

Comparing Concentration and Separation

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Comparing Concentration and Separation

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Abstract

Interpreting the concentration of an allocation as a special case of its separation from the uniform benchmark provides a starting point for extending the measurement of concentration and separation to multiple allocations and for organizing the concepts and measures used in this area within a common methodological framework. We introduce a comparison relation and conditions for numerical representations consistent with it, thereby proposing an apparatus analogous to majorization and the Lorenz order in inequality theory. We study several classes of consistent numerical representations, including locally additive functionals encompassing, among others, the Herfindahl–Hirschman index, and a family of partition-based indices such as the Kolkata index and an index equal to the optimal probability of correct Bayesian classification under equal priors. Selected constructions are extended to an arbitrary number of allocations. For one allocation, the proposed preorder reduces to majorization; for two, it coincides with the Blackwell order and zonotope inclusion; from three allocations onward, these relations cease to be equivalent. We characterize the preorder through attainable sets, their support functions, and the optimal probability of correct identification for all prior distributions. Applying the proposed methods to the global production of copper, lithium, cobalt, and nickel illustrates their value as a complement to classical tools for analyzing concentration and dependence by revealing how separation analysis enriches the description of the joint geographical distribution of these structures.

Keywords: separation order, majorization, Blackwell order, Herfindahl–Hirschman index, Kolkata index, total variation distance

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1 Introduction

Concentration and separation are widely used concepts in economics, statistics, and other fields, but they do not have a single precise and universally accepted definition. What exactly does it mean for one allocation to be more concentrated than another? How should the concentration of several allocations defined on the same set of agents be compared, and how should their mutual position be described? In practice, the answers depend on the application: separate indices are used, axioms are tailored to a particular problem, or tools are borrowed from inequality theory. The boundary between concentration and inequality is also blurred at the level of the indices themselves. The Herfindahl–Hirschman index (HHI) is a standard measure of market concentration (Grullon et al. 2019; U.S. Department of Justice and Federal Trade Commission 2023), yet the same functional has also been classified as an inequality measure (Gastwirth 1975; Eeckhout 2021); conversely, the Kolkata index was introduced and has been developed primarily as an inequality measure, although its construction admits a natural concentration interpretation (Ghosh et al. 2014; Chatterjee et al. 2017). Thus, there are many indices and applications, but no common comparison criterion that precedes the choice of a particular numerical scale.

This contrasts with inequality theory, where Lorenz curves, majorization, the Gini index, and broad classes of Schur-convex functionals are embedded in a developed axiomatic framework (Atkinson 1970; Cowell 2011; Marshall et al. 2011; Kvålseth 2025). There, one first specifies when one allocation is more unequal than another and then studies indices consistent with that relation. The present paper follows an analogous order of construction, but extends it through a relational view of concentration. The classical question of concentration of a single structure can be interpreted as a comparison with the uniform benchmark; moving from such a pair to two arbitrary structures leads to the question of their mutual separation, and then to the analogous problem for any number of allocations. The separation preorder formalizes this comparison relation, and only then do we consider its monotone numerical representations. An index should respect the relation defined beforehand: if one configuration dominates another according to the preorder, the index should not assign it a lower value.

Moving from one to multiple allocations is more than a technical increase in dimension. Separate values $C(A_1), \dots, C(A_m)$ describe each structure individually, but they do not reveal whether their masses are concentrated on the same units or on disjoint groups. Nor does a collection of pairwise relations necessarily recover the joint position of the entire configuration. The proposed construction therefore treats $(A_1, \dots, A_m)$ as a single object. The underlying idea is to partition the set of agents, possibly fractionally, into m parts, one associated with each allocation, and to compare configurations by the degree of separation attainable through such partitions. For a single structure paired with the uniform benchmark, this yields the classical information about concentration; for two or more structures, the same construction has a natural interpretation in terms of separation. In this sense, classical concentration is a special case of a broader relational problem.

The construction also preserves the classical boundary cases. For one allocation compared with the uniform benchmark, the proposed relation reduces exactly to majorization. For two allocations, it coincides with the Blackwell order and with zonotope

inclusion. Only from three allocations onward do these relations diverge: the Blackwell order is stronger than the proposed preorder, whereas the zonotope order and the separation preorder are generally incomparable. The multi-allocation setting is therefore not merely a change in the number of arguments; it gives rise to a distinct problem of jointly separating several structures.

The multi-allocation setting also creates a bridge between the economic concept of concentration and statistical decision theory. Theorem 3.8 shows that dominance under the proposed preorder is equivalent to dominance in the optimal probability of correctly identifying the state for every prior distribution. The family of Bayesian functionals indexed by the prior is classical in the comparison of experiments and in quantitative information flow (Torgersen 1991; Alvim et al. 2020); the specific contribution of the present construction is its equivalent representation through the geometry of attainable sets generated by common partitions and the interpretation of these sets in terms of concentration and separation.

The contribution of the paper is fourfold. First, we propose a relational view of concentration in which the classical concentration of a single structure relative to the uniform benchmark is a special case of a broader separation relation, with the corresponding preorder preceding the choice of a particular index. Second, we obtain geometric and decision-theoretic characterizations of the preorder through attainable sets, their support functions, and the optimal probability of correct identification for all priors, and we establish its relations with existing informational and geometric orders. Third, we develop a theory of numerical representations that includes a family of separating functionals, their positive mixtures, pairwise-projection representations, and a class of partition-based indices. Fourth, we obtain a representation of the Kolkata index as the value of a fractional max–min problem; this identification leads to a natural extension from one allocation to two and then to an arbitrary number of allocations, and, together with the arithmetic index, determines the endpoints of a natural family of partition-based representations.

A case study of global copper, lithium, cobalt, and nickel production illustrates the distinction between concentration of individual structures and their mutual separation. In particular, it shows that high concentration of individual commodities can coexist with changes in the geographical location of their production centers, while multi-allocation separation provides information that is not contained in separate concentration indices or in pairwise comparisons alone.

Section 2 organizes different meanings of concentration and separation and presents the relational view adopted in the paper. Section 3 defines the preorder and studies its relations with majorization, the Blackwell order, the zonotope order, and the multivariate Lorenz order. Section 4 develops its numerical representations, while Section 5 analyzes the partition-based family and its particular indices. The case study is presented in Section 6, and Section 7 discusses the significance of the construction, its limitations, and directions for further research. Technical results and supplementary proofs are collected in the Appendix.

Readers approaching the paper from different perspectives may use different routes through the argument. Readers primarily interested in the order-theoretic construction may focus on Sections 2 and 3, where the relational interpretation, attainable-set

construction, and relations to majorization, the Blackwell order, and zonotope inclusion are developed. Readers interested in numerical measurement may proceed to Sections 4 and 5, which study consistent numerical representations and partition-based indices, including the HHI and Kolkata-type constructions. Readers interested in the decision-theoretic and information-theoretic connections may focus on the corresponding parts of Sections 3 and 4, while Section 6 is intended primarily as an illustration of the framework.

2 Concentration and Separation

The concepts of concentration and separation are used across many research traditions, but they do not have a single generally accepted formal meaning. In economics, concentration may refer to the concentration of market shares in the hands of a few firms; in inequality theory, to unevenness of a distribution; while research on segregation, dissimilarity, and information theory employs distinct notions of separation, distinguishability, and exposure. These traditions are adjacent but not identical.

In economics, the development of concentration theory has to a large extent been the development of its measures. Early axiomatic discussions explicitly distinguished industrial concentration measures from inequality measures and examined the properties required of concentration indices (Marfels 1971). Hall and Tideman (1967) formulate properties required of a measure of industrial concentration, Hannah and Kay (1977) relate concentration to the increasing control of economic activity by large firms, and Encaoua and Jacquemin (1980) analyze properties of concentration indices in the context of monopoly power and market entry; the HHI and CR_k ratios remain standard practical tools. At the same time, even for these classical indices there is no full agreement on the desirable properties of the scale; for example, the *value validity* criterion of Kvålseth (2022) questions the interpretation of numerical differences as direct differences in concentration itself. For a single structure of shares, inequality theory provides a particularly natural framework: transferring mass from a smaller share to a larger one leads to majorization, the Lorenz order, and Schur-convex functions (Marshall et al. 2011). With a fixed number of units, concentration and inequality therefore become closely related in order terms, although they need not be identical on a domain that allows the number of units to vary. A different recent line of work extends inequality measurement beyond the classical univariate setting through Fourier-based constructions, including multidimensional formulations (Toscani 2022; Giudici et al. 2025). These approaches address multidimensional inequality, whereas the present framework concerns the relational concentration and separation of several allocation structures.

The boundary between concentration and separation is equally blurred. In segregation theory, classical dissimilarity indices measure the extent to which groups can be separated across units, whereas Massey and Denton (1988) distinguish several dimensions of segregation, including *evenness*, *exposure*, and *concentration*; this distinction was subsequently re-examined empirically by Massey et al. (1996). Contemporary literature also shows that, even within one field, different measures may answer different questions (Reardon and Firebaugh 2002; Hutchens 2004; Frankel and Volij 2011). A

similar issue arises when several allocations or distributions are compared: correlation, divergence, segregation, informativeness, and separability describe different aspects of their mutual position.

We therefore do not assume that there is one universal axiomatization of concentration and separation. Our aim is to formalize one particularly natural intuition: the more easily the masses of several structures can be separated by a common partition of the units, the greater their mutual separation. For a single structure compared with the uniform benchmark, the same construction has the interpretation of classical concentration.

Let A denote an allocation structure and U the uniform structure. A classical concentration index has the form $C(A)$ and describes a property of a single structure. The relational approach replaces it with a functional $F(A, B)$ of two structures. When the second argument is the uniform benchmark, we obtain a measure of the concentration of A relative to equal distribution,

$$C(A) = F(A, U).$$

This is not a universal representation of every notion of concentration found in the literature, but a scheme covering a broad class of measures based on deviation from equality.

Moving from (A, U) to an arbitrary pair (A, B) changes the interpretation. In the first case, we ask how far the structure A departs from equality; in the second, how strongly two structures can be separated from one another. For arbitrary $m \geq 2$ structures, we shall therefore speak of *m-separation*.

We treat concentration as a special case of this construction obtained by adding the uniform benchmark. If F_m denotes a measure of separation of m structures, then the corresponding m -concentration is defined by

$$C(A_1, \dots, A_m) = F_{m+1}(A_1, \dots, A_m, U).$$

In particular, classical concentration is the 2-separation of (A, U) , biconcentration of two allocations is the 3-separation of (A_1, A_2, U) , and triconcentration is the 4-separation of (A_1, A_2, A_3, U) . For $m \geq 3$, separation does not reduce to a sum or a collection of pairwise relations; it describes the joint possibility of separating all structures by a single partition of the units.

The benchmark mechanism has direct analogues in other areas of measurement theory. Total variation between a structure A and the uniform benchmark U is affinely equivalent to the Pietra index, whereas for an arbitrary pair A, B it becomes a classical measure of statistical distinguishability. Similarly, Kullback–Leibler divergence from the uniform benchmark leads to classical inequality measures in the entropy family, whereas for two arbitrary distributions it retains its interpretation as a statistical divergence (Csiszár 1967; Cowell 2011).

These examples suggest a more general principle: the same function may act as a concentration index once a benchmark is fixed and as a separation measure once an arbitrary second argument is admitted. Interpretation therefore depends not only on

the formula, but also on the domain and on the comparison relation represented by the formula.

To separate the concept from its particular numerical scale, we adopt the perspective of measurement theory: first specify a comparison relation on the objects of interest, and only then its numerical representations (Debreu 1954; Fishburn 1970; Krantz et al. 1971). The scheme is

$$\begin{array}{ccccc} \text{concept of concentration} & & \longrightarrow & \text{comparison} & \longrightarrow & \text{numerical} \\ \text{and separation} & & & \text{relation} & & \text{representation} \end{array} .$$

For one allocation compared with the uniform benchmark, the relation should preserve the classical majorization order. For two or more arbitrary structures, it describes their separation and, for $m \geq 3$, the possibility of jointly separating all structures by a single partition of the units. Under the convention adopted above, m -concentration is not a separate relation, but a special case of $(m+1)$ -separation after the uniform benchmark U has been added.

This organization separates two questions. The first is: when is one configuration of structures at least as concentrated or separated as another? The second is: by what number should this relation be represented? The next section addresses the first question by introducing a preorder based on common partitions; the subsequent sections study its numerical representations.

3 The Separation Preorder

3.1 Allocations, Structures, and Concentration of a Single Allocation

An allocation will mean the assignment of a given amount of goods to a given group of agents and will be denoted by $A = (N, w, a)$, where $N = \{1, 2, \dots, n\}$ is a set of $n \in \mathbb{N}$ agents, $w > 0$ is the total amount of goods being allocated, and $a = (a_1, a_2, \dots, a_n)$ is the allocation structure, that is, a vector of nonnegative numbers satisfying $\sum_{i=1}^n a_i = 1$. The set of all structures is denoted by

$$\Delta^{n-1} := \left\{ a \in \mathbb{R}_+^n : \sum_{i=1}^n a_i = 1 \right\},$$

the standard simplex in $\mathbb{R}^n$. More generally, Δ^{k-1} will denote the standard simplex in $\mathbb{R}^k$; depending on the context, the parameter k may therefore refer either to the number of agents or to the number of allocations. The coordinate w separates the scale of an allocation from its normalized structure a . When two or more allocations are compared, we assume by default that they are defined on the same set of agents.

