-m}.$$

Since $B_m = zY^m$, extraction of the remaining power of z gives (A.2).

The resulting distribution agrees with the unit-weight limit in (7.17). Figure 6 displays the finite-length probabilities and that limit.

Table 1: Numbers of unit-weight service cycles of length n , classified by switch level. The row sums agree with $[z^n]z^3M(z)^2$.

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	Total
3	1	0	0	0	1
4	2	0	0	0	2
5	4	1	0	0	5
6	8	4	0	0	12
7	17	12	1	0	30
8	38	32	6	0	76
9	89	82	24	1	196

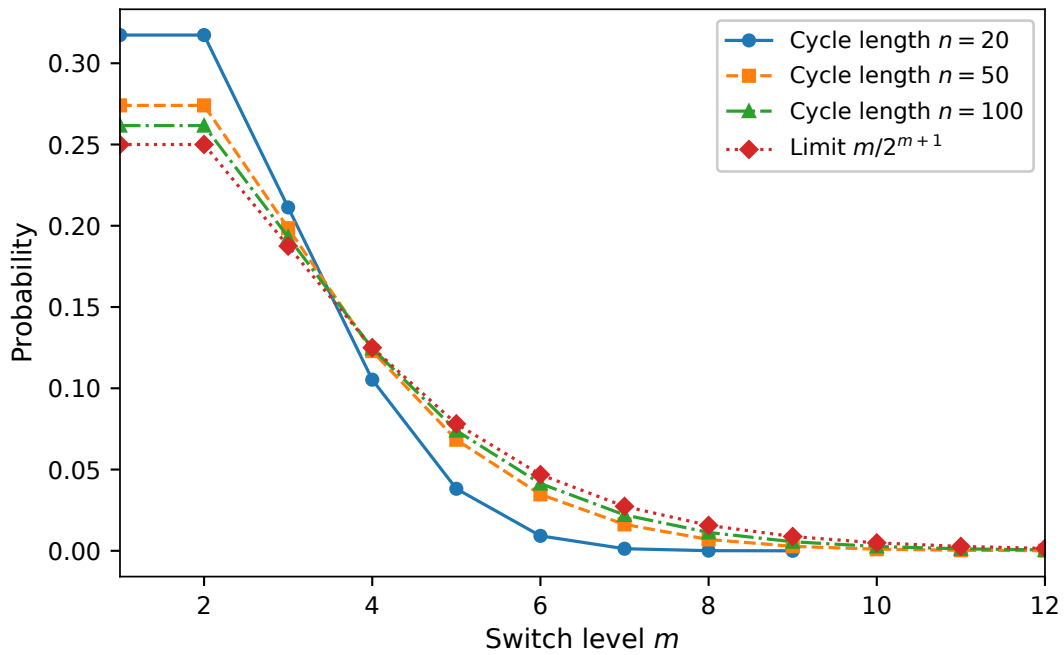


Fig. 6 Switch-level probabilities for unit-weight service cycles. The limiting masses are $m/2^{m+1}$. The theorem also supplies the exponential-moment control needed for convergence of the mean and variance.

A.3 Unit-weight renewal counts

For full unit-weight renewal paths ending at zero, with $h_b^0 = 1$ and no catastrophes,

$$U_M(z) = \frac{1}{1 - z(1 - z)M(z)} = 1 + z + z^2 + 2z^3 + 5z^4 + 13z^5 + 34z^6 + 90z^7 + \dots \quad (\text{A.3})$$

With both catastrophe weights also equal to one, let

$$J(z) = z\{\mathcal{H}(z) - 1\}, \quad U_D(z) = \frac{1}{1 - z - z^3M(z)^2 - J(z)}.$$

This gives

$$U_D(z) = 1 + z + 2z^2 + 7z^3 + 23z^4 + 75z^5 + 246z^6 + 811z^7 + \dots \quad (\text{A.4})$$

Table 2 gives the first thirteen coefficients of the two renewal series.

Table 2: Full unit-weight renewal paths ending at zero. Stays at zero have weight one in both columns.

n	$[z^n]U_M$ (no catastrophes)	$[z^n]U_D$ (unit catastrophes)
0	1	1
1	1	1
2	1	2
3	2	7
4	5	23
5	13	75
6	34	246
7	90	811
8	241	2682
9	652	8889
10	1779	29508
11	4889	98071
12	13518	326233

A short initial segment can suggest a false identification. If $A(n)$ denotes OEIS A023425(OEIS Foundation Inc., 2026c), then $[z^n]U_M = A(n + 2)$ for $0 \leq n \leq 9$, but the next coefficients are 1779 and 1780, respectively. The two sequences separate at this coefficient.

The ratio plot in Appendix E illustrates these two growth constants.

