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A Nonmatching-Grid Discontinuous Galerkin Method for Nonlinear Reactive Transport in Hybrid-Dimensional Fractured Media

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Abstract

We study a structural question in hybrid-dimensional transport: when can independently generated matrix-trace and fracture meshes be coupled conservatively without a multiplier or a projection stability condition? For Robin transfer laws, the interface contribution is already a coercive physical $L^2(\gamma)$ product, so it can be integrated directly on the geometric common refinement. We analyze this primal discontinuous Galerkin (DG) coupling for nonlinear advection–diffusion–reaction transport. Explicit SIP thresholds yield coercivity, strong monotonicity and finite-dimensional well-posedness. Under fixed coefficient bounds and a stated mesh-ratio-uniform concentration-range assumption for the nonlinear error estimate, we prove a quasi-optimal energy bound whose constant contains no factor h/H or H/h . We also prove that, for constant transfer coefficients, the direct exchange dominates its L^2 -projected counterpart by the exact gap $\sum_s \kappa^s \|(I - \Pi_h^s)v_{\gamma,H}\|_{0,\gamma}^2$. Fixed-ratio refinements with $H/h = 4, 1, 1/4$ retain first-order energy convergence. In the benign regime $\kappa = 4$ direct and projected couplings are indistinguishable, whereas at $\kappa = 10^4$ the projected energy error is 21% larger on a 16×23 pair and stagnates under further fracture refinement with a fixed matrix trace. Neutral and interpolated Newton starts converge under the same relative-plus-absolute stopping rule. Source code, data and figures are supplied for complete regeneration.

Keywords: discontinuous Galerkin method; hybrid-dimensional model; fractured porous media; nonmatching grids; reactive transport; interior penalty; nonlinear reaction; matrix–fracture coupling.

Mathematics Subject Classification: 65N30, 65N12, 65N15, 76S05.

1 Introduction

Thin fractures strongly alter transport in porous media because they may act as high-conductivity channels, low-permeability barriers, or chemically active surfaces. Resolving the physical fracture aperture by a conforming equal-dimensional mesh may be prohibitively expensive, especially when many fractures are present. Hybrid-dimensional models instead replace a thin d -dimensional inclusion by a $(d-1)$ -dimensional manifold embedded in the matrix and couple the two geometric components through exchange conditions. This strategy is now standard in flow and transport modeling in fractured media; representative developments include reduced interface models, nonmatching mixed finite elements, polytopic DG methods, and discrete-fracture formulations [1–5].

DG methods are attractive in this setting because they are locally conservative, naturally accommodate coefficient jumps, and allow independent local approximation spaces. Classical foundations for interior-penalty and transport DG schemes are given in [6–10]; recent porous-media transport work also includes the adaptive space–time DG study of Corallo and Wieners [11] and hybrid-dimensional flow/transport approximation by Zhao and Rui [12]. More recently, DG formulations have been developed for hybrid-dimensional Darcy flow, two-phase flow, Darcy–Forchheimer flow, and advection–diffusion transport [13–16]. The 2026 *Electronic Research Archive* paper of Chen and Jin [15] is particularly relevant: it combines DG in space with backward Euler in time for a hybrid-dimensional advection–diffusion model, proves optimal discrete H^1 estimates, and treats single and intersecting fractures. In that construction, however, the matrix meshes are assumed to coincide on the fracture so that a unique fracture partition is inherited from the matrix trace. This assumption is convenient analytically but restrictive in applications in which matrix and fracture grids are produced by different meshing pipelines or refined according to different physical scales.

Nonmatching grids have a long history in reduced fracture models. Mixed finite-element couplings with nonmatching trace meshes were studied, for example, by Frih et al. [2], while Fumagalli and Scotti [17] developed reduced transport ideas with nonmatching grids. More broadly, Fumagalli, Keilegavlen and Scialò [18] compare conforming, non-conforming and non-matching coupling strategies in discrete-fracture simulations. The terminology *common refinement* is classical in conservative transfer between nonmatching interface meshes; Jiao and Heath [19] construct conservative common-refinement transfer operators for multiphysics surface data. Our use of the geometric intersections is related in spirit, but the object transferred here is not a projected trace field: the Robin law enters directly as the primal $L^2(\gamma)$ exchange product of the matrix and fracture unknowns.

For DG flow problems, polytopic formulations can accommodate general element geometry [3, 4], and adaptive/hybridized DG methods have been developed for fractured Darcy systems [13]. Unfitted HDG formulations can be geometrically more general still: Fu and Yang [20] treat Darcy flow on meshes unfitted to conductive and blocking fractures. The present construction does *not* compete with such methods in geometric generality; it isolates a different question—conservative nonlinear transport/reaction coupling when a geometrically represented lower-dimensional interface and its matrix trace are partitioned independently. In reactive transport, DG discretizations are well established on heterogeneous and fractured grids [10, 21], and mixed-dimensional reactive-flow models have been proposed in [5]. Nevertheless, the combination of (i) a genuinely hybrid-dimensional transport equation, (ii) independent matrix-trace and fracture meshes, (iii) a direct conservative DG exchange operator on the common refinement, and (iv) a monotone nonlinear reaction deserves a separate analysis.

A different established route to nonmatching interfaces is the mortar family introduced by Bernardi, Maday and Patera [22]. Dual multiplier bases can make the mortar constraint locally supported and computationally efficient [23], while interpolation-based alternatives such as INTERNODES couple nonconforming subdomains without a classical mortar multiplier [24]. The present construction does not claim that mortar methods are intrinsically expensive or inaccurate. Its narrower advantage is that the Robin exchange term is already a primal physical $L^2(\gamma)$ product, so the method can integrate that product directly on the common refinement without introducing an additional multiplier space or master/slave interpolation operator.

The main contribution is not simply the use of a nonmatching grid. It is a conservative primal common-refinement exchange operator together with a stability and energy-error analysis whose constants do not require a matching trace partition. More specifically:

- (i) We formulate a nonlinear hybrid-dimensional advection–diffusion–reaction problem in which the matrix and fracture equations are approximated by independently generated DG spaces.
- (ii) We introduce a mortar-free exchange operator evaluated directly on $I = F \cap E$. The operator is symmetric, nonnegative, and conservative because the identical physical transfer integral is assembled with opposite signs in the two balances.
- (iii) We give explicit sufficient SIP penalty thresholds and derive a mesh-ratio-uniform strong-monotonicity constant. A separate finite-dimensional residual argument establishes existence; uniqueness follows from strong monotonicity.
- (iv) We make the quasi-optimal energy estimate the principal analytical result. The common-refinement contribution is recombined into the physical $L^2(\gamma)$ interface norm before Cauchy–Schwarz is applied, so no factor h/H or H/h enters the stability constant.
- (v) For constant transfer coefficients we order the direct and L^2 -projected couplings by an exact positive-semidefinite gap, $C_{hH}^{\text{dir}}(v, v) - C_{hH}^{\text{proj}}(v, v) = \sum_s \kappa^s \|(I - \Pi_h^s)v_{\gamma, H}\|_{0, \gamma}^2$. This identifies both the equality regime and the information discarded by projection.
- (vi) We provide a reproducible two-dimensional Q_1/P_1 implementation and report convergence, fixed-ratio refinement, transfer-coefficient, nonlinear-reaction, conditioning, direct-versus-projected coupling, matching-versus-nonmatching cost, and non-standard-geometry coupling tests. The kinked/intersecting tests audit the coupling layer itself; a full PDE convergence study on general fracture networks is left for future work.

Figure 1 illustrates the central geometric difference from a matching-grid treatment. Matrix and fracture meshes are generated independently; exchange integrals are computed only after constructing the one-dimensional common refinement along the fracture.

The rest of the paper is organized as follows. Section 2 states the hybrid-dimensional model and assumptions. Section 3 defines the independent meshes and common-refinement coupling. Section 4 gives the complete DG formulation. Sections 5 and 6 establish well-posedness and a priori estimates. Section 7 records conservation and nonlinear-solver properties. Section 8 presents the numerical verification. Implementation details and reproducibility information are collected in the appendices.

2 Hybrid-dimensional nonlinear transport model

For clarity we write the model for a single codimension-one fracture that separates two matrix subdomains; the coupling construction extends face-by-face to a finite fracture network. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be bounded and polygonal/polyhedral. Let γ be a Lipschitz $(d - 1)$ -manifold such that

$$\overline{\Omega} = \overline{\Omega^-} \cup \overline{\Omega^+}, \quad \gamma = \partial\Omega^- \cap \partial\Omega^+.$$

The unit normals n^- and n^+ point outward from Ω^- and Ω^+ , respectively, so that $n^+ = -n^-$ on γ .