Because the total amount of goods allocated is secondary to the problem studied here, and the indices considered in the paper are homogeneous of degree zero, we henceforth identify an allocation A with its structure a . In particular, the equal

allocation U is identified with the structure $u = (1/n, 1/n, \dots, 1/n)$. Whenever no ambiguity arises, the same symbols will be used for allocations and their corresponding structures.

A classical tool for comparing the concentration of a single structure is majorization (Marshall et al. 2011). For $a, b \in \Delta^{n-1}$, after arranging their coordinates in nonincreasing order, we write $a \succeq_{\text{maj}} b$ if $\sum_{i=1}^k a_{(i)} \geq \sum_{i=1}^k b_{(i)}$ for every $k = 1, \dots, n-1$. With equal totals, this is equivalent to the corresponding Lorenz dominance (Cowell 2011). Majorization is a common reference point for much of classical inequality and concentration theory; later in the paper it appears as the single-allocation special case of the broader separation preorder.

The construction can be understood as a partition problem. Given m allocation structures, the set of agents is divided, possibly fractionally, into m parts, one associated with each allocation. Each partition generates a profile describing how much mass of the corresponding allocation is captured by each part. The preorder compares configurations through the set of all profiles attainable in this way: one configuration dominates another when every separation profile attainable under the latter can also be attained under the former.

3.2 The Attainable Set for Two Allocations

For two structures $A, B \in \Delta^{n-1}$, consider a fractional partition $p = (p_1, \dots, p_n) \in [0, 1]^n$: a fraction p_i of unit i is assigned to allocation A , while the remaining fraction $1 - p_i$ is assigned to allocation B , yielding the outcome $(\sum_i p_i a_i, \sum_i (1 - p_i) b_i)$.

Definition 3.1 (Attainable set) We define

$$\mathcal{R}_2(A, B) = \left\{ \left(\sum_i p_i a_i, \sum_i (1 - p_i) b_i \right) : p \in [0, 1]^n \right\}.$$

Its downward comprehensive closure is

$$\mathcal{D}_2(A, B) = \left\{ z \in [0, 1]^2 : z \leq r \text{ coordinatewise for some } r \in \mathcal{R}_2(A, B) \right\}.$$

The set of points that are undominated under the coordinatewise order is called the *Pareto frontier* (see (Miettinen 1999)). For nondecreasing generators, the maximum of F over $\mathcal{R}_m$, and hence the value C_F , can be sought on this frontier. Later in Section 5 we consider, in particular, the minimum, geometric, and arithmetic generators, which lead respectively to the indices K , C_G , and C_A .

For the example in Figure 1, $A = (0.55, 0.30, 0.15)$ and $B = (0.10, 0.35, 0.55)$, the points maximizing the minimum, geometric, and arithmetic generators are respectively $(0.711538, 0.711538)$, $(0.660714, 0.770833)$, and $(0.55, 0.90)$, giving $K(A, B) \approx 0.7115$, $C_G(A, B) \approx 0.7137$, and $C_A(A, B) = 0.7250$.

The set $\mathcal{R}_2(A, B)$ is compact and convex; as shown below, it is an affine image of the zonotope generated by the vectors (a_i, b_i) . Links between zonotope inclusion and the comparison of experiments, dissimilarity, and the multivariate Lorenz order are well known (Blackwell 1951, 1953; Koshevoy and Mosler 1996; Bertschinger and Rauh 2014; Andreoli and Zoli 2023).

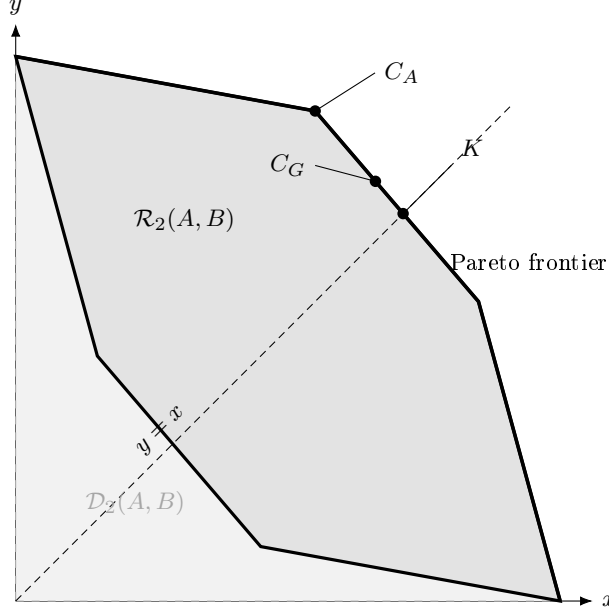


Figure 1 The attainable set $\mathcal{R}_2(A, B)$, its downward comprehensive closure $\mathcal{D}_2(A, B)$, and the Pareto frontier for $A = (0.55, 0.30, 0.15)$ and $B = (0.10, 0.35, 0.55)$. The points K , C_G , and C_A correspond to the optima of the minimum, geometric-mean, and arithmetic-mean generators discussed in Section 5.

3.3 Zonotope Representation

For a pair (A, B) , define the zonotope

$$Z(A, B) = \left\{ \left(\sum_i p_i a_i, \sum_i p_i b_i \right) : p \in [0, 1]^n \right\}.$$

The attainable set is its affine image under $T(x, y) = (x, 1 - y)$, that is, $\mathcal{R}_2(A, B) = T(Z(A, B))$.

Proposition 3.2 (Equivalence with zonotope inclusion) *For any two pairs of allocations (A, B) and (C, D) ,*

$$\mathcal{D}_2(C, D) \subseteq \mathcal{D}_2(A, B) \iff \mathcal{R}_2(C, D) \subseteq \mathcal{R}_2(A, B) \iff Z(C, D) \subseteq Z(A, B).$$

Proof The second equivalence follows immediately from the bijectivity of the map $T(x, y) = (x, 1 - y)$. Inclusion of the sets $\mathcal{R}_2$ directly implies inclusion of their downward comprehensive closures.

For the reverse direction, use the support function $h_K(c) = \sup_{z \in K} c \cdot z$ of a compact convex set K . If $c \in \mathbb{R}_+^2$, downward closure does not change the maximum of $c \cdot z$, so $h_{\mathcal{D}_2(A, B)}(c) = h_{\mathcal{R}_2(A, B)}(c)$. Inclusion of the sets $\mathcal{D}_2$ therefore gives the required support-function inequality for $c \geq 0$. For $c \leq 0$, the same inequality follows from central symmetry

of $\mathcal{R}_2$ about $(1/2, 1/2)$. If the coordinates of c have opposite signs, the maximum is attained at one of the common points $(1, 0)$ or $(0, 1)$. Hence the support-function inequality holds for every $c \in \mathbb{R}^2$, which characterizes inclusion of compact convex sets. $\square$

Remark 3.3 (Related geometric orders) As a consequence of Proposition 3.2, for two allocations the separation preorder is exactly the criterion of zonotope inclusion, after an affine change of coordinates (see (Bertschinger and Rauh 2014; Andreoli and Zoli 2023)). This places the proposed construction within a broader class of geometric orders comparing experiments and information structures. Related approaches are developed, among others, in geometric characterizations of the Blackwell order (Wu 2023) and in contemporary information economics (Brooks et al. 2024). Zonotope geometry also has applications beyond information theory, including the measurement of firm heterogeneity and efficiency (Cococcioni et al. 2022).

3.4 The 2-Separation Preorder

Definition 3.4 (2-separation preorder) For two pairs of structures (A, B) and (A', B') , write

$$(A, B) \succeq_c (A', B') \iff \mathcal{D}_2(A, B) \supseteq \mathcal{D}_2(A', B').$$

The relation is reflexive and transitive, but generally incomplete. The interpretation is direct: $(A, B) \succeq_c (A', B')$ means that every mass vector that can be guaranteed by a suitable partition for (A', B') can be at least matched by some partition for (A, B) .

A simple example.

The distinction between concentration of a single allocation and relational information already appears with three agents. Consider

$$A = (1/2, 1/2, 0), \quad B = (1/2, 1/2, 0), \quad B' = (0, 1/2, 1/2).$$

All three structures have the same single-allocation concentration: B' is merely a permutation of A , so every symmetric index consistent with majorization assigns them the same value. The pairs (A, B) and (A, B') , however, are relationally different. In the first pair the two masses are distributed identically and the pair belongs to the minimal class. In the second, the allocations are partially separated: assigning the first agent to A and the second and third agents to B' yields the point $(1/2, 1)$. Single-allocation indices cannot distinguish these two situations because they do not use information about the position of one structure relative to the other.

The minimal case occurs when $A = B$. The maximal case occurs when the supports are disjoint, that is, $\text{supp } A \cap \text{supp } B = \emptyset$, because then the point $(1, 1)$ is attainable.

3.5 Single-Allocation Restriction and Majorization

Concentration of one allocation is always defined relative to the uniform structure: at the level of the relation, $A \succeq_c B \Leftrightarrow (A, U) \succeq_c (B, U)$.

Proposition 3.5 (Single-allocation restriction) *For $A, B \in \Delta^{n-1}$, the relation $\succeq_C$ is equivalent to majorization:*

$$A \succeq_C B \iff A \succeq_{\text{maj}} B.$$

Proof sketch The pair (A, U) generates a two-dimensional object that, after an affine change of coordinates, is equivalent to the classical Lorenz representation of the structure A . For a one-dimensional allocation, inclusion of the corresponding zonotopes is equivalent to Lorenz dominance, which in turn—under a common total—is equivalent to majorization. See [Koshevoy and Mosler \(1996\)](#); [Koshevoy \(1998\)](#) for the Lorenz-zonoid-majorization connection and [Andreoli and Zoli \(2023\)](#) for relations among zonotope inclusion, dissimilarity, and majorization orders. $\square$

Thus, in the single-allocation case the proposed preorder does not differ from the Lorenz order: its representation reduces exactly to majorization. The common restriction to majorization explains why, on the single-allocation domain, concentration and inequality functionals may induce order-equivalent rankings.

3.6 Extension to m Allocations

Let

$$A_j = (a_{j1}, \dots, a_{jn}) \in \Delta^{n-1}, \quad j = 1, \dots, m.$$

A fractional partition of the common set of units is described by numbers $p_{ij} \geq 0$ satisfying $\sum_{j=1}^m p_{ij} = 1$ for every i ; the mass of allocation A_j captured by its part of the partition is $x_j = \sum_{i=1}^n p_{ij} a_{ji}$.

Definition 3.6 (Attainable set for m allocations)

$$\mathcal{R}_m(A_1, \dots, A_m) = \left\{ (x_1, \dots, x_m) : x_j = \sum_i p_{ij} a_{ji}, p_{ij} \geq 0, \sum_j p_{ij} = 1 \right\}.$$

We denote its downward comprehensive closure by $\mathcal{D}_m(A_1, \dots, A_m)$.

Definition 3.7 (m -separation preorder) For $\mathbf{A} = (A_1, \dots, A_m)$ and $\mathbf{B} = (B_1, \dots, B_m)$, write

$$\mathbf{A} \succeq_C \mathbf{B}$$

if and only if

$$\mathcal{D}_m(\mathbf{A}) \supseteq \mathcal{D}_m(\mathbf{B}).$$

We retain the symbol $\succeq_C$ for notational continuity; throughout the remainder of the paper it denotes the separation preorder, with the number of compared structures understood from the arguments.

Following the convention in [Section 2](#), for arbitrary $m \geq 2$ structures we speak of m -separation. m -concentration is the special case obtained by adding the uniform benchmark U , and therefore corresponds to the $(m+1)$ -separation of $(A_1, \dots, A_m, U)$.

The decision-theoretic part of the following characterization has classical counterparts in the theory of comparison of experiments and in quantitative information flow; the specific feature of the present construction is its connection to the geometry of the attainable sets $\mathcal{D}_m$ and its interpretation in terms of concentration and separation.

Theorem 3.8 (Functional and decision-theoretic characterization) *Let $\mathbf{A} = (A_1, \dots, A_m)$ and $\mathbf{B} = (B_1, \dots, B_m)$, where $A_j = (a_{j1}, \dots, a_{jn})$ and $B_j = (b_{j1}, \dots, b_{jn'})$. The following conditions are equivalent:*

(i) $\mathbf{A} \succeq_{\mathcal{C}} \mathbf{B}$;

(ii)

$$h_{\mathcal{D}_m(\mathbf{A})}(c) \geq h_{\mathcal{D}_m(\mathbf{B})}(c) \quad \text{for every } c \in \mathbb{R}_+^m;$$

(iii)

$$\sum_i \max_{1 \leq j \leq m} c_j a_{ji} \geq \sum_i \max_{1 \leq j \leq m} c_j b_{ji} \quad \text{for every } c \in \mathbb{R}_+^m;$$

(iv) for every prior $\pi \in \Delta^{m-1}$, the optimal probability of correctly identifying the state under 0-1 loss satisfies

$$P_{\text{corr}}^*(\mathbf{A}; \pi) \geq P_{\text{corr}}^*(\mathbf{B}; \pi),$$

where

$$P_{\text{corr}}^*(\mathbf{A}; \pi) = \sum_i \max_j (\pi_j a_{ji}).$$

Moreover, the optimum in (iv) can be attained by the deterministic rule

$$j^*(i) \in \arg \max_j \{\pi_j a_{ji}\}.$$

Proof First observe that, for a compact, convex, downward comprehensive set $D \subseteq \mathbb{R}_+^m$, inclusion can be tested by the support function only in nonnegative directions. Indeed, if $x \notin D$, the separation theorem gives a normal vector separating x from D ; downward comprehensiveness allows its negative coordinates to be replaced by zero without weakening the separation. Hence, for such sets,

$$D_1 \supseteq D_2 \iff h_{D_1}(c) \geq h_{D_2}(c) \quad \forall c \in \mathbb{R}_+^m,$$

which proves the equivalence of (i) and (ii).

