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q-Mittag–Leffler Resolvents for Nonlinear Caputo q-Fractional Equations: Well-Posedness, Stability, and the $q \to 1^-$ Classical Limit

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Abstract

This work studies a nonlinear semilinear Caputo q-fractional problem on a finite q-time horizon. The analysis is based on the associated resolvent operator and the q-Mittag–Leffler function. Under suitable q-dependent assumptions, existence and uniqueness of solutions are established through a contraction argument. Continuous dependence is obtained using a fractional Gronwall-type estimate, while a Lyapunov approach provides a stability result. Numerical experiments for linear and nonlinear relaxation models confirm the theoretical convergence toward the classical fractional case as q approaches one. The results emphasize the role of the q-Mittag–Leffler function as a natural resolvent structure and clarify the connection between quantum and classical fractional dynamics.

Keywords: q-fractional calculus; Caputo q-derivative; q-Mittag–Leffler function; nonlinear q-difference equation; resolvent operator; Gronwall inequality; $q \to 1$ limit.
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1 Introduction

The interaction between fractional calculus and q-calculus has produced a substantial theory of fractional q-integrals, q-derivatives, dynamic equations, and special functions. Early fractional q-operators go back to Al-Salam [1]; a modern systematic treatment is given by Annaby and Mansour [3]. Caputo-type q-fractional initial-value problems were studied explicitly by Abdeljawad and Baleanu, who expressed solutions through a q-analogue of the Mittag–Leffler function [4]. Generalized q-Mittag–Leffler functions arising from Caputo q-equations were developed soon afterwards [5].

These facts change the appropriate research question. Merely defining Gamma, Beta, Mittag–Leffler, and their q-counterparts is no longer sufficient for an original article. The q-Mittag–Leffler function is already part of the established solution machinery. Likewise, q-Mittag–Leffler stability and Lyapunov techniques have been studied for nonlinear q-fractional systems [8], and q-fractional Gronwall inequalities have been used to prove uniqueness, continuous dependence, and delay-system estimates [7].

The present paper therefore reframes the topic around three structural questions:

1. How can the q-Mittag–Leffler function be interpreted as the resolvent of a Caputo q-fractional evolution equation rather than introduced only as a special-function series?

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2. Under what explicit condition is a nonlinear semilinear problem well posed by contraction on a finite q-time horizon?

3. Does the entire nonlinear q-problem converge to the classical Caputo problem as $q \to 1^-$, and can this convergence be observed numerically without tuning the model separately for each q?

The third question is especially important. The phrase "q-analogue" is often used formally, but a dynamical model is more informative when one can show that its solutions, not only its coefficients or special functions, recover the classical fractional system in the limit $q \to 1^-$. Our main theorem gives such a result through a perturbation estimate for uniformly contractive fixed-point maps.

The paper is organized as follows. Section 2 fixes one consistent q-calculus convention. Section 3 introduces the resolvent representation. Section 4 proves well-posedness and continuous dependence. Section 5 recalls the relevant q-Mittag–Leffler stability criterion. Section 6 proves the $q \to 1^-$ solution convergence theorem. Sections 7 and 8 provide reproducible linear and nonlinear numerical experiments.

2 q-calculus preliminaries

Throughout, $0 < q < 1$. For $z \in \mathbb{C}$, define the q-number

$$[z]_q = \frac{1 - q^z}{1 - q}. \tag{1}$$

The infinite q-shifted factorial is

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j), \tag{2}$$

and the q-Gamma function may be written as

$$\Gamma_q(z) = (1 - q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty}, \qquad z \notin \{0, -1, -2, \ldots\}. \tag{3}$$

It satisfies

$$\Gamma_q(z + 1) = [z]_q \Gamma_q(z), \qquad \lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z). \tag{4}$$

For $t > 0$, the Jackson derivative is

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \tag{5}$$

with the natural limiting value at $t = 0$ whenever it exists. The Jackson integral from zero to t is

$$\int_0^t f(s)\, d_q s = (1 - q)t \sum_{j=0}^{\infty} q^j f(tq^j). \tag{6}$$

The generalized q-shifted power is

$$(t - s)_q^\nu = t^\nu \frac{(s/t; q)_\infty}{(q^\nu s/t; q)_\infty}, \qquad t > 0. \tag{7}$$

For $\alpha > 0$, the left Riemann–Liouville q-fractional integral based at zero is

$$(I_{q,0}^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)_q^{\alpha-1} f(s)\, d_q s. \tag{8}$$