For (A.3), expansion at $1/3$ yields $[z^n]U_M \sim (3\sqrt{3}/\sqrt{\pi})3^n n^{-3/2}$. In the catastrophe case the first-return series $z + z^3M^2 + J$ increases from zero and diverges as $z \uparrow 1/3$ because $\mathcal{H}$ does. It therefore equals one at a unique $\rho_D \in (0, 1/3)$, with positive derivative. Consecutive return lengths exclude other poles of the same modulus. Hence

$$[z^n]U_D \sim \frac{\rho_D^{-n}}{\rho_D C'_D(\rho_D)}, \quad \rho_D \simeq 0.299602595657,$$

with growth constant about 3.3377547942. Catastrophe-induced renewal poles were studied by Banderier and Wallner (2017).

B Cycle maxima and joint occupation times

B.1 Maximum in a return cycle

Assume the normalized finite-mean hypotheses of Section 4, and retain its roots a, t_-, t_+ . Let M be the largest level in a slot first-return cycle, so that $M = 0$ for an immediate stay. Let $L \geq 1$ be a threshold. For red $1 \leq m < L$, let $u_m^{(L)}$ be the probability of reaching L before service to zero or a catastrophe. It satisfies

$$(1 - h_r)u_m^{(L)} = p_r u_{m+1}^{(L)} + q_r u_{m-1}^{(L)}, \quad u_0^{(L)} = 0, \quad u_L^{(L)} = 1.$$

Solving the second-order recurrence gives

$$u_m^{(L)} = \frac{t_-^{-m} - t_+^{-m}}{t_-^{-L} - t_+^{-L}}. \quad (\text{B.1})$$

The blue part either reaches L itself or switches at some $m < L$. Therefore

$$\mathbb{P}(M \geq L) = p_b^0 \left[a^{L-1} + \frac{\gamma}{1 - h_b} \sum_{m=1}^{L-1} a^{m-1} u_m^{(L)} \right]. \quad (\text{B.2})$$

For a cycle conditioned on departure, omit p_b^0 . The sum in (B.2) is finite, so it also gives a direct overflow calculation.

Since $t_+ > t_- > 0$, rewriting (B.1) as

$$u_m^{(L)} = t_-^{L-m} \frac{1 - (t_-/t_+)^m}{1 - (t_-/t_+)^L}$$

shows how the two geometric factors compete. Bounding the last fraction above and below by positive constants and summing the remaining geometric progression proves

$$\mathbb{P}(M \geq L) \asymp \begin{cases} a^L, & a > t_-, \\ t_-^L, & a < t_-, \\ La^L, & a = t_-. \end{cases}$$

The notation $\asymp$ means that the ratio is bounded above and below by fixed positive constants for all sufficiently large L .

B.2 Joint positive-blue and red occupation

In this subsection the transition weights are normalized as in (2.1). Start at blue one immediately after an arrival to an empty system. Let T_b, T_r be the numbers of subsequent slots whose source states are positive blue and red, up to and including the transition to zero. A switch contributes one blue slot; a service return contributes one red slot. Define

$$\Phi(u, v) = \mathbb{E}[u^{T_b} v^{T_r}],$$

where u marks positive-blue slots and v red slots. In a defective-return case this is the subprobability transform on finite termination; the moment assertions below require the finite-mean conditions of Section 4.

For red level m , let $R_m(v)$ be the generating function of its absorption time, allowing either service or catastrophe. Define

$$A_r(v) = \frac{\xi_r v}{1 - (1 - \xi_r)v}.$$

When $\xi_r = 0$, use $A_r(v) = 0$ and take any endpoint value by continuity. First-step decomposition gives

$$R_m(v) = A_r(v) + (1 - A_r(v))x_1(v)^m. \quad (\text{B.3})$$

The constant term in the level recurrence accounts for a catastrophe from any positive height. Substitution verifies (B.3), including boundary value $R_0(v) = 1$; finite-time coefficients establish uniqueness.

Proposition B.1. *The joint transform is*

$$\begin{aligned} \Phi(u, v) = & \frac{\xi_b u}{1 - u(h_b + p_b)} \\ & + \gamma u \left[\frac{A_r(v)}{1 - u(h_b + p_b)} + \frac{(1 - A_r(v))x_1(v)}{1 - u(h_b + p_b x_1(v))} \right]. \end{aligned} \quad (\text{B.4})$$

Proof. Before the first switch or blue catastrophe, the path is a word of blue up-steps and stays. If it ends by a blue catastrophe, summing over this word gives the first term. If it ends by a switch, each preceding up-step increases the red starting height by one. Substitute (B.3) for the subsequent red duration. Its constant part sums the blue word with up-weight p_b , while its x_1^m part sums it with up-weight $p_b x_1(v)$. The initial height one supplies the remaining factor $x_1(v)$. This gives (B.4). $\square$

Let T_0 count slots with source zero in a slot first-return cycle. Then $T_0 = 1$. Its trivariate occupation transform is

$$\mathcal{C}(y_0, u, v) = y_0 \{h_b^0 + p_b^0 \Phi(u, v)\}, \quad \mathcal{C}(z, z, z) = C_D(z). \quad (\text{B.5})$$

The second identity is an important check on all initial and final factors.

