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Battery-Aware Finsler Geometry for Energy-Optimal Routing of Electric Vehicles

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Abstract: The increasing adoption of electric vehicles (EVs) has created a need for route-planning strategies that account not only for travel distance but also for battery consumption, acceleration tendency and road-gradient effects. Conventional shortest-path methods generally represent transportation cost through scalar edge weights and may therefore overlook directional differences in EV energy demand. In this work, a battery-aware Finsler-geometric framework is developed for energy-aware EV route optimization. The road environment is represented as a differentiable manifold, and a direction-dependent Finsler-type cost function is constructed from a Riemannian travel component and normalized energetic corrections associated with battery demand, desired-speed variation, and uphill motion.

A physically interpretable EV energy model is first formulated by considering inertial, rolling-resistance, aerodynamic and gravitational forces together with drive-train efficiency and regenerative braking. The physical battery energy, acceleration-related energy, and uphill gravitational energy are then evaluated independently for the resulting trajectories. The proposed directional acceleration term is interpreted as a first-order surrogate based on the directional variation of a prescribed desired-speed field, rather than as the instantaneous vehicle acceleration itself.

A numerical experiment compares distance-based, physical-energy-based and Finsler-based routing strategies on the same directed grid. The results show that the shortest route may require substantially greater battery energy, whereas the proposed Finsler-based route provides a direction-dependent compromise between travel distance, battery demand, acceleration tendency and road-gradient effects. The proposed framework is presented as a numerical Finsler-type route-selection model whose regularity and positivity are verified on the computational domain used in the experiment.

Keywords: Finsler geometry; electric vehicles; battery energy optimization; energy-efficient routing; vehicle dynamics; road gradient; acceleration cost; state of charge; autonomous navigation; intelligent transportation systems.

1. INTRODUCTION

The rapid development of electric vehicles (EVs) has created a growing need for energy-efficient and battery-aware transportation strategies. Unlike conventional vehicles, EVs are strongly constrained by the available battery capacity and the relatively limited driving range. Consequently, the selection of an appropriate route is not determined solely by the shortest travel distance. The energy required by an EV depends on several interacting factors, including vehicle speed, acceleration, road gradient, vehicle mass, aerodynamic resistance, rolling resistance and powertrain efficiency. In particular, road topography and acceleration can substantially modify

the energy required to travel between two locations. Existing energy-aware routing models have therefore incorporated topography, speed, acceleration, braking and powertrain characteristics into electric-vehicle route planning [1, 5].

Road gradient is particularly important in EV transportation. During uphill motion, additional energy is required to overcome the gravitational component of the vehicle weight, whereas downhill motion can reduce the required traction power and may allow a portion of the kinetic and potential energy to be recovered through regenerative braking. Recent studies have demonstrated that incorporating road gradients and regenerative braking can significantly influence EV routing decisions. Similarly, energy-consumption models have incorporated acceleration and braking effects together with topography and speed in order to obtain more realistic estimates of EV energy demand [1, 5]. These developments indicate that an energy-optimal route may differ considerably from the geometrically shortest route [6].

Although existing approaches provide useful energy-consumption models and optimization procedures, most route-planning formulations represent a road network through scalar edge weights or direction-independent costs. Such a representation may not fully capture the intrinsic directional nature of EV motion. For example, travelling along a particular road in the uphill direction may require substantially more battery energy than travelling along the same road in the opposite direction. Likewise, acceleration and deceleration can produce different energetic consequences depending on the direction of motion and the local road geometry. This motivates the development of a geometric framework in which the cost of motion depends simultaneously on the position of the vehicle and its instantaneous direction.

Finsler geometry provides a natural mathematical framework for such direction-dependent optimization problems [2, 3]. In a Finsler space, the fundamental function is generally written as $F = F(x, y)$ where, x represents the position of the vehicle and y represents a tangent vector describing its instantaneous direction of motion. In contrast to a Riemannian metric, a Finsler metric need not satisfy the reversibility condition $F(x, y) = F(x, -y)$. This property is particularly attractive for EV routing because the energetic cost of travelling along a road need not be identical in opposite directions. Consequently, the geometry itself can encode directional effects arising from road slope, vehicle acceleration and battery-energy consumption [3, 4].

In this work, we develop a battery-aware Finsler optimization framework for electric vehicles. The proposed formulation combines three major energetic components: battery usage, acceleration cost and road-slope effects. Let $x \in M$ denote the position of an EV and $y \in T_x M$ denote its instantaneous velocity or direction vector. The proposed framework represents the local EV travel cost by a direction-dependent Finsler function constructed from an underlying Riemannian travel component and normalized energetic corrections. The detailed mathematical construction of this function is presented in Section 4.

The central idea is to regard the energy cost of an EV as a geometric quantity defined on the tangent bundle of the road domain. In this formulation, the position of the vehicle determines the local road and environmental conditions, whereas the tangent vector determines the direction and magnitude of motion. Therefore, the resulting cost can distinguish between different directions of travel on the same road segment. In particular, uphill and downhill movements can naturally acquire

different costs, while acceleration-dependent effects can be incorporated through the velocity and its local variation.

For a feasible vehicle trajectory $\gamma : [t_0, t_1] \rightarrow M$, the proposed optimization problem is formulated through the path functional

$$\mathcal{J}[\gamma] = \int_{t_0}^{t_1} F_{\text{EV}}(\gamma(t), \dot{\gamma}(t)) dt.$$

The optimal EV trajectory is then obtained by minimizing this functional over the admissible set of trajectories connecting a prescribed origin and destination:

$$\gamma^* = \arg \min_{\gamma \in \mathcal{A}} \mathcal{J}[\gamma].$$

Thus, the proposed framework converts battery-aware EV route planning into a Finsler-geometric path optimization problem.

The main contributions of this study are summarized as follows:

- A Finsler-geometric model is developed for energy-aware electric vehicle navigation in which the travel cost depends on both position and direction.
- Battery-energy consumption, acceleration effects, and road-slope effects are incorporated into a unified geometric cost function.
- The proposed formulation naturally permits directional asymmetry, allowing uphill and downhill motion to have different energetic costs.
- A variational formulation is established for determining energy-efficient EV trajectories through Finsler geodesics.
- A numerical framework is developed to compare the proposed Finsler-optimal route with conventional distance-based and energy-based routing strategies.
- The framework is discussed for electric passenger cars, electric buses, and autonomous electric vehicles, where energy-aware trajectory selection is particularly important.

The remainder of the paper is organized as follows. Section 2 introduces the mathematical preliminaries and formulates the EV navigation problem. Section 3 develops the physical energy-consumption model by incorporating vehicle dynamics, battery usage, acceleration, regenerative braking, and road-gradient effects. Section 4 constructs the proposed battery-aware Finsler metric and establishes its positive homogeneity and directional dependence. Section 5 formulates the associated Finsler-geometric optimization problem and derives the corresponding optimality conditions. Section 6 presents the numerical experiment and compares distance-based, energy-based, and Finsler-based routing strategies. Section 7 discusses applications to electric passenger cars, electric buses, and autonomous electric vehicles. Section 8 summarizes the main findings and outlines possible future developments.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we introduce the mathematical setting required for the development of the proposed Finsler-geometric framework for electric vehicle (EV) navigation. The road environment is represented as a differentiable manifold, while the position and direction of the vehicle are described by points and tangent vectors, respectively. This formulation allows the energetic cost of an EV to depend not only on its location but also on its direction of motion.

