Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Nonlinear Nabla Caputo q-Fractional Difference Equations on Quantum Time Scales: Well-Posedness, Order Preservation, and the Classical q → 1− Limit

Taylan Demir

1996.matematik.1996@gmail.com

Ankara University https://orcid.org/0000-0001-7402-4468

Mohammad Esmael Samei

Bu-Ali Sina University https://orcid.org/0000-0002-5450-3127

Research Article

Keywords: nabla q-difference; Caputo q-fractional operator; quantum time scale; comparison principle; q-Mittag–Leffler function; fixed point; classical limit.

Posted Date: September 22nd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11103827/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: The authors declare no competing interests.

Nonlinear Nabla Caputo q-Fractional Difference Equations on Quantum Time Scales: Well-Posedness, Order Preservation, and the Classical $q \to 1^-$ Limit

Taylan Demir[1] Mohammad Esmael Samei[2]

[1]Department of Mathematics, Çankaya University, Ankara, Türkiye
[2]Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran
E-mail: demir.taylan96@gmail.com, mesamei@gmail.com

Abstract

This work develops a nonlinear initial-value framework for Caputo-type q-fractional dynamics on the quantum time scale. The analysis is carried out through an equivalent Jackson–Volterra formulation, which allows us to establish existence, uniqueness, and continuous dependence of solutions under suitable contraction assumptions. The positivity of the fractional q-integral is then used to derive an order-preservation principle for nonlinear terms that are nondecreasing in the state variable. We also examine the connection between the quantum model and its classical fractional counterpart as q approaches one. Numerical examples involving linear and nonlinear relaxation problems support the theoretical results and illustrate the convergence behavior of the proposed framework. The study extends earlier linear q-Mittag–Leffler constructions to a nonlinear setting and provides a unified perspective on well-posedness, order structure, and the classical-limit behavior of q-fractional systems.

Keywords: nabla q-difference; Caputo q-fractional operator; quantum time scale; comparison principle; q-Mittag–Leffler function; fixed point; classical limit.
2020 Mathematics Subject Classification: 26A33; 39A12; 39A70; 34A08.

1 Introduction

Discrete fractional calculus and quantum calculus provide two complementary routes for introducing memory into nonclassical evolution equations. The ordinary backward difference acts on an additive lattice, whereas the nabla q-difference acts on a multiplicative grid. The latter is especially natural on the quantum time scale $\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where the backward point associated with t is qt. Standard references on q-calculus and q-fractional equations include Kac and Cheung [1] and the monograph of Annaby and Mansour [2].

Caputo-type q-fractional initial-value problems were developed by Abdeljawad and Baleanu, who also introduced a q-analogue of the Mittag–Leffler function and solved linear equations in this setting [3]. The relation between delta and nabla Caputo fractional differences, including dual identities and discrete Mittag–Leffler functions, was studied by Abdeljawad [4]. The (q, h) framework of Čermák and Nechvátal provides a broader setting connecting quantum and additive difference calculi [7], while Du et al. investigated asymptotic and monotonicity properties of nabla fractional (q, h)-difference equations [8]. Comparison principles remain subtle in discrete fractional problems; a recent result for the nabla fractional difference operator is given by Jonnalagadda [9].

The preliminary conference abstract from which the present paper develops was largely expository: it proposed to review Grünwald–Letnikov, Riemann–Liouville, and Caputo definitions and then adapt them to q-difference operators. That objective is no longer sufficient for a research

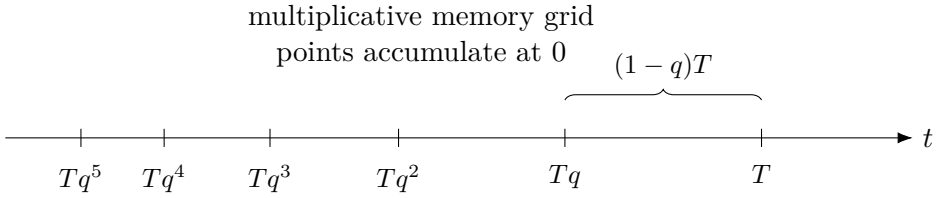


Figure 1: The nabla q-grid is multiplicative rather than uniformly spaced.

article because the core definitions and several linear solution formulas are already established in the literature [3, 2]. We therefore focus on a different question:

> What nonlinear solution theory can be obtained directly from the positive Jackson fractional integral, and how does the resulting quantum dynamics converge to its classical Caputo counterpart as $q \to 1^-$?