The matrix concentration c^s ($s \in \{-, +\}$) satisfies

$$-\nabla \cdot (D^s \nabla c^s) + \nabla \cdot (\beta^s c^s) + r^s(c^s) = f^s \quad \text{in } \Omega^s, \quad (1)$$

$$(-D^s \nabla c^s + \beta^s c^s) \cdot n^s = \kappa^s(c^s - c_\gamma) \quad \text{on } \gamma. \quad (2)$$

We assume for the analysis that the normal advective component vanishes on γ , $\beta^s \cdot n^s = 0$, so that the exchange law represents the entire normal transfer. This includes the common situation in

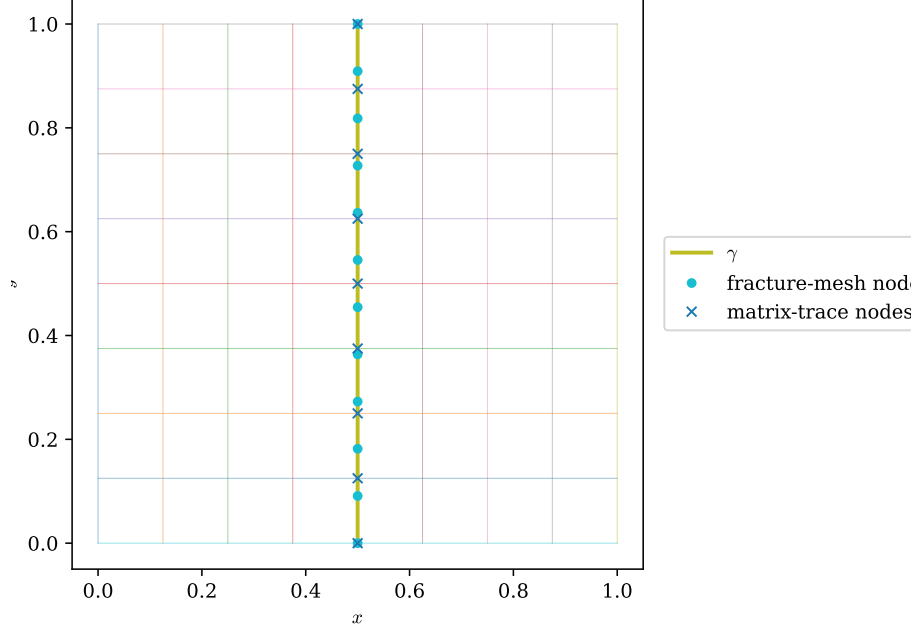


Figure 1: Independent matrix and fracture discretizations. The matrix trace partition at $x = 1/2$ and the fracture partition do not coincide. Coupling is integrated on all nonempty intersections of trace faces and fracture elements.

which matrix advection is predominantly tangential to a reduced fracture. If $\beta^s \cdot n^s \neq 0$, the normal advective flux must be included in the interface operator (and in the inflow/outflow bookkeeping), so the Robin term alone no longer represents the complete transfer. The present coercivity argument therefore does not cover that case without modifying the interface form; we list this as a limitation rather than silently absorbing it into the same theorem. The fracture concentration c_γ solves

$$-\nabla_\tau \cdot (D_\gamma \nabla_\tau c_\gamma) + \nabla_\tau \cdot (\beta_\gamma c_\gamma) + r_\gamma(c_\gamma) + \sum_{s \in \{-, +\}} \kappa^s (c_\gamma - c^s|_\gamma) = f_\gamma \quad \text{on } \gamma. \quad (3)$$

Dirichlet data g^s are prescribed on $\Gamma_D^s \subset \partial\Omega^s \setminus \gamma$ and g_γ on $\partial\gamma$; homogeneous notation is used in the analysis after a standard lifting. Mixed inflow/outflow data can be incorporated in the upwind form without changing the interior coupling argument.

Assumption 2.1 (Coefficients). There are constants $0 < D_0 \leq D_1$, $0 < D_{\gamma,0} \leq D_{\gamma,1}$, and $0 < \kappa_0 \leq \kappa_1$ such that

$$D_0 |\xi|^2 \leq \xi^T D^s(x) \xi \leq D_1 |\xi|^2, \quad D_{\gamma,0} |\zeta|^2 \leq \zeta^T D_\gamma(x) \zeta \leq D_{\gamma,1} |\zeta|^2,$$

and $\kappa_0 \leq \kappa^s \leq \kappa_1$. The advection fields are piecewise $W^{1,\infty}$, divergence-free within each matrix subdomain and tangentially divergence-free on γ for the basic coercivity statement.

Assumption 2.2 (Reaction). For $q \in \{-, +, \gamma\}$, $r_q : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and globally strongly monotone: there exists $m_r \geq 0$ such that

$$(r_q(a) - r_q(b))(a - b) \geq m_r |a - b|^2 \quad \forall a, b \in \mathbb{R}.$$

Moreover r'_q is locally bounded. This global formulation is used only for the finite-dimensional existence theorem below; the local error estimate requires boundedness of r'_q only on an interval containing the exact and discrete solutions. A model example satisfying the assumption is

$$r_q(c) = \lambda_q c + \mu_q c^3, \quad \lambda_q > 0, \quad \mu_q \geq 0.$$

Assumption 2.3 (Uniform concentration range for the nonlinear error estimate). There exists $M > 0$, independent of h , H , and the ratio h/H , such that the exact and discrete concentrations satisfy

$$\|c_q\|_{L^\infty(q)} \leq M, \quad \|c_{hH,q}\|_{L^\infty(q)} \leq M, \quad q \in \{\Omega^-, \Omega^+, \gamma\}.$$

This assumption is invoked only in the a priori error theorem, not in strong monotonicity or finite-dimensional existence. It makes the local Lipschitz constant $L_r = \max_q \sup_{|z| \leq M} |r'_q(z)|$ finite and, crucially, independent of the relative mesh ratio. For $r_q(z) = \lambda_q z + \mu_q z^3$, one may take $L_r \leq \max_q (\lambda_q + 3\mu_q M^2)$. Establishing such an L^∞ bound uniformly from the DG energy estimate is outside the scope of the present paper.

Define

$$V = H^1(\Omega^-) \times H^1(\Omega^+) \times H^1(\gamma)$$

with homogeneous traces on the external Dirichlet boundaries. For $u = (u^-, u^+, u_\gamma)$ and $v = (v^-, v^+, v_\gamma)$, the continuous exchange form is

$$C(u, v) = \sum_{s \in \{-, +\}} \int_\gamma \kappa^s (u^s - u_\gamma)(v^s - v_\gamma) \, ds. \quad (4)$$

The sign convention in (2) is chosen precisely so that $C(v, v) \geq 0$.

Proposition 2.4 (Continuous energy identity). *If the data are homogeneous on the external boundary and Assumptions 2.1–2.2 hold, then any sufficiently regular solution satisfies*

$$\begin{aligned} \sum_s (D^s \nabla c^s, \nabla c^s)_{\Omega^s} + (D_\gamma \nabla_\tau c_\gamma, \nabla_\tau c_\gamma)_\gamma + C(c, c) + \sum_q (r_q(c_q), c_q)_q \\ = \sum_s (f^s, c^s)_{\Omega^s} + (f_\gamma, c_\gamma)_\gamma - \mathcal{B}_{\text{adv}}(c), \end{aligned} \quad (5)$$

where $\mathcal{B}_{\text{adv}}$ reduces to the standard inflow/outflow boundary contribution when the velocity fields are divergence-free.

Proof. Multiply (1) and (3) by the corresponding concentrations and integrate by parts. The matrix boundary term on γ is $\int_\gamma \kappa^s (c^s - c_\gamma) c^s$, whereas the fracture exchange contribution is $\int_\gamma \kappa^s (c_\gamma - c^s) c_\gamma$. Their sum is $\int_\gamma \kappa^s |c^s - c_\gamma|^2$. The advection contribution follows from the identity $(\nabla \cdot (\beta c), c) = \frac{1}{2} \int_{\partial\Omega} \beta \cdot n c^2$ for divergence-free β . $\square$

3 Independent meshes and the common-refinement coupling

Let $\mathcal{T}_h^s$ be a shape-regular mesh of Ω^s with local diameter h_K . The face-by-face analysis permits separate matrix-side trace partitions. The supplied implementation and all full-PDE experiments below use the same matrix trace partition on the two sides of γ ; the independently generated *fracture* partition is the nonmatching component tested numerically. We therefore make no numerical claim

here about mutually nonconforming Ω^-/Ω^+ traces. Independently, let $\mathcal{G}_H$ be a shape-regular mesh of the fracture with diameter H_E . We do *not* assume that the trace partition

$$\mathcal{F}_h^s = \{F = \partial K \cap \gamma : K \in \mathcal{T}_h^s, |F| > 0\}$$

coincides with $\mathcal{G}_H$.

For a trace face $F \in \mathcal{F}_h^s$ and fracture element $E \in \mathcal{G}_H$, define an intersection cell

$$I = F \cap E,$$

and retain it whenever $|I| > 0$. The collection of all such nonempty intersections is the common refinement

$$\mathcal{I}_{hH}^s = \{F \cap E : F \in \mathcal{F}_h^s, E \in \mathcal{G}_H, |F \cap E| > 0\}.$$

In two dimensions, each I is an interval. In three dimensions it is a polygonal surface cell. No additional unknown is introduced on I .

For polynomial degrees $k \geq 1$ and $\ell \geq 1$, set

$$\begin{aligned} V_h^s &= \{v \in L^2(\Omega^s) : v|_K \in \mathbb{P}_k(K), K \in \mathcal{T}_h^s\}, \\ W_H &= \{w \in L^2(\gamma) : w|_E \in \mathbb{P}_\ell(E), E \in \mathcal{G}_H\}, \\ X_{hH} &= V_h^- \times V_h^+ \times W_H. \end{aligned}$$

The discrete nonmatching exchange form is

$$C_{hH}(u_h, v_h) = \sum_{s \in \{-, +\}} \sum_{I \in \mathcal{I}_{hH}^s} \int_I \kappa^s (u_h^s - u_{\gamma, H})(v_h^s - v_{\gamma, H}) \, ds. \quad (6)$$

Because the intersection cells exactly partition each participating trace face and fracture element,

$$C_{hH}(v_h, v_h) = \sum_s \|(\kappa^s)^{1/2} (v_h^s - v_{\gamma, H})\|_{0, \gamma}^2 \geq 0, \quad (7)$$

independently of the ratio h/H .

Remark 3.1 (Why no mortar variable is required). A mortar space is useful when one wishes to enforce a constraint by a Lagrange multiplier. Here the physical interface law is already a Robin-type transfer law. Its variational contribution is an $L^2(\gamma)$ product, so direct integration on $F \cap E$ is sufficient. The discrete exchange is therefore conservative without projecting one primal unknown onto the grid of the other.