Moreover,

$$\mathcal{R}_m(\mathbf{A}) = \sum_i \text{conv}\{a_{1i}e_1, \dots, a_{mi}e_m\}.$$

For $c \geq 0$, passing to the downward comprehensive closure does not change the maximum of $c^\top x$, and additivity of support functions under Minkowski addition yields

$$h_{\mathcal{D}_m(\mathbf{A})}(c) = h_{\mathcal{R}_m(\mathbf{A})}(c) = \sum_i \max_j (c_j a_{ji}).$$

This proves the equivalence of (ii) and (iii).

For $c \neq 0$, write $c = (\sum_j c_j) \pi$. By positive homogeneity, condition (iii) then reduces to $\pi \in \Delta^{m-1}$. Under prior π and 0-1 loss, the contribution of observation i to the probability of a correct decision is maximized by choosing $j^*(i) \in \arg \max_j \{\pi_j a_{ji}\}$. Randomization cannot

improve this value because, for fixed i , the objective is linear in the decision probabilities. Thus the optimum is deterministic and

$$P_{\text{corr}}^*(\mathbf{A}; \pi) = \sum_i \max_j (\pi_j a_{ji}),$$

which proves the equivalence of (iii) and (iv). $\square$

The theorem also has a simple interpretation outside the language of support functions. If the unit index is treated as an observation and the A_j as alternative structures, then greater m -separation means a greater ability to identify which structure generated the observed mass. In this sense, part of the relational information encoded by the preorder can be read as statistical distinguishability of allocations rather than merely as their marginal concentration.

Remark 3.9 (Relation to quantitative information flow) Interpreting the rows A_j as signal distributions, condition (iv) is the order induced by posterior Bayes vulnerability, that is, the optimal probability of correctly guessing the hidden state after observing the signal; in the QIF literature this relation is known as the strong min-entropy order. The family of functionals $\sum_i \max_j (\pi_j a_{ji})$, as the prior π ranges over the simplex, also has a classical counterpart in the theory of comparison of experiments (Torgersen 1991; McIver et al. 2014; Alvim et al. 2020). This order is based on one particular decision problem—correctly guessing the state—whereas the Blackwell order considers all decision problems. The specific feature of Theorem 3.8 is the equivalent characterization of this order through inclusion of the downward comprehensive attainable sets $\mathcal{D}_m$, generated by a single common partition problem, and through their support functions.

3.7 Blackwell, Zonotope, and Multivariate Lorenz Orders

The proposed preorder has natural reference points in the theory of comparison of experiments and in multivariate geometric orders. This subsection compares it with the Blackwell order, the zonotope-inclusion order, and the Koshevoy–Mosler multivariate Lorenz order. These results locate the construction relative to existing comparison relations.

By *garbling* we mean a common stochastic post-processing of the observed signal that may remove information but cannot create information about the hidden state.

Table 1 summarizes the main relations. It is useful to distinguish the single-allocation restriction of our preorder from orders on multistate experiments: comparison of one allocation with the uniform benchmark reduces to majorization, whereas the Blackwell order becomes the relevant reference point for two allocations. Detailed statements and proofs of the relations in the table are provided in Appendix A.1.

In the last row, the condition $UK = U$ means that the common stochastic channel K does not change the population benchmark used in the Lorenz zonoid: the uniform allocation remains uniform after processing. For square K , this condition is satisfied in particular by bistochastic matrices; the corresponding implication is formalized in Proposition A.4.

Table 1 Relations among four orders for comparing multivariate structures. The relation $\succeq_1 \subseteq \succeq_2$ means that dominance under $\succeq_1$ implies dominance under $\succeq_2$.

Pair of orders	Relation	Condition
$\succeq_B, \succeq_Z$	$\succeq_B \subseteq \succeq_Z$ (equality for $m = 2$)	always
$\succeq_B, \succeq_C$	$\succeq_B \subseteq \succeq_C$ (equality for $m = 2$)	always
$\succeq_Z, \succeq_C$	incomparable	$m \geq 3$
$\succeq_{KM}, \succeq_Z$	$\succeq_{KM} \Rightarrow \succeq_Z$	always
$\succeq_{KM}, \succeq_C$	$\succeq_{KM} \Rightarrow \succeq_C$	$m = 2$; unresolved here for $m \geq 3$
$\succeq_B, \succeq_{KM}$	$\succeq_B \Rightarrow \succeq_{KM}$	when the garbling preserves the population benchmark $UK = U$

For a row-stochastic matrix $\mathbf{A} = (a_{ji})$, write $\mathbf{A} \succeq_B \mathbf{B}$ if there exists a stochastic matrix K such that $\mathbf{B} = \mathbf{A}K$. This is the classical Blackwell preorder: $\mathbf{B}$ can be obtained from $\mathbf{A}$ by garbling (Blackwell 1951, 1953).

Let $v_i = (a_{1i}, \dots, a_{mi})^\top$ denote the i th column of $\mathbf{A}$ and define

$$Z(\mathbf{A}) = \sum_i [0, v_i].$$

We write $\mathbf{A} \succeq_Z \mathbf{B}$ when $Z(\mathbf{A}) \supseteq Z(\mathbf{B})$. This is the *zonotope order* known from the theory of channel comparison; for two states it coincides with the Blackwell order, but from three states onward it is weaker (Bertschinger and Rauh 2014).

The order $\succeq_Z$ must be distinguished from the Koshevoy–Mosler order. For a configuration $\mathbf{A}$ on n units, define the Lorenz zonoid using the augmented vectors

$$\bar{v}_i = \left(\frac{1}{n}, a_{1i}, \dots, a_{mi} \right)^\top, \quad LZ(\mathbf{A}) = \sum_i [0, \bar{v}_i],$$

and write

$$\mathbf{A} \succeq_{KM} \mathbf{B} \iff LZ(\mathbf{A}) \supseteq LZ(\mathbf{B}).$$

The order $\succeq_{KM}$ appears in the literature mainly as the *multivariate Lorenz order* (also the *Lorenz-zonoid order*); inclusion of Lorenz zonoids is interpreted as an increase in multivariate dispersion and is related to directional majorization (Koshevoy and Mosler 1996, 2007). The additional first coordinate is essential: $\succeq_Z$ and $\succeq_{KM}$ are not the same order.

The relations summarized in Table 1 have two particularly important boundary cases. In the single-allocation restriction with the uniform benchmark, the proposed preorder reduces to majorization, in accordance with Proposition 3.5. For two allocations, it coincides with both the Blackwell order and zonotope inclusion. From $m = 3$ onward, the relations diverge: the Blackwell order remains stronger than $\succeq_C$, whereas the zonotope order and the separation preorder are incomparable. The Koshevoy–Mosler multivariate Lorenz order implies separation dominance for $m = 2$; for $m \geq 3$,

we do not fully characterize the relation between these orders. Formal statements, counterexamples, and proofs are moved to Appendix A.1.

The geometric criterion underlying the dissimilarity order of Andreoli and Zoli and their multigroup Gini segregation index is based on zonotope inclusion and therefore corresponds to the order $\succeq_Z$ considered here (Andreoli and Zoli 2023, 2026). For two groups, $\succeq_Z = \succeq_B = \succeq_C$, so the two approaches induce the same order in that case. From three groups onward, the situation is fundamentally different: $\succeq_Z$ and $\succeq_C$ are formally incomparable. The numerical representations are also constructed differently: the Andreoli–Zoli Gini segregation index is based on zonotope volume, whereas the partition-based indices studied below arise from optimizing a generator over the attainable set associated with $\succeq_C$.

From a decision-theoretic perspective, the distinction from Blackwell is transparent. The preorder $\succeq_C$ tests robustness of the comparison to the choice of prior under a fixed 0–1 loss function, whereas the Blackwell order requires dominance when the entire decision problem is allowed to vary. Therefore the two criteria are equivalent for $m = 2$, but from three states onward the Blackwell order becomes strictly stronger.

3.8 Properties of the Attainable Set and Extreme Classes

Allowing fractional partitions implies that $\mathcal{R}_m$ is a compact and convex affine image of a product of simplices. This is simpler than the situation for deterministic partitions of general vector measures; for nonatomic measures, convexity of ranges and equivalence between randomization and deterministic rules are related to the classical Dvoretzky–Wald–Wolfowitz theorem (Dvoretzky et al. 1951), while Gouweleeuw (1995) studies convexity conditions for partition ranges in the presence of atoms.

Proposition 3.10 (Extreme classes of m -separation) *The set $\mathcal{R}_m(A_1, \dots, A_m)$ contains the standard simplex Δ^{m-1} . Moreover, $\mathcal{D}_m$ is minimal if and only if $A_1 = \dots = A_m$, whereas $\mathcal{D}_m = [0, 1]^m$ if and only if the supports of $A_1, \dots, A_m$ are pairwise disjoint.*

Proof The standard simplex is obtained from partitions that split every unit in the same proportions among all sides. The maximal sum of coordinates of an attainable point is

$$\max_{x \in \mathcal{R}_m} \sum_{j=1}^m x_j = \sum_{i=1}^n \max_{1 \leq j \leq m} a_{ji}.$$

This value equals 1 exactly when, at every i , all positive masses overlap in the manner implied by identical structures; then the attainable set does not extend beyond the simplex and we obtain the minimal class. Conversely, $(1, \dots, 1)$ is attainable exactly when every positive mass can be assigned to its own side without loss of mass from any of the other allocations, which is equivalent to pairwise disjoint supports. The downward comprehensive closure is then the whole cube. $\square$

These classes justify the later axioms identifying the minimum and maximum of an index. Their numerical values, however, are not part of the order structure: for naturally normalized generators the minimum is usually $1/m$ and the maximum 1;

a common $[0, 1]$ scale can be obtained through the transformation $\tilde{C}_m = (mC_m - 1)/(m - 1)$.

Remark 3.11 (Invariance to proportional refinement of units) If unit i is replaced by any number of units having proportionally identical shares in all allocations $A_1, \dots, A_m$, the set $\mathcal{R}_m$ remains unchanged: every partition of the new units can be recombined into a partition of the original unit, and conversely. Consequently, neither the preorder $\succeq_C$ nor any functional monotone with respect to it responds to such splitting of mass across a larger number of units. This property distinguishes the proposed relation from some classical measures of industrial concentration, such as the HHI or Hall–Tideman index, which also respond to the number of firms. Consistency of the HHI with majorization should therefore be understood on a simplex of fixed dimension. Invariance to proportional refinement may also be viewed as a special case of the broader monotonicity under common data processing discussed in Subsection 3.9.

3.9 Data-Processing Inequality

The preorder $\succeq_C$ is monotone under common stochastic processing of all allocations. This property is a direct counterpart of the data-processing inequality for measures of distinguishability and of monotonicity under garbling in Blackwell’s comparison of experiments (Blackwell 1951, 1953).

Let $K = (K_{ii'})$ be an $n \times n'$ stochastic matrix:

$$K_{ii'} \geq 0, \quad \sum_{i'=1}^{n'} K_{ii'} = 1.$$

For $A_j \in \Delta^{n-1}$, define

$$A_j K = \left(\sum_i a_{ji} K_{i1}, \dots, \sum_i a_{ji} K_{in'} \right).$$

Theorem 3.12 (Data-processing inequality for the m -separation preorder) *For arbitrary $A_1, \dots, A_m$ and any common stochastic channel K ,*

$$(A_1, \dots, A_m) \succeq_C (A_1 K, \dots, A_m K).$$

Proof Let $(x_1, \dots, x_m) \in \mathcal{R}_m(A_1 K, \dots, A_m K)$ be implemented by a partition $p_{i'j}$ on the target units and define $\tilde{p}_{ij} := \sum_{i'=1}^{n'} K_{ii'} p_{i'j}$. Then $\tilde{p}_{ij} \geq 0$, $\sum_j \tilde{p}_{ij} = 1$, and $\sum_i \tilde{p}_{ij} a_{ji} = x_j$. Hence every outcome attainable after processing is also attainable before processing, that is, $\mathcal{R}_m(A_1 K, \dots, A_m K) \subseteq \mathcal{R}_m(A_1, \dots, A_m)$. Inclusion passes to downward comprehensive closures, proving the claim. $\square$

This property has a direct interpretation in applications. If the units are, for example, regions, sectors, or categories of observations, their common aggregation cannot artificially increase the measured separation among several structures. Likewise, common noise or a loss of data resolution may reduce the distinguishability of allocations

but cannot improve it. In this sense, this property makes the preorder consistent with a natural economic requirement of monotonicity under common aggregation of data. This is related to the long-standing requirement of consistency in aggregation in the theory of economic indices (Blackorby and Primont 1980).