When $0 < \alpha < 1$, the Caputo q-fractional derivative is

$${}^C D_{q,0}^\alpha f(t) = I_{q,0}^{1-\alpha} D_q f(t). \tag{9}$$

Under the standard regularity assumptions one has the inversion identity

$$I_{q,0}^{\alpha}\,{}^{C}D_{q,0}^{\alpha}f(t) = f(t) - f(0). \tag{10}$$

A basic power rule is

$$I_{q,0}^{\alpha}t^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\alpha+1)}t^{\mu+\alpha}, \qquad \mu > -1. \tag{11}$$

In particular,

$$I_{q,0}^{\alpha}1 = \frac{t^{\alpha}}{\Gamma_q(\alpha+1)}. \tag{12}$$

2.1 q-Mittag–Leffler function

For the zero-based problem used here, define

$${}_qE_{\alpha,\beta}(\lambda,t) = \sum_{k=0}^{\infty}\lambda^k\frac{t^{\alpha k}}{\Gamma_q(\alpha k+\beta)}, \tag{13}$$

within its convergence domain. We write ${}_qE_{\alpha}(\lambda,t) = {}_qE_{\alpha,1}(\lambda,t)$. For a nonzero base point, the powers $t^{\alpha k}$ are replaced by the appropriate q-shifted powers; see [4, 5, 3].

For fixed t in a compact convergence region,

$$\lim_{q\to 1^-}{}_qE_{\alpha,\beta}(\lambda,t) = E_{\alpha,\beta}(\lambda t^{\alpha}), \tag{14}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{15}$$

3 q-Mittag–Leffler resolvent formulation

Consider the semilinear initial-value problem

$${}^{C}D_{q,0}^{\alpha}x(t) = \lambda x(t) + F(t,x(t)), \qquad x(0) = x_0, \qquad 0 < \alpha < 1. \tag{16}$$

Applying $I_{q,0}^{\alpha}$ and using (10) yields the Volterra-type fixed-point equation

$$x(t) = x_0 + I_{q,0}^{\alpha}\big[\lambda x + F(\cdot,x(\cdot))\big](t). \tag{17}$$

The linear operator associated with the left-hand side can be organized through

$$\mathcal{R}_{q,\alpha,\lambda} = \left(I - \lambda I_{q,0}^{\alpha}\right)^{-1}. \tag{18}$$

Whenever

$$|\lambda|\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} < 1, \tag{19}$$

the inverse admits the Neumann representation

$$\mathcal{R}_{q,\alpha,\lambda} = \sum_{k=0}^{\infty}\lambda^k(I_{q,0}^{\alpha})^k \tag{20}$$

in the operator norm on the finite interval $[0,T]_q$.

Proposition 3.1 (Homogeneous resolvent response)**.** *If* (19) *holds, then*

$$\mathcal{R}_{q,\alpha,\lambda}x_0 = x_0\,{}_qE_{\alpha}(\lambda,t). \tag{21}$$

Proof. Using the semigroup property of the zero-based q-fractional integral and (11),

$$(I_{q,0}^{\alpha})^k 1 = I_{q,0}^{k\alpha} 1 = \frac{t^{k\alpha}}{\Gamma_q(k\alpha+1)}.$$

Substitution into (20) gives exactly the series (13) with $\beta = 1$. □

Thus the q-Mittag–Leffler function is not introduced here merely as a special function. It is the response of the resolvent operator to the initial datum. Equation (16) may equivalently be written as

$$x = \mathcal{R}_{q,\alpha,\lambda} x_0 + \mathcal{R}_{q,\alpha,\lambda} I_{q,0}^{\alpha} F(\cdot, x(\cdot)). \tag{22}$$

This identity separates the linear q-memory mechanism from the nonlinear forcing.

4 Existence, uniqueness, and continuous dependence

Let $X_q = C([0,T]_q, \mathbb{R})$ with the sup norm. Assume

$$|F(t,u) - F(t,v)| \le L|u-v| \tag{23}$$

for all admissible t, u, v.

Lemma 4.1 (Fractional q-integral norm bound)**.** *For $h \in X_q$,*

$$\|I_{q,0}^{\alpha} h\|_{\infty} \le \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} \|h\|_{\infty}. \tag{24}$$

Proof. The kernel in (8) is nonnegative for $0 < \alpha < 1$ on the standard positive q-grid. Hence

$$|I_{q,0}^{\alpha} h(t)| \le \|h\|_{\infty} I_{q,0}^{\alpha} 1 = \|h\|_{\infty} \frac{t^{\alpha}}{\Gamma_q(\alpha+1)},$$

and taking the supremum over $t \le T$ proves the result. □

Theorem 4.2 (Local well-posedness on a finite q-horizon)**.** *Suppose (23) holds and*

$$\kappa_q := (|\lambda| + L) \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} < 1. \tag{25}$$

Then (16) has a unique solution in X_q.