Lemma 3.2 (Exact cancellation of internal transfer). *Let $u_h \in X_{hH}$. On every $I \in \mathcal{I}_{hH}^s$, the matrix equation receives the transfer*

$$+ \int_I \kappa^s (u_h^s - u_{\gamma, H}) v_h^s,$$

while the fracture equation receives

$$- \int_I \kappa^s (u_h^s - u_{\gamma, H}) v_{\gamma, H}.$$

Consequently, choose the matrix test equal to one on the parent cell K of F and the fracture test equal to one on the parent element E ; their traces are then both one on I . Summing the two tested balance equations yields exact cancellation of the internal transfer on I .

Proof. This is immediate from expanding the integrand in (6). No matching-node assumption is used. $\square$

Lemma 3.3 (Ordering of direct and projected Robin couplings). *Assume for this comparison that κ^s is constant on each side of γ . Let T_h^s denote the trace space of V_h^s on γ , and let $\Pi_h^s : L^2(\gamma) \rightarrow T_h^s$ be the L^2 -orthogonal projection. Define the projected coupling by replacing $v_{\gamma,H}$ with $\Pi_h^s v_{\gamma,H}$ in the fracture contribution while retaining the same matrix and cross terms. Then, for every $v_h \in X_{hH}$,*

$$C_{hH}^{\text{dir}}(v_h, v_h) - C_{hH}^{\text{proj}}(v_h, v_h) = \sum_s \kappa^s \|(I - \Pi_h^s)v_{\gamma,H}\|_{0,\gamma}^2 \geq 0. \quad (8)$$

Equality holds for a given v_h precisely when $v_{\gamma,H} \in T_h^s$ for both sides; equality for every $v_h \in X_{hH}$ is equivalent to $W_H \subseteq T_h^- \cap T_h^+$.

Proof. Because $v_h^s|_\gamma \in T_h^s$, orthogonality gives $(v_h^s, \Pi_h^s v_{\gamma,H})_\gamma = (v_h^s, v_{\gamma,H})_\gamma$. Hence the matrix and cross terms of the two couplings coincide. The only difference is $\kappa^s(\|v_{\gamma,H}\|_0^2 - \|\Pi_h^s v_{\gamma,H}\|_0^2)$, which equals $\kappa^s\|(I - \Pi_h^s)v_{\gamma,H}\|_0^2$ by the Pythagorean identity for the orthogonal projection. $\square$

Remark 3.4 (What is and is not new). Integrating a known product over geometric intersections is elementary and common-refinement technology is established. The point of (6) is instead the consequence specific to a Robin transfer law: the physical interface term is already coercive, so no multiplier inf-sup condition and no projection-stability estimate enters the mesh-ratio-uniform energy argument. Lemma 3.3 also quantifies the exact price paid by an L^2 -projected alternative when fracture information lies outside the matrix trace space.

4 DG formulation

4.1 Diffusive interior-penalty terms

For an interior face $e = \partial K^- \cap \partial K^+$ in a matrix mesh, fix a normal n_e from K^- to K^+ . For scalar v define

$$\llbracket v \rrbracket = v^- - v^+, \quad \{\{D\nabla v \cdot n_e\}\} = \frac{1}{2}(D^- \nabla v^- \cdot n_e + D^+ \nabla v^+ \cdot n_e).$$

The symmetric interior-penalty form in Ω^s is

$$\begin{aligned} a_{D,h}^s(u, v) &= \sum_{K \in \mathcal{T}_h^s} \int_K D^s \nabla u \cdot \nabla v - \sum_{e \in \mathcal{E}_{h,0}^s} \int_e \{\{D^s \nabla u \cdot n_e\}\} \llbracket v \rrbracket \\ &\quad - \sum_{e \in \mathcal{E}_{h,0}^s} \int_e \{\{D^s \nabla v \cdot n_e\}\} \llbracket u \rrbracket + \sum_{e \in \mathcal{E}_{h,0}^s} \int_e \frac{\sigma_e}{h_e} \llbracket u \rrbracket \llbracket v \rrbracket. \end{aligned} \quad (9)$$

External Dirichlet boundaries are treated by the symmetric Nitsche analogue. Faces contained in γ are excluded from the SIP set because the normal matrix-fracture interaction is represented by C_{hH} , not by an artificial matrix-to-matrix numerical flux.

On the fracture mesh, the corresponding $(d-1)$ -dimensional SIP form is

$$\begin{aligned} a_{D,H}^\gamma(u, v) &= \sum_{E \in \mathcal{G}_H} \int_E D_\gamma \nabla_\tau u \cdot \nabla_\tau v - \sum_{e \in \mathcal{E}_{H,0}^\gamma} \int_e \{\{D_\gamma \nabla_\tau u \cdot n_e\}\} \llbracket v \rrbracket \\ &\quad - \sum_{e \in \mathcal{E}_{H,0}^\gamma} \int_e \{\{D_\gamma \nabla_\tau v \cdot n_e\}\} \llbracket u \rrbracket + \sum_{e \in \mathcal{E}_{H,0}^\gamma} \int_e \frac{\sigma_{\gamma,e}}{H_e} \llbracket u \rrbracket \llbracket v \rrbracket. \end{aligned} \quad (10)$$

In $d = 2$, the fracture is one-dimensional and the interior-face terms are evaluated at fracture nodes in the usual one-dimensional DG sense.

4.2 Upwind transport terms

For a divergence-free matrix velocity β^s , define the conservative upwind form

$$a_{A,h}^s(u, v) = - \sum_{K \in \mathcal{T}_h^s} \int_K \beta^s u \cdot \nabla v + \sum_{e \in \mathcal{E}_{h,0}^s} \int_e (\beta^s \cdot n_e) u^{\text{up}} \llbracket v \rrbracket + \int_{\Gamma_{\text{out}}^s} (\beta^s \cdot n) uv, \quad (11)$$

with the known inflow state moved to the right-hand side. The fracture advection form $a_{A,H}^\gamma$ is defined analogously using β_γ and the tangential normal to fracture-element interfaces.

The standard identity

$$a_{A,h}^s(v, v) = \frac{1}{2} \sum_{e \in \mathcal{E}_{h,0}^s} \int_e |\beta^s \cdot n_e| \llbracket v \rrbracket^2 + \frac{1}{2} \int_{\partial\Omega^s} |\beta^s \cdot n| v^2 \quad (12)$$

holds for homogeneous inflow data; the corresponding identity holds on γ .

4.3 Nonlinear discrete problem

Let

$$\begin{aligned} A_{hH}(u, v) &= \sum_s (a_{D,h}^s(u^s, v^s) + a_{A,h}^s(u^s, v^s)) + a_{D,H}^\gamma(u_\gamma, v_\gamma) + a_{A,H}^\gamma(u_\gamma, v_\gamma) + C_{hH}(u, v), \\ R(u; v) &= \sum_s (r^s(u^s), v^s)_{\Omega^s} + (r_\gamma(u_\gamma), v_\gamma)_\gamma. \end{aligned}$$

The discrete problem is: find $c_{hH} \in X_{hH}$ such that

$$A_{hH}(c_{hH}, v_{hH}) + R(c_{hH}; v_{hH}) = F(v_{hH}) \quad \forall v_{hH} \in X_{hH}, \quad (13)$$

where F contains the volume sources and the Nitsche/upwind boundary data.

Definition 4.1 (Energy norm). For $v = (v^-, v^+, v_\gamma) \in X_{hH}$, define

$$\begin{aligned} \|v\|_{E,hH}^2 &= \sum_s \left(\sum_{K \in \mathcal{T}_h^s} \|(D^s)^{1/2} \nabla v^s\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h^s} \frac{\sigma_e}{h_e} \|\llbracket v^s \rrbracket\|_{0,e}^2 \right) \\ &\quad + \sum_{E \in \mathcal{G}_H} \|D_\gamma^{1/2} \nabla_\tau v_\gamma\|_{0,E}^2 + \sum_{e \in \mathcal{E}_H^\gamma} \frac{\sigma_{\gamma,e}}{H_e} \|\llbracket v_\gamma \rrbracket\|_{0,e}^2 \\ &\quad + \sum_s \|(\kappa^s)^{1/2} (v^s - v_\gamma)\|_{0,\gamma}^2 + \mathcal{U}_h(v)^2, \end{aligned} \quad (14)$$

where $\mathcal{U}_h$ is the upwind jump/outflow seminorm induced by (12).

5 Stability and well-posedness

Lemma 5.1 (SIP coercivity with an explicit sufficient threshold). *Let N_∂ and $N_{\partial,\gamma}$ bound the number of SIP/Nitsche faces incident on one matrix and fracture element, respectively. Assume the patchwise discrete trace bounds*

$$\|\{D^s \nabla v \cdot n\}\|_{0,e}^2 \leq C_{\text{tr}}^2 \frac{D_1^2}{D_0} h_e^{-1} \sum_{K \subset \omega_e} \|(D^s)^{1/2} \nabla v\|_{0,K}^2, \quad (15)$$

$$\|\{D_\gamma \nabla_\tau w \cdot n\}\|_{0,e}^2 \leq C_{\text{tr},\gamma}^2 \frac{D_{\gamma,1}^2}{D_{\gamma,0}} H_e^{-1} \sum_{E \subset \omega_e} \|D_\gamma^{1/2} \nabla_\tau w\|_{0,E}^2. \quad (16)$$

It is sufficient to choose

$$\sigma_e \geq \sigma_*^m := 4N_\partial C_{\text{tr}}^2 \frac{D_1^2}{D_0}, \quad (17)$$

$$\sigma_{\gamma,e} \geq \sigma_*^\gamma := 4N_{\partial,\gamma} C_{\text{tr},\gamma}^2 \frac{D_{\gamma,1}^2}{D_{\gamma,0}}. \quad (18)$$

Then

$$\sum_s a_{D,h}^s(v^s, v^s) + a_{D,H}^\gamma(v_\gamma, v_\gamma) \geq \frac{1}{2} \|v\|_{D,hH}^2,$$

where $\|\cdot\|_{D,hH}$ is the diffusive and penalty part of (14). The threshold is sufficient rather than optimal and is independent of h/H .