Remark 3.13 The theorem is a property of the preorder itself and precedes the definition of an index. For a square bistochastic matrix, it yields a special case related to majorization, because $AK \preceq_{\text{maj}} A$. The full Theorem 3.12 is stronger: it requires only row stochasticity and concerns common processing of multiple allocations.

4 Numerical Representations of the Separation Preorder

The preorder $\succeq_{\mathcal{C}}$ from Definition 3.7 provides a qualitative comparison of configurations of allocations, but it does not select a unique numerical scale. The natural problem is therefore not to compare two prespecified classes of indices, but to ask a more general question:

how can functions monotone with respect to $\succeq_{\mathcal{C}}$ be characterized?

Below we separate the properties required of an index from several mechanisms for constructing monotone functionals. We do not claim that the mechanisms listed exhaust the full class of representations.

4.1 Separation Index and Monotonicity

Let $\mathfrak{A}_m$ denote the class of ordered m -tuples of allocation structures on a common finite set of units. An element $\mathbf{A} = (A_1, \dots, A_m) \in \mathfrak{A}_m$ can equivalently be represented by a matrix whose rows are $A_1, \dots, A_m$.

Definition 4.1 (*m*-separation index) A function $C_m : \mathfrak{A}_m \rightarrow \mathbb{R}$ is called an *m*-separation index if:

(C1) **Monotonicity with respect to the preorder:**

$$\mathbf{A} \succeq_{\mathcal{C}} \mathbf{B} \implies C_m(\mathbf{A}) \geq C_m(\mathbf{B}).$$

(C2) **Identification of the minimal class:** C_m attains its minimum if and only if $A_1 = \dots = A_m$.

(C3) **Identification of the maximal class:** C_m attains its maximum if and only if the supports of $A_1, \dots, A_m$ are pairwise disjoint.

Condition (C1) requires the numerical scale to be consistent with the comparison relation and may be viewed as an order-theoretic implementation of the *value validity* requirement in the sense of Kvålseth (2022). Conditions (C2) and (C3) play the role of boundary conditions analogous to complete integration and complete segregation

in the axiomatic segregation literature (Hutchens 2004; Frankel and Volij 2011); in the present construction, however, the corresponding extreme classes follow from the geometry of the preorder rather than being postulated independently.

Two alternative conditions will be useful when particular representations are compared. The first strengthens (C1):

$$\mathbf{A} \succ_C \mathbf{B} \implies C_m(\mathbf{A}) > C_m(\mathbf{B}). \quad (\text{C1}')$$

We call (C1') *strict monotonicity*. The second modifies the meaning of the maximal class:

$$C_m(\mathbf{A}) = C_m^{\max} \iff \bigcap_{j=1}^m \text{supp } A_j = \emptyset. \quad (\text{C3}')$$

Condition (C3') is not a strengthening of (C3). For $m = 2$ the two conditions are equivalent, whereas for $m \geq 3$ the empty-common-intersection condition admits into the maximal class configurations whose supports still overlap pairwise. It therefore corresponds to a different, weaker semantics of complete separation: no unit is common to all allocations simultaneously, but observing a unit need not uniquely identify a single allocation.

The numerical values of the minimum and maximum are matters of normalization. The definition does not require them to equal 0 and 1. Conditions (C2)–(C3) tie the extreme values to the classes characterized in Proposition 3.10, rather than to an arbitrarily chosen numerical range.

We define the strict part of the preorder by $\mathbf{A} \succ_C \mathbf{B}$ if and only if $\mathbf{A} \succeq_C \mathbf{B}$ and $\neg(\mathbf{B} \succeq_C \mathbf{A})$. Condition (C1') is not included in the basic definition of an index: many natural representations, including C_A and the Kolkata index, preserve the direction of the preorder while assigning the same value to distinct comparable classes.

Corollary 4.2 (DPI for m -separation indices) *Every index satisfying (C1) obeys*

$$C_m(A_1 K, \dots, A_m K) \leq C_m(A_1, \dots, A_m)$$

for any common stochastic channel K .

The corollary follows directly from Theorem 3.12. This creates a natural analogy between separation indices and statistical divergences: in both cases, common data processing should not increase the measured degree of distinguishability. For two allocations the analogy is especially close because the proposed preorder coincides with the Blackwell order; divergences satisfying DPI are therefore natural candidates for indices of separation between two structures. This connection is developed in Subsection 4.2.

4.2 Boundary Cases: Majorization and the Blackwell Order

As in the theory of inequality and divergence, useful representations need not satisfy an identical set of properties: symmetry, strict monotonicity, DPI, and the identification of extreme classes depend on the purpose of measurement. Conditions (C1)–(C3)

define a basic class, whereas the boundary cases below show how natural functionals may satisfy only some of these requirements or variants thereof.

The single-allocation restriction reduces the problem to classical inequality theory. By Proposition 3.5, for fixed n ,

$$A \succeq_C B \iff A \succeq_{\text{maj}} B.$$

Consequently, Schur-convex functionals are natural monotone representations of this restriction. In particular, the Herfindahl–Hirschman index $\text{HHI}(A) = \sum_i a_i^2$ is strictly Schur-convex and therefore strictly monotone under strict majorization. In the single-allocation restriction, where A is compared with the uniform benchmark U , the HHI satisfies analogues of (C1)–(C3): it is monotone with respect to $\succeq_C$, attains its minimum $1/n$ exactly at $A = U$, and its maximum 1 exactly at structures concentrated on a single unit. It is therefore a concentration index on a fixed simplex Δ^{n-1} . The qualification “for fixed n ” is important: the HHI changes under proportional refinement of units, whereas the preorder is invariant to such refinement; see Remark 3.11.

For two allocations the situation is equally classical, but the relevant reference point is the theory of experiments. Theorem A.1 gives

$$\succeq_C = \succeq_B \quad (m = 2).$$

Thus, if a bivariate functional $D(A, B)$ satisfies the data-processing inequality

$$D(AK, BK) \leq D(A, B),$$

then it automatically satisfies (C1) for $m = 2$. Conditions (C2) and (C3), however, must be checked separately: DPI alone does not guarantee the correct identification of the extreme classes.

An example is the squared Hellinger distance

$$H^2(A, B) = 1 - \sum_i \sqrt{a_i b_i} = \frac{1}{2} \sum_i (\sqrt{a_i} - \sqrt{b_i})^2.$$

It is a symmetric f -divergence satisfying DPI (Csiszár 1967); its links to Bayes error and the classical Bhattacharyya–Chernoff bounds are discussed, among others, by Nielsen (2014). Moreover, $H^2(A, B) = 0$ if and only if $A = B$, and $H^2(A, B) = 1$ if and only if the supports are disjoint. Hence

$$C_H(A, B) = \frac{1 + H^2(A, B)}{2}$$

is a 2-separation index in the sense of Definition 4.1.

A natural multi-argument extension leads to the generalized Bhattacharyya coefficient (Bhattacharyya 1943). More broadly, power-divergence families have a long statistical tradition, including goodness-of-fit testing (Cressie and Read 1984). A

broader theory of multi-distribution f -divergences and their links to Bayes risk in multiclass classification is developed by [García-García and Williamson \(2012\)](#):

$$BC_m(A_1, \dots, A_m) = \sum_i \left(\prod_{j=1}^m a_{ji} \right)^{1/m}$$

and the functional $H_m^2 = 1 - BC_m$. The inequality between the geometric and arithmetic means gives $0 \leq H_m^2 \leq 1$, with $H_m^2 = 0$ if and only if $A_1 = \dots = A_m$, whereas

$$H_m^2 = 1 \iff \bigcap_{j=1}^m \text{supp } A_j = \emptyset.$$

After normalization, $\tilde{C}_{H,m} = m^{-1} + (1 - m^{-1})H_m^2$ therefore identifies the minimal class in (C2), but its maximum corresponds to the alternative condition (C3') rather than (C3). For $m = 2$ the distinction disappears; for $m \geq 3$, the maximal value means only that no unit is common to all allocations simultaneously, rather than full pairwise disjointness. Monotonicity of H_m^2 with respect to $\succeq_C$ for $m \geq 3$ remains an open question, so without an additional result we do not classify it as an m -separation index satisfying (C1)–(C3).

The directed Pearson divergence $\chi^2(A\|B)$ also satisfies DPI, and

$$\chi^2(A\|U) = n \text{HHI}(A) - 1,$$

which establishes a direct link between the HHI and divergence theory; because of asymmetry and the possibility of infinite values, it is more convenient to treat it as a monotone functional or to apply a suitable increasing normalization.

Thus, the two lowest-dimensional cases have familiar counterparts: for one allocation, monotonicity reduces to consistency with majorization; for two, to consistency with the Blackwell order. The genuinely new representation problem appears for $m \geq 3$, when $\succeq_C$ is weaker than the Blackwell order and cannot be reduced to a single standard pairwise divergence.

4.3 A Separating Family for the Preorder

Theorem 3.8 gives a direct answer to part of the representation problem. For $c \in \mathbb{R}_+^m$, define

$$V_c(\mathbf{A}) := \sum_i \max_{1 \leq j \leq m} c_j a_{ji}.$$

Then

$$\boxed{\mathbf{A} \succeq_C \mathbf{B} \iff V_c(\mathbf{A}) \geq V_c(\mathbf{B}) \text{ for every } c \in \mathbb{R}_+^m.}$$

A single functional V_c need not represent the entire preorder, but the full family $\{V_c\}_{c \geq 0}$ is *separating*: every strict difference in the preorder is detected in at least one direction c . After normalizing c to the simplex Δ^{m-1} , the functional V_c is the optimal probability of correct identification under prior c . Families of Bayesian functionals indexed by the prior are a standard tool for representing and comparing statistical

experiments (Torgersen 1991); here this family arises directly as the family of support functions of the geometric object defining the separation preorder.

Using this family, one can construct single monotone representations of the preorder.

Proposition 4.3 (Positive mixtures of separating functionals) *Let μ be a finite nonnegative measure on Δ^{m-1} and*

$$J_\mu(\mathbf{A}) = \int_{\Delta^{m-1}} V_\pi(\mathbf{A}) d\mu(\pi).$$

Then J_μ is monotone with respect to $\succeq_C$. If μ has full support on the simplex, then

$$\mathbf{A} \succ_C \mathbf{B} \implies J_\mu(\mathbf{A}) > J_\mu(\mathbf{B}).$$

After an affine normalization of its values on the minimal and maximal classes, this yields a strictly monotone m -separation index.

The proof is given in Appendix B.1.

The result shows that the classical family of Bayesian functionals indexed by the prior can be aggregated into a single index preserving the separation preorder. Full support of the measure μ yields a property stronger than (C1), namely strict monotonicity. The preorder is therefore not tied to one particular scale: it can be recovered from the full family $\{V_c\}$, while positive mixtures of that family provide a systematic mechanism for constructing its numerical representations.

4.4 Mechanisms for Constructing Consistent Representations

The first mechanism follows directly from Proposition 4.3. Since

$$J_\mu(\mathbf{A}) = \sum_i \underbrace{\int_{\Delta^{m-1}} \max_j \{\pi_j a_{ji}\} d\mu(\pi)}_{\Phi_\mu(a_{1i}, \dots, a_{mi})},$$

we obtain the *locally additive* family

$$J_\mu(\mathbf{A}) = \sum_i \Phi_\mu(a_{1i}, \dots, a_{mi}).$$

Monotonicity is not postulated from the local shape of Φ_μ ; it follows from the characterization of the preorder. A special case is the uniform prior:

$$V_{(1/m, \dots, 1/m)}(\mathbf{A}) = \frac{1}{m} \sum_i \max_j a_{ji} = C_A(\mathbf{A}).$$

The arithmetic index therefore belongs simultaneously to the locally additive class and to the partition-based class developed in the next section. The two constructions are not disjoint.

The second mechanism uses projections onto pairs of allocations. The characterization through V_c implies that

$$\mathbf{A} \succeq_c \mathbf{B} \implies (A_j, A_\ell) \succeq_c (B_j, B_\ell) \quad \text{for every } j < \ell,$$

because one may choose a vector c with only two nonzero coordinates. If D is a monotone 2-separation index, then every nonnegative combination $\sum_{j < \ell} w_{j\ell} D(A_j, A_\ell)$ is monotone for arbitrary m . With positive weights and proper normalization it also inherits the identification of the minimal and maximal classes. For example,

$$\overline{H}_m^2 = \frac{2}{m(m-1)} \sum_{j < \ell} H^2(A_j, A_\ell)$$

leads to the symmetric index

$$C_H^{(m)} = \frac{1}{m} + \left(1 - \frac{1}{m}\right) \overline{H}_m^2.$$

This construction, however, is reducible to the complete set of pairwise comparisons and therefore does not exploit higher-order information contained in the full attainable set.