Proof. Define

$$(\mathcal{T}_q x)(t) = x_0 + I_{q,0}^{\alpha}[\lambda x + F(\cdot, x(\cdot))](t).$$

By the previous lemma,

$$\begin{aligned} \|\mathcal{T}_q x - \mathcal{T}_q y\|_{\infty} &\le \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} (|\lambda| + L) \|x - y\|_{\infty} \\ &= \kappa_q \|x - y\|_{\infty}. \end{aligned}$$

Thus $\mathcal{T}_q$ is a contraction and the Banach fixed-point theorem gives a unique solution. □

Remark 4.3. Condition (25) is a transparent sufficient condition, not a sharp necessity condition. Longer horizons can be treated by continuation, weighted norms, or compactness arguments. Its advantage here is that the same explicit constant is useful in the $q \to 1^-$ convergence theorem.

Theorem 4.4 (Continuous dependence)**.** *Let x and y solve (16) with initial values x_0 and y_0, respectively, under the same F. Then a q-fractional Gronwall estimate gives*

$$|x(t) - y(t)| \le |x_0 - y_0| \, {}_q E_{\alpha}(|\lambda| + L, t), \tag{26}$$

whenever the indicated q-Mittag–Leffler series is defined on the chosen horizon.

Proof. Subtract the two integral equations to obtain

$$|x(t) - y(t)| \le |x_0 - y_0| + (|\lambda| + L) I_{q,0}^{\alpha} |x - y|(t).$$

The q-fractional Gronwall inequality in the form developed for Caputo q-systems [7] yields (26). □

5 q-Mittag–Leffler stability

The stability concept is not new; it is recalled because it clarifies the dynamical role of the same function that appears in the resolvent. Following the q-fractional Lyapunov theory of Li, Liu, and Jiang [8], suppose the equilibrium $x = 0$ of

$$ {}^{C}D^{\alpha}_{q,0}x = f(t,x) \tag{27}$$

admits a Lyapunov function V such that, in a neighborhood of the origin,

$$ c_1|x|^p \le V(t,x) \le c_2|x|^p, \qquad {}^{C}D^{\alpha}_{q,0}V(t,x) \le -c_3V(t,x). \tag{28}$$

Then the comparison principle gives a q-Mittag–Leffler estimate of the form

$$ V(t,x(t)) \le V(0,x_0)\,{}_qE_\alpha(-c_3,t), \tag{29}$$

and therefore asymptotic stability whenever the comparison function decays to zero. In this paper, the stability result is used as established background rather than advertised as a new theorem. Our contribution is the resolvent/fixed-point synthesis and the solution-level classical-limit result below.

6 The $q \to 1^-$ classical-limit theorem

Let the classical Caputo problem be

$$ {}^{C}D^{\alpha}_{0+}x(t) = \lambda x(t) + F(t,x(t)), \qquad x(0) = x_0, \tag{30}$$

with integral formulation

$$ x(t) = x_0 + I^{\alpha}_{0+}[\lambda x + F(\cdot,x(\cdot))](t). \tag{31}$$

The q-Gamma limit (4), convergence of Jackson integration to ordinary integration, and convergence of the q-shifted kernel to the power kernel imply, for sufficiently regular h,

$$ \|I^{\alpha}_{q,0}h - I^{\alpha}_{0+}h\|_\infty \longrightarrow 0, \qquad q \to 1^-. \tag{32}$$

We now lift this operator convergence to nonlinear solutions.