For the uniform rectangular Q_1 / interval P_1 implementation used in Section 8, the normal derivative is constant in the normal coordinate, so the elementwise trace scaling used above is satisfied with the safe normalization $C_{\text{tr}} = C_{\text{tr},\gamma} = 1$. Taking $N_\partial = 4$, $N_{\partial,\gamma} = 2$, $D_m = 1$ and $D_\gamma = 0.25$ gives $\sigma_*^m = 16$ and $\sigma_*^\gamma = 2$. The code uses the multiplier $\sigma = 18$ in $\sigma D/h$; hence the effective coefficients in (17)–(18) are $\sigma_e = 18$ and $\sigma_{\gamma,e} = 18D_\gamma = 4.5$, respectively, both above these sufficient bounds. This instantiation is specific to the uniform Q_1/P_1 test meshes and is not asserted for arbitrary element families.

Proof. For one matrix face, Cauchy–Schwarz followed by Young’s inequality gives, for every $\varepsilon > 0$,

$$2 \left| \int_e \{D^s \nabla v \cdot n\} [v] \right| \leq \varepsilon \sum_{K \subset \omega_e} \|(D^s)^{1/2} \nabla v\|_{0,K}^2 + \frac{C_{\text{tr}}^2 D_1^2}{\varepsilon D_0} h_e^{-1} \|[v]\|_{0,e}^2,$$

where (15) was used in the second term. Choose $\varepsilon = (2N_\partial)^{-1}$. After summing faces, each element-gradient contribution is used at most N_∂ times, so at most one half of the volume energy is consumed. To retain at least one half of every penalty term it is enough that

$$\frac{C_{\text{tr}}^2 D_1^2}{\varepsilon D_0} \leq \frac{\sigma_e}{2},$$

which is exactly (17). The same argument in one lower dimension, with $N_{\partial,\gamma}$ and (16), yields (18). Summing matrix and fracture estimates proves the claim. The factor $4N_\partial$ therefore comes directly from the Young split and the requirement that one half of both the volume and penalty energies remain. $\square$

Lemma 5.2 (Coupled coercivity). *Under Assumption 2.1,*

$$C_{hH}(v, v) = \sum_s \|(\kappa^s)^{1/2}(v^s - v_\gamma)\|_{0,\gamma}^2.$$

Hence the coupling constant does not deteriorate when the matrix and fracture meshes have different sizes.

Proof. Sum the intersection contributions in (6). The nonempty intersections form a partition up to sets of measure zero, so the sum recovers the full interface integral. No inverse estimate involving h/H occurs. $\square$

Theorem 5.3 (Strong monotonicity). *Let Assumptions 2.1–2.2 hold and let the penalties satisfy Lemma 5.1. Define the nonlinear discrete operator $\mathcal{A}_{hH} : X_{hH} \rightarrow X'_{hH}$ by*

$$\langle \mathcal{A}_{hH}(u), v \rangle = A_{hH}(u, v) + R(u; v).$$

With the normalization (14), one may take directly

$$\alpha = 1/2,$$

because Lemma 5.1 supplies the coefficient 1/2 on the diffusive/penalty part, whereas the exchange and upwind-dissipation contributions enter with coefficient 1. Consequently,

$$\langle \mathcal{A}_{hH}(u) - \mathcal{A}_{hH}(w), u - w \rangle \geq \alpha \|u - w\|_{E,hH}^2 + m_r \|u - w\|_0^2. \quad (19)$$

The constant α is independent of the relative matrix/fracture mesh ratio.

Proof. Set $z = u - w$. Lemma 5.1 contributes at least $\frac{1}{2} \|z\|_{D,hH}^2$. The upwind contribution equals the nonnegative seminorm from (12); the exchange contribution is exactly the nonnegative interface term of Lemma 5.2; and strong monotonicity of the reaction gives

$$\sum_q (r_q(u_q) - r_q(w_q), z_q)_q \geq m_r \sum_q \|z_q\|_{0,q}^2.$$

Taking the smallest coefficient among the components of (14) gives $\alpha = 1/2$ and proves (19). $\square$

Lemma 5.4 (Continuity and coercivity of the discrete residual). *Define $\mathcal{F}_{hH} : X_{hH} \rightarrow X'_{hH}$ by*

$$\langle \mathcal{F}_{hH}(u), v \rangle = A_{hH}(u, v) + R(u; v) - F(v).$$

Then $\mathcal{F}_{hH}$ is continuous. Moreover,

$$\frac{\langle \mathcal{F}_{hH}(u), u \rangle}{\|u\|_{E,hH}} \longrightarrow +\infty \quad \text{as } \|u\|_{E,hH} \rightarrow \infty.$$

Consequently the discrete problem has at least one solution.

Proof. The linear DG forms are continuous on the finite-dimensional space X_{hH} . Since each $r_q \in C^1(\mathbb{R})$, the associated finite-dimensional Nemytskii map is continuous; for the cubic model it is polynomial in the coefficient vector. Thus $\mathcal{F}_{hH}$ is continuous. Using Theorem 5.3 with $w = 0$ and absorbing the fixed value $\mathcal{A}_{hH}(0)$ into a bounded linear functional gives

$$\langle \mathcal{F}_{hH}(u), u \rangle \geq \alpha \|u\|_{E,hH}^2 - C_0 \|u\|_{E,hH},$$

where C_0 depends on the data but not on h/H . Hence the residual points outward on a sufficiently large sphere in the finite-dimensional coefficient space. Brouwer's theorem (equivalently, the standard finite-dimensional coercive-map existence lemma) gives a zero of $\mathcal{F}_{hH}$ inside that sphere. $\square$

Theorem 5.5 (Existence and uniqueness). *Under the assumptions of Theorem 5.3, problem (13) has a unique solution $c_{hH} \in X_{hH}$.*

Proof. Existence follows from Lemma 5.4. If u and w are two solutions, subtract their equations and test with $u - w$. The left-hand side is bounded below by (19), while the right-hand side is zero. Therefore $u = w$. $\square$

Theorem 5.6 (A priori stability). *For homogeneous boundary data, the discrete solution satisfies*

$$\|c_{hH}\|_{E,hH}^2 + m_r \|c_{hH}\|_0^2 \leq C \|F\|_{X'_{hH}}^2,$$

with a constant independent of h/H .

Proof. Let $r_q^0 = r_q(0)$ and define the shifted reaction $\tilde{r}_q(z) = r_q(z) - r_q^0$. Move the fixed functional $\sum_q(r_q^0, v_q)_q$ to the right-hand side and denote the resulting load by $\tilde{F} \in X'_{hH}$. Testing the discrete equation with c_{hH} and applying Theorem 5.3 to c_{hH} and 0 gives

$$\alpha \|c_{hH}\|_{E,hH}^2 + m_r \|c_{hH}\|_0^2 \leq \langle \tilde{F}, c_{hH} \rangle.$$

Here the dual norm is the one induced by $\|v\|_X^2 = \|v\|_{E,hH}^2 + \|v\|_0^2$; the broken Poincaré–Friedrichs inequality with the Dirichlet penalties controls $\|v\|_0$ by the DG energy terms on the present boundary configuration. Therefore $|\langle \tilde{F}, c_{hH} \rangle| \leq C \|\tilde{F}\|_{X'_{hH}} \|c_{hH}\|_{E,hH}$, and Young’s inequality yields the stated estimate after absorbing the fixed reaction offset into the load norm. The constants depend on the domain, shape regularity and fixed penalty/coefficient bounds, but not on h/H . $\square$

6 Consistency and a priori error analysis

Let Π_h^s and Π_H^γ be local polynomial projections or suitable DG interpolants. Define

$$\Pi_{hH}c = (\Pi_h^- c^-, \Pi_h^+ c^+, \Pi_H^\gamma c_\gamma).$$

We assume the standard local estimates for $1 \leq m \leq k+1$,

$$\|v - \Pi_h v\|_{0,K} + h_K |v - \Pi_h v|_{1,K} \leq C h_K^m |v|_{m,K}, \quad (20)$$

$$\|w - \Pi_H w\|_{0,E} + H_E |w - \Pi_H w|_{1,E} \leq C H_E^m |w|_{m,E}. \quad (21)$$

The trace estimate

$$\|v - \Pi_h v\|_{0,F}^2 \leq C (h_K^{-1} \|v - \Pi_h v\|_{0,K}^2 + h_K |v - \Pi_h v|_{1,K}^2)$$

then yields the required interface control.

Lemma 6.1 (Consistency). *Let c be a sufficiently smooth solution of (1)–(3). Then*

$$A_{hH}(c, v_{hH}) + R(c; v_{hH}) = F(v_{hH}) \quad \forall v_{hH} \in X_{hH}.$$

Proof. Integrate the matrix and fracture strong equations by parts elementwise. On ordinary interior faces, the physical diffusive and advective fluxes are single-valued, and the DG consistency terms reconstruct them. On γ , insert the exchange law (2); the matrix and fracture interface terms combine exactly into (6). Since the common-refinement cells partition γ , nonmatching meshes introduce no consistency defect. $\square$

Definition 6.2 (Starred DG norm). For a piecewise sufficiently regular triple v , let

$$\begin{aligned} \|v\|_{E,hH,*}^2 := & \|v\|_{E,hH}^2 + \sum_s \sum_{e \in \mathcal{E}_h^s} h_e \|\{D^s \nabla v^s \cdot n_e\}\|_{0,e}^2 + \sum_{e \in \mathcal{E}_H^\gamma} H_e \|\{D_\gamma \nabla_\tau v_\gamma \cdot n_e\}\|_{0,e}^2 \\ & + \sum_s \sum_{e \in \mathcal{E}_h^s} \| |\beta^s \cdot n_e|^{1/2} v^s \|_{0,e}^2 + \sum_{e \in \mathcal{E}_H^\gamma} \| |\beta_\gamma \cdot n_e|^{1/2} v_\gamma \|_{0,e}^2. \end{aligned} \quad (22)$$

The last two sums are the standard trace terms needed to bound the upwind consistency contributions. They introduce no comparison between h and H .