The third mechanism is based directly on the geometry of $\mathcal{R}_m$ and $\mathcal{D}_m$. For a coordinatewise nondecreasing function $F : [0, 1]^m \rightarrow \mathbb{R}$, one can maximize F over the attainable set. Inclusion of the downward comprehensive closures then guarantees monotonicity of the resulting functional with respect to $\succeq_c$. This class is particularly important below because multi-allocation interaction is built into one common partition problem rather than reconstructed from marginal or pairwise functionals. Section 5 develops this construction in detail.

5 Partition-Based m -Separation Indices

5.1 Partition Construction and Conditions on the Generator

Definition 5.1 (Partition functional generated by F) Let $F : [0, 1]^m \rightarrow [0, 1]$ be a function aggregating the outcome of a partition. Define

$$C_F(A_1, \dots, A_m) = \sup_{x \in \mathcal{R}_m(A_1, \dots, A_m)} F(x).$$

The functional in Definition 5.1 is the main representation mechanism analyzed in this section. If F is coordinatewise nondecreasing, then C_F satisfies (C1). The remaining conditions are not intended as a universal axiomatization of concentration or separation; rather, they define a natural class of numerical representations of the preorder considered here. For arbitrary $m \geq 2$, we consider generators satisfying:

- (F1) monotonicity with respect to the coordinatewise order;
- (F2) symmetry under permutations of the arguments;

- (F3) diagonal normalization $F(t, \dots, t) = t$;
(F4) preference for balance,

$$F(x_1, \dots, x_m) \leq F(\bar{x}, \dots, \bar{x}), \quad \bar{x} = \frac{1}{m} \sum_{j=1}^m x_j.$$

Condition (F1) makes the generator consistent with the coordinatewise order, (F2) treats allocations symmetrically, (F3) fixes the scale on the diagonal, and (F4) favors balance in the performance of a common partition. We take this class as the baseline because it matches the interpretation of concentration and separation adopted in the paper; asymmetric or differently normalized generators remain possible as alternative measurement conventions.

Two canonical generators are

$$F_{\min}^{(m)}(x) = \min_j x_j, \quad F_A^{(m)}(x) = \frac{1}{m} \sum_{j=1}^m x_j,$$

and they lead respectively to the indices K_m and $C_{A,m}$.

Theorem 5.2 (Bounds for the natural class) *If F satisfies (F1), (F3), and (F4), then for every $x \in [0, 1]^m$,*

$$\min_j x_j \leq F(x) \leq \frac{1}{m} \sum_{j=1}^m x_j.$$

Consequently,

$$\boxed{K_m(A_1, \dots, A_m) \leq C_F(A_1, \dots, A_m) \leq C_{A,m}(A_1, \dots, A_m).}$$

Proof Let $\underline{x} = \min_j x_j$ and $\bar{x} = m^{-1} \sum_j x_j$. By (F1) and (F3), $F(x) \geq F(\underline{x}, \dots, \underline{x}) = \underline{x}$, while by (F4) and (F3), $F(x) \leq F(\bar{x}, \dots, \bar{x}) = \bar{x}$. Maximizing both sides over the same attainable set yields the second part of the claim. $\square$

The theorem shows that the interval between the weakest-side criterion and the arithmetic criterion is not an artifact of the binary case; it persists for an arbitrary number of allocations.

5.2 Arithmetic Generator and Total Variation Distance

Let $F_A(x, y) = (x + y)/2$. Then

$$\begin{aligned} C_A(A, B) &= \frac{1}{2} \max_{p \in [0,1]^n} \left[\sum_i p_i a_i + \sum_i (1 - p_i) b_i \right] \\ &= \frac{1}{2} \left[1 + \max_p \sum_i p_i (a_i - b_i) \right] \\ &= \frac{1}{2} \left[1 + \sum_{a_i > b_i} (a_i - b_i) \right]. \end{aligned}$$

Since

$$\text{TVD}(A, B) = \frac{1}{2} \sum_i |a_i - b_i| = \sum_{a_i > b_i} (a_i - b_i),$$

we obtain

$$C_A(A, B) = \frac{1 + \text{TVD}(A, B)}{2}.$$

This is an important point of contact between concentration theory and statistical distinguishability. Total variation is not identified here with classical single-allocation concentration. Rather, it appears as an affinely equivalent representation of the separation of two structures.

For one allocation, $C_A(A, U) = \frac{1 + \Pi(A)}{2}$, where $\Pi(A)$ denotes the Pietra index, also known as the Schutz or Hoover index (Cowell 2011).

5.3 The Kolkata Index: From the Lorenz Curve to a Partition Problem

The classical Kolkata index $k(A)$ is defined as the solution to $L_A(k) = 1 - k$, where L_A is the Lorenz curve of allocation A (Ghosh et al. 2014). Unlike exogenous thresholds such as the top 1% or top 10% used in the study of top incomes (Atkinson et al. 2011), the threshold k is determined endogenously by the allocation structure itself.

The following representation is central to the move from one to multiple allocations.

Proposition 5.3 (Alternative representation of the Kolkata index) *For every allocation A on n agents,*

$$k(A) = \max_{p \in [0,1]^n} \min \left\{ \sum_i p_i a_i, \sum_i (1 - p_i) u_i \right\}.$$

Proof Fix $s = \sum_i (1 - p_i) u_i$, the share of the uniform mass assigned to the second side. For fixed s , the largest value of $\sum_i p_i a_i$ is obtained by assigning to the first side the richest fraction of the population of size $1 - s$. With fractional partitioning this value is exactly

$1 - L_A(s)$. The problem therefore reduces to $\max_{s \in [0,1]} \min\{1 - L_A(s), s\}$. The function $1 - L_A(s)$ is nonincreasing and s is increasing, so the maximum of their minimum occurs at their intersection, $1 - L_A(s) = s$, that is, at $s = k(A)$. $\square$

Proposition 5.3 does not require the Lorenz curve to be treated as the primitive object and removes the granularity effect of a finite set of agents. It also leads directly to two arbitrary allocations:

$$K(A, B) = \max_{p \in [0,1]^n} \min \left\{ \sum_i p_i a_i, \sum_i (1 - p_i) b_i \right\},$$

with $K(A, U) = k(A)$ and $K(A, B) = K(B, A)$. This is the partition-based index of separation of two structures generated by $F_{\min}^{(2)}(x, y) = \min\{x, y\}$, so $K = K_2$.

Proposition 5.4 (Extreme values of the binary Kolkata index) *For any two structures A, B ,*

$$\frac{1}{2} \leq K(A, B) \leq 1,$$

with

$$K(A, B) = \frac{1}{2} \iff A = B, \quad K(A, B) = 1 \iff \text{supp } A \cap \text{supp } B = \emptyset.$$

Proof Choosing $p_i = 1/2$ for every i gives the lower bound. If $A \neq B$, then as distinct probability vectors A and B are linearly independent: a dependence $B = cA$, together with equality of their sums, would force $c = 1$. Hence the linear map $q \mapsto (q \cdot A, q \cdot B)$ has rank 2, and there exists q such that $q \cdot A = 1$ and $q \cdot B = -1$. After scaling q sufficiently, the vector $p = \frac{1}{2}\mathbf{1} + q$ belongs to $[0, 1]^n$ and both values inside the minimum exceed $1/2$. Thus $K(A, B) = 1/2$ only when $A = B$. Conversely, the value 1 requires simultaneously $p_i = 1$ on the support of A and $p_i = 0$ on the support of B , which is possible exactly when the supports are disjoint. $\square$

Proposition 5.4 shows in particular that the binary Kolkata index identifies exactly the extreme classes required by (C2)–(C3).

Theorem 5.2 gives $K(A, B) \leq C_A(A, B) = (1 + \text{TVD}(A, B))/2$, and hence $2K(A, B) - 1 \leq \text{TVD}(A, B)$. In the special case $B = U$, we obtain $2k(A) - 1 \leq \Pi(A)$, the known Kolkata–Pietra inequality (Ghosh and Chakrabarti 2026). Broader relations among the Kolkata, Pietra, and Gini indices are discussed, among others, by Chatterjee et al. (2017); Ghosh and Chakrabarti (2026); here the inequality follows directly from the ordering of the generators.

5.4 Intermediate Families

Conditions (F1)–(F4) allow parameterized families of indices with different sensitivity to imbalance. The first family is generated by

$$F_\lambda^{(m)}(x) = (1 - \lambda) \min_j x_j + \lambda \frac{1}{m} \sum_{j=1}^m x_j, \quad 0 \leq \lambda \leq 1.$$

For $\lambda = 0$ we obtain K_m , and for $\lambda = 1$ the arithmetic index. Pointwise,

$$K_m \leq C_{F_\lambda^{(m)}} \leq C_{A,m},$$

and moreover

$$C_{F_\lambda^{(m)}}(\mathbf{A}) \leq (1 - \lambda)K_m(\mathbf{A}) + \lambda C_{A,m}(\mathbf{A}).$$

For $0 < \lambda < 1$, equality holds if and only if there exists a point in the attainable set that simultaneously maximizes $F_{\min}^{(m)}$ and $F_A^{(m)}$. Failure of equality therefore reflects not only a difference between the generators but also a divergence between their maximizing points.

The second family consists of symmetric power means

$$M_r^{(m)}(x) = \begin{cases} \left(m^{-1} \sum_{j=1}^m x_j^r \right)^{1/r}, & r \neq 0, \\ \left(\prod_{j=1}^m x_j \right)^{1/m}, & r = 0. \end{cases}$$

For $r < 0$ we use the standard continuous extension to the boundary of $[0, 1]^m$; in particular, $M_r^{(m)}(x) = 0$ whenever at least one coordinate is zero. For $r \leq 1$, these functions satisfy (F1)–(F4), whereas for $r > 1$ the preference for balance (F4) is violated. The technical verification is given in Appendix B.2.

Monotonicity of power means in the parameter yields

$$K_m = C_{M_{-\infty}^{(m)}} \leq C_{M_{-1}^{(m)}} \leq C_{M_0^{(m)}} \leq C_{M_1^{(m)}} = C_{A,m}.$$

The parameter r can be interpreted as the degree of tolerance for imbalance: the smaller r , the more weight is placed on the weakest component of the partition profile.

5.5 Asymmetric Generators and Directional Functionals

Conditions (F1)–(F4) define a convenient class of symmetric representations, but they need not be treated as universal requirements. Depending on the application, one may deliberately weaken or omit a selected condition while preserving the remaining properties of the generator. The simplest example is dropping symmetry (F2), which leads to directional functionals. If the two allocations play different roles, one may consider the generator $F_\alpha(x, y) = \alpha x + (1 - \alpha)y$, $0 < \alpha < 1$. The corresponding functional is

$$C_\alpha(A, B) = 1 - \alpha + \sum_i [\alpha a_i - (1 - \alpha)b_i]_+ = \frac{1}{2} + \frac{1}{2} \sum_i |\alpha a_i - (1 - \alpha)b_i|.$$

It is monotone with respect to the preorder. For $\alpha \neq 1/2$, it is not symmetric and does not identify the minimal class $A = B$; lack of symmetry by itself does not violate (C1), but it shows that symmetry must be imposed separately if it is desired.

Proposition 5.5 (Linear directional functional) *For $0 < \alpha < 1$, the minimum value of C_α is $\max\{\alpha, 1 - \alpha\}$ and the maximum is 1. The maximum is attained exactly for disjoint supports, whereas the minimum identifies the class $A = B$ exactly when $\alpha = 1/2$. Consequently, only the case $\alpha = 1/2$ is a 2-separation index satisfying the full set (C1)–(C3); the remaining parameter values yield directional functionals consistent with (C1).*

The proof is given in Appendix B.3. Proposition 5.5 shows that abandoning symmetry may preserve monotonicity with respect to the preorder, but takes us outside the class of full indices satisfying (C1)–(C3). The basic cases are summarized in Table 2.

Table 2 Basic generators of partition-based separation functionals

Generator	Index / family	Evaluation rule	Main feature
$\min\{x, y\}$	K	weaker side	guaranteed level; strong penalty for imbalance
F_λ	C_{F_λ}	average with penalty	transition parameter between K and C_A
$\sqrt{xy}$	C_G	multiplicative compromise	intermediate sensitivity to imbalance
$M_r, r < 1$	power family	nonlinear mean	flexible tolerance for imbalance
$(x + y)/2$	C_A	total captured mass	affine relation to TVD; closed form
$F_\alpha, \alpha \neq 1/2$	directional functional	weighted mass	preserves monotonicity but abandons symmetry
$\max\{x, y\}$	degenerate case	stronger side	value 1 for every pair

5.6 An Arbitrary Number of Allocations

For m allocations, the two canonical generators are

$$F_A^{(m)}(x_1, \dots, x_m) = \frac{1}{m} \sum_{j=1}^m x_j,$$

and

$$F_{\min}^{(m)}(x_1, \dots, x_m) = \min_j x_j.$$

For the arithmetic mean,

$$C_A(A_1, \dots, A_m) = \frac{1}{m} \max_p \sum_{i,j} p_{ij} a_{ji}.$$

The optimization separates across units, so each unit should be assigned to the allocation with the largest share:

$$C_A(A_1, \dots, A_m) = \frac{1}{m} \sum_{i=1}^n \max_{j=1, \dots, m} a_{ji}.$$

For brevity, we also denote this value by $C_{A,m}$, in analogy with K_m for the minimum generator. The same formula simultaneously shows the local additivity of C_A :

$$C_A(A_1, \dots, A_m) = \sum_i \Phi_A(a_{1i}, \dots, a_{mi}), \quad \Phi_A(x_1, \dots, x_m) = \frac{1}{m} \max_j x_j.$$

Thus the index C_A lies at the intersection of two representation mechanisms described in Section 4.