The main contributions are:

1. an explicit contraction criterion for nonlinear nabla Caputo q-fractional initial-value problems;
2. continuous-dependence estimates in the sup norm;
3. an order-preservation and comparison theorem based on positivity of the q-fractional integral;
4. a solution-level $q \to 1^-$ convergence theorem under a uniform contraction hypothesis;
5. a reproducible q-Mittag–Leffler convergence experiment and a nonlinear order-preserving numerical test.

2 q-calculus preliminaries

2.1 Quantum grid and nabla q-difference

Fix $0 < q < 1$ and $T > 0$. We work on

$$\mathbb{T}_q(T) = \{Tq^n : n \in \mathbb{N}_0\} \cup \{0\}.$$

For $t \neq 0$, the nabla q-difference is

$$\nabla_q x(t) = \frac{x(t) - x(qt)}{(1-q)t}. \tag{1}$$

As $q \to 1^-$ and x is differentiable,

$$\nabla_q x(t) \longrightarrow x'(t).$$

2.2 q-shifted powers and the q-gamma function

For $a \in \mathbb{C}$,

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

For $\nu \in \mathbb{R}$ and $0 \leq s < t$, define

$$(t-s)_q^\nu = t^\nu \frac{(s/t;q)_\infty}{(q^\nu s/t;q)_\infty}, \tag{2}$$

whenever the quotient is well defined. The q-gamma function is

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}, \qquad z > 0. \tag{3}$$

It satisfies

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z), \qquad [z]_q = \frac{1-q^z}{1-q},$$

and

$$\Gamma_q(z) \longrightarrow \Gamma(z) \qquad (q \to 1^-).$$

2.3 Jackson integral and fractional q-integral

The Jackson integral on $[0,t]$ is

$$\int_0^t g(s)\, d_q s = (1-q)t \sum_{j=0}^{\infty} q^j g(tq^j). \tag{4}$$

For $\alpha > 0$, define

$$(I_{q,0}^{\alpha} g)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\alpha-1} g(s)\, d_q s. \tag{5}$$

The operator is positive, and

$$I_{q,0}^{\alpha} 1 = \frac{t^{\alpha}}{\Gamma_q(\alpha+1)}. \tag{6}$$

For $0 < \alpha < 1$, the Caputo-type nabla q-fractional operator is

$${}^{C}\nabla_{q,0}^{\alpha} x(t) = I_{q,0}^{1-\alpha} \nabla_q x(t). \tag{7}$$

3 Nonlinear initial-value problem

Consider

$${}^{C}\nabla_{q,0}^{\alpha} x(t) = F(t, x(t)), \qquad x(0) = x_0, \qquad 0 < \alpha < 1. \tag{8}$$

Under the standard inversion relation for the Caputo q-operator, (8) is equivalent to

$$x(t) = x_0 + I_{q,0}^{\alpha}\big[F(\cdot, x(\cdot))\big](t). \tag{9}$$

Let

$$X = C(\mathbb{T}_q(T), \mathbb{R}), \qquad \|x\|_{\infty} = \sup_{t \in \mathbb{T}_q(T)} |x(t)|.$$

Define

$$(\mathcal{T}x)(t) = x_0 + I_{q,0}^{\alpha}\big[F(\cdot, x(\cdot))\big](t). \tag{10}$$

4 Well-posedness

Theorem 4.1 (Existence and uniqueness)**.** *Assume that $F : \mathbb{T}_q(T) \times \mathbb{R} \to \mathbb{R}$ is continuous in the quantum-time variable and globally Lipschitz in its second argument:*

$$|F(t,u) - F(t,v)| \le L|u - v|.$$

If

$$\Theta_{q,\alpha}(T) := \frac{LT^{\alpha}}{\Gamma_q(\alpha+1)} < 1, \tag{11}$$

then problem (8) has a unique solution in X.

Proof. For $x, y \in X$, positivity of the kernel and (6) give

$$\begin{aligned}|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\le I^{\alpha}_{q,0}\left(|F(\cdot,x) - F(\cdot,y)|\right)(t)\\ &\le L\|x-y\|_{\infty} I^{\alpha}_{q,0}1(t)\\ &\le \frac{LT^{\alpha}}{\Gamma_q(\alpha+1)}\|x-y\|_{\infty}.\end{aligned}$$

Hence $\mathcal{T}$ is a contraction when (11) holds. Banach's fixed-point theorem gives a unique fixed point. □