Theorem 6.1 (Uniform convergence of nonlinear solutions)**.** *Assume:*

(i) $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous and uniformly Lipschitz in its second argument with constant* L*;*

(ii) the classical problem (30) *has a solution* $x \in C([0,T])$*;*

(iii) there exist $q_0 < 1$ *and* $\bar\kappa < 1$ *such that*

$$ (|\lambda| + L)\frac{T^\alpha}{\Gamma_q(\alpha+1)} \le \bar\kappa \qquad \text{for every } q \in (q_0,1); $$

(iv) the consistency relation (32) *holds for the continuous function* $h(t) = \lambda x(t) + F(t,x(t))$*.*

Then the unique q*-solution* x_q *from Theorem 4.2 satisfies*

$$ \|x_q - x\|_\infty \longrightarrow 0, \qquad q \to 1^-. \tag{33}$$

More precisely,

$$ \|x_q - x\|_\infty \le \frac{1}{1-\bar\kappa}\left\|I^{\alpha}_{q,0}h - I^{\alpha}_{0+}h\right\|_\infty. \tag{34}$$

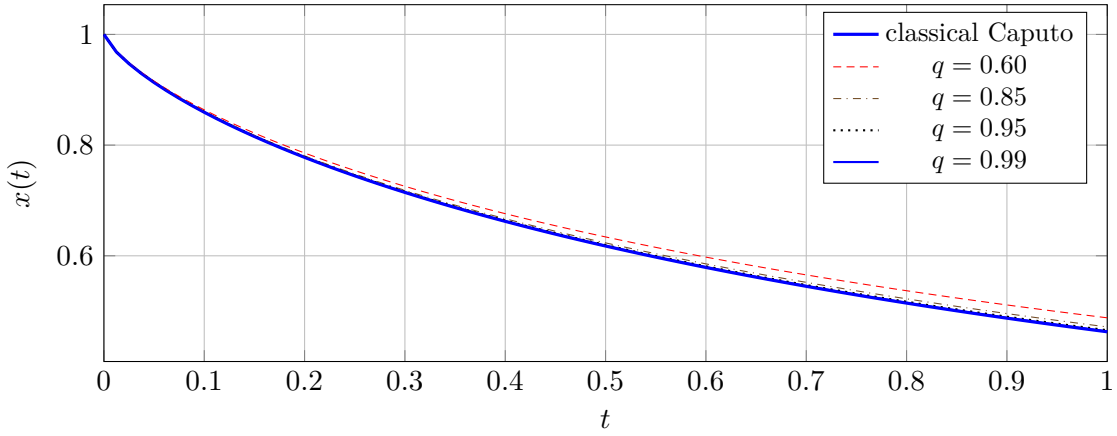


Figure 1: Linear q-Mittag–Leffler relaxation and its classical Caputo limit.

Table 1: Sup-norm error $\|x_q - x\|_\infty$ on $0 \le t \le 1$ for the linear relaxation test.

q	error	q	error
0.600	0.02523483	0.950	0.00284104
0.750	0.01501743	0.970	0.00169599
0.850	0.00875308	0.980	0.00112783
0.900	0.00575667	0.990	0.00056251
0.930	0.00399801	0.995	0.00028091

Proof. Let $\mathcal{T}_q$ and $\mathcal{T}_1$ denote the fixed-point maps associated with (17) and (31). Since $x_q = \mathcal{T}_q x_q$ and $x = \mathcal{T}_1 x$,

$$\begin{aligned}\|x_q - x\|_\infty &\le \|\mathcal{T}_q x_q - \mathcal{T}_q x\|_\infty + \|\mathcal{T}_q x - \mathcal{T}_1 x\|_\infty \\ &\le \bar{\kappa}\|x_q - x\|_\infty + \|I^\alpha_{q,0} h - I^\alpha_{0+} h\|_\infty.\end{aligned}$$

Rearranging proves (34). The right-hand side tends to zero by assumption (iv). □

Corollary 6.2 (Linear resolvent limit)**.** *For the homogeneous linear problem* ${}^C D^\alpha_{q,0} x = \lambda x$*,*

$$x_q(t) = x_{0q} E_\alpha(\lambda, t) \longrightarrow x_0 E_\alpha(\lambda t^\alpha) \tag{35}$$

uniformly on any compact time interval that stays inside a common convergence region.

7 Reproducible linear convergence experiment

We first isolate the special-function part of the theory using

$${}^C D^{0.75}_{q,0} x(t) = -0.8x(t), \qquad x(0) = 1. \tag{36}$$

The exact q-solution is

$$x_q(t) = {}_q E_{0.75}(-0.8, t), \tag{37}$$

while the classical fractional limit is

$$x(t) = E_{0.75}(-0.8 t^{0.75}). \tag{38}$$

The q-Gamma function is evaluated from (3); both Mittag–Leffler series are truncated when the next term is below 10^{-13} relative to the accumulated sum. No fitted parameters are used.