Theorem 6.3 (Quasi-optimal energy error). *Suppose $c^s \in H^{k+1}(\Omega^s)$ and $c_\gamma \in H^{\ell+1}(\gamma)$. Under the assumptions of Theorem 5.3 and Assumption 2.3, there exists $C > 0$, independent of h/H but allowed to depend on the fixed coefficient bounds in Assumption 2.1 (in particular on κ_1), such that*

$$\|c - c_{hH}\|_{E,hH} \leq C \left(\sum_s \sum_{K \in \mathcal{T}_h^s} h_K^{2k} |c^s|_{k+1,K}^2 + \sum_{E \in \mathcal{G}_H} H_E^{2\ell} |c_\gamma|_{\ell+1,E}^2 \right)^{1/2}. \quad (23)$$

For Q_1/P_1 discretization this predicts first-order energy convergence.

Proof. Write $e = c - c_{hH} = \eta + \xi$, where $\eta = c - \Pi_{hH}c$ and $\xi = \Pi_{hH}c - c_{hH}$. Consistency yields

$$\langle \mathcal{A}_{hH}(\Pi_{hH}c) - \mathcal{A}_{hH}(c_{hH}), \xi \rangle = \langle \mathcal{A}_{hH}(\Pi_{hH}c) - \mathcal{A}_{hH}(c), \xi \rangle.$$

By Theorem 5.3, the left-hand side is at least $\alpha \|\xi\|_{E,hH}^2$. Split the right-hand side into diffusion, advection, exchange, and reaction terms. Standard SIP and upwind continuity, with (22), gives

$$|T_D + T_A| \leq C \|\eta\|_{E,hH,*} \|\xi\|_{E,hH}.$$

For the nonlinear reaction, with L_r defined by Assumption 2.3, the mean-value theorem gives $|T_R| \leq L_r \|\eta\|_0 \|\xi\|_0$. The two L^2 factors are then controlled separately. For $\xi \in X_{hH}$ we use the broken Poincaré–Friedrichs inequality with the DG jumps and Dirichlet penalties; for the piecewise- H^1 interpolation error η the corresponding piecewise inequality applies with the same boundary control [25]. Consequently

$$|T_R| \leq CL_r \|\eta\|_{E,hH,*} \|\xi\|_{E,hH}.$$

Here C depends on the domain, shape-regularity constants and the fixed lower penalty bound, but not on the relative matrix/fracture mesh ratio. The nonmatching term is the decisive point. Because the common-refinement cells partition the physical interface,

$$\begin{aligned} T_C &= \sum_s \sum_{I \in \mathcal{I}_{hH}^s} \int_I \kappa^s (\eta^s - \eta_\gamma) (\xi^s - \xi_\gamma) \\ &= \sum_s \int_\gamma \kappa^s (\eta^s - \eta_\gamma) (\xi^s - \xi_\gamma). \end{aligned}$$

Therefore Cauchy–Schwarz is applied *after recombination*:

$$|T_C| \leq \left(\sum_s \|(\kappa^s)^{1/2} (\eta^s - \eta_\gamma)\|_{0,\gamma}^2 \right)^{1/2} \left(\sum_s \|(\kappa^s)^{1/2} (\xi^s - \xi_\gamma)\|_{0,\gamma}^2 \right)^{1/2}.$$

The second factor is part of $\|\xi\|_{E,hH}$ and the first is part of $\|\eta\|_{E,hH,*}$. No cellwise inverse estimate is used, so no factor h/H or H/h appears. Combining the four bounds gives

$$\alpha\|\xi\|_{E,hH}^2 \leq C\|\eta\|_{E,hH,*}\|\xi\|_{E,hH}, \quad \|\xi\|_{E,hH} \leq \frac{C}{\alpha}\|\eta\|_{E,hH,*}.$$

The approximation estimates (20)–(21), the matrix trace inequality, and its fracture analogue imply

$$\|\eta\|_{E,hH,*} \leq C \left(\sum_s \sum_K h_K^{2k} |c^s|_{k+1,K}^2 + \sum_E H_E^{2\ell} |c_\gamma|_{\ell+1,E}^2 \right)^{1/2}.$$

Finally use $e = \eta + \xi$ and the triangle inequality. $\square$

Remark 6.4 (Independence of the mesh ratio). The estimate does not say that the total error is unchanged when one mesh is coarsened. It says that the stability constant does not become singular solely because H/h is large or small. The approximation error still contains separate h^k and H^ℓ contributions. Nor is this a κ -uniform theorem: throughout Assumption 2.1, $0 < \kappa_0 \leq \kappa^s \leq \kappa_1$ is fixed. Inspection of the proof localizes the κ_1 dependence to the coupling estimate T_C . The interface interpolation contribution in the starred norm is of higher trace order, $O(h^{k+1/2}) + O(H^{\ell+1/2})$, than the leading $O(h^k) + O(H^\ell)$ energy terms. Thus the coefficient dependence is asymptotically subdominant for smooth solutions, although its preasymptotic constant can be visible on nonmatching partitions. Section 8.4 measures precisely that effect. The ratio-uniform statement should therefore be read only as uniformity with respect to the relative discretization scales under fixed coefficient bounds.

Remark 6.5 (Formal adjoint prediction for the L^2 error). Assume, in addition, piecewise H^2 regularity for the adjoint linearization about the exact solution and sufficient boundedness of the second derivative of the reaction on the relevant concentration interval. A formal nonlinear duality calculation for $k = \ell = 1$ predicts

$$\|c - c_{hH}\|_{0,\Omega \cup \Omega \cup \Gamma} = O(h^2 + H^2).$$

The common-refinement part of that calculation can again be recombined into a physical-interface integral before it is estimated. However, closing the nonlinear quadratic remainder uniformly in the relative mesh ratio requires additional adjoint/interface control. We therefore do *not* claim a ratio-uniform nonlinear Aubin–Nitsche theorem here. Table 1 provides numerical evidence for the approximately second-order matrix L^2 rate in the smooth manufactured benchmark.

7 Conservation and nonlinear algebra

7.1 Local and global transfer balance

DG schemes are often chosen for transport because numerical fluxes give an elementwise balance. The nonmatching coupling preserves this advantage.

Proposition 7.1 (Conservative matrix–fracture transfer). *For every common-refinement cell $I \in \mathcal{I}_{hH}^s$, define*

$$Q_I^s = \int_I \kappa^s (c_h^s - c_{\gamma,H}) \, ds.$$

The contribution of I to the matrix residual is $+Q_I^s$ when the matrix test is one, and its contribution to the fracture residual is $-Q_I^s$ when the fracture test is one. Therefore the assembled transfer is exactly conservative to algebraic precision.

The property is important computationally: a fine fracture grid may intersect a coarse matrix face several times, but the sum of those intersection transfers equals the transfer seen by the matrix face. Conversely, a coarse fracture element may overlap several matrix trace faces with equal-and-opposite bookkeeping.

7.2 Newton linearization

For the cubic reaction used in the experiments,

$$r_q(c) = \lambda_q c + \mu_q c^3, \quad r'_q(c) = \lambda_q + 3\mu_q c^2 > 0.$$

Given $c^{(m)}$, Newton's method solves

$$\left[A_{hH} + M_{r'}(c^{(m)}) \right] \delta c^{(m)} = -\mathcal{R}(c^{(m)}), \quad c^{(m+1)} = c^{(m)} + \alpha_m \delta c^{(m)}, \quad (24)$$

where $M_{r'}$ is assembled by element quadrature and $\alpha_m \in (0, 1]$ is chosen by residual backtracking.

Proposition 7.2 (Jacobian nonsingularity). *Under the assumptions of Theorem 5.3, the Jacobian in (24) is nonsingular at every iterate contained in the admissible concentration interval.*

Proof. The symmetric part contains the coercive SIP diffusion, nonnegative exchange, nonnegative upwind dissipation, and the positive reaction derivative. If a Jacobian null vector existed, testing the linearized equation by that vector and taking the symmetric part would force its energy norm to vanish. The Dirichlet Nitsche/penalty terms make the energy seminorm a norm on X_{hH} , hence the vector is zero. $\square$

7.3 Cost of the common-refinement construction

For a straight two-dimensional interface with ordered interval partitions, the common refinement can be generated by a two-pointer sweep rather than by testing every pair (F, E) ; linear-complexity nonmatching-grid projection algorithms use the same ordering principle [26]. For piecewise-linear or network geometries, mandatory geometric vertices are retained and the same sweep is applied segment-by-segment. If N_F is the number of matrix-trace faces, N_E the number of fracture elements, and N_I the number of nonempty emitted intersection cells, the bookkeeping work is

$$O(N_F + N_E + N_I),$$

with $O(N_I)$ local coupling integrations. The full-PDE production class presently generates uniform trace and fracture partitions from their element counts and applies the two-pointer sweep to those ordered nodes. The supplementary geometry routine accepts explicit ordered segment partitions and applies the same sweep segment-by-segment; thus the algorithmic complexity statement concerns the ordered-partition operation, while arbitrary nonuniform full-PDE meshes remain an implementation extension. The $O(N_F + N_E + N_I)$ statement concerns only interface construction; matrix/fracture volume assembly and the sparse direct solve are separate costs. Storage of the interface layer is $O(N_I)$ before sparse compression. Overall sparse-factorization memory is implementation- and ordering-dependent because SuperLU fill-in can dominate the original matrix storage; no linear-complexity claim is made for that stage. Near-degenerate intersections with measure below a geometry tolerance are discarded consistently as zero-measure cells. The supplied geometry stress test records N_I , compares the common-refinement integral with a direct reference integral, and independently assembles a transfer block whose constant-test balance can fail if signs or indices are wrong.