2.1. Road Geometry and Vehicle State. Let M be a two-dimensional smooth manifold representing the accessible road domain. A local coordinate system on M is denoted by $x = (x^1, x^2)$, where x^1 and x^2 represent the spatial coordinates of the vehicle. A vehicle trajectory is described by a smooth curve

$$\gamma : [t_0, t_1] \rightarrow M, \quad t \mapsto \gamma(t),$$

with

$$\gamma(t_0) = x_S, \quad \gamma(t_1) = x_G,$$

where, x_S and x_G denote the prescribed source and destination points, respectively.

The instantaneous velocity of the vehicle is represented by the tangent vector

$$y = \dot{\gamma}(t) = \frac{d\gamma}{dt} = (\dot{x}^1, \dot{x}^2) \in T_x M.$$

The complete state of the vehicle is therefore represented by the pair $(x, y) \in TM$, where TM denotes the tangent bundle of the road manifold M . The speed of the vehicle is given by $v = \|y\|$, where the norm is determined by the underlying spatial metric. Thus, the tangent vector y contains both the magnitude and direction of the vehicle's motion. For a vehicle moving with velocity y , the directional rate of change of elevation is given by

$$D_y h(x) = \nabla h(x) \cdot y.$$

Consequently, $D_y h(x) > 0$ corresponds to an uphill component of motion, while $D_y h(x) < 0$ corresponds to a downhill component. The corresponding road-gradient angle can be represented by

$$\theta(x, y) = \tan^{-1} \left(\frac{\nabla h(x) \cdot y}{\|y\|} \right),$$

under an appropriate local coordinate representation. This quantity provides a directional measure of the road slope experienced by the vehicle.

It is important to note that the same road segment can produce different energetic effects depending on the direction of travel. In particular,

$$D_{-y} h(x) = -D_y h(x),$$

which demonstrates the directional character of the slope effect.

Let $g = g_{ij}(x) dx^i \otimes dx^j$ be a positive-definite Riemannian metric describing the underlying spatial geometry of the road domain. The corresponding norm of a tangent vector $y \in T_x M$ is

$$\alpha(x, y) = \sqrt{g_{ij}(x) y^i y^j}.$$

For a vehicle trajectory $\gamma(t)$, the Riemannian length is given by

$$L[\gamma] = \int_{t_0}^{t_1} \alpha(\gamma(t), \dot{\gamma}(t)) dt.$$

The quantity $\alpha(x, y)$ provides the direction-independent geometric component of the travel cost. However, it cannot by itself distinguish between opposite directions of travel, since

$$\alpha(x, y) = \alpha(x, -y).$$

Therefore, an additional direction-dependent structure is required to model battery consumption, acceleration, and slope effects.

2.2. Finsler Representation and EV Route Optimization. A Finsler metric is a positive function $F : TM \rightarrow [0, \infty)$ that is smooth on the slit tangent bundle $TM \setminus \{0\}$, positively homogeneous in the tangent variable and strongly convex under the standard Finsler regularity conditions [2].

In the present problem, we seek a Finsler function of the form

$$F_{\text{EV}}(x, y) = F_{\text{EV}}(x^1, x^2, y^1, y^2),$$

which represents the local cost of EV motion.

The principal distinction from a conventional Riemannian formulation is that the proposed cost may satisfy [4]

$$F_{\text{EV}}(x, y) \neq F_{\text{EV}}(x, -y).$$

Thus, travelling along the same road in opposite directions can result in different costs. This property is particularly relevant for EVs because uphill motion, downhill motion, acceleration, and regenerative braking can produce different battery-energy requirements.

Let $\mathcal{A}(x_S, x_G)$ denote the set of admissible trajectories connecting the source point x_S to the destination point x_G . Thus,

$$\mathcal{A}(x_S, x_G) = \{\gamma : [t_0, t_1] \rightarrow M : \gamma(t_0) = x_S, \gamma(t_1) = x_G\}.$$

In practical EV navigation, the admissible set may also be restricted by obstacles, road boundaries, maximum vehicle speed, acceleration limits, and battery-state constraints. Accordingly, we may write

$$\mathcal{A}_{\text{EV}} \subseteq \mathcal{A}(x_S, x_G)$$

for the set of physically feasible EV trajectories.

For example, the velocity and acceleration constraints can be expressed as

$$0 \leq v(t) \leq v_{\max},$$

and

$$a_{\min} \leq a(t) \leq a_{\max}.$$

If $\text{SOC}(t)$ denotes the battery state of charge, a feasible trajectory must also satisfy

$$\text{SOC}(t) \geq \text{SOC}_{\min}.$$

The objective of the proposed framework is to determine a feasible trajectory that minimizes the combined cost associated with travelling from x_S to x_G . For a trajectory $\gamma \in \mathcal{A}_{\text{EV}}$, we define the general cost functional as

$$\mathcal{J}[\gamma] = \int_{t_0}^{t_1} F_{\text{EV}}(\gamma(t), \dot{\gamma}(t)) dt.$$

The optimal trajectory is therefore obtained from

$$\gamma^* = \arg \min_{\gamma \in \mathcal{A}_{\text{EV}}} \mathcal{J}[\gamma].$$

The proposed formulation differs from a conventional shortest-path problem. Instead of minimizing only the geometric distance $L[\gamma]$, the objective incorporates the energetic characteristics of the vehicle and the road environment.

The proposed framework is designed to incorporate three principal direction-dependent effects: battery-energy demand, acceleration tendency, and road-gradient effects. These quantities will be derived from the physical longitudinal dynamics of the electric vehicle and subsequently incorporated into the Finsler structure.

To preserve the mathematical consistency of the Finsler formulation, the directional dependence is introduced through the normalized tangent vector

$$\hat{y} = \frac{y}{\alpha(x, y)}. \quad (2.1)$$

Since

$$\hat{y}(x, \mu y) = \hat{y}(x, y), \quad \mu > 0, \quad (2.2)$$

the directional energetic quantities constructed from $\hat{y}$ are independent of the magnitude of the tangent vector. The complete battery-aware Finsler function, including the battery, acceleration and road-gradient contributions, is defined in Section 4.

The resulting route-optimization problem therefore combines the geometric structure of the road manifold with physically motivated, direction-dependent energy information. This formulation provides the mathematical basis for energy-aware EV navigation while preserving the positive homogeneity required for a Finsler metric.

3. PHYSICAL ENERGY MODEL OF AN ELECTRIC VEHICLE

The construction of a battery-aware Finsler metric requires a physically interpretable model of the energy demand of an electric vehicle [5]. In this section, the longitudinal vehicle dynamics are formulated by considering inertial, rolling-resistance, aerodynamic, and road-gradient forces. The resulting traction power is subsequently related to battery power through the drivetrain efficiency.

3.1. Longitudinal Vehicle Dynamics and Traction Forces. Consider an electric vehicle of mass m moving along a road with velocity $v(t)$. Let $a(t) = \frac{dv}{dt}$ denote the longitudinal acceleration and let $\theta(x)$ denote the local road inclination. The total longitudinal traction force required at the wheels is expressed as

$$F_{\text{tr}} = F_{\text{acc}} + F_{\text{roll}} + F_{\text{aero}} + F_{\text{slope}}. \quad (3.1)$$

The individual components are considered below. The inertial force required to accelerate the vehicle is

$$F_{\text{acc}} = ma, \quad (3.2)$$

where, m is the vehicle mass and a is the longitudinal acceleration. For positive acceleration,

$$F_{\text{acc}} > 0, \quad (3.3)$$

and additional traction power is required. During deceleration, the force can become negative, allowing regenerative braking when the operating conditions permit energy recovery. The rolling-resistance force is approximated by

$$F_{\text{roll}} = C_r mg \cos \theta, \quad (3.4)$$

where, C_r is the coefficient of rolling resistance and g is the gravitational acceleration.