For the minimum generator,

$$K_m(A_1, \dots, A_m) = \max_p \min_j \sum_i p_{ij} a_{ji}.$$

This is the largest mass that one common partition can guarantee simultaneously to each of the m allocations. The egalitarian max-min problem for divisible goods and additive valuations is well known in fair-division theory (Dubins and Spanier 1961; Moulin 2003; Alon 1987); in the present construction it serves as a representation of the separation preorder and leads to a multi-allocation extension of the Kolkata index.

5.7 Computational Properties

For the arithmetic generator, computing $C_A(A_1, \dots, A_m) = m^{-1} \sum_i \max_j a_{ji}$ requires $O(mn)$ operations and, for fixed m , is linear in the number of agents. The linear directional functional for two allocations has the same complexity.

For $K(A, B)$, the special structure of the problem can be exploited.

Proposition 5.6 (Linear-time algorithm for the binary Kolkata index) *The index $K(A, B)$ can be computed in $O(n)$ time in the random-access machine (RAM) model, without constructing the entire boundary of $\mathcal{R}_2(A, B)$.*

The proof is given in Appendix B.4.

For $m > 2$, the index K_m can be written as the linear program

$$\max_{p, t} t$$

subject to

$$\sum_i p_{ij} a_{ji} \geq t \quad (j = 1, \dots, m), \quad \sum_{j=1}^m p_{ij} = 1, \quad p_{ij} \geq 0.$$

The program has size polynomial in m and n ; computational details and the contrast with naive enumeration of deterministic assignments are given in Appendix B.4. For the geometric generator, equivalently, one maximizes the sum of logarithms of captured masses; this is a standard convex optimization problem (Boyd and Vandenberghe 2004).

It is also useful to control the difference between K_m and the computationally simpler $C_{A,m}$.

Proposition 5.7 (Certificate for the gap between $C_{A,m}$ and K_m) *Let y_j denote the mass of side j under a partition attaining $C_{A,m}$. Then, for*

$$\gamma_m := C_{A,m} - K_m,$$

we have

$$0 \leq \gamma_m \leq \frac{1}{m} \sum_{j=1}^m y_j - \min_j y_j.$$

Proof For the partition under consideration, $C_{A,m} = m^{-1} \sum_{j=1}^m y_j$. The same partition is feasible in the problem defining K_m , so $K_m \geq \min_j y_j$. Subtracting yields the claimed inequality. The lower bound $\gamma_m \geq 0$ follows from Theorem 5.2. $\square$

The right-hand side of the inequality in Proposition 5.7 is obtained without any additional optimization. A small value provides a direct certificate of approximation quality, while a large value indicates that solving the full linear program may be worthwhile.

The value of the index may remain continuous even when the optimal partition changes discontinuously. For $B = (1/2, 1/2)$ and $A_\varepsilon = (1/2 + \varepsilon, 1/2 - \varepsilon)$, the optimal side changes at $\varepsilon = 0$, while $C_A(A_\varepsilon, B) = (1 + |\varepsilon|)/2$.

5.8 Statistical Interpretation

The partition construction has a direct interpretation in the classical problem of discriminating among multiple simple hypotheses. Let $H_j : X \sim A_j$, $j = 1, \dots, m$, where $A_1, \dots, A_m$ are distributions on a common finite observation space N . A decision rule assigns each observation to one of the hypotheses. For a randomized rule, p_{ij} is the probability of choosing H_j after observing i , while a deterministic rule corresponds to a partition $N = S_1 \sqcup \dots \sqcup S_m$, where $i \in S_j$ leads to choosing H_j . Formulating classification and multiple-hypothesis testing through decision regions and optimal partitions of the observation space is classical (Devroye et al. 1996; Hill and Tong 1989).

Under equal prior probabilities, $\Pr(H_j) = 1/m$, and 0–1 loss, the probability of a correct decision for a rule $p = (p_{ij})$ is

$$P_{\text{corr}}(p) = \frac{1}{m} \sum_{j=1}^m \sum_{i \in N} p_{ij} a_{ji}.$$

This is exactly the value of the arithmetic generator applied to the masses captured by the respective parts of the common partition.

Corollary 5.8 (Equal prior and arithmetic m -separation) *Under equal prior probabilities and 0–1 loss,*

$$C_A(A_1, \dots, A_m) = P_{\text{corr}}^*,$$

where P_{corr}^* is the maximal probability of correctly identifying one of m simple hypotheses. Consequently,

$$1 - C_A(A_1, \dots, A_m) = P_e^*,$$

where P_e^* is the minimum Bayes error probability. Moreover, the optimum can be attained by a deterministic rule, so

$$C_A(A_1, \dots, A_m) = \max_{N=S_1 \sqcup \dots \sqcup S_m} \frac{1}{m} \sum_{j=1}^m A_j(S_j).$$

Proof This is the case $\pi_j = 1/m$ of Theorem 3.8. The determinism of an optimal rule established there gives the representation directly through a partition $N = S_1 \sqcup \dots \sqcup S_m$. $\square$

For $m = 2$, we recover the classical discrimination problem

$$H_1 : X \sim A \quad \text{and} \quad H_2 : X \sim B.$$

Then

$$C_A(A, B) = \frac{1 + \text{TVD}(A, B)}{2}, \quad P_e^* = \frac{1 - \text{TVD}(A, B)}{2},$$

and hence

$$C_A(A, B) = 1 - P_e^*.$$

For two hypotheses, total variation therefore gives a closed-form expression for the optimal classification error. For $m > 2$, the index C_A retains the exact interpretation as the optimal accuracy of the multiclass Bayes rule, although it no longer reduces to a single standard pairwise distance.

This interpretation shows that the link between multi-allocation separation and hypothesis testing is not merely formal. The same object can be read statistically as a collection of decision regions or economically as a common partition of agents in which each part captures as much as possible of the mass of its corresponding allocation. The construction therefore provides a common language for two problems with different interpretations.

Remark 5.9 (Other generators as alternative classification criteria) The statistical interpretation is not restricted to the arithmetic generator. For

$$F_{\min}(x_1, \dots, x_m) = \min_j x_j$$

we obtain

$$K_m(A_1, \dots, A_m) = \max_{N=S_1 \sqcup \dots \sqcup S_m} \min_j A_j(S_j),$$

that is, a rule maximizing the smallest class-specific probability of a correct decision. This is a bottleneck criterion: instead of average accuracy, it maximizes the guarantee for the least well-recognized hypothesis. Similarly, the geometric generator

$$F_G(x_1, \dots, x_m) = \left(\prod_{j=1}^m x_j \right)^{1/m}$$

prefers rules with more balanced probabilities of correct recognition across hypotheses.

Moving beyond average accuracy toward criteria that account for class-specific performance is an active topic in the literature on imbalanced and multiclass classification (He and Garcia 2009; Feng et al. 2021; Han et al. 2023; Chen et al. 2024). We do not claim that the generators $F_{\min}$ or F_G are new statistical criteria in themselves. Rather, the proposed construction shows that average accuracy, balanced criteria, and the weakest-side principle can be treated as different objective functions on the same attainable set of class-specific accuracies.

The connection works in both directions. From statistical decision theory, one can bring into concentration analysis unequal prior probabilities, general loss functions, minimax criteria, and risk bounds (Tsybakov 2009). Conversely, the partition interpretation shows that the classical Bayes criterion is one way of aggregating class-specific probabilities of correct decisions and suggests systematic study of alternative generators F as evaluation criteria for the same decision problem.

6 Case Study: Concentration and Separation of Metal-Production Structures

6.1 Research Question, Data, and Interpretation

Analyses of critical-mineral supply risk often assess the geographical concentration of each commodity separately, using, among other tools, the Herfindahl–Hirschman index (HHI), shares of the largest producers, or related concentration measures. The International Energy Agency (IEA) points out that supply chains for key minerals remain highly concentrated and that diversification of supply is one of the basic conditions for improving their resilience (International Energy Agency 2026). At the same time, Bucciarelli et al. (2025) emphasize limitations of standard concentration measures in assessing metal criticality. Abdelkawy and Alshamy (2026) move even further away from single-market analysis: they argue that static single-commodity measures reveal little about inter-market dependencies and the propagation of disruptions, and combine seven mineral markets in a multilayer criticality index. Their network approach differs from the one adopted below, but rests on a similar motivation: analyzing

each commodity separately may be insufficient when the object of interest is the joint organization of multiple markets.

In practice, it is therefore useful to distinguish at least two questions. The first concerns how strongly production of a single commodity is concentrated in a small group of countries. The second concerns the mutual position of several production structures: are the same countries production centers for different commodities, or are these centers geographically separated? The second question is especially relevant for technologies and supply chains that require several commodities simultaneously. In the present construction, $C_A(X, U)$ describes concentration of a single structure relative to the uniform benchmark, whereas $C_A(X, Y)$ and $C_A(X_1, \dots, X_m)$ measure separation of two and multiple structures, respectively. A high separation index is not by itself a measure of supply risk; rather, it describes a feature of the geographical organization of production that may matter for such risk.

We analyze the structures of world population P , nominal gross domestic product (GDP) G , and production of copper Cu , lithium Li , cobalt Co , and nickel Ni for 2005, 2010, 2015, 2020, and 2025. Production data come from the United States Geological Survey (USGS) ([U.S. Geological Survey 2026](#)), while population and GDP structures come from international databases used to construct the panel. It is worth noting that USGS data from the same source family are also used in recent analyses of multi-mineral criticality ([Abdelkawy and Alshamy 2026](#)).

The raw data determine both the scale of production $w(A)$, measured in physical units, and—after normalization—its geographical structure $a = A/w(A)$. The indices analyzed below depend on the latter: they compare the distribution of production across countries, not its global level. An increase in total production of a commodity therefore need not change the index if its geographical structure remains unchanged.

Countries outside the detailed list are closed by the aggregate ALL OTHER COUNTRIES. Because the number of explicitly distinguished units changes across years, values relative to the benchmark U should be interpreted primarily as illustrations of structural change. Relational comparisons of structures within the same year, in contrast, do not depend on the choice of the uniform benchmark.

6.2 Changes over Time

Table 3 reports selected series so as to show both the dynamics of concentration of individual structures and changes in their mutual position.

There is no single common trend in the concentration of production of the individual metals. Copper production concentration relative to the uniform allocation is almost constant: $C_A(Cu, U)$ remains close to 0.84 throughout the period. Lithium is even more concentrated, with values around 0.94–0.96 for most of the period; the decline to 0.9333 in 2025 does not alter the fact that production remains geographically highly concentrated. Cobalt and nickel fluctuate more strongly, with local minima around 2015, but by 2025 their concentration is again high: 0.9321 and 0.9053, respectively.

A different direction of change appears for the economic-demographic structures. $C_A(P, G)$ declines in every successive observed year, from 0.7732 to 0.7000. Under the adopted interpretation, this means a gradual decline in the separation of the

Table 3 Selected values of the partition-based index C_A

Measure	2005	2010	2015	2020	2025
$C_A(Cu, U)$	0.8478	0.8436	0.8472	0.8458	0.8432
$C_A(Li, U)$	0.9377	0.9568	0.9591	0.9561	0.9333
$C_A(Co, U)$	0.9094	0.8922	0.8786	0.9033	0.9321
$C_A(Ni, U)$	0.8805	0.8927	0.8596	0.8803	0.9053
$C_A(P, G)$	0.7732	0.7323	0.7078	0.7034	0.7000
$C_A(P, Cu)$	0.8646	0.8474	0.8353	0.8412	0.8308
$C_A(Li, Co)$	0.8686	0.9382	0.9416	0.9685	0.9781
$C_A(Co, Ni)$	0.7935	0.8271	0.8032	0.8724	0.8774
$C_A(Li, Co, Ni)$	0.7420	0.8105	0.8104	0.8643	0.8742

global population and nominal-GDP structures, and hence greater similarity in their geographical distributions. A similar, though weaker, decline occurs for $C_A(P, Cu)$, from 0.8646 to 0.8308. The structure of copper production therefore remains clearly separated from the population structure, but less so than at the beginning of the period.

The most pronounced changes, however, occur not in concentration of individual metals but in separation of their production structures. $C_A(Li, Co)$ rises from 0.8686 in 2005 to 0.9781 in 2025. This means that the production structures of lithium and cobalt become increasingly easy to separate into two groups of countries capturing large shares of the respective allocations. The increase is especially informative when compared with $C_A(Li, U)$ and $C_A(Co, U)$: both commodities were already highly concentrated, while what changed over time was primarily *where* that concentration was located relative to the other commodity.

An analogous, though weaker, process is observed for cobalt and nickel: $C_A(Co, Ni)$ rises from 0.7935 to 0.8774. In 2025, this interpretation has a direct geographical counterpart: the dominant center of cobalt production is the Democratic Republic of the Congo, whereas nickel production is highly concentrated in Indonesia ([U.S. Geological Survey 2026](#)).