Algorithm 1 Nonmatching-grid DG assembly and nonlinear solve used in the supplied code

Require: independent matrix/fracture meshes; coefficients and manufactured data; $\varepsilon_{\text{rel}} = 10^{-10}$, $\varepsilon_{\text{abs}} = 10^{-12}$; maximum 20 Newton corrections

Ensure: coefficient vector U and residual history

- 1: Generate $\mathcal{T}_h^-$, $\mathcal{T}_h^+$ and $\mathcal{G}_H$ independently.
 - 2: Assemble SIP diffusion, upwind advection, linear reaction, sources, and boundary terms.
 - 3: Traverse the ordered trace and fracture partitions with a two-pointer sweep and emit every nonempty $I = F \cap E$.
 - 4: **for** each emitted common-refinement cell I **do**
 - 5: integrate $\kappa(u_m - u_\gamma)(v_m - v_\gamma)$ on I and add the four matrix/fracture blocks.
 - 6: **end for**
 - 7: Initialize U with the declared start: exact interpolant for convergence tables, or zero/constant state for the solver-start audit.
 - 8: **repeat**
 - 9: assemble the cubic reaction residual and exact Jacobian; set $F = AU + r(U) - b$;
 - 10: **if** $\|F\|_2 \leq \varepsilon_{\text{abs}} + \varepsilon_{\text{rel}}\|F_0\|_2$ **then**
 - 11: **break**
 - 12: **end if**
 - 13: solve $(A + J_r)d = -F$ by SciPy `spsolve` (SuperLU);
 - 14: set $\alpha = 1$ and repeatedly replace $\alpha \leftarrow \alpha/2$ down to $1/64$ until $\|F(U + \alpha d)\|_2 < \|F(U)\|_2$;
 - 15: accept the first residual-reducing step (the supplied benchmarks do not trigger the documented full-step fallback).
 - 16: **until** convergence or 20 corrections
-

8 Numerical experiments

All tabulated values in this section are generated by the supplied Python programs; the one-command driver `code/reproduce_all.py` regenerates the numerical data and figures and records hashes in `data/generated_files_manifest.json`. Calculations use IEEE-754 binary64 arithmetic (unit roundoff $u = 2^{-53} \approx 1.11 \times 10^{-16}$). Newton convergence is declared when

$$\|F_m\|_2 \leq 10^{-12} + 10^{-10}\|F_0\|_2,$$

so the stopping rule scales with the initial nonlinear residual while retaining an absolute floor. Sparse SciPy matrices, SuperLU through `spsolve`, tensor Gauss quadrature, bilinear Q_1 DG matrix elements, linear P_1 DG fracture elements, SIP diffusion and conservative upwinding are used throughout. The exact software versions and reference hardware are recorded in `ENVIRONMENT.txt`; wall-clock times are implementation diagnostics, not cross-machine benchmarks.

8.1 Manufactured nonmatching-grid benchmark

Take $\Omega = (0, 1)^2$, $\gamma = \{1/2\} \times (0, 1)$, and

$$c_\gamma(y) = 1 + 0.2 \sin(\pi y).$$

The matrix exact solution is chosen as

$$c^-(x, y) = c_\gamma(y) + J_- - \frac{\kappa J_-}{D_m}(x - 1/2), \quad (25)$$

$$c^+(x, y) = c_\gamma(y) + J_+ + \frac{\kappa J_+}{D_m}(x - 1/2), \quad (26)$$

so that the Robin transfer laws are satisfied exactly. We use

$$D_m = 1, \quad D_\gamma = 0.25, \quad \beta_m = (0, 0.35)^T, \quad \beta_\gamma = 0.25,$$

$$\lambda_m = 1, \quad \mu_m = 0.2, \quad \lambda_\gamma = 0.8, \quad \mu_\gamma = 0.15, \quad \kappa = 4, \quad J_- = 0.04, \quad J_+ = -0.03.$$

The source terms are obtained by substituting the exact solution into (1) and (3); exact Dirichlet values are imposed weakly by Nitsche terms.

The matrix uses an $n \times n$ grid while the fracture uses n_f intervals. The sequence

$$(n, n_f) = (4, 5), (8, 11), (16, 23), (32, 47), (64, 95)$$

is intentionally nonmatching at every level. The reported $\|e\|_E$ is evaluated from Definition 4.1: the code includes diffusion-weighted broken gradients, SIP/Nitsche penalty terms, the matrix–fracture exchange term, and the upwind jump/outflow seminorm. Thus the numerical energy column is the same discrete norm used in the analysis, rather than a reduced gradient-only diagnostic. Table 1 shows the computed errors. The matrix L^2 order tends to 2, the matrix broken H^1 order tends to 1, and the coupled energy error tends to order 1. The fracture errors show the corresponding P_1 behavior. The interface contribution reported below is $e_\Gamma = (\sum_s \int_\gamma \kappa^s |(c_h^s - c_{\gamma,H}) - (c^s - c_\gamma)|^2 ds)^{1/2}$. Observed orders are computed explicitly as $p = \log(e_h/e_{h/2})/\log 2$. The interface term converges at rates 2.30, 2.03, 2.01, 2.00 for this smooth benchmark, faster than the conservative trace-order estimate used in Theorem 6.3. We report this only as an observed high rate for this manufactured solution; no structural superconvergence theorem is claimed or used in the proof.

Table 1: Convergence on deliberately nonmatching matrix/fracture meshes.

n	n_f	DOFs	$\ e_m\ _{L^2}$	rate	$ e_m _{H_h^1}$	rate	$\ e\ _E$	rate
4	5	74	4.3610×10^{-3}	–	1.0059×10^{-1}	–	0.1174	–
8	11	278	1.0759×10^{-3}	2.019	5.0367×10^{-2}	0.998	0.05554	1.080
16	23	1070	2.6757×10^{-4}	2.008	2.5184×10^{-2}	1.000	0.02713	1.034
32	47	4190	6.6744×10^{-5}	2.003	1.2592×10^{-2}	1.000	0.01341	1.016
64	95	16574	1.6680×10^{-5}	2.001	6.2958×10^{-3}	1.000	0.006671	1.008

Table 2: Fracture and interface errors for the same nonmatching sequence.

n	n_f	$\ e_\gamma\ _{L^2}$	rate	$ e_\gamma _{H_H^1}$	rate	e_Γ	rate
4	5	4.1559×10^{-3}	–	8.0073×10^{-2}	–	1.2968×10^{-2}	–
8	11	9.1328×10^{-4}	2.186	3.6582×10^{-2}	1.130	2.6241×10^{-3}	2.305
16	23	2.1713×10^{-4}	2.073	1.7513×10^{-2}	1.063	6.4340×10^{-4}	2.028
32	47	5.3083×10^{-5}	2.032	8.5723×10^{-3}	1.031	1.5964×10^{-4}	2.011
64	95	1.3114×10^{-5}	2.015	4.2412×10^{-3}	1.015	3.9777×10^{-5}	2.005

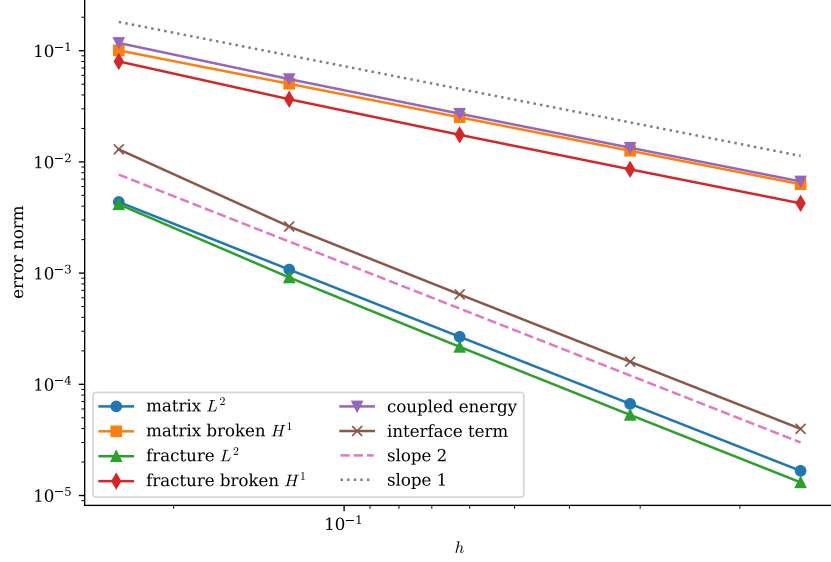


Figure 2: Measured convergence on independent matrix and fracture grids. The coupled energy and interface terms are shown explicitly; reference slopes one and two are offset from the data for visual comparison.

8.2 Representative two-dimensional field

To show the fully coupled computation rather than only norm tables, we increase tangential transport and nonlinear reaction, use $D_\gamma = 0.08$, $\kappa = 8$, and discretize the 24×24 matrix grid with 35 fracture intervals. Figure 3 shows the matrix field and the independently approximated fracture trace. The fracture DOFs are not inherited from the matrix trace. The computed fracture solution tracks the manufactured exact trace while matrix concentrations retain the prescribed transfer jumps.