For relatively small road gradients,

$$\cos \theta \simeq 1,$$

and therefore

$$F_{\text{roll}} \simeq C_r mg. \quad (3.5)$$

The aerodynamic drag force is given by

$$F_{\text{aero}} = \frac{1}{2} \rho C_D A_f v^2, \quad (3.6)$$

where, ρ is the air density, C_D is the aerodynamic drag coefficient and A_f is the frontal area of the vehicle. The corresponding aerodynamic power is

$$P_{\text{aero}} = F_{\text{aero}}v = \frac{1}{2}\rho C_D A_f v^3. \quad (3.7)$$

This cubic dependence on velocity demonstrates why aerodynamic losses become increasingly important at higher vehicle speeds.

Let $h(x)$ be the elevation of the road. The local road-gradient contribution is determined by the directional derivative of the elevation. For a trajectory $\gamma(t)$, the rate of change of elevation is

$$\frac{dh}{dt} = \nabla h(\gamma(t)) \cdot \dot{\gamma}(t). \quad (3.8)$$

The corresponding gravitational force along the direction of motion is

$$F_{\text{slope}} = mg \sin \theta. \quad (3.9)$$

For small road angles,

$$\sin \theta \simeq \tan \theta, \quad (3.10)$$

and the local slope can be approximated using

$$\tan \theta = \frac{dh}{ds}, \quad (3.11)$$

where, s denotes travelled distance. Hence,

$$F_{\text{slope}} \simeq mg \frac{dh}{ds}. \quad (3.12)$$

An uphill segment has $\frac{dh}{ds} > 0$, whereas a downhill segment has $\frac{dh}{ds} < 0$. Combining the individual contributions gives

$$F_{\text{tr}} = ma + C_r mg \cos \theta + \frac{1}{2}\rho C_D A_f v^2 + mg \sin \theta. \quad (3.13)$$

The corresponding wheel power is

$$P_{\text{w}} = F_{\text{tr}}v. \quad (3.14)$$

Therefore,

$$P_{\text{w}} = \left[ma + C_r mg \cos \theta + \frac{1}{2}\rho C_D A_f v^2 + mg \sin \theta \right] v. \quad (3.15)$$

This equation provides the physical basis for estimating the instantaneous energy requirement of the vehicle.

3.2. Battery Power and Energy Consumption. During propulsion, the wheel power is positive: $P_{\text{w}} > 0$. The positive traction-power component is defined as

$$P_{\text{w}}^+ = \max(P_{\text{w}}, 0). \quad (3.16)$$

Assuming a drivetrain efficiency η_d , the corresponding battery power required for propulsion is

$$P_{\text{bat}}^+ = \frac{P_{\text{w}}^+}{\eta_d}. \quad (3.17)$$

The positive battery energy consumed over a trajectory is consequently

$$E_B^+ = \int_{t_0}^{t_1} P_{\text{bat}}^+(t) dt. \quad (3.18)$$

When $P_w < 0$, the vehicle is decelerating or travelling under conditions where the wheel power becomes negative. A fraction of this mechanical energy can potentially be recovered through regenerative braking [1].

Let η_r denote the regenerative-braking efficiency. The recovered power is defined as

$$P_{\text{reg}} = \eta_r \max(-P_w, 0). \quad (3.19)$$

In the present numerical experiment, the regenerative-braking efficiency η_r is treated as a constant effective efficiency. Battery state-of-charge limits, battery-temperature restrictions, motor-generator power limits, and detailed regenerative-braking control logic are not modelled explicitly. Consequently, the recovered energy represents an idealized effective regenerative contribution under the assumed operating conditions. These practical limitations can be incorporated in a more detailed vehicle-control model in future work. The total recovered energy is therefore

$$E_{\text{reg}} = \int_{t_0}^{t_1} P_{\text{reg}}(t) dt. \quad (3.20)$$

The net battery-energy consumption is then

$$E_{\text{net}} = E_B^+ - E_{\text{reg}}. \quad (3.21)$$

In a practical EV, regenerative braking is subject to limits imposed by battery state of charge, motor-generator capacity, battery temperature and braking-system control. These limitations can be incorporated into the numerical model when required. For Finsler-geometric route construction, it is useful to introduce an energy cost per unit travelled distance. Since $ds = vdt$, the propulsion energy per unit distance can be written as

$$e_B^+ = \frac{P_{\text{bat}}^+}{v}, \quad v > 0. \quad (3.22)$$

Using Equation (3.18), we obtain

$$e_B^+ = \frac{P_w^+}{\eta_d v}. \quad (3.23)$$

Similarly, the regenerative contribution per unit distance is

$$e_{\text{reg}} = \frac{P_{\text{reg}}}{v}. \quad (3.24)$$

Thus, the net energy cost per unit distance becomes

$$e_{\text{net}} = e_B^+ - e_{\text{reg}}. \quad (3.25)$$

This quantity provides the physical energy scale used in the battery-dependent component of the Finsler formulation.

3.3. Acceleration- and Slope-Related Energy. The acceleration contribution to the wheel power is

$$P_{\text{acc}} = mav. \quad (3.26)$$

Since only positive acceleration requires additional propulsion effort, we define

$$P_{\text{acc}}^+ = ma^+v, \quad (3.27)$$

where,

$$a^+ = \max(a, 0). \quad (3.28)$$

The corresponding acceleration-related energy is

$$E_{\text{acc}} = \int_{t_0}^{t_1} ma^+(t)v(t)dt. \quad (3.29)$$

This quantity is used as an independent performance measure for evaluating the smoothness and energetic demand of the optimized trajectories. The quantity E_{acc} is treated as an acceleration-related physical performance measure rather than as an additional term to be directly added to E_{net} . This distinction prevents double counting because the inertial contribution ma is already included in the total traction force and therefore contributes to the battery-energy calculation through P_{w} .

The gravitational contribution to the wheel power is

$$P_{\text{slope}} = mg \sin \theta v. \quad (3.30)$$

Using

$$\frac{dh}{dt} = v \sin \theta,$$

the gravitational power can equivalently be expressed as

$$P_{\text{slope}} = mg \frac{dh}{dt}. \quad (3.31)$$

The uphill component is therefore

$$P_{\text{slope}}^+ = mg \left[\frac{dh}{dt} \right]_+. \quad (3.32)$$

The corresponding uphill gravitational energy is

$$E_{\text{slope}} = mg \int_{t_0}^{t_1} \left[\frac{dh}{dt} \right]_+ dt. \quad (3.33)$$

Equivalently,

$$E_{\text{slope}} = mg \int_{\gamma} \left[\frac{dh}{ds} \right]_+ ds. \quad (3.34)$$

This expression directly connects the physical slope-energy requirement with the geometry of the travelled path. The quantity E_{slope} is likewise retained as an independent diagnostic measure of the energetic influence of uphill motion. It is not added separately to E_{net} , since the gravitational force $mg \sin \theta$ is already included in the total traction force (3.13).