For the positive part, T_b is geometric on $1, 2, \dots$ with success probability $\theta_b = \gamma + \xi_b$. Thus

$$\mathbb{E}T_b = \theta_b^{-1}, \quad \text{Var}(T_b) = (1 - \theta_b)\theta_b^{-2}.$$

For a slot first return, positive-blue occupation is zero on an immediate stay, so its mean and variance become p_b^0/θ_b and $p_b^0(2 - \theta_b - p_b^0)/\theta_b^2$. The two cycle conventions therefore give different occupation moments. Other moments follow from derivatives of the joint transform; for example $\mathbb{E}[T_b T_r] = \partial_u \partial_v \Phi(1, 1)$ for the positive part whenever it is finite, retaining the dependence between T_b and T_r .

C Switch-level limit proofs and finite-length correction

C.1 Proof of Theorem 7.3

Retain the weights, $Y(z)$ and branch constants from Section 7.3. The probability law is conditioned on service return at exactly n slots, as in (7.7).

Proof. First suppose $\Delta_1 > 0$. Substitution of (7.10) into (7.9) shows that its square-root coefficient is

$$-\frac{\gamma p_b^0}{p_b} z_1 \beta_* \frac{w}{(1 - w y_*)^2}.$$

Dividing the coefficient asymptotic by its value at $w = 1$ gives

$$\mathbb{E}[w^{L_n}] \longrightarrow \frac{w(1 - y_*)^2}{(1 - w y_*)^2}. \quad (\text{C.1})$$

For the moment assertion, the following tail control is sufficient. Choose $1 < w_0 < 1/y_*$. Positivity gives $|Y(z)| \leq y_*$ on $|z| \leq z_1$, so $1 - wY(z)$ has no zero there for $|w| \leq w_0$. Also $1 - h_b z$ has no zero on this disk. Since $h_r > 0$, the other red branch point has strictly larger modulus, or is absent. The algebraic expansion therefore holds uniformly for w in this disk and z in a sufficiently small dented Δ -domain at z_1 in the sense of the singularity-analysis transfer theorems of Flajolet and Sedgewick (Flajolet and Sedgewick, 2009, Chapter VI). Transfer at w_0 proves $\sup_{n \geq 3} \mathbb{E}[w_0^{L_n}] < \infty$. This controls all polynomial moments. Expanding (C.1) proves (7.11); differentiation gives (7.12).

At $\Delta_1 = 0$, write $(l)_k = l(l-1) \cdots (l-k+1)$ for the falling factorial, where $k \geq 1$ is fixed. Differentiate the marked series before extracting coefficients:

$$\left. \frac{\partial^k}{\partial w^k} \mathcal{B}(z, w) \right|_{w=1} = \frac{\gamma p_b^0}{p_b} z \frac{k! Y(z)^k}{(1 - Y(z))^{k+1}}.$$

Its leading singular exponent is $-(k+1)/2$. Write Γ for the Euler gamma function. Dividing its coefficient equivalent by that of $\mathcal{B}(z, 1)$ yields

$$\begin{aligned} \mathbb{E}[(L_n)_k] &\sim \frac{k! \sqrt{\pi}}{\beta_*^k \Gamma((k+1)/2)} n^{k/2} \\ &= \left(\frac{2}{\beta_*} \right)^k \Gamma(1 + k/2) n^{k/2}. \end{aligned} \quad (\text{C.2})$$

The same limits hold for ordinary moments after division by $n^{k/2}$, since lower falling-factorial terms have smaller orders. The right side is the k th moment of the density (7.13). That law has finite exponential moments and is moment-determinate, that is, uniquely determined by its moments. The moment method proves convergence in distribution as well as convergence of each scaled moment. In particular, the first two limits give (7.14).

Finally suppose $\Delta_1 < 0$. The root z_2 lies strictly below both z_1 and $1/h_b$ (with $1/h_b = \infty$ if $h_b = 0$). The coefficients of Y are nonnegative and its coefficients of degrees two and three are positive. Define a random block length J by the probability law

$$\mathbb{P}(J = k) = [z^k] Y(z) z_2^k, \quad k \geq 2.$$

These probabilities sum to one. Its masses at two and three are positive, so the law is nondegenerate; it also has an exponential moment because z_2 is inside the analytic disk of Y . Its mean and variance are exactly m_B and v_B in (7.15); in particular $v_B > 0$.