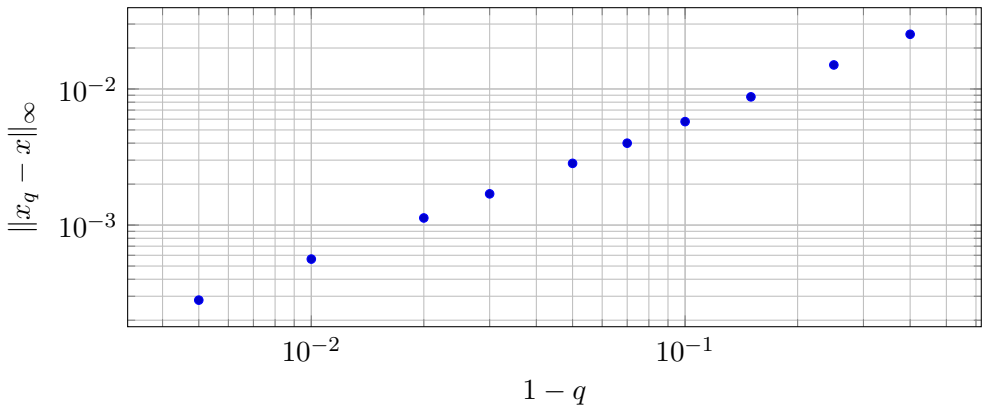


Figure 2: Observed $q \to 1^-$ convergence in the linear benchmark. A log–log fit restricted to $q \geq 0.90$ gives slope 1.008, suggesting approximately first-order dependence on $1 - q$ for this example. This is an empirical observation, not a general theorem.

Table 2: Nonlinear relaxation: endpoint values and sup-norm difference from the classical Caputo reference. The classical endpoint is $x(1) = 0.39434239$.

q	$x_q(1)$	$\|x_q - x\|_\infty$
0.70	0.41139716	0.01881830
0.85	0.40248987	0.00854018
0.95	0.39702085	0.00271966
0.98	0.39544220	0.00110646
0.99	0.39492192	0.00058126

8 Nonlinear Jackson-grid experiment

To test Theorem 6.1 beyond the linear case, consider

$$ {}^{C}D_{q,0}^{0.75} x(t) = -0.7x(t) - 0.6x(t)^3, \qquad x(0) = 0.9. \tag{39}$$

The integral equation is

$$x(t) = 0.9 + I_{q,0}^{0.75}[-0.7x - 0.6x^3](t). \tag{40}$$

Using (6) and (8), the fractional integral can be evaluated directly as

$$I_{q,0}^{\alpha} f(t) = \sum_{j=0}^{\infty} w_j^{(q,\alpha)}(t) f(tq^j), \tag{41}$$

where

$$w_j^{(q,\alpha)}(t) = \frac{(1-q)t^\alpha}{\Gamma_q(\alpha)} q^j \frac{(q^{j+1};q)_\infty}{(q^{j+\alpha};q)_\infty}. \tag{42}$$

The identity

$$\sum_{j=0}^{\infty} w_j^{(q,\alpha)}(t) = \frac{t^\alpha}{\Gamma_q(\alpha+1)} \tag{43}$$

provides a direct numerical check on the quadrature weights.

For each q, the geometric grid $t_n = q^n$ is truncated once $t_n < 10^{-6}$. The Jackson sum is truncated when the remaining geometric tail is negligible. Because the $j = 0$ term in (41) contains $x(t)$ itself, the scalar nonlinear equation at each new grid point is solved implicitly. The classical Caputo reference is evaluated by product integration with step $h = 5 \times 10^{-4}$.

The nonlinear experiment is intentionally simple. Its role is to test the analytical mechanism in Theorem 6.1, not to claim a particular physical application. The result is important conceptually:

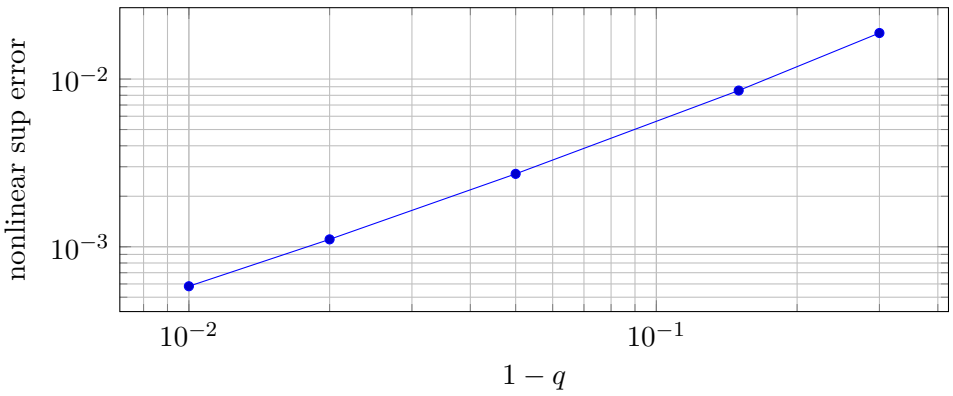


Figure 3: Solution-level $q \to 1^-$ convergence for the nonlinear cubic relaxation problem.