3.4. State of Charge. Let E_{bat} denote the usable battery capacity and let SOC_0 be the initial state of charge. The state of charge during the trip can be approximated by

$$\text{SOC}(t) = \text{SOC}_0 - \frac{E_{\text{net}}(t)}{E_{\text{bat}}}, \quad 0 \leq \text{SOC}(t) \leq 1. \quad (3.35)$$

When reported as a percentage, the state of charge is given by

$$\text{SOC}_{f\%} = 100 \text{SOC}_f. \quad (3.36)$$

The vehicle must satisfy

$$\text{SOC}(t) \geq \text{SOC}_{\text{min}} \quad (3.37)$$

throughout the trip. The final state of charge is

$$\text{SOC}_f = \text{SOC}_0 - \frac{E_{\text{net}}}{E_{\text{bat}}}. \quad (3.38)$$

The SOC constraint is subsequently incorporated into the Finsler-based route optimization problem.

The physical energy model provides the local energetic information required by the Finsler formulation. In particular, the net energy per unit distance e_{net} provides the reference scale for the battery contribution, while the directional road-gradient and desired-speed fields determine the direction-dependent slope and acceleration terms.

Accordingly, the Finsler model does not replace the physical EV energy model. Instead, it provides a geometric framework for incorporating direction-dependent energetic information into route optimization. The physical battery energy E_{net} , acceleration-related energy E_{acc} and slope-related energy E_{slope} are retained as independent physical performance measures for evaluating the optimized trajectories.

The quantity E_{net} represents the net physical energy drawn from the battery after accounting for both propulsion energy consumption and regenerative recovery. It is therefore used as the principal physical measure of battery demand in the numerical evaluation. The Finsler formulation introduced later does not replace this quantity; rather, it uses a normalized direction-dependent representation of the local energetic demand for route optimization.

4. BATTERY-AWARE FINSLER METRIC

The physical energy model developed in Section 3 provides the energetic quantities required for constructing a direction-dependent geometric cost. The main objective of this section is to incorporate battery demand, acceleration tendency, and road-gradient effects into a unified Finsler framework.

Let M denote the two-dimensional road manifold and let $x \in M$ represent the position of the electric vehicle. A non-zero tangent vector

$$y \in T_x M \setminus \{0\}$$

represents the local direction and magnitude of vehicle motion.

4.1. Directional Components of the Battery-Aware Metric. Let $g_{ij}(x)$ denote the Riemannian metric associated with the geometric structure of the road environment. The corresponding norm is defined by

$$\alpha(x, y) = \sqrt{g_{ij}(x) y^i y^j}. \quad (4.1)$$

The normalized tangent direction is introduced as

$$\hat{y} = \frac{y}{\alpha(x, y)}. \quad (4.2)$$

For every $\mu > 0$,

$$\hat{y}(x, \mu y) = \frac{\mu y}{\alpha(x, \mu y)} = \frac{y}{\alpha(x, y)} = \hat{y}(x, y), \quad (4.3)$$

where, we have used

$$\alpha(x, \mu y) = \mu \alpha(x, y).$$

Thus, $\hat{y}$ describes the direction of motion independently of the magnitude of the tangent vector. Let $e_{\text{net}}(x, \hat{y})$ denote the local net battery-energy demand per unit distance. This quantity is obtained from the physical EV energy model developed

in Section 3. To obtain a dimensionless quantity, we introduce a positive reference energy-per-distance scale e_{ref} and define

$$B(x, \hat{y}) = \frac{e_{\text{net}}(x, \hat{y})}{e_{\text{ref}}}. \quad (4.4)$$

The function B represents the relative battery demand associated with traveling through the position x in the direction $\hat{y}$. Because road elevation and other environmental characteristics may vary with direction, in general

$$B(x, \hat{y}) \neq B(x, -\hat{y}). \quad (4.5)$$

The quantity B is allowed to reflect the net battery-energy demand obtained from the physical model, including the effect of regenerative braking. Consequently, its sign is determined by the underlying energy model rather than imposed by the geometric construction.

Let $h : M \rightarrow \mathbb{R}$ be the road-elevation function. The directional derivative of the elevation along $\hat{y}$ is

$$D_{\hat{y}}h(x) = \nabla h(x) \cdot \hat{y}. \quad (4.6)$$

The uphill component is defined by

$$[D_{\hat{y}}h(x)]^+ = \max\{\nabla h(x) \cdot \hat{y}, 0\}. \quad (4.7)$$

The corresponding normalized slope contribution is defined as

$$S(x, \hat{y}) = \frac{mg}{e_{\text{ref}}} [\nabla h(x) \cdot \hat{y}]^+. \quad (4.8)$$

For opposite directions,

$$\hat{y}(x, -y) = -\hat{y}(x, y), \quad (4.9)$$

and consequently

$$S(x, \hat{y}) \neq S(x, -\hat{y}) \quad (4.10)$$

whenever the local road gradient is non-zero. Hence, the proposed formulation naturally distinguishes uphill and downhill motion. The slope term is introduced as a directional penalty in the route-selection metric, while the actual gravitational contribution to vehicle energy remains determined by the physical energy model of Section 3. The actual vehicle acceleration

$$a(t) = \frac{dv}{dt} \quad (4.11)$$

is a second-order quantity and therefore cannot be inserted directly into a standard first-order Finsler function.

To retain a first-order directional formulation, let

$$v_d : M \rightarrow \mathbb{R}^+$$

denote a prescribed or estimated desired-speed field. Its directional rate of change is

$$D_{\hat{y}}v_d(x) = \nabla v_d(x) \cdot \hat{y}. \quad (4.12)$$

The positive directional acceleration demand is defined by

$$a_d(x, \hat{y}) = [\nabla v_d(x) \cdot \hat{y}]^+, \quad [q]^+ = \max\{q, 0\}. \quad (4.13)$$

We introduce the corresponding normalized acceleration contribution as

$$A(x, \hat{y}) = \frac{m a_d(x, \hat{y})}{e_{\text{ref}}}. \quad (4.14)$$

This term represents the tendency of the local speed environment to require positive acceleration. It is a directional surrogate used in the geometric route-selection cost and should not be interpreted as the actual instantaneous vehicle acceleration.

The actual acceleration energy of the final trajectory is evaluated independently using

$$E_{\text{acc}} = \int_{t_0}^{t_1} m a^+(t) v(t) dt, \quad (4.15)$$

where, $a^+(t) = \max\{a(t), 0\}$. Thus, the directional acceleration contribution in the Finsler function does not replace the physical acceleration-energy calculation.

4.2. Battery-Aware Finsler Function. The proposed battery-aware Finsler function is defined by

$$F_{\text{EV}}(x, y) = \alpha(x, y) [1 + \lambda_B B(x, \hat{y}) + \lambda_A A(x, \hat{y}) + \lambda_S S(x, \hat{y})]. \quad (4.16)$$

where, λ_B , λ_A , λ_S are non-negative weighting parameters corresponding to battery, acceleration and slope effects, respectively. For convenience, define the directional correction factor

$$\Psi(x, \hat{y}) = 1 + \lambda_B B(x, \hat{y}) + \lambda_A A(x, \hat{y}) + \lambda_S S(x, \hat{y}). \quad (4.17)$$

Then

$$F_{\text{EV}}(x, y) = \alpha(x, y) \Psi(x, \hat{y}). \quad (4.18)$$

The proposed formulation separates the geometric magnitude $\alpha(x, y)$ from the direction-dependent energetic correction $\Psi(x, \hat{y})$. Since the battery contribution may represent net battery demand, including regenerative effects, positivity of the Finsler function requires the admissibility condition

$$\Psi(x, \hat{y}) > 0 \quad (4.19)$$

for all admissible $(x, \hat{y})$ considered in the route optimization.