The same two consecutive powers imply that z_2 is the only solution of $Y(z) = 1$ on the circle $|z| = z_2$. The analytic implicit function theorem gives a root $z(w)$ of $wY(z(w)) = 1$ near $w = 1$, with $z(1) = z_2$. Choose a circle with radius strictly between z_2 and the next zero or singularity. On this circle the denominator stays nonzero, uniformly for w in a sufficiently small complex neighborhood of one. The implicit root is then the only pole inside that circle. There are $\eta \in (0, 1)$ and an analytic nonzero function

$$A(w) = \frac{\gamma p_b^0}{p_b w Y'(z(w))}$$

such that, uniformly in this neighborhood,

$$[z^n] \mathcal{B}(z, w) = A(w) z(w)^{-n} \{1 + O(\eta^n)\}.$$

Put $\Psi(s) = \log z_2 - \log z(e^s)$. The moment generating function of L_n is therefore an analytic factor times $\exp(n\Psi(s))$, with a uniform exponentially small relative error. Differentiating $e^s Y(z(e^s)) = 1$ gives

$$\Psi'(0) = 1/m_B, \quad \Psi''(0) = v_B/m_B^3.$$

The analytic factor contributes only $O(1)$ to the mean and variance. Taylor expansion at $s = it/\sqrt{\sigma_L^2 n}$ proves convergence of the centered characteristic functions to $e^{-t^2/2}$. This proves (7.16). $\square$

C.2 Proof of Proposition 7.4

In this subsection $p_b = 2$ and the other service-cycle weights equal one. The central trinomial coefficients T_k are those of (7.18).

Proof. For these weights, put $V(z) = 1 - Y(z)$. The quadratic identity for Y gives

$$V(z) = \frac{\sqrt{(1-3z)(1+z)}}{1-z}, \quad \mathcal{B}(z, 1) = \frac{z}{2}\{V(z)^{-1} - 1\}.$$

Differentiating the height marker before extracting coefficients gives

$$\partial_w \mathcal{B}(z, w)|_{w=1} = \frac{z}{2}\{V(z)^{-2} - V(z)^{-1}\}.$$

Let $d_k = [z^k]V(z)^{-1} = \mathsf{T}_k - \mathsf{T}_{k-1}$ for $k \geq 1$. Also

$$V(z)^{-2} = \frac{(1-z)^2}{(1-3z)(1+z)} = -\frac{1}{3} + \frac{1}{3(1-3z)} + \frac{1}{1+z}.$$

Consequently $[z^k]V(z)^{-2} = 3^{k-1} + (-1)^k$ for $k \geq 1$. The ratio of the differentiated and unmarked coefficients proves (7.19). Its denominator is positive for $n \geq 3$, since $d_{n-1} = 2[z^n]\mathcal{B}(z, 1)$.

For the asymptotic correction, the expansion at $z = 1/3$ is

$$V(z)^{-1} = \frac{1}{\sqrt{3}}(1-3z)^{-1/2} \left\{ 1 + \frac{5}{8}(1-3z) + O((1-3z)^2) \right\}.$$

Coefficient transfer, including the first correction for $(1-3z)^{-1/2}$, gives

$$d_k = \frac{3^k}{\sqrt{3\pi k}} \left\{ 1 - \frac{7}{16k} + O(k^{-2}) \right\}.$$

The other branch point, at $z = -1$, has larger modulus and does not change these orders. Substitute $k = n - 1$ into (7.19); its alternating numerator term is exponentially negligible. Expansion in powers of $1/n$ proves (7.20). $\square$