Proposition 4.2 (Continuous dependence on initial data)**.** *Let x and y solve (8) with initial data x_0 and y_0, respectively, under the assumptions of Theorem 4.1. Then*

$$\|x-y\|_{\infty} \le \frac{|x_0 - y_0|}{1 - \Theta_{q,\alpha}(T)}. \tag{12}$$

5 Order preservation and comparison

Theorem 5.1 (Order preservation)**.** *Assume the hypotheses of Theorem 4.1 and suppose, in addition, that $F(t,\cdot)$ is nondecreasing for every $t \in \mathbb{T}_q(T)$. Let x and y solve (8) with $x_0 \le y_0$. Then*

$$x(t) \le y(t) \qquad \text{for every } t \in \mathbb{T}_q(T).$$

Proof. Start the Picard sequences from the ordered constants

$$x^{(0)}(t) = x_0, \qquad y^{(0)}(t) = y_0.$$

If $x^{(n)} \le y^{(n)}$, monotonicity of F and positivity of $I^{\alpha}_{q,0}$ imply

$$x^{(n+1)}(t) \le y^{(n+1)}(t).$$

The contraction theorem gives uniform convergence of both Picard sequences to the unique fixed points, so the inequality survives in the limit. □

Corollary 5.2 (Two-nonlinearity comparison)**.** *Suppose $F(t,u) \le G(t,u)$ for all (t,u) and both nonlinearities are nondecreasing and satisfy the same contraction condition. If $x_0 \le y_0$, where*

$${}^{C}\nabla^{\alpha}_{q,0}x = F(t,x), \qquad {}^{C}\nabla^{\alpha}_{q,0}y = G(t,y),$$

then $x(t) \le y(t)$ on $\mathbb{T}_q(T)$.

6 The classical limit $q \to 1^{-}$

Let

$$(I^{\alpha}_0 g)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\,ds.$$

The classical Caputo problem is

$${}^{C}D^{\alpha}_0 x(t) = F(t,x(t)), \qquad x(0) = x_0.$$

Lemma 6.1 (q-fractional integral limit)**.** *If g is continuous on $[0,T]$, then*

$$\|I^{\alpha}_{q,0}g - I^{\alpha}_0 g\|_{\infty} \to 0 \qquad (q \to 1^{-}).$$

Theorem 6.2 (Solution-level classical limit)**.** *Assume F is continuous and globally Lipschitz with constant L. Suppose there exist $q_0 < 1$ and $\theta < 1$ such that*

$$\frac{LT^\alpha}{\Gamma_q(\alpha+1)} \le \theta \qquad (q_0 < q < 1).$$

Let x_q solve the quantum problem and let x solve the classical Caputo problem. Then

$$\|x_q - x\|_\infty \to 0 \qquad (q \to 1^-),$$

and

$$\|x_q - x\|_\infty \le \frac{\|I^\alpha_{q,0} F(\cdot, x(\cdot)) - I^\alpha_0 F(\cdot, x(\cdot))\|_\infty}{1-\theta}. \tag{13}$$

7 Linear relaxation and the q-Mittag–Leffler function

For

$${}^C\nabla^\alpha_{q,0} x(t) = -\lambda x(t), \qquad x(0) = 1,$$

define

$$E_{\alpha,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k + 1)}.$$

Then

$$x_q(t) = E_{\alpha,q}(-\lambda t^\alpha),$$

and

$$E_{\alpha,q}(-\lambda t^\alpha) \to E_\alpha(-\lambda t^\alpha) \qquad (q \to 1^-).$$

8 Numerical scheme

On a geometric grid,

$$I^\alpha_{q,0} g(t) = t^\alpha \sum_{j=0}^{\infty} w_j^{(\alpha,q)} g(tq^j), \tag{14}$$

where

$$w_j^{(\alpha,q)} = \frac{(1-q)q^j}{\Gamma_q(\alpha)} \frac{(q^{j+1};q)_\infty}{(q^{j+\alpha};q)_\infty}. \tag{15}$$

Thus the Picard iteration is

$$x^{(m+1)}(t) = x_0 + t^\alpha \sum_{j=0}^{J} w_j^{(\alpha,q)} F(tq^j, x^{(m)}(tq^j)),$$

with the tail truncated below a fixed tolerance.

9 Reproducible numerical experiments

9.1 $q \to 1^-$ convergence

Take

$$\alpha = 0.75, \qquad \lambda = 0.6, \qquad T = 1.$$

The reference solution is

$$x(t) = E_{0.75}(-0.6t^{0.75}),$$

while the quantum solution is $E_{0.75,q}(-0.6t^{0.75})$.