4.3. Finsler Regularity and Directional Properties. A fundamental requirement of a Finsler function is positive homogeneity in the tangent variable. Since

$$\alpha(x, \mu y) = \mu \alpha(x, y), \quad \mu > 0, \quad (4.20)$$

and

$$\hat{y}(x, \mu y) = \hat{y}(x, y), \quad (4.21)$$

the directional quantities satisfy

$$\mathcal{B}(x, \mu y) = \mathcal{B}(x, y), \quad \mu > 0. \quad (4.22)$$

$$\mathcal{A}(x, \mu y) = \mathcal{A}(x, y), \quad \mu > 0. \quad (4.23)$$

$$\mathcal{S}(x, \mu y) = \mathcal{S}(x, y), \quad \mu > 0. \quad (4.24)$$

Therefore,

$$\begin{aligned} F_{\text{EV}}(x, \mu y) &= \mu \alpha(x, y) [1 + \lambda_B B(x, \hat{y}) + \lambda_A A(x, \hat{y}) + \lambda_S S(x, \hat{y})] \\ &= \mu F_{\text{EV}}(x, y). \end{aligned} \quad (4.25)$$

Hence,

$$F_{\text{EV}}(x, \mu y) = \mu F_{\text{EV}}(x, y), \quad \mu > 0. \quad (4.26)$$

Thus, the proposed cost is positively homogeneous with respect to the tangent vector. The underlying Riemannian component is reversible:

$$\alpha(x, y) = \alpha(x, -y). \quad (4.27)$$

However, the directional energetic contributions need not be reversible. Therefore,

$$F_{\text{EV}}(x, y) \neq F_{\text{EV}}(x, -y) \quad (4.28)$$

whenever at least one of the directional contributions changes under reversal. In particular, the slope contribution satisfies

$$S(x, \hat{y}) = \frac{mg}{e_{\text{ref}}} [\nabla h(x) \cdot \hat{y}]^+, \quad (4.29)$$

while

$$S(x, -\hat{y}) = \frac{mg}{e_{\text{ref}}} [-\nabla h(x) \cdot \hat{y}]^+. \quad (4.30)$$

Consequently,

$$F_{\text{EV}}(x, y) \neq F_{\text{EV}}(x, -y) \quad (4.31)$$

in general. This non-reversibility is particularly relevant for electric vehicles because travelling uphill and downhill along the same road segment can have substantially different energy requirements. The fundamental tensor associated with the proposed Finsler function is defined, on the smooth directional regions, by

$$g_{ij}^F(x, y) = \frac{1}{2} \frac{\partial^2 F_{\text{EV}}^2(x, y)}{\partial y^i \partial y^j}. \quad (4.32)$$

For a regular Finsler structure, the fundamental tensor must be positive definite, namely,

$$g_{ij}^F(x, y) \xi^i \xi^j > 0 \quad (4.33)$$

for every non-zero

$$\xi \in T_x M.$$

The present formulation contains positive-part operators of the form

$$[q]^+ = \max\{q, 0\}.$$

Consequently, the directional correction is piecewise smooth and may have transition surfaces at which the corresponding directional quantity changes from zero to positive values. The regularity and strong-convexity condition is therefore assessed on the smooth directional regions and verified numerically for the parameter set used in the computational implementation.

In particular, for the selected values of λ_B , λ_A , and λ_S , the numerical evaluation of the fundamental tensor is used to determine its eigenvalues over the admissible computational domain and travel directions. The minimum eigenvalue,

$$\lambda_{\min}^F = \min_{(x, y) \in \mathcal{D}_{\text{adm}}} \lambda_{\min} [g_{ij}^F(x, y)], \quad (4.34)$$

provides the corresponding numerical diagnostic. A strictly positive value of $\lambda_{\min}^F$ over the tested smooth directional regions confirms positive definiteness of the fundamental tensor for the numerical parameter set.

The numerical convexity check is therefore treated as part of the admissibility verification of the proposed direction-dependent metric rather than assumed solely from the algebraic form of F_{EV} .

The proposed Finsler function has three distinct directional corrections:

$$\lambda_B B, \quad \lambda_A A, \quad \lambda_S S. \quad (4.35)$$

The first penalizes locally expensive battery-energy conditions, the second penalizes directional acceleration demand, and the third penalizes uphill motion.

Importantly, the proposed geometric formulation does not replace the physical battery model developed in Section 3. Instead, the Finsler function provides the direction-dependent geometric objective used for route selection, while the physical quantities E_{net} , E_{acc} , E_{slope} , SOC are independently evaluated for the resulting trajectory. This separation allows the proposed framework to combine direction-dependent geometric optimization with physically interpretable electric-vehicle performance measures.

5. FINSLER-GEOMETRIC OPTIMIZATION OF EV TRAJECTORIES

The optimal EV trajectory is obtained by minimizing the Finsler length functional associated with the battery-aware metric defined in Section 4. Let $L(x, \dot{x}) = F_{\text{EV}}(x, \dot{x})$. For a regular Finsler structure, the stationary trajectories satisfy the Euler–Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial F_{\text{EV}}}{\partial \dot{x}^i} \right) - \frac{\partial F_{\text{EV}}}{\partial x^i} = 0, \quad i = 1, \dots, n. \quad (5.1)$$

Equivalently, the Finsler geodesics can be written in spray form as

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = 0, \quad (5.2)$$

where, G^i are the spray coefficients associated with F_{EV} . The fundamental tensor is

$$g_{ij}^F = \frac{1}{2} \frac{\partial^2 F_{\text{EV}}^2}{\partial y^i \partial y^j}. \quad (5.3)$$

The constrained optimization problem is

$$\begin{aligned} & \underset{\gamma}{\text{minimize}} \quad \mathcal{J}_F[\gamma] = \int_{t_0}^{t_1} F_{\text{EV}}(\gamma, \dot{\gamma}) dt, \\ & \text{subject to} \quad \gamma(t_0) = x_S, \\ & \quad \quad \quad \gamma(t_1) = x_G, \\ & \quad \quad \quad \gamma(t) \in \mathcal{D} \setminus \mathcal{O}, \\ & \quad \quad \quad 0 < v(t) \leq v_{\text{max}}, \\ & \quad \quad \quad \text{SOC}(t) \geq \text{SOC}_{\text{min}}. \end{aligned} \quad (5.4)$$

where, $\mathcal{D}$ denotes the accessible road domain and $\mathcal{O} \subset \mathcal{D}$ denotes the union of obstacle regions.

For the numerical implementation, the continuous Finsler variational problem is discretized on a directed spatial grid. Each admissible grid edge represents a local tangent direction, and its weight is obtained by evaluating the Finsler function $F_{\text{EV}}(x, y)$ along that edge. The direction of the edge is retained explicitly so that opposite directions on the same spatial segment may have different costs. The resulting directed weighted graph is then solved using a shortest-path algorithm. The graph-based computation therefore provides a discrete approximation to the continuous Finsler-geometric optimization problem.

6. NUMERICAL EXPERIMENT AND ROUTE OPTIMIZATION

To demonstrate the performance of the proposed battery-aware Finsler framework, a numerical route-optimization experiment is considered. The objective is to compare a conventional distance-based route, a physical-energy-based route, and the proposed Finsler-optimal route under the same road and vehicle conditions.

6.1. Numerical Setup and Computational Environment. The computational road domain is defined as

$$D = [0, 10] \times [0, 10] \text{km}. \quad (6.1)$$

The initial and terminal positions of the electric vehicle are selected as

$$S = (1, 1) \text{km}, \quad G = (9, 9) \text{km}. \quad (6.2)$$

The domain contains three rectangular obstacle regions representing inaccessible road segments. The vehicle is required to remain within the admissible road domain while avoiding these obstacles.