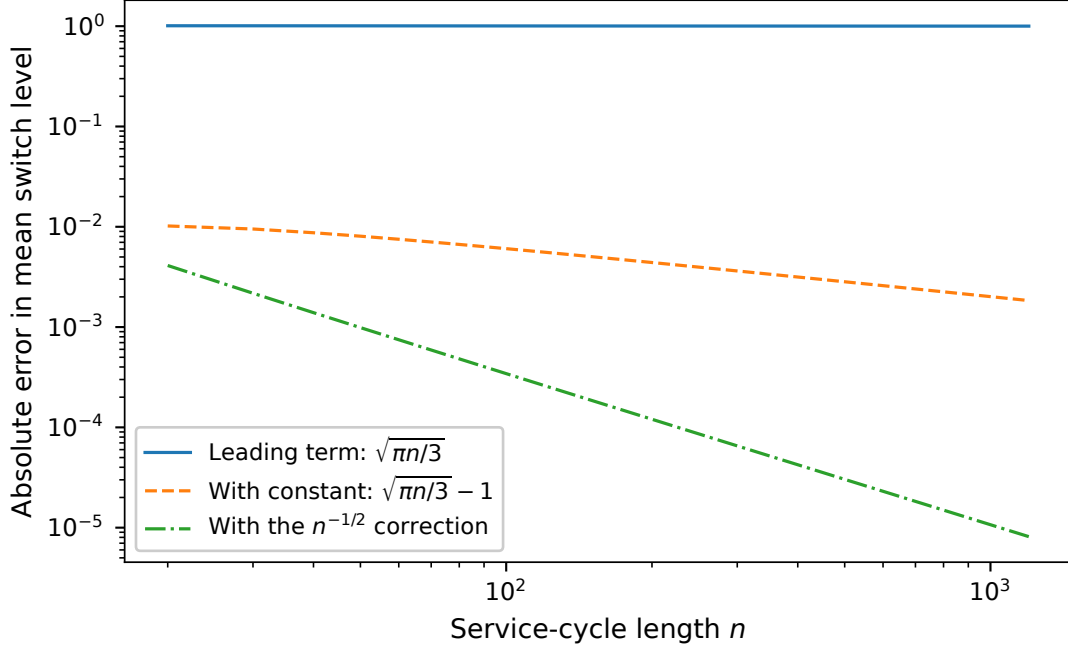


Fig. 7 Absolute errors of three approximations to the mean critical switch level. The exact values are computed from (7.19). The leading square-root term has an error tending to one; including the constant and then the $n^{-1/2}$ correction reduces that error. The plot illustrates the finite-length improvement provided by (7.20).

D Continuous-time transforms and previous queueing models

D.1 The continuous-time positive part and its ending cause

To compare directly with earlier formulas, this subsection allows different arrival rates $\lambda_b, \lambda_r > 0$ and catastrophe rates $\delta_b, \delta_r \geq 0$. The red service rate is $\mu > 0$ and the switch rate is $\nu > 0$. These are local rate names for the same two-regime model. Let $T_b^{\text{ct}}, T_r^{\text{ct}}$ be the elapsed times in positive blue and red, respectively, from the first arrival until the next return to zero. For $\alpha, \beta \geq 0$, define their joint Laplace transform on finite returns by

$$\hat{\Phi}(\alpha, \beta) = \mathbb{E}\left[e^{-\alpha T_b^{\text{ct}} - \beta T_r^{\text{ct}}}; T_b^{\text{ct}} + T_r^{\text{ct}} < \infty\right].$$

The red downward-passage transform is

$$\eta(\beta) = \frac{\beta + \lambda_r + \mu + \delta_r - \sqrt{(\beta + \lambda_r + \mu + \delta_r)^2 - 4\lambda_r\mu}}{2\lambda_r}. \quad (\text{D.1})$$

It is the transform of one downward red passage before catastrophe. With $d_b(\alpha) = \alpha + \nu + \delta_b$ and $A = \delta_r/(\beta + \delta_r)$, the joint transform of positive-blue and red durations is

$$\hat{\Phi}(\alpha, \beta) = \frac{\delta_b}{d_b(\alpha)} + \nu \left[\frac{A}{d_b(\alpha)} + \frac{(1-A)\eta(\beta)}{d_b(\alpha) + \lambda_b(1-\eta(\beta))} \right]. \quad (\text{D.2})$$

For $\delta_r = 0$ use $A = 0$ and extend to $\beta = 0$ by continuity in the stable case. Substituting $u = R/(R+\alpha)$ and $v = R/(R+\beta)$ into (B.4), with the corresponding uniformized probabilities, proves (D.2): R cancels from each term. The cancellation of the quadratic gives (D.1).

For a common Laplace argument $s > 0$, define $L_{C_b}(s)$, $L_{\text{serv}}(s)$ and $L_{C_r}(s)$ to be the partial transforms of the positive part ending by blue catastrophe, service, and red catastrophe,

respectively. Then

$$L_{C_b}(s) = \frac{\delta_b}{s + \nu + \delta_b}, \quad (\text{D.3})$$

$$L_{\text{serv}}(s) = \frac{\nu \eta(s)}{s + \nu + \delta_b + \lambda_b(1 - \eta(s))}, \quad (\text{D.4})$$

$$L_{C_r}(s) = \frac{\delta_r}{s + \delta_r} \left[\frac{\nu}{s + \nu + \delta_b} - L_{\text{serv}}(s) \right]. \quad (\text{D.5})$$

One may derive them without uniformization: condition on the first blue terminating time, whose hazard is $\nu + \delta_b$, and sum the Poisson arrivals accumulated before that time. The service factor from a switch level m is $\eta(s)^m$. The red catastrophe factor is $\delta_r(1 - \eta(s)^m)/(s + \delta_r)$. Integrating yields the three formulas. Their sum is (D.2) with equal arguments. The idle-plus-positive transform has the extra factor $\lambda_b/(\lambda_b + s)$.

These single-argument transforms are boundary specializations of the earlier two-mode busy-period work (Kleiner et al., 2023b). For unit batches and exponential red service, take exponential blue service with rate tending to zero. This yields no blue service in the limit. The cycle convention is the positive part after the first arrival. We use these earlier single-argument laws as consistency checks for the discrete and joint-occupation formulas.