The classical value is

$$x(1) = 0.5509133848.$$

A least-squares fit over $q \ge 0.90$ gives the observed convergence slope

$$p_{\text{obs}} \approx 1.008.$$

Table 1: Uniform error in the linear q-Mittag–Leffler relaxation problem.

q	$\|x_q - x\|_\infty$	$x_q(1)$
0.600	2.027137e-02	0.57118475
0.700	1.469293e-02	0.56560631
0.800	9.495203e-03	0.56040859
0.900	4.613653e-03	0.55552704
0.950	2.275879e-03	0.55318926
0.980	9.032136e-04	0.55181660
0.990	4.504400e-04	0.55136382
0.995	2.249303e-04	0.55113832

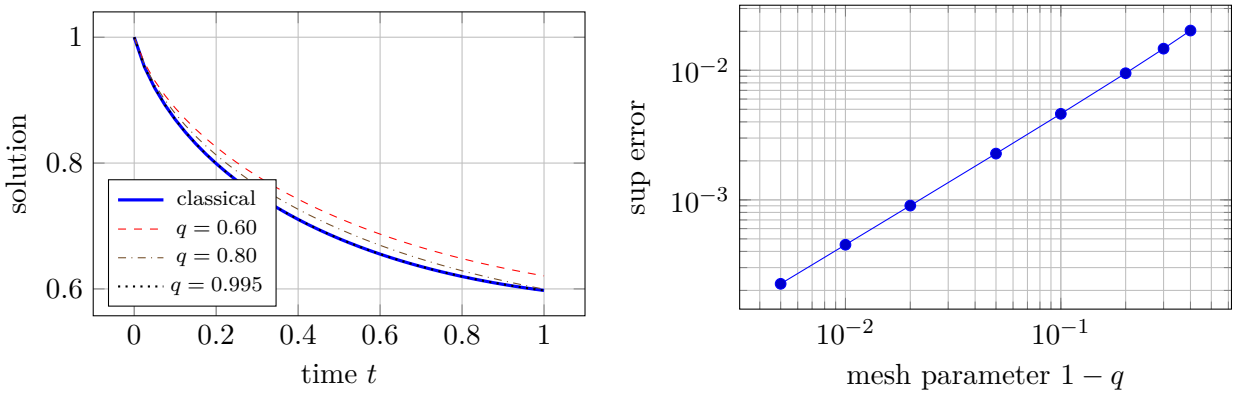


Figure 2: Left: quantum relaxation converges to the classical Caputo solution. Right: the finest-grid errors are nearly first order in $1 - q$.

9.2 Nonlinear order-preserving example

Consider

$$ {}^{C}\nabla_{q,0}^{0.75} x(t) = 0.25 \tanh x(t) + 0.05 \sin(2\pi t), \qquad q = 0.90. $$

Since

$$ 0 \le \frac{\partial}{\partial x} \left(0.25 \tanh x + 0.05 \sin(2\pi t)\right) \le 0.25, $$

the nonlinearity is nondecreasing and has Lipschitz constant $L = 0.25$. Moreover,

$$ \Gamma_q(1.75) \approx 0.923526, \qquad \Theta_{q,\alpha}(1) \approx 0.2707 < 1. $$

The comparison theorem therefore applies.

For $x_0 = 0.3$ and $y_0 = 0.8$, Picard iteration reaches tolerance 10^{-11} in 11 and 9 iterations, respectively, and

$$ x(1) \approx 0.390975, \qquad y(1) \approx 0.994107. $$

10 Discussion

The novelty of the present framework should be read relative to established q-fractional calculus. Linear Caputo q-fractional initial-value problems and q-Mittag–Leffler functions are not new [3]. Likewise, delta/nabla Caputo identities and linear discrete Mittag–Leffler solutions are established [4]. The present contribution is therefore not the definition of another operator.

Instead, the analysis isolates three features directly useful for nonlinear quantum dynamics. First, the positive Jackson fractional integral provides an explicit contraction constant and continuous-dependence estimate. Second, the same positivity produces an order structure when