The computational grid spacing is chosen as

$$\Delta x = \Delta y = 0.2 \text{km}. \quad (6.3)$$

A directed grid graph is constructed from the computational domain. Each admissible edge represents a possible local direction of vehicle motion.

To incorporate road-gradient effects, the elevation field is represented by the smooth function

$$h(x, y) = 0.8 \sin(0.45x) + 0.6 \cos(0.38y) + 0.35 \sin(0.30x + 0.25y). \quad (6.4)$$

Where, x and y are expressed in kilometres, while $h(x, y)$ represents road elevation in metres. The corresponding spatial derivatives are converted consistently when evaluating the physical slope and energy quantities. The corresponding gradient is

$$\frac{\partial h}{\partial x} = 0.36 \cos(0.45x) + 0.105 \cos(0.30x + 0.25y), \quad (6.5)$$

$$\frac{\partial h}{\partial y} = -0.228 \sin(0.38y) + 0.0875 \cos(0.30x + 0.25y). \quad (6.6)$$

For a unit travel direction $\hat{y} = (\hat{y}_1, \hat{y}_2)$, the directional slope is calculated as

$$D_{\hat{y}}h = \nabla h(x, y) \cdot \hat{y}. \quad (6.7)$$

The positive component of this quantity contributes to the uphill energy demand.

A spatially varying desired-speed field is introduced to represent local changes in driving conditions. It is defined by

$$v_d(x, y) = 12 - 3 \exp \left[-\frac{(x - 5.2)^2 + (y - 5.3)^2}{3} \right] - 1.5 \exp \left[-\frac{(x - 7)^2 + (y - 3.5)^2}{1.5} \right]. \quad (6.8)$$

The desired-speed field represents the locally preferred vehicle speed associated with the road environment. Its spatial variation provides a first-order measure of the tendency of the vehicle to increase or decrease its speed along a given direction. For a unit travel direction

$$\hat{y} = (\hat{y}^1, \hat{y}^2), \quad (6.9)$$

the directional rate of change of the desired speed is given by

$$D_{\hat{y}}v_d(x, y) = \nabla v_d(x, y) \cdot \hat{y}. \quad (6.10)$$

Following the directional acceleration definition introduced in Eq. (4.12), the positive acceleration demand is defined as

$$a_d(x, \hat{y}) = [\nabla v_d(x, y) \cdot \hat{y}]^+ \quad (6.11)$$

This definition is consistent with the Finsler construction in Section 4 and preserves the dependence of the acceleration-related cost on the instantaneous direction of motion. The corresponding normalized acceleration contribution used in the Finsler function is therefore

$$A(x, \hat{y}) = \frac{ma_d(x, \hat{y})}{e_{\text{ref}}}. \quad (6.12)$$

Thus, directions along which the desired-speed field increases are assigned a larger acceleration-related cost, whereas directions associated with non-positive directional changes in the desired speed do not contribute to the positive acceleration penalty. The actual acceleration energy of each optimized trajectory is subsequently evaluated independently from the physical vehicle dynamics using Eq. (4.15). A representative electric passenger vehicle is considered with mass

$$m = 1800\text{kg}. \quad (6.13)$$

The principal vehicle parameters are selected as

$$C_r = 0.012, \quad C_D = 0.29, \quad A_f = 2.2 \text{ m}^2, \quad (6.14)$$

with air density $\rho = 1.225 \text{ kgm}^{-3}$ and gravitational acceleration $g = 9.81 \text{ ms}^{-2}$. The drivetrain efficiency is taken as $\eta_d = 0.90$, while the regenerative-braking efficiency is represented by $\eta_r = 0.70$. A reference energy-per-distance scale is selected as $e_{\text{ref}} = 180 \text{ Jm}^{-1}$ and the initial battery state of charge is taken as $SOC_0 = 1$, corresponding to a fully charged battery. The minimum admissible state of charge is imposed as a feasibility constraint during route evaluation. For the numerical experiment, the weighting parameters of the proposed Finsler metric are

$$\lambda_B = 0.3, \quad \lambda_A = 2.0, \quad \lambda_S = 1.0. \quad (6.15)$$

These parameters represent a balanced preference for battery efficiency, smooth acceleration, and avoidance of energetically expensive road gradients.

6.2. Routing Strategies and Optimization Procedure. Three routing strategies are considered.

6.2.1. Distance-Based Optimization. The first strategy minimizes only the geometric path length:

$$J_D[\gamma] = \int_{\gamma} ds. \quad (6.16)$$

This represents a conventional shortest-path strategy.

6.2.2. Energy-Based Optimization. The second strategy minimizes the physical battery-energy requirement:

$$J_E[\gamma] = E_{\text{net}}[\gamma]. \quad (6.17)$$

The energy model developed in Section 3 is used to evaluate the energetic demand of each admissible route.

6.2.3. *Finsler-Based Optimization.* The third strategy minimizes the proposed directional Finsler functional:

$$J_F[\gamma] = \int_{\gamma} F_{EV}(x, \dot{x}) dt. \quad (6.18)$$

The three optimization problems are solved over the same directed grid and under identical obstacle constraints. A shortest-path search is performed on the resulting weighted directed graph.

6.3. **Numerical Results and Comparison.** The optimized trajectories are evaluated using the physical quantities introduced in Section 3. The principal results are summarized in Table 1.

TABLE 1. Comparison of the three routing strategies.

Route	Distance (km)	E_{net} (MJ)	E_{acc} (MJ)	E_{slope} (MJ)	J_F	SOC _f (%)
Distance-based	11.31	17.76	5.44	6.49	137.41	91.78
Energy-based	14.01	6.80	0.20	0.91	32.57	96.85
Finsler-based	14.13	6.82	0.16	0.93	32.38	96.84

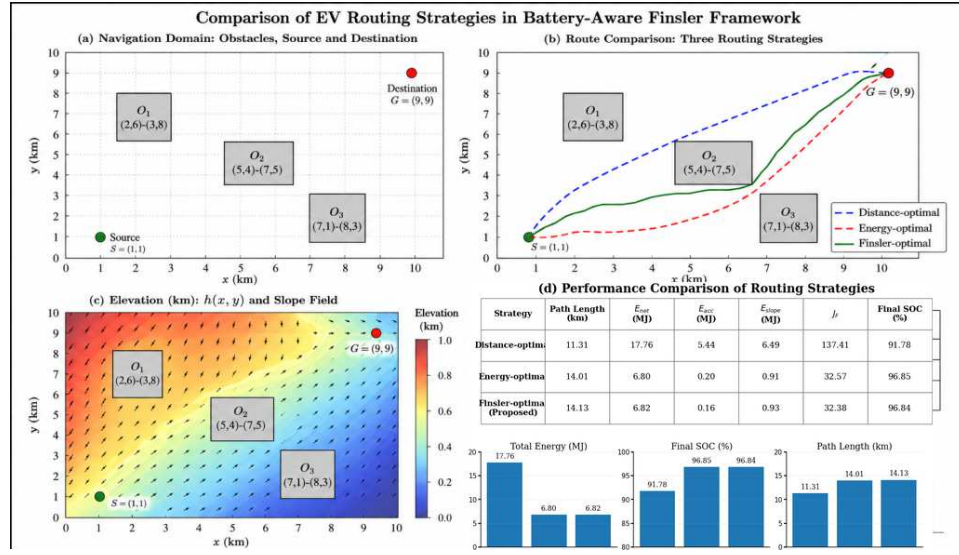


FIGURE 1. Numerical comparison of routing strategies for the electric vehicle navigation problem. (a) Computational domain $D = [0, 10] \times [0, 10]$ km with three rectangular obstacles, source $S = (1, 1)$, and destination $G = (9, 9)$. (b) Comparison of the distance-optimal, energy-optimal, and proposed Finsler-optimal routes. (c) Elevation field $h(x, y)$ and corresponding road-slope field. (d) Comparison of path length, total energy consumption, and final state of charge (SOC).