For completeness, the whole vacation up to a successful switch, including repeated empty intervals after blue catastrophes, has transform

$$L_V(s) = \frac{\lambda_b \nu}{(\lambda_b + s)(s + \nu + \delta_b) - \lambda_b \delta_b}. \quad (\text{D.6})$$

To verify this, each attempt has an idle transform $\lambda_b/(\lambda_b + s)$, followed by either a switch factor $\nu/(s + \nu + \delta_b)$ or a catastrophe factor $\delta_b/(s + \nu + \delta_b)$ restarting the attempt. Summing the geometric series gives (D.6). Thus (D.6) groups all failed attempts and the final successful switch.

D.2 A common-rate resolvent for comparison of clocks

Return to the common rates $\lambda, \mu, \nu, \delta$ and initial blue zero state of Section 5. Assume $\delta > 0$ or, if $\delta = 0$, $\lambda < \mu$. The functions $\hat{p}_i^c(s)$ are the state-probability transforms defined before (5.3). In (D.1), take $\lambda_r = \lambda$ and $\delta_r = \delta$ to obtain $\eta(s)$. Put

$$\ell_b = \lambda + \nu + \delta, \quad \tilde{a}(s) = \frac{\lambda}{s + \ell_b}, \quad \tilde{t}(s) = \frac{\lambda \eta(s)}{\mu}, \quad \tilde{D}(s) = s + \ell_b - \lambda \eta(s). \quad (\text{D.7})$$

The tilde distinguishes these continuous-time functions from the slot series $a(z), t(z), \Delta_b(z)$.

Proposition D.1. *With these definitions,*

$$\hat{p}_0^b(s) = \frac{\tilde{D}(s)(1 - \tilde{t}(s))}{s(s + \ell_b)}, \quad (\text{D.8})$$

$$\hat{p}_i^b(s) = \tilde{a}(s)^i \hat{p}_0^b(s), \quad i \geq 1, \quad (\text{D.9})$$

$$\hat{p}_i^r(s) = \frac{\nu \tilde{t}(s)(1 - \tilde{t}(s))}{s(s + \ell_b)} \sum_{j=0}^{i-1} \tilde{a}(s)^j \tilde{t}(s)^{i-1-j}, \quad i \geq 1. \quad (\text{D.10})$$

If $p^b(t) = \sum_{i \geq 0} p_i^b(t)$ is the probability of the whole vacation regime, then

$$\hat{p}^b(s) = \frac{\tilde{D}(s)(1 - \tilde{t}(s))}{s(s + \nu + \delta)}. \quad (\text{D.11})$$

Proof. Use the bridge (5.3), derived by integrating the Poisson sum. Substitution into Theorem 6.1 gives (D.8)–(D.10); in particular $x_1(R/(s+R)) = \eta(s)$ and $\Delta_b(R/(s+R)) = \tilde{D}(s)/(s+R)$. The uniformization rate cancels. Since $0 < \tilde{a}(s) < 1$ for $s > 0$, summing (D.9) over all blue levels proves (D.11). $\square$

These formulas put the local and aggregate comparisons in physical time. The red branch point is at $s = -\ell_r$, where

$$\ell_r = \delta + (\sqrt{\mu} - \sqrt{\lambda})^2.$$

The fixed-blue-level denominator has its possible pole at $s = -\ell_b$, whereas the aggregate blue denominator has its possible pole at $s = -(\nu + \delta)$. A numerator cancellation must still be checked before using any of these as an asymptotic term. The equality $\ell_b = \ell_r$ gives the fixed-state boundary ν_{loc} in (6.8).

The no-vacation limit also checks the normalization. Let $\nu \rightarrow \infty$ with the other rates fixed. Then

$$\hat{p}_0^b(s) \rightarrow \frac{1 - \tilde{t}(s)}{s}, \quad \hat{p}_i^r(s) \rightarrow \frac{(1 - \tilde{t}(s))\tilde{t}(s)^i}{s}, \quad \hat{p}_i^b(s) \rightarrow 0 \quad (i \geq 1).$$

These are the one-regime transforms from an empty system, with catastrophes retained. Without catastrophes, the associated red edge is $(\sqrt{\mu} - \sqrt{\lambda})^2$, the classical physical-time $M/M/1$ relaxation rate discussed by Kijima (1989). Kijima also compares it with a relaxation measure based on an arrival-count clock. Equations (D.8)–(D.11) specify the level, regime and initial law needed for comparisons with the earlier vacation models (Keilson and Servi, 1987; Takagi, 1990).