For two structures, a natural complement to the separation index is the degree of their geographical overlap,

$$O(A, B) = \sum_i \min\{a_i, b_i\} = 1 - \text{TVD}(A, B) = 2(1 - C_A(A, B)).$$

The value $O = 1$ indicates identical distributions of the two structures, while $O = 0$ indicates complete disjointness. For Co and Ni , this coefficient falls from about 0.413 in 2005 to 0.245 in 2025, indicating a marked decline in the common geographical base of production of the two metals. Under equal priors, half of this value equals the minimum Bayes error probability; this is the two-argument case of the interpretation in Corollary 5.8.

Table 4 shows that correlation and overlap describe different properties of the data. Correlation measures linear comovement of country shares and changes irregularly over the period. By contrast, the common part of the Co and Ni structures clearly

Table 4 Separation, overlap, and correlation of cobalt and nickel production structures

Year	$C_A(Co, Ni)$	$O(Co, Ni)$	$\rho(Co, Ni)$
2005	0.7935	0.4130	0.2640
2010	0.8271	0.3458	0.1327
2015	0.8032	0.3935	0.1207
2020	0.8724	0.2552	0.0231
2025	0.8774	0.2451	0.1653

declines, while their geographical separation increases. The index C_A is therefore not an alternative scale for correlation, but a measure of a different aspect of the mutual position of the structures.

6.3 The Role of Multi-Allocation Separation

Moving from two to three structures shows that the observed process is not confined to one particular pair of metals. Separation of lithium, cobalt, and nickel production structures, measured by the three-allocation index $C_A(Li, Co, Ni)$, increases from 0.7420 in 2005 to 0.8742 in 2025. Thus increasingly large shares of all three production structures can simultaneously be assigned to distinct groups of countries through one common partition.

For systems using lithium, cobalt, and nickel simultaneously, this means that dependence on several highly concentrated commodities need not imply exposure to the same group of countries. The geographical exposures of individual links in the chain may concern different production centers, which can matter for risk assessment and the design of diversification strategies. The need to consider multiple minerals jointly also appears in recent network analyses of criticality ([Abdelkawy and Alshamy 2026](#)); here we focus on a different dimension of the problem—the mutual geographical position of production structures.

This result does not follow from the separate values $C_A(Li, U)$, $C_A(Co, U)$, and $C_A(Ni, U)$, nor from pairwise comparisons alone. The index $C_A(Li, Co, Ni)$ is based on one common partition of countries and captures the simultaneous position of all three structures. Multi-allocation separation therefore provides information about their joint geographical organization that is absent from separate concentration measures.

The choice of numerical representation additionally distinguishes different ways of valuing the same partitioning possibility. For Co and Ni in 2025,

$$C_A(Co, Ni) = 0.87743, \quad K(Co, Ni) = 0.83538,$$

whereas for the three structures,

$$C_{A,3}(Li, Co, Ni) = 0.8742, \quad K_3(Li, Co, Ni) = 0.8201.$$

The index $C_{A,m}$ measures the average performance of the best common partition, whereas K_m , based on the representation in Proposition 5.3, applies the weakest-side criterion and measures the level guaranteed to every structure. High average separation therefore need not imply equally high separation under the max–min criterion.

For *Li, Co, Ni* in 2025,

$$\gamma_3 = C_{A,3} - K_3 = 0.0541.$$

The difference is moderate: the computationally simpler arithmetic index and the three-allocation Kolkata index lead to a similar overall picture, although the max–min criterion indicates a somewhat lower level of separation guaranteed to each structure. The example also illustrates the practical role of γ_m and of the certificate in Proposition 5.7 when assessing whether the additional computational cost of K_m leads to a materially different conclusion.

This interpretation concerns the geographical distribution of production, not economic independence of markets. Co-production and technological links may connect metal supplies even when their geographical centers differ; the Cu–Co–Ni system is an example, as a substantial share of cobalt is produced as a by-product of copper or nickel mining (Ryter et al. 2025).

The case study therefore illustrates both the distinction between concentration of a single structure and separation of several structures, and the importance of moving from pairwise to multi-allocation separation. Over the period studied, production concentration of individual metals generally remains high, while multi-allocation separation of lithium, cobalt, and nickel production structures rises markedly. This pattern may matter for the analysis of multi-commodity exposures and for supply-chain diversification strategies.

7 Discussion

7.1 Concentration, Separation, and Representations

The starting point of this paper is a relational view of concentration. In the classical case, a single structure is compared with the uniform benchmark U ; moving from the pair (A, U) to an arbitrary pair (A, B) leads instead to separation of two structures, and the further extension to multi-allocation separation. This perspective opens several additional research avenues and invites a reconsideration of the relations among concentration, inequality, segregation, and other comparative concepts. We discuss several of them below. The preorder $\succeq_C$ formalizes this mode of comparison by ordering configurations according to their potential for common separation. It therefore does not cover all meanings of concentration found in the literature. In particular, it is invariant to proportional refinement of units, which distinguishes it from some classical measures of industrial concentration; see Remark 3.11.

The connection with inequality theory is exact on the single-allocation restriction. At fixed dimension, comparing a structure with the uniform benchmark reduces to majorization, so concentration and inequality functionals may induce equivalent

rankings. With multiple allocations, however, the object of comparison becomes their mutual position and the possibility of jointly separating them, rather than the deviation of a single structure from equality. The Koshevoy–Mosler multivariate Lorenz order describes a different aspect of multivariate dispersion; formal relations with $\succeq_C$ are given in Subsection 3.7.

The separation interpretation connects the proposed construction to the segregation literature (Massey and Denton 1988; Reardon and Firebaugh 2002; Hutchens 2004; Frankel and Volij 2011), but the connection is structural rather than definitional. Segregation, polarization, and concentration have their own axiomatic traditions and emphasize different aspects of distribution (Esteban and Ray 1994; Kakwani and Son 2022). The common methodological lesson is that the distinctiveness of a concept is better grounded in the structure of the comparison relation than in the formula of a single index.

A natural reference point is the multigroup Gini segregation index of Andreoli and Zoli, constructed within an order based on zonotope inclusion (Andreoli and Zoli 2023, 2026). Their approach arises from a similar need to compare the mutual position of multiple structures, but it is based on a different order relation and leads to a different class of numerical representations. The difference is visible, among other things, in the treatment of extreme configurations: in our construction, condition (C3) identifies the maximum with families having pairwise disjoint supports, whereas the zonotope construction does not lead to the same extreme class. The distinction can also be seen at the level of numerical values. Consider

$$\mathbf{A}^0 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}, \quad \mathbf{A}^1 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In both cases the rows are linearly dependent, so the three-dimensional volume of the corresponding zonotope, and hence the volume-based Gini segregation index, is zero (Andreoli and Zoli 2026). The partition-based indices nevertheless distinguish the configurations:

$$C_A(\mathbf{A}^0) = K_3(\mathbf{A}^0) = \frac{1}{3}, \quad C_A(\mathbf{A}^1) = \frac{2}{3}, \quad K_3(\mathbf{A}^1) = \frac{1}{2}.$$

The first configuration belongs to the minimal class, whereas in the second the third structure is completely separated from the other two. Zero zonotope volume can therefore coexist with nontrivial information about partial separation of the configuration, which the partition-based representations retain.

The preorder does not determine a unique numerical scale. The family $\{V_c\}$ separates the relation, while locally additive, partition-based, and pairwise-projection representations may emphasize different aspects of the same comparison. Multiplicity of representations therefore does not mean that the relation itself is underdetermined; rather, it requires separation of two levels: the comparison relation and the way it is numerically aggregated.

Within the partition-based class, this role is played by the generator F . It does not define concentration or separation; instead, it determines how points of the same attainable set are valued. Symmetry, normalization, and a preference for balance lead to different classes of indices that share the common requirement of monotonicity with respect to the preorder. This raises a natural question about robustness of rankings to the choice of representation, analogous to research on robustness of composite indicators ([Foster et al. 2013](#); [Seth and McGillivray 2018](#)).

7.2 Decision Theory, Data Processing, and Geometry

The proposed construction links three perspectives: geometric, decision-theoretic, and informational. Inclusion of attainable sets describes the possibility of jointly separating structures, the functionals V_c have a decision-theoretic interpretation, and the data-processing inequality guarantees monotonicity under common stochastic processing of all allocations.

For two structures, the relation $\succeq_c$ coincides with both the Blackwell order and the zonotope order. For $m \geq 3$, these relations diverge: the Blackwell order remains stronger, whereas the zonotope order and $\succeq_c$ are incomparable. Thus, from three structures onward, competing notions of informativeness and separation differ already at the level of comparison relations, not only at the level of index choice.

Theorem 3.12 gives a multi-allocation data-processing inequality, and Corollary 4.2 transfers it to every index satisfying (C1): common stochastic processing of all structures cannot increase their separation. In applications, this means that aggregation, noise, or loss of resolution cannot artificially increase the measured relational information.

The link to decision theory is equally direct. Under a uniform prior, the arithmetic index $C_{A,m}$ equals, by Corollary 5.8, the maximal probability of correct classification, while other generators correspond to alternative ways of aggregating class-specific accuracies: max-min, geometric, or directional. The common attainable set can therefore be interpreted both as a geometric object and as a set of attainable decision profiles.

The construction also reveals an additional metric aspect. For any metric d , the Gromov product with base point o is defined as

$$(x \mid y)_o^d = \frac{1}{2} [d(x, o) + d(y, o) - d(x, y)]$$

([Gromov 1987](#)). Since

$$C_A(X, Y) = \frac{1 + \text{TVD}(X, Y)}{2},$$

for $d = \text{TVD}$ we obtain

$$\begin{aligned} (A \mid B)_U^{\text{TVD}} &= \frac{1}{2} [\text{TVD}(A, U) + \text{TVD}(B, U) - \text{TVD}(A, B)] \\ &= C_A(A, U) + C_A(B, U) - C_A(A, B) - \frac{1}{2}. \end{aligned}$$

This quantity measures the common component of the distances of the two structures from the uniform benchmark. The relation points to an additional connection between partition-based indices and metric geometry, but it does not imply stronger geometric properties such as Gromov hyperbolicity. Analogous constructions may be considered for other metrics, in particular the Gini metric.

7.3 Limitations and Further Directions

Beyond the perspectives discussed above, further development of the proposed construction requires answers to a number of open questions. These concern both the completion of the representation theory and its relations with other orders, as well as the computational and statistical operationalization of the construction and the expansion of its range of applications. We indicate the most important directions below.

At the level of representation theory, Section 4 identifies a separating family of functionals and several mechanisms for constructing monotone indices, but it does not provide necessary and sufficient conditions for the entire class. Open questions include a full characterization of partition generators F , local functions Φ_m , and their relations to multi-argument generalizations of f -divergences (García-García and Williamson 2012; Balsubramani 2026). At this level, it also remains open whether the multi-argument Hellinger functional H_m^2 is monotone with respect to $\succeq_C$ for $m \geq 3$.

Relations with other multivariate orders raise a related theoretical question. For $m = 2$,

$$\succeq_B = \succeq_Z = \succeq_C,$$

whereas for $m \geq 3$ the equivalence breaks down. Further analysis is needed, in particular, of conditions under which multivariate majorization or directional majorization implies dominance under $\succeq_C$ (Koshevoy and Mosler 1996, 2007; Andreoli and Zoli 2023, 2026). Beyond the sufficient condition in Proposition A.6, the relation with the Koshevoy–Mosler order for $m \geq 3$ is not characterized in full generality.

A second direction concerns the operationalization of the construction: discrete versions, computational questions, and statistical properties. The basic construction allows fractional partitioning, which guarantees convex attainable sets and a convenient optimization theory. Versions based on deterministic partitions raise questions about the cost of integrality and links to the theory of measure partitioning and necklace splitting (Alon 1987; Alon and Graur 2020; Elton et al. 1986). From an applied perspective, another open problem is the development of statistical inference for the indices C_F , in particular their estimation properties. A natural computational question is the choice between the simple $C_{A,m}$ and the more demanding K_m ; the coefficient $\gamma_m = C_{A,m} - K_m$ provides a simple starting point for assessing approximation quality.

A third direction concerns applications. In multigroup segregation, the simultaneous distribution of several groups across the same organizational or spatial units is of interest; the existing literature shows that moving from two to multiple groups is a measurement problem in its own right (Reardon and Firebaugh 2002; Frankel and Volij 2011; del Río and Alonso-Villar 2022). The preorder $\succeq_C$ may provide an alternative order-theoretic perspective based on the possibility of jointly separating structures.

In competition policy, not only concentration of shares but also the relative location of firms may matter. Results of [Benson et al. \(2024\)](#) for bank mergers show limitations of screening based solely on the HHI when information is carried by geographical proximity of the merging banks, while [Taragin and Taylhardat \(2025\)](#) propose a distance-based algorithm for defining geographic markets. The proposed construction does not encode geographical distance itself, but with appropriately defined spatial units it may complement such approaches by measuring co-location or separation of several share structures.