The distance-based strategy produces the shortest route, with a total distance of approximately 11.31 km. However, this route requires a substantially larger physical battery-energy consumption of approximately 17.76 MJ. The energy-based strategy increases the travel distance to approximately 14.01 km while reducing the physical battery-energy requirement to approximately 6.80 MJ.

The Finsler-based strategy produces a route of approximately 14.13 km with a physical battery-energy consumption of approximately 6.82 MJ. Its acceleration-related diagnostic is lower than that of the energy-based route, and its weighted Finsler objective is slightly smaller. However, the Finsler objective J_F is a dimensionally normalized geometric route cost and must not be interpreted as a direct physical battery-energy measurement. The physical battery demand is represented separately by E_{net} .

Consequently, the numerical experiment demonstrates that the proposed Finsler-type route-selection criterion can identify a trajectory that is energetically competitive with the physical-energy-optimal route while also accounting for directional acceleration tendency and road-gradient effects. The result should be interpreted as a weighted multi-criteria compromise, whose behaviour depends on the selected parameters $\lambda_B, \lambda_A, \lambda_S$.

6.4. Discussion and Physical Interpretation. The numerical experiment highlights the principal advantage of the proposed direction-dependent formulation. The shortest geometric path is not necessarily the most energy-efficient path because it may contain energetically expensive acceleration and elevation changes.

The energy-optimal route successfully reduces battery consumption but does not explicitly balance all directional effects included in the Finsler metric. In contrast, the proposed Finsler route considers battery demand, acceleration tendency, and road-gradient effects simultaneously.

The difference between the energy-optimal and Finsler-optimal routes is small in terms of battery consumption, while the Finsler route achieves a lower combined directional cost. This indicates that the proposed framework can provide a useful compromise between battery conservation and smooth, terrain-aware navigation.

The numerical results also demonstrate that route length alone is an insufficient criterion for energy-aware EV navigation. A modest increase in distance can result in a substantial reduction in battery consumption when the additional distance avoids energetically expensive road segments.

The present numerical example is intended as a controlled demonstration of the proposed mathematical framework. In practical applications, the elevation field, speed field, vehicle parameters, traffic conditions, and charging infrastructure can be replaced by real-world data.

The numerical comparison demonstrates that minimizing geometric distance alone is not sufficient for energy-aware electric-vehicle navigation. The distance-based strategy produces the shortest trajectory, but it requires substantially greater battery energy because the selected route is associated with higher acceleration and road-gradient costs.

The energy-based strategy substantially reduces the physical battery-energy requirement by accepting a longer travel distance. The proposed Finsler-based strategy provides a similar level of battery efficiency while simultaneously incorporating directional acceleration and road-gradient information into the route-selection criterion.

The close agreement between the physical energy consumption of the energy-based and Finsler-based routes indicates that the proposed Finsler formulation can identify energetically competitive trajectories without reducing the optimization problem to battery energy alone. In particular, the Finsler formulation provides a unified geometric representation of several direction-dependent factors that influence EV navigation.

These results support the central premise of the proposed framework: an energy-aware EV route should be evaluated not only by its geometric length, but also by the direction-dependent energetic characteristics of the road environment.

7. APPLICATIONS TO ELECTRIC CARS, BUSES, AND AUTONOMOUS VEHICLES

The proposed battery-aware Finsler framework provides a general mathematical model for energy-efficient navigation in electric transportation systems. Since the local cost depends on position, direction, road gradient, vehicle dynamics, and battery-related characteristics, the framework can be adapted to different classes of electric vehicles. In this section, three important applications are considered: electric passenger cars, electric buses, and autonomous electric vehicles.

7.1. Electric Passenger Cars. Electric passenger cars are the most direct application of the proposed framework. For a passenger EV, the vehicle state is described by

$$(x, y, v, \text{SOC}),$$

where x denotes the position, y the direction of motion, v the vehicle speed, and SOC the battery state of charge.

The battery-aware Finsler function is given by

$$F_{\text{car}}(x, y) = \alpha(x, y) [1 + \lambda_B \mathcal{B}_{\text{car}}(x, \hat{y}) + \lambda_A \mathcal{A}_{\text{car}}(x, \hat{y}) + \lambda_S \mathcal{S}_{\text{car}}(x, \hat{y})]. \quad (7.1)$$

The corresponding optimal route satisfies

$$\gamma_{\text{car}}^* = \arg \min_{\gamma \in \mathcal{A}_{\text{car}}} \int_{\gamma} F_{\text{car}}(x, \dot{x}) dt. \quad (7.2)$$

For passenger cars, the weighting parameters can be selected according to the driver's preference [9]. For example, a driver primarily concerned with extending driving range may select a relatively large value of λ_B , whereas a driver interested in smoother driving may increase λ_A .

The model is particularly useful when multiple routes connect the same origin and destination but have different elevation profiles. A slightly longer route may become preferable if it avoids a steep uphill segment and therefore requires substantially less battery energy.

7.2. Electric Buses. Electric buses present a more demanding application because their larger mass, frequent stops, passenger loading, and urban driving conditions can result in substantial energy consumption. Let $m_b = m_0 + m_p$ denote the effective bus mass, where m_0 is the unloaded bus mass and m_p represents the passenger-load contribution. The tractive force becomes

$$F_{\text{tr}}^{\text{bus}} = m_b a + C_r m_b g \cos \theta + \frac{1}{2} \rho C_D A_f v^2 + m_b g \sin \theta. \quad (7.3)$$

Consequently, the battery-aware Finsler function for an electric bus can be written as

$$F_{\text{bus}}(x, y) = \alpha(x, y) [1 + \lambda_B \mathcal{B}_{\text{bus}}(x, \hat{y}) + \lambda_A \mathcal{A}_{\text{bus}}(x, \hat{y}) + \lambda_S \mathcal{S}_{\text{bus}}(x, \hat{y})]. \quad (7.4)$$

Because the slope-related force is proportional to vehicle mass, we have

$$F_{\text{slope}}^{\text{bus}} = m_b g \sin \theta. \quad (7.5)$$

Thus, road-gradient effects can become particularly important for electric buses operating in hilly urban regions. In addition, buses frequently experience acceleration and deceleration because of traffic signals, bus stops, pedestrian crossings, and congestion. The acceleration-related contribution can therefore be represented by

$$E_A^{\text{bus}} = \int_{t_0}^{t_1} m_b a^+(t) v(t) dt. \quad (7.6)$$

The proposed framework can consequently be used to identify routes that balance battery consumption with operational requirements such as route length and scheduled travel time.

For a bus route with multiple mandatory stops

$$P_0, P_1, \dots, P_N,$$

the optimization can be performed segment by segment:

$$\mathcal{J}_{\text{bus}} = \sum_{k=0}^{N-1} \mathcal{J}_F[P_k, P_{k+1}]. \quad (7.7)$$

This provides a natural extension from point-to-point EV navigation to public-transport route optimization.

7.3. Autonomous Electric Vehicles. Autonomous electric vehicles provide an important application because route selection and vehicle control can be performed automatically. In this setting, the proposed Finsler framework can be incorporated into a hierarchical navigation system.