D.3 Cycle, spatial and local critical curves

For comparison, take the continuous-time queue of Section 5 without catastrophes. Choose R strictly above both exit rates, so that $h_r > 0$. Uniformization with rate R gives $p_b = p_r = \lambda/R$, $q_r = \mu/R$, $\gamma = \nu/R$, $h_b = 1 - (\lambda + \nu)/R$ and $h_r = 1 - (\lambda + \mu)/R$. Substitution into (7.4) gives

$$\nu_{\text{time}} = \mu - \sqrt{\lambda\mu}.$$

By contrast, the no-stay event-chain condition $\bar{p}_b = 2\bar{p}_r$ of Paper I, with $\bar{p}_b = \lambda/(\lambda + \nu)$ and $\bar{p}_r = \lambda/(\lambda + \mu)$, gives

$$\nu_{\text{event}} = \frac{\mu - \lambda}{2}.$$

The stationary spatial-tail equality $a = t_-$ gives $\nu_{\text{space}} = \mu - \lambda$. Therefore, for $0 < \lambda < \mu$,

$$\nu_{\text{event}} < \nu_{\text{time}} < \nu_{\text{space}}. \quad (\text{D.12})$$

These boundaries belong to different observables. The idle-plus-positive cycle can also have an idle-time singularity, so its tail need not share the service-cycle boundary.

Set $\rho = \lambda/\mu$ and $\varepsilon = 1 - \rho$. A Taylor expansion gives

$$\nu_{\text{time}}/\mu = \varepsilon/2 + \varepsilon^2/8 + \varepsilon^3/16 + O(\varepsilon^4), \quad \nu_{\text{event}}/\mu = \varepsilon/2, \quad \nu_{\text{space}}/\mu = \varepsilon.$$

This Taylor expansion compares the three critical curves as $\rho \uparrow 1$. With a common catastrophe rate δ , the spatial equality instead gives

$$\nu_{\text{space}} = \frac{\mu - \lambda - \delta + \sqrt{(\lambda + \delta - \mu)^2 + 4\delta\mu}}{2}.$$

The full return transform includes all ending causes through C_D . In the unit-weight renewal example of Appendix A.3, this changes the dominant singularity from the Motzkin branch point to a renewal pole.

The fixed-state and aggregate comparison appears in Figure 3.

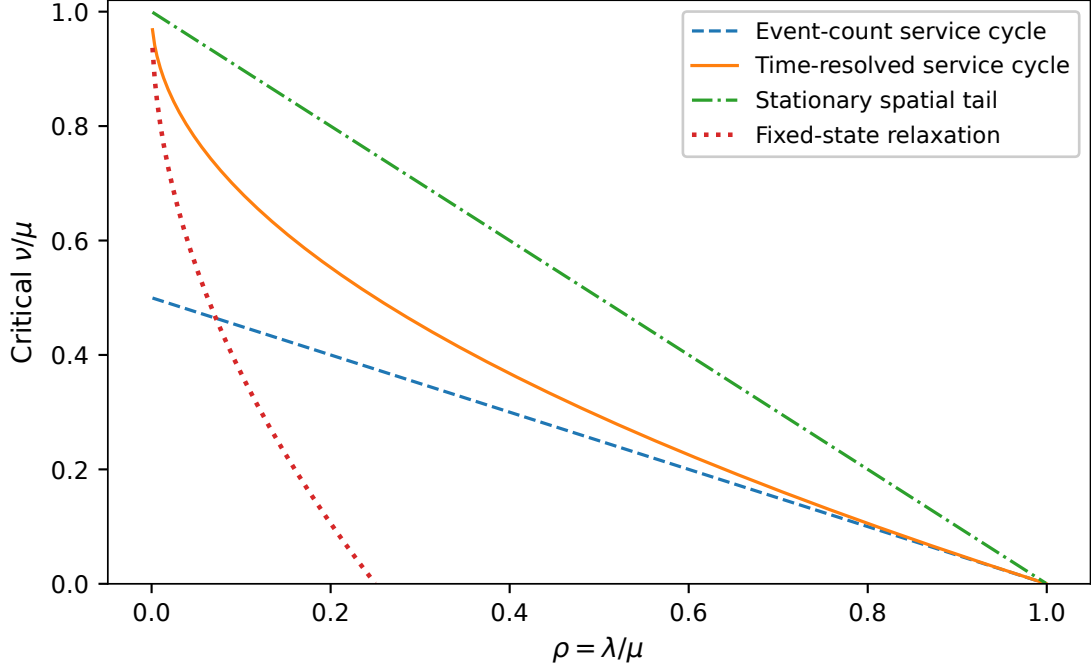


Fig. 8 Critical curves for four observables without catastrophes. The fixed-state curve concerns the local relaxation of a specified level; only its nonnegative part is shown.