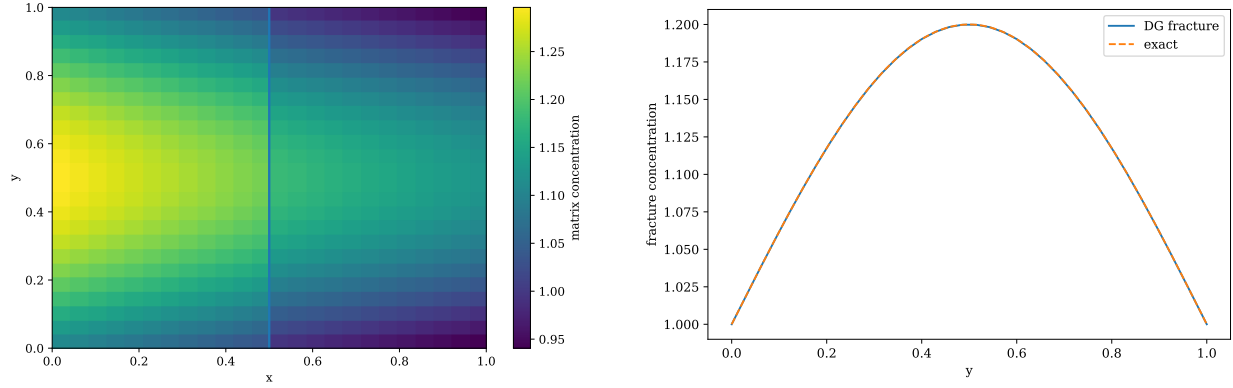


Figure 3: Representative nonlinear nonmatching-grid solution: matrix field (left) and one-dimensional fracture concentration (right).

8.3 Robustness with respect to the mesh-size ratio

We use three complementary ratio studies. Most importantly, Table 3 performs genuine refinement at the fixed ratios $H/h = 4, 1, 1/4$; all three sequences retain first-order energy convergence and therefore test directly the central mesh-ratio claim. A second study fixes the 16×16 matrix grid while changing $n_f = 9$ to 37. Finally, Table 5 spans four decades, $H/h = 10^{-2}$ to 10^2 , by varying the tangential matrix and fracture partitions while holding $n_x = 8$. The last sweep is intentionally anisotropic and is used only to expose algebraic failure under severe scale separation, not as a shape-regular convergence test.

Table 3: Fixed-ratio refinement test. Each sequence doubles matrix resolution while preserving the indicated H/h ; the energy rate is computed between consecutive rows.

H/h	(n, n_f)	DOFs	$\ e\ _E$	rate
4	(8, 2)	260	0.122926	–
4	(16, 4)	1032	0.057659	1.092
4	(32, 8)	4112	0.028346	1.024
1	(8, 8)	272	0.058109	–
1	(16, 16)	1056	0.028588	1.023
1	(32, 32)	4160	0.014184	1.011
1/4	(8, 32)	320	0.052806	–
1/4	(16, 64)	1152	0.025866	1.030
1/4	(32, 128)	4352	0.012809	1.014

Table 4: Independent fracture refinement at fixed 16×16 matrix grid.

n_f	H/h	DOFs	$ e_m _{H_h^1}$	$ e_\gamma _{H_H^1}$	$\ e\ _E$
9	1.778	1042	0.025190	0.044724	0.034125
13	1.231	1050	0.025185	0.030974	0.030002
17	0.941	1058	0.025184	0.023690	0.028285
23	0.696	1070	0.025184	0.017513	0.027127
31	0.516	1086	0.025184	0.012997	0.026484
37	0.432	1098	0.025184	0.010891	0.026245

Table 5: Four-decade mesh-ratio stress test with fixed normal matrix resolution $n_x = 8$. Extreme rows are anisotropic and are interpreted only as an algebraic diagnostic.

n_y	n_f	H/h	DOFs	final residual	$\ e\ _E$
2	200	10^{-2}	464	2.58×10^{-12}	0.308
4	40	10^{-1}	208	6.51×10^{-12}	0.117
8	8	1	272	4.82×10^{-14}	0.0581
40	4	10	1288	2.83×10^{-13}	0.0527
100	1	10^2	3202	1.14×10^{-12}	0.346

The condition estimates vary by more than two orders of magnitude even though the converged residuals are all tiny. We therefore describe these rows as solvability/conditioning diagnostics and do not infer error control from the residual alone.

Table 6: One-norm condition estimates of the converged nonlinear Jacobian at representative mesh-ratio rows. A small residual alone is not used as a proxy for algebraic accuracy.

H/h	(n_y, n_f)	DOFs	$\text{cond}_1(J)$ (est.)	$\ F\ _2/\ b\ _2$
10^{-2}	(2, 200)	464	1.13×10^5	9.60×10^{-15}
1	(8, 8)	272	5.18×10^2	1.12×10^{-13}
10^2	(100, 1)	3202	3.33×10^4	1.86×10^{-15}

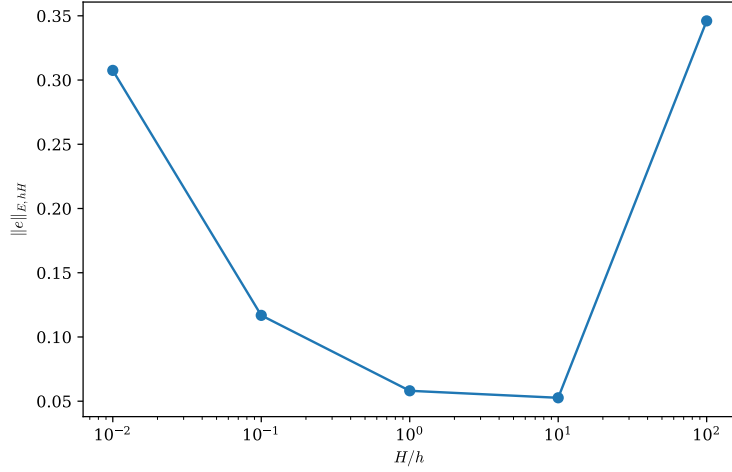


Figure 4: Four-decade algebraic stress test for the tangential matrix/fracture mesh-size ratio. The energy error is not expected to remain constant because the approximation spaces change strongly across the sweep.

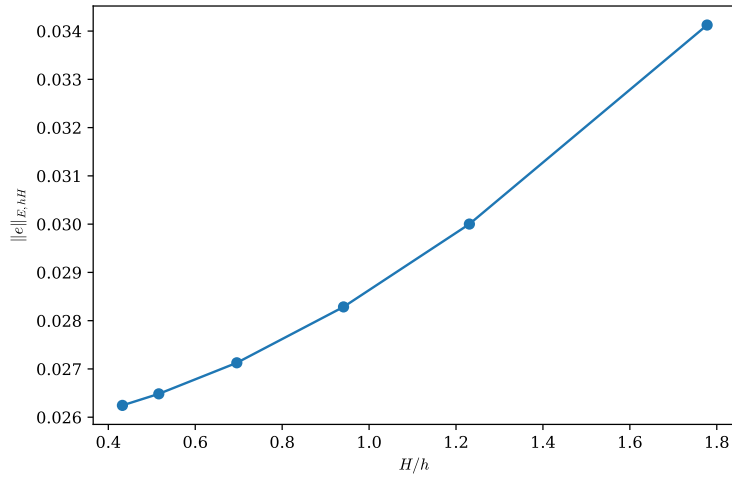


Figure 5: Coupled energy error as the fracture/matrix mesh-size ratio is varied independently.

8.4 Exchange coefficient and nonlinear reaction strength

The exchange-coefficient study is designed with $\kappa J_{\pm}$ fixed, so the exact normal slopes are held fixed while the weight of the exchange term changes. The original narrow sweep $\kappa \in [0.5, 32]$ showed little movement and should not be interpreted as a conditioning result. The revised audit therefore compares matching and nonmatching partitions at the same 16×16 matrix resolution and extends the coefficient range to 10^5 . Table 7 isolates a clear effect: the matching energy error is essentially invariant, whereas the nonmatching case develops a larger preasymptotic constant at very large κ . This behavior is consistent with Remark 6.4: the coefficient enters through the higher-order interface contribution and does not change the asymptotic energy order.

Table 7: Exchange-coefficient audit at fixed 16×16 matrix resolution with rescaled jumps. Values for the extended audit were independently reproduced during the R5 verification. Six significant digits are retained because the variation over $\kappa \in [0.5, 32]$ occurs beyond the fourth digit.

κ	0.5	4	32	10^4	10^5
matching $n_f = 16$, $\ e\ _E$	0.0285882	0.0285883	0.0285883	0.0285883	0.0285883
nonmatching $n_f = 23$, $\ e\ _E$	0.0271206	0.0271272	0.0271796	0.0346091	0.0465624

Over $\kappa \in [0.5, 32]$ the relative spread is about 6.4×10^{-6} for the matching partition and 2.2×10^{-3} for the nonmatching partition. Along the nonmatching refinement sequence, the ratio $\|e\|_E(32)/\|e\|_E(0.5)$ decreases from 1.0469 at $(n, n_f) = (4, 5)$ to 1.00055 at $(32, 47)$, i.e. the excess decreases by approximately a factor four per refinement. Thus the measured effect is a preasymptotic constant rather than a loss of the first-order energy rate.

Table 8: Budget of the squared discrete energy norm at $(n, n_f) = (16, 23)$. Percentages sum the terms appearing in Definition 4.1.

component	$\kappa = 4$	$\kappa = 10^4$
matrix gradient	86.19%	54.45%
fracture gradient	10.42%	17.86%
SIP/Nitsche penalty	3.34%	7.83%
interface exchange	0.056%	19.85%
upwind	0.0001%	0.0065%

At the baseline $\kappa = 4$ the headline energy norm is dominated by the matrix broken-gradient error and the exchange term contributes only 0.056% of $\|e\|_E^2$. The high-transfer regime is therefore essential for exposing the coupling itself: at $\kappa = 10^4$ the exchange contribution rises to 19.85%.