Let $x(t)$ represent the vehicle position and let $y(t) = \dot{x}(t)$ represent its velocity vector. The navigation layer determines an energy-efficient reference trajectory

$$\gamma^* = \arg \min_{\gamma \in \mathcal{A}_{\text{AV}}} \int F_{\text{EV}}(x, \dot{x}) dt. \quad (7.8)$$

The resulting trajectory is then provided to a lower-level motion-control system.

A general autonomous EV navigation architecture can be represented by the following sequence:

Road Information $\rightarrow$ Finsler Cost Construction $\rightarrow$ Optimal Route $\rightarrow$ Trajectory Generation $\rightarrow$ Vehicle Control.

The road-information layer can contain

$$\mathcal{I} = \{h(x), v_{\text{lim}}(x), \rho_{\text{traffic}}(x), \text{SOC}, \text{roadcondition}\}.$$

The Finsler cost can then be updated according to the current environment:

$$F_{\text{EV}} = F_{\text{EV}}(x, y; \mathcal{I}). \quad (7.9)$$

This allows the navigation system to adapt the optimal route when road conditions change.

7.4. Dynamic and Charging-Aware Navigation. In a practical autonomous EV, the battery state changes continuously during operation. Therefore, the Finsler metric can be made dependent on the current state of charge:

$$F_{\text{EV}} = F_{\text{EV}}(x, y, \text{SOC}). \quad (7.10)$$

For example, when the battery level becomes low, the battery weighting parameter can be increased according to

$$\lambda_B = \lambda_B(\text{SOC}). \quad (7.11)$$

A simple adaptive form is

$$\lambda_B(\text{SOC}) = \lambda_{B0} \left[1 + \kappa_B \left(1 - \frac{\text{SOC}}{\text{SOC}_0} \right) \right], \quad (7.12)$$

where, λ_{B0} is the nominal battery weight and κ_B controls the sensitivity to battery depletion. Thus, as the battery state decreases, the optimization can increasingly favor energy-efficient routes. The framework can also be extended to EV Charging infrastructure into the proposed optimization framework [8]. Let

$$\mathcal{C} = \{C_1, C_2, \dots, C_N\}$$

denote the available charging stations.

A route passing through a charging station C_j can be assigned a charging cost

$$J_C(C_j) = \lambda_C T_C(C_j), \quad (7.13)$$

where, $T_C(C_j)$ denotes the estimated charging time and λ_C is the corresponding weighting parameter.

The complete route objective can then be written as

$$\mathcal{J}_{\text{total}} = \mathcal{J}_F + \lambda_C \sum_{C_j \in \gamma} T_C(C_j). \quad (7.14)$$

The optimization problem becomes

$$\gamma^* = \arg \min_{\gamma} \left[\mathcal{J}_F[\gamma] + \lambda_C \sum_{C_j \in \gamma} T_C(C_j) \right], \quad (7.15)$$

subject to

$$\text{SOC}(t) \geq \text{SOC}_{\min}. \quad (7.16)$$

This extension allows the proposed framework to address not only energy consumption but also the practical problem of selecting routes and charging locations jointly [7]. Beyond the three primary applications considered above, the proposed framework can be adapted to several other electric-transportation scenarios: energy-efficient urban mobility, electric delivery vehicles, electric taxis and ride-sharing fleets, autonomous electric shuttles, electric freight transportation, fleet-level energy-aware route planning and emergency EV navigation under limited battery availability.

The common mathematical structure in these applications is the dependence of transportation cost on both the spatial environment and the direction of motion. This makes the Finsler formulation suitable for extending the proposed model to increasingly complex transportation networks.

8. CONCLUSION AND FUTURE PERSPECTIVES

In this study, a battery-aware Finsler-geometric framework has been developed for energy-efficient navigation and route optimization of electric vehicles. The principal motivation of the proposed approach is that the energetic cost of EV motion is not determined solely by the geometric length of a road. Battery consumption depends on several interacting factors, including vehicle acceleration, road gradient, aerodynamic resistance, rolling resistance, vehicle mass and powertrain efficiency.

To capture these effects within a unified mathematical framework, the road environment was represented as a differentiable manifold and the vehicle state was described by a position x and a tangent vector y . A battery-aware Finsler function was then introduced in the form

$$F_{\text{EV}}(x, y) = \alpha(x, y) [1 + \lambda_B \mathcal{B}(x, \hat{y}) + \lambda_A \mathcal{A}(x, \hat{y}) + \lambda_S \mathcal{S}(x, \hat{y})]. \quad (8.1)$$

where α represents the underlying geometric travel cost, while B , A , and S account for battery usage, acceleration-related effects, and road-slope effects, respectively.

An important feature of the proposed formulation is its direction dependence. Unlike a purely Riemannian shortest-path model, the proposed cost can satisfy

$$F_{\text{EV}}(x, y) \neq F_{\text{EV}}(x, -y).$$

This property provides a natural mathematical representation of the asymmetric energetic behavior of EVs. In particular, uphill and downhill motion along the same road segment can have different costs because of gravitational energy requirements and regenerative braking.

The physical energy model was constructed from the longitudinal vehicle dynamics by incorporating acceleration, rolling resistance, aerodynamic drag and road-gradient forces. The resulting mechanical power was connected to battery power through drivetrain efficiency, while regenerative braking was included through a separate recovery term. The battery state of charge was subsequently incorporated as a feasibility constraint.

The route-selection problem was formulated as a Finsler-geometric variational problem,

$$\gamma^* = \arg \min_{\gamma \in \mathcal{A}_{\text{EV}}} \int F_{\text{EV}}(\gamma(t), \dot{\gamma}(t)) dt,$$

subject to constraints on vehicle speed, acceleration, battery state of charge, road boundaries, and obstacles. The corresponding Euler–Lagrange equations and Finsler geodesic equations provide the mathematical description of the optimal trajectories.

The numerical experiment demonstrates that minimizing geometric distance alone may lead to substantially higher physical battery-energy consumption. The physical-energy-based route reduces battery demand by accepting a longer travel distance, while the proposed Finsler-based route provides a comparable battery performance together with a lower weighted direction-dependent cost.

The proposed formulation should be interpreted as a Finsler-type geometric route-selection framework. Its battery, acceleration and slope terms combine normalized directional information into a unified optimization criterion, whereas the physical quantities E_{net} , E_{acc} , E_{slope} , and SOC are evaluated independently for the resulting trajectories.

The acceleration term is a first-order surrogate based on the directional variation of a prescribed desired-speed field. It does not replace the actual vehicle acceleration model. Similarly, the slope term acts as a direction-dependent route-selection penalty, while the physical gravitational energy remains included in the longitudinal vehicle-energy model.

The present numerical example is a controlled demonstration based on a prescribed elevation field, desired-speed field, vehicle parameters and directed grid. Real-world deployment would require measured road data, traffic information, vehicle-control laws, regenerative-braking constraints and additional sensitivity studies for the weighting parameters. These extensions are left for future work.

Data Availability

No external datasets were used in this study. All results presented in this work are based on analytical calculations and numerical computations described in the manuscript. The corresponding computational details are available from the authors upon reasonable request.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Author Contributions S.K. conceptualized the study, developed the theoretical framework, performed the analytical calculations and numerical analysis and prepared the original manuscript. P.K.D. contributed to the formulation of the theoretical model, interpretation of the results and critical revision of the manuscript. C.K.M. contributed to the mathematical analysis, discussion of the results, and critical revision of the manuscript. All authors have read and approved the final version of the manuscript.

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