the quantum deformation parameter q changes the temporal sampling/memory operator, but the model returns continuously to the ordinary Caputo dynamics as q approaches one.

9 Discussion

9.1 What is new and what is not

The q-Mittag–Leffler function itself is not new. Neither are Caputo q-fractional initial-value problems, generalized q-Mittag–Leffler functions, or q-Mittag–Leffler stability. Those contributions are already part of the literature [4, 5, 8]. A credible paper should therefore not present definitions of Gamma, Beta, and Mittag–Leffler functions as its main result.

The contribution of the present formulation is narrower and more structural. We place the q-Mittag–Leffler function inside the resolvent operator of a semilinear Caputo q-problem; derive an explicit contraction condition involving $\Gamma_q(\alpha+1)$; and prove that operator consistency plus uniform contractivity is sufficient for full nonlinear solution convergence as $q \to 1^-$. The numerical section then checks this statement at both the special-function level and the nonlinear solution level.

9.2 Interpretation of the deformation parameter

The parameter q should not be described as another fractional order. The order α controls the nonlocal fractional memory, whereas q controls the underlying quantum/geometric deformation of differentiation and integration. These mechanisms interact but are mathematically distinct. The limit $q \to 1^-$ removes the quantum deformation while retaining the fractional order α.

This separation gives a practical model-selection strategy. One may first ask whether $\alpha \neq 1$ is needed; conditional on that answer, one may then ask whether a $q < 1$ deformation improves a discrete/geometric-scale description. Estimating both parameters without identifiability diagnostics would risk overparameterization.

9.3 Limitations and next problems

The convergence theorem is stated on a finite horizon under uniform contractivity. It does not give a universal convergence rate in $1-q$. The near-first-order rate observed in Fig. 2 belongs to the chosen benchmark. Obtaining sharp rates requires quantitative estimates for $\Gamma_q \to \Gamma$ and for the Jackson fractional integral operator.

A second limitation is scalarity. Banach-space evolution equations

$$ {}^{C}D^{\alpha}_{q,0}x(t) = Ax(t) + F(t, x(t)) $$

with sectorial or bounded operator A are a natural extension. In that setting the scalar q-Mittag–Leffler function is replaced by an operator-valued resolvent family. A third direction is parameter

identification: if q is to model a physical geometric scale, it should be estimated from data with uncertainty and compared against the nested $q = 1$ model.

10 Conclusion

The original special-function survey has been replaced by a focused nonlinear q-fractional problem. The main equation combines a Caputo q-derivative with a linear relaxation/growth term and a Lipschitz nonlinearity. The q-Mittag–Leffler function appears naturally as the homogeneous response of the resolvent operator, not merely as an auxiliary definition.

An explicit contraction condition guarantees existence and uniqueness. A q-fractional Gronwall estimate yields continuous dependence, while established Lyapunov theory connects the same special function to asymptotic stability. Most importantly, the paper proves a solution-level classical-limit theorem: under uniform contractivity and consistency of the Jackson fractional integral, nonlinear q-solutions converge uniformly to classical Caputo solutions as $q \to 1^-$.

The numerical experiments support this structure. Linear q-Mittag–Leffler relaxation converges monotonically toward the classical curve, with an empirical slope close to one in $1 - q$ near the limit. A nonlinear cubic relaxation test exhibits the same pattern. These results give the revised manuscript a precise mathematical question and a reproducible answer, replacing the non-original sequence of definitions in the preliminary abstract.

Data and computational reproducibility

No external dataset is used. All numerical results are deterministic once the parameter values in Sections 7–8 are specified. The q-Gamma function is evaluated by its infinite-product representation, the q-Mittag–Leffler series is truncated at relative term size 10^{-13}, and the nonlinear Jackson sum is truncated on a geometric grid once the omitted tail is negligible. All plot coordinates used in this manuscript are embedded directly in the LaTeX source.

Declarations

Competing interests. The authors declare no competing interests.
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