E Reductions and computational checks

E.1 Limiting models

The formulas retain the reductions to earlier models. Setting $h_b = h_r = h_b^0 = 0$ and $p_b^0 = 1$ gives the preceding event-clock path formulas with the same switch and catastrophe markers. For probability weights, deleting holding steps renormalizes each effective-event row by its total non-stay probability. Setting $\xi_b = \xi_r = 0$ recovers the disaster-free Motzkin model. Its stationary decomposition and continuous-time positive-part transforms agree with the corresponding vacation-queue results (Fuhrmann and Cooper, 1985; Kleiner et al., 2023a,b).

At $\gamma = 0$, a process initialized at blue zero stays in the blue regime. For $p_b^0, p_b > 0$, it is a birth/stay/reset chain: a positive catastrophe rate gives a finite mean return time, whereas zero catastrophe rate leads to escape through increasing levels. At $p_b^0 = 0$, blue zero is absorbing. In continuous time, letting $\nu \rightarrow \infty$ gives the one-regime queue. Under $\delta > 0$, or $\delta = 0$ with $\lambda < \mu$, its stationary probabilities are $(1 - t_-)t_-^i$, $i \geq 0$, where $t_- \in (0, 1)$ solves

$$\mu t_-^2 - (\lambda + \mu + \delta)t_- + \lambda = 0.$$

For $\delta = 0$ this is the stable $M/M/1$ distribution.

E.2 Coefficient and numerical verification

The accompanying code separates exact coefficient checks from floating-point calculations. For finite paths, a forward recursion in the state, regime and switch marker is compared with extraction from (7.6). A second integer recursion accumulates the first two switch-level moments and checks (7.19). The limiting statements are proved in Appendix C.1; the finite computations provide coefficient-level checks of the displayed formulas.

For the one-hour example, the forward recursion computes the complete finite-support distributions through slot N , and then applies (5.2). Changing the uniformization rate

changes these discrete distributions, but not their Poisson sum. The missing mass is the Poisson tail stated in Section 5. A separate calculation builds the continuous-time generator and solves its resolvent. The height-truncated generator kills a trajectory when it attempts to cross the upper boundary.

The effect of killing above level H admits an explicit bound. From an empty initial state, reaching $H + 1$ requires at least $H + 1$ arrivals. Before an independent exponential time of rate s , the number of arrivals has tail $(\lambda/(\lambda + s))^{H+1}$. Hence the height-truncation error of any one of these Laplace transforms, or of the aggregate transform, is at most

$$\frac{1}{s} \left(\frac{\lambda}{\lambda + s} \right)^{H+1}. \quad (\text{E.1})$$

This bound concerns height truncation; numerical roundoff in a matrix solve is a separate error.

E.3 Unit-weight growth illustration

Figure 9 shows the ratios of successive coefficients of the two renewal series in Appendix A.3. The growth constants shown by the horizontal reference lines are obtained analytically from the dominant singularities.

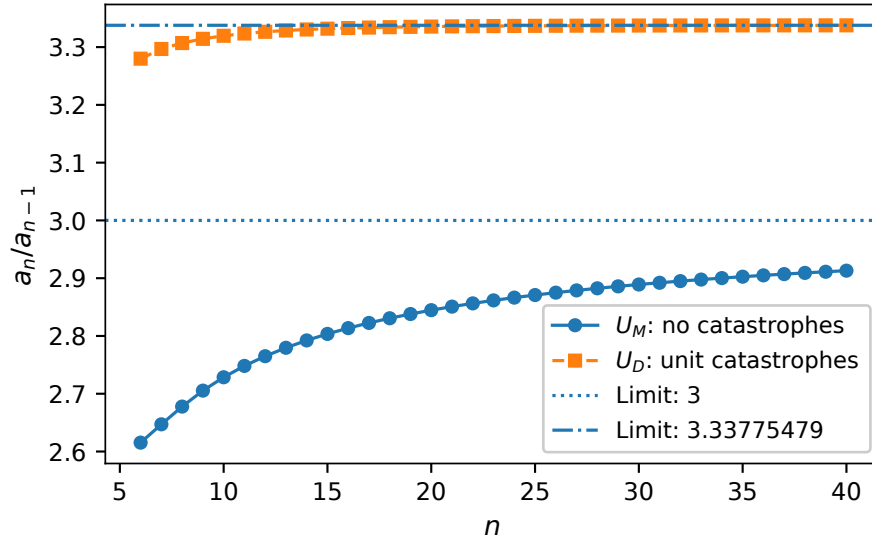


Fig. 9 Ratios of successive unit-weight renewal counts. Without catastrophes the limiting growth constant is 3; with unit catastrophe weights the earlier renewal pole gives growth about 3.3377547942.

Statements and Declarations

Funding

No specific funding was received for this work.

Competing Interests

The author declares no competing interests.

Data and code availability

No new empirical data sets were generated. Verification code and scripts for reproducing the numerical examples and data-based figures are provided as Online Resource 1.

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