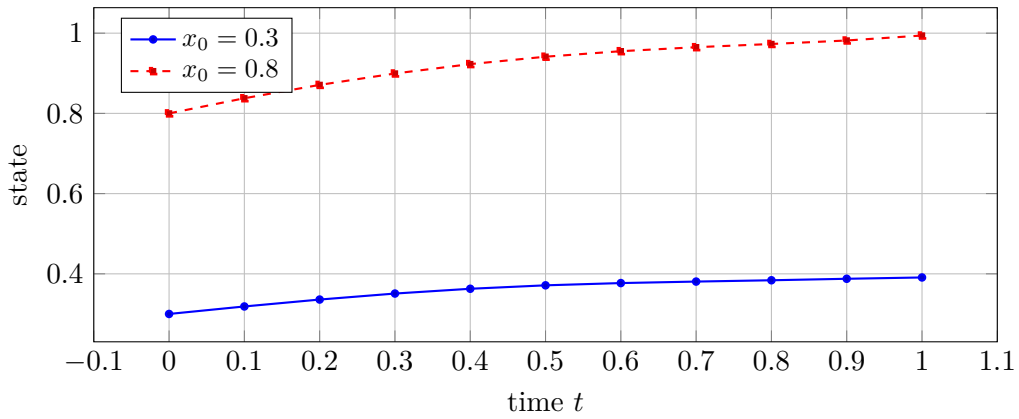


Figure 3: Nonlinear order-preserving test for $q = 0.90$ and $\alpha = 0.75$.

the nonlinearity is monotone. Third, the solution map is shown to converge to the classical Caputo solution as the multiplicative quantum grid becomes dense.

The numerical results are consistent with this picture. In the linear relaxation test, q acts as a mesh/deformation parameter rather than as a second fractional order. The observed convergence is close to first order in $1 - q$ over the finest grids, but this rate is reported as an empirical observation, not a universal theorem. The nonlinear example serves a different purpose: it verifies that the fixed-point computation respects the comparison theorem.

The contraction condition is sufficient rather than necessary and may be restrictive on long intervals. Local continuation, monotone iterations, topological fixed-point methods, or measure-of-noncompactness arguments may extend the theory beyond this regime. The (q, h) framework is another natural direction because it connects quantum and additive difference calculi in one parameterized setting [7, 8].

11 Conclusion

A conference-level survey of nabla Caputo-type q-fractional operators has been reformulated as a nonlinear research problem on quantum time scales. The resulting theory gives a transparent contraction criterion, a continuous-dependence bound, an order-preservation theorem, and a solution-level classical limit. The linear q-Mittag–Leffler experiment quantifies the approach to the Caputo solution, while the nonlinear example confirms ordered fixed-point dynamics.

The framework is deliberately conservative about originality: established q-fractional definitions and linear special-function solutions are treated as background, whereas the paper concentrates on nonlinear well-posedness, comparison structure, and convergence of solution operators.

Data and computational reproducibility

No external data are used. Infinite products defining Γ_q are truncated when the next logarithmic contribution is below 10^{-14}, while the fractional Jackson weights are truncated at 10^{-12}. All numerical values and plotted coordinates are embedded directly in this `main.tex` file.

Declarations

Competing interests. The authors declare no competing interests.
Funding. To be completed by the authors before submission.
Author contributions. To be finalized and approved by both authors before submission.

References

[1] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[2] M. H. Annaby and Z. S. Mansour, *q-Fractional Calculus and Equations*, Lecture Notes in Mathematics, Vol. 2056, Springer, Berlin, 2012. doi:10.1007/978-3-642-30898-7.

[3] T. Abdeljawad and D. Baleanu, Caputo q-fractional initial value problems and a q-analogue Mittag–Leffler function, *Communications in Nonlinear Science and Numerical Simulation* 16(12) (2011), 4682–4688. doi:10.1016/j.cnsns.2011.01.026.

[4] T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, *Discrete Dynamics in Nature and Society* 2013 (2013), Article ID 406910. doi:10.1155/2013/406910.

[5] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proceedings of the American Mathematical Society* 137(3) (2009), 981–989. doi:10.1090/S0002-9939-08-09626-3.

[6] F. M. Atici and P. W. Eloe, Discrete fractional calculus with the nabla operator, *Electronic Journal of Qualitative Theory of Differential Equations*, Special Edition I, No. 3 (2009), 1–12.

[7] J. Čermák and L. Nechvátal, On (q,h)-analogue of fractional calculus, *Journal of Nonlinear Mathematical Physics* 17(1) (2010), 51–68. doi:10.1142/S1402925110000593.

[8] F. Du, L. Erbe, B. Jia and A. Peterson, Two asymptotic results of solutions for nabla fractional (q,h)-difference equations, *Turkish Journal of Mathematics* 42(5) (2018), 2214–2242. doi:10.3906/mat-1802-49.

[9] J. M. Jonnalagadda, A comparison result for the nabla fractional difference operator, *Foundations* 3(2) (2023), 181–198. doi:10.3390/foundations3020016.

[10] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer, Cham, 2015.