A same-resolution isolation test removes the resolution confound from the cost comparison. At $n = 16$, changing only the fracture partition from matching $n_f = 16$ (1056 DOFs) to nonmatching $n_f = 17$ (1058 DOFs) changes $\|e\|_E$ from 0.0285883 to 0.0282850 at $\kappa = 4$; at $\kappa = 10^4$ the corresponding values are 0.0285883 and 0.0510329. This test makes explicit when the price of nonmatching transfer becomes observable.

We next vary μ_m from 0 to 2 with $\mu_\gamma = 0.75\mu_m$ and repeat each case from three declared starts: the exact interpolant, the zero vector, and a constant state. Failure is defined a priori as not satisfying $\|F_m\|_2 \leq 10^{-12} + 10^{-10}\|F_0\|_2$ within 20 corrections. All 18 solves converge. Table 9 shows that the interpolated start needs one to three corrections, while the neutral starts require at most five. The energy error is unchanged across starts, so this is a solver-initialization audit rather than a physical sensitivity claim.

Table 9: Newton corrections versus reaction strength and initialization. All entries converged under the same relative-plus-absolute stopping rule.

μ_m	interpolant	zero	constant
0.0	1	1	1
0.1	2	3	3
0.2	2	3	3
0.5	2	4	3
1.0	3	4	3
2.0	3	5	4

8.5 Matching versus nonmatching accuracy and assembly cost

To quantify the practical overhead of common refinement, Table 10 compares matching fracture partitions ($n_f = n$) with deliberately nonmatching partitions of nearly the same total number of unknowns. Assembly times are the median of three complete linear-assembly calls after one discarded warm-up; the accompanying CSV also reports the interquartile range and extrema. Wall-clock values are machine-dependent and are reported only to expose the order of the overhead. The $n = 16, n_f = 17$ row isolates nonmatching from resolution with only two additional DOFs. The reproducible quantities are the degrees of freedom, intersection-cell counts, and errors.

Table 10: Matching versus nonmatching grids. Times are illustrative wall-clock seconds from the supplied script; N_I is the number of common-refinement intervals on one physical fracture.

n	n_f	type	DOFs	N_I	assembly (s)	$\ e_m\ _{L^2}$	$\ e\ _E$
8	8	matching	272	8	0.106 (IQR = 0.004)	1.12×10^{-3}	0.0581
8	11	nonmatching	278	18	0.107 (IQR = 0.001)	1.08×10^{-3}	0.0555
16	16	matching	1056	16	0.439 (IQR = 0.004)	2.80×10^{-4}	0.0286
16	23	nonmatching	1070	38	0.440 (IQR = 0.002)	2.68×10^{-4}	0.0271
16	17	nonmatching (isolated)	1058	32	0.441 (IQR = 0.002)	2.77×10^{-4}	0.0283
32	32	matching	4160	32	1.807 (IQR = 0.046)	6.99×10^{-5}	0.0142
32	47	nonmatching	4190	78	1.790 (IQR = 0.029)	6.67×10^{-5}	0.0134

The nonmatching cases emit more interface cells, as expected. The refreshed timing protocol and its dispersion are reported in the machine-readable CSV rather than overinterpreted in the manuscript; at these resolutions volume assembly remains the dominant contribution. The comparison is not intended as a universal performance benchmark; it demonstrates that independent fracture refinement does not introduce an obvious assembly-time penalty in this implementation.

8.6 Direct versus L^2 -projected exchange: the separation regime

Lemma 3.3 predicts exactly when the direct and projected couplings should agree and when they should separate. We therefore compare the same two implementations over both the benign baseline and a strong-transfer regime, keeping $\kappa J_{\pm}$ fixed as in the coefficient audit. Table 11 reports the directly regenerated energy errors; both methods use the same PDE, meshes, quadrature and nonlinear solver.

At $\kappa = 4$ the two methods are indistinguishable, exactly as the projection-gap identity predicts when the unresolved component is weakly weighted. At $\kappa = 10^4$ the projected error is 21% larger on (16, 23). More importantly, with the matrix trace held fixed at $n = 8$, refining the fracture grid

Table 11: Direct common-refinement coupling versus the L^2 -projection competitor. The final column is $(e_{\text{proj}} - e_{\text{dir}})/e_{\text{dir}}$.

n	n_f	κ	direct $\ e\ _E$	projected $\ e\ _E$	relative gap
16	23	4	0.02712719	0.02712720	2.0×10^{-7}
16	23	10^3	0.02871186	0.02894358	0.0081
16	23	10^4	0.03460907	0.04189102	0.210
8	32	10^4	0.05771883	0.12390578	1.147
8	64	10^4	0.05750350	0.12409692	1.158
8	128	10^4	0.05732972	0.12411112	1.165

from $n_f = 32$ to 128 slightly improves the direct result while the projected result stagnates near 0.124: projection discards the part of the refined fracture field outside the fixed trace space. This is a qualitative separation, not merely a timing difference.

As a direct algebraic check of Lemma 3.3, the fracture-block difference $C^{\text{dir}} - C^{\text{proj}}$ has numerical eigenvalue ranges $[-7.5 \times 10^{-15}, 9.76 \times 10^{-2}]$ for $(16, 23)$ and $[-3.4 \times 10^{-14}, 6.25 \times 10^{-2}]$ for $(8, 64)$, with no eigenvalue below -10^{-11} . The tiny negative values are at roundoff scale and support the predicted positive-semidefinite ordering.

8.7 Kinked and intersecting common-refinement stress tests

The analysis is written face-by-face, but the full PDE convergence experiment above deliberately isolates a straight fracture. To test the geometry layer separately, the supplementary script constructs a kinked polyline and an intersecting X-network, retains the geometric vertices, independently partitions every segment on the matrix-trace and fracture sides, and then emits their common-refinement cells. Representative configurations are shown in Figures 6–7.

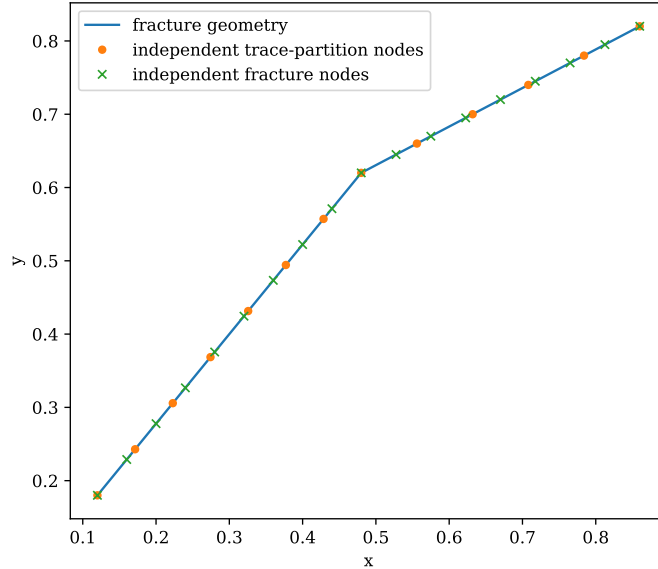


Figure 6: Geometry-only stress test for a kinked fracture. Matrix-trace and fracture nodes are generated independently on each geometric segment; the kink itself is retained by both meshes.

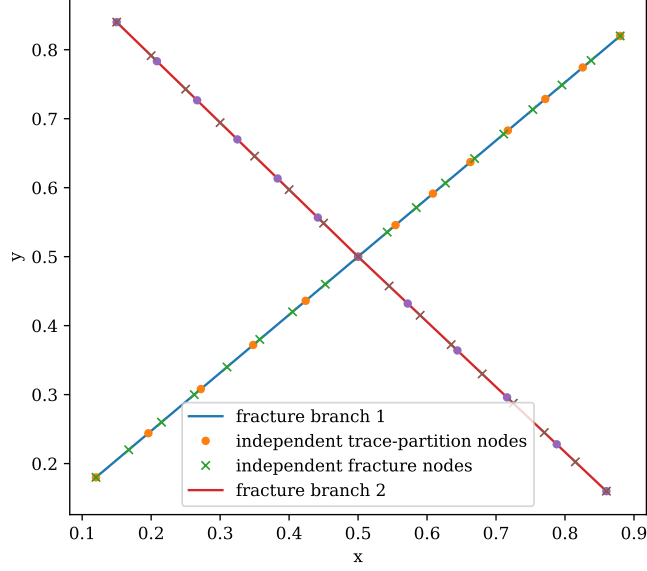


Figure 7: Geometry-only stress test for an intersecting X-network. Junction vertices are retained while partitions away from the junction are independent; no physical junction/transmission law is imposed or tested at the crossing.

Table 12: Common-refinement geometry audit. The final three columns report an independently assembled balance residual, an independent moment test of the cell-to-cell pairing, and the common-refinement integral error against a direct high-order reference integral.

case	branches	N_F	N_E	N_I	balance	pairing	partition
kinked	1	12	17	27	1.11×10^{-16}	1.39×10^{-16}	1.11×10^{-16}
intersecting X	2	23	32	51	2.22×10^{-16}	1.39×10^{-16}	0

These are coupling-assembly tests, not substitutes for a full PDE convergence study on curved or intersecting networks. Their purpose is narrower: they audit segment-by-segment common refinement, partition recovery as a tessellation, the cell-to-cell pairing through an independent moment test, and the assembled equal-and-opposite transfer balance for non-standard polyline geometries. They do not test intersection of an arbitrary fracture with a two-dimensional background mesh.

8.8 What the experiments establish—and what they do not

The full PDE experiments establish four concrete points: (a) independently partitioned matrix/fracture traces can be coupled directly by common-refinement quadrature; (b) the Q_1/P_1 energy rate remains first order on deliberately nonmatching sequences and on three fixed-ratio refinement paths; (c) the observed matrix L^2 rate is close to two for the smooth manufactured solution; and (d) the