Multi-stage and multi-commodity supply chains provide a particularly natural field of application. Separate concentration measures do not reveal whether geographical bottlenecks occur in the same countries or regions. Recent network analyses of mineral criticality likewise move away from evaluating individual markets toward simultaneous analysis of multiple commodities and cascading dependencies ([Abdelkawy and Alshamy 2026](#)). Multi-allocation separation answers a narrower question about the joint geographical position of structures, but may complement such analyses. A natural extension of the case study would move from several commodities to several stages of the same chain—mining, refining, component production, and demand—and examine whether their structures are co-located or spatially separated.

As noted in Section 6, geographical separation is not equivalent to economic independence of markets; co-production and technological links may create dependencies that are invisible in the geographical distribution of extraction alone.

In this sense, the main object of the proposed construction is not a single index but a comparison relation that contains classical concentration as a special case and extends it to separation of multiple structures.

A Technical Results on Relations among Orders

A.1 Proofs of Relations among the Blackwell, Zonotope, and Lorenz Orders

Theorem A.1 (Relations among information orders and separation) *For two states,*

$$\succeq_B = \succeq_Z = \succeq_C .$$

For at least three states, the inclusions

$$\succeq_B \subsetneq \succeq_Z, \quad \succeq_B \subsetneq \succeq_C$$

are strict, whereas $\succeq_Z$ and $\succeq_C$ are incomparable:

$$\succeq_Z \not\subseteq \succeq_C, \quad \succeq_C \not\subseteq \succeq_Z .$$

Proof The implications $\succeq_B \Rightarrow \succeq_Z$ and $\succeq_B \Rightarrow \succeq_C$ follow, respectively, from zonotope inclusion under garbling and from the data-processing inequality in Theorem 3.12. For two states, $\succeq_B = \succeq_Z$ is classical, and Proposition 3.2 identifies the same order with $\succeq_C$ ([Bertschinger and Rauh 2014](#)).

Strictness of $\succeq_B \subsetneq \succeq_Z$ for three states follows from known counterexamples for the zonotope order ([Bertschinger and Rauh 2014](#)). Strictness of $\succeq_B \subsetneq \succeq_C$ follows from QIF counterexamples in which posterior Bayes vulnerability is ordered for all priors despite the absence of composition refinement ([Espinoza Becerra 2014](#); [McIver et al. 2014](#); [Alvim et al. 2020](#)).

For incomparability, we give two counterexamples. First, let

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{4} \end{pmatrix}.$$

We verify $Z(\mathbf{B}) \subseteq Z(\mathbf{A})$ directly through support functions. Since the sums of the generators of both zonotopes equal $(1, 1, 1)^\top$, one can use the representation, where $v_i^{\mathbf{X}}$ denotes the i th column of matrix $\mathbf{X}$,

$$h_{Z(\mathbf{X})}(c) = \frac{1}{2} \left(c^\top \mathbf{1} + \sum_i |c^\top v_i^{\mathbf{X}}| \right).$$

For the matrices above, direct verification on the finite collection of cones of linearity determined by the hyperplanes $c^\top v_i = 0$ —equivalently, on cells of the direction-space partition on which the signs of all $c^\top v_i$ are fixed—gives $h_{Z(\mathbf{A})}(c) \geq h_{Z(\mathbf{B})}(c)$ for every $c \in \mathbb{R}^3$. Hence $Z(\mathbf{B}) \subseteq Z(\mathbf{A})$ and $\mathbf{A} \succeq_Z \mathbf{B}$. Under the uniform prior, however,

$$P_{\text{corr}}^*(\mathbf{A}) = \frac{5}{12} < \frac{4}{9} = P_{\text{corr}}^*(\mathbf{B}),$$

so $\mathbf{A} \not\succeq_C \mathbf{B}$.

For the reverse direction, one may use a standard counterexample for the strong min-entropy order. For

$$\mathbf{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix},$$

we have $\mathbf{C} \succeq_C \mathbf{D}$ for all priors (Espinoza Becerra 2014). However, the second column of $\mathbf{D}$ does not belong to $Z(\mathbf{C})$: solving

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{3}{4} \end{pmatrix}^\top = \lambda_1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^\top + \lambda_2 \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^\top + \lambda_3 \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}^\top$$

requires $\lambda_1 = -1/12$. Hence $\mathbf{C} \not\succeq_Z \mathbf{D}$. $\square$

In summary, our order tests robustness of a comparison to beliefs about which allocation is the true state—that is, to an arbitrary prior—but not to the decision maker’s loss function. Thus $\succeq_C$ preserves comparisons that are robust to the choice of prior under a fixed 0–1 loss. This is precisely how its decision-theoretic scope differs from the Blackwell order, which also requires dominance when the decision problem itself changes.

Two additional technical facts needed for a complete picture of the relations among the orders—projection of the Lorenz zonoid onto the zonotope order and a sufficient condition based on matrix majorization—are given in Appendix A (Lemma A.5 and Proposition A.6).

Proposition A.2 (Multivariate Lorenz dominance and separation for two allocations) *For $m = 2$ and arbitrary n, n' (not necessarily equal),*

$$\mathbf{A} \succeq_{KM} \mathbf{B} \quad \implies \quad \mathbf{A} \succeq_C \mathbf{B}.$$

Proof For arbitrary $u, v \in \mathbb{R}$, $\max(u, v) = v + (u - v)_+$. Hence, for $c = (c_1, c_2)$,

$$\sum_i \max(c_1 a_{1i}, c_2 a_{2i}) = c_2 + \sum_i (c_1 a_{1i} - c_2 a_{2i})_+,$$

because $\sum_i a_{2i} = 1$. The second term is the support function of the Lorenz zonoid in a direction with zero population coordinate:

$$\sum_i (c_1 a_{1i} - c_2 a_{2i})_+ = h_{LZ(\mathbf{A})}(0, c_1, -c_2).$$

Therefore,

$$\sum_i \max(c_1 a_{1i}, c_2 a_{2i}) = c_2 + h_{LZ(\mathbf{A})}(0, c_1, -c_2).$$

If $\mathbf{A} \succeq_{KM} \mathbf{B}$, then the support function of the Lorenz zonoid is no smaller for $\mathbf{A}$ in every direction, in particular in $(0, c_1, -c_2)$. Theorem 3.8 therefore yields $\mathbf{A} \succeq_C \mathbf{B}$. $\square$

Remark A.3 Proposition A.2 uses the special structure of the maximum of two arguments. For $m > 2$, this reduction is unavailable. Proposition A.6, however, provides a simple common sufficient condition for multivariate Lorenz dominance and separation dominance; a full characterization of the relation between $\succeq_{KM}$ and $\succeq_C$ in higher dimension is not needed for the remaining results.

Proposition A.4 (Blackwell dominance and multivariate Lorenz dominance under preservation of the population benchmark) *If $\mathbf{B} = \mathbf{A}K$ for a stochastic matrix K and simultaneously $UK = U$ (in particular, for a square bistochastic matrix), then*

$$\mathbf{A} \succeq_{KM} \mathbf{B} \quad \text{and} \quad \mathbf{A} \succeq_C \mathbf{B}.$$

Proof The second implication is Theorem 3.12. For the first, the augmented matrix with the population row first satisfies

$$\begin{pmatrix} U \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} U \\ \mathbf{A} \end{pmatrix} K,$$

because $UK = U$. Garbling implies inclusion of the corresponding zonotopes, hence inclusion of the Lorenz zonoids. $\square$

Lemma A.5 (Projection of the Lorenz zonoid) *Let*

$$\pi : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m, \quad \pi(t, x_1, \dots, x_m) = (x_1, \dots, x_m).$$

Then for every configuration $\mathbf{A}$,

$$\pi(LZ(\mathbf{A})) = Z(\mathbf{A}).$$

Consequently,

$$\mathbf{A} \succeq_{KM} \mathbf{B} \quad \implies \quad \mathbf{A} \succeq_Z \mathbf{B}.$$

Proof Linear projection preserves Minkowski sums, hence

$$\pi \left(\sum_i [0, \bar{v}_i] \right) = \sum_i [0, \pi(\bar{v}_i)] = \sum_i [0, v_i] = Z(\mathbf{A}).$$

Inclusion of convex sets is preserved by linear maps. $\square$

Proposition A.6 (Matrix majorization as a common sufficient condition) *Suppose $\mathbf{A}$ and $\mathbf{B}$ have the same number of units and there exists a bistochastic matrix D such that*

$$\mathbf{B} = \mathbf{A}D.$$

Then

$$\mathbf{A} \succeq_{KM} \mathbf{B} \quad \text{and} \quad \mathbf{A} \succeq_{\mathcal{C}} \mathbf{B}.$$

Proof The first implication is the standard relation between uniform (matrix) majorization and directional/multivariate Lorenz majorization (Koshevoy and Mosler 1996; Andreoli and Zoli 2023). For the second, for every $c \in \mathbb{R}_+^m$,

$$\begin{aligned} \sum_i \max_j c_j b_{ji} &= \sum_i \max_j \sum_k d_{ki} c_j a_{jk} \\ &\leq \sum_i \sum_k d_{ki} \max_j c_j a_{jk} \\ &= \sum_k \max_j c_j a_{jk}, \end{aligned}$$

because D is bistochastic. Theorem 3.8 then gives $\mathbf{A} \succeq_{\mathcal{C}} \mathbf{B}$. $\square$

B Technical Results on Representations and Computation

B.1 Positive Mixtures of Separating Functionals

Proof of Proposition 4.3 If $\mathbf{A} \succeq_{\mathcal{C}} \mathbf{B}$, then $V_{\pi}(\mathbf{A}) \geq V_{\pi}(\mathbf{B})$ for every π , so the inequality is preserved after integration with respect to a nonnegative measure. If the dominance is strict, the support-function characterization yields a direction π with a strict inequality. The difference $V_{\pi}(\mathbf{A}) - V_{\pi}(\mathbf{B})$ is continuous in π , so it remains positive on a neighborhood of that direction; full support of μ assigns positive measure to this neighborhood. On the minimal class $A_1 = \dots = A_m$, we have $V_{\pi} = \max_j \pi_j$, independently of the common structure. On the maximal class, with pairwise disjoint supports, $V_{\pi} = 1$. With full support, these equalities identify the minimum and maximum after integration, respectively. $\square$

B.2 Power Means as Partition Generators

For $r \leq 1$, the power mean $M_r^{(m)}$ is nondecreasing in each coordinate and symmetric, and on the diagonal satisfies $M_r^{(m)}(t, \dots, t) = t$, yielding (F1)–(F3), respectively. Condition (F4) follows from the classical power-mean inequality: for fixed arithmetic mean $\bar{x}$,

$$M_r^{(m)}(x_1, \dots, x_m) \leq M_1^{(m)}(x_1, \dots, x_m) = \bar{x} = M_r^{(m)}(\bar{x}, \dots, \bar{x})$$

for every $r \leq 1$. For $r > 1$, the inequality relative to the arithmetic mean reverses for nonconstant vectors, so (F4) fails. For $r < 0$, on the boundary of $[0, 1]^m$ we use the continuous extension: if at least one coordinate is zero, then $M_r^{(m)} = 0$.

B.3 Asymmetric Directional Functional

Proof of Proposition 5.5 Let $d_i = \alpha a_i - (1 - \alpha)b_i$. From $\sum_i d_i = 2\alpha - 1$ and the triangle inequality, $C_\alpha \geq \max\{\alpha, 1 - \alpha\}$. For $\alpha \neq 1/2$, equality may hold for many distinct pairs, whereas for $\alpha = 1/2$ we have $C_{1/2} = (1 + \text{TVD}(A, B))/2$, whose minimum occurs exactly at $A = B$. Moreover, each positive term is bounded by the corresponding weighted mass, so $C_\alpha \leq 1$; equality holds if and only if positive masses of the two allocations do not occur on the same agents. $\square$

B.4 Computing the Kolkata Index

For general m , the index K_m is the value of the linear program

$$\max_{p, t} t \quad \text{subject to} \quad \sum_i p_{ij} a_{ji} \geq t \ (j = 1, \dots, m), \quad \sum_{j=1}^m p_{ij} = 1 \ (i = 1, \dots, n), \quad p_{ij} \geq 0.$$

The program has $mn + 1$ variables, m guarantee constraints, n assignment equalities, and nonnegativity constraints. Its size is therefore polynomial in m and n , even though the number of deterministic assignments of units to m classes is m^n .

Proof of Proposition 5.6 At an optimum the two sides may be taken to be equal: $\sum_i p_i a_i = \sum_i (1 - p_i) b_i = K(A, B)$. Otherwise, continuously shifting suitable p_i in the direction that improves the weaker value leads to equality without reducing the minimum. Hence $\sum_i (a_i + b_i) p_i = 1$, and the problem is equivalent to

$$K(A, B) = \max \left\{ \sum_i a_i p_i : \sum_i (a_i + b_i) p_i = 1, \ 0 \leq p_i \leq 1 \right\}.$$

This is a fractional knapsack problem (Dantzig 1957) with values a_i , weights $a_i + b_i$, and capacity 1. Full sorting of the ratios $a_i/(a_i + b_i)$ gives $O(n \log n)$, but only the threshold element is needed for the optimum value. It can be found with a linear-time selection algorithm (Blum et al. 1973). $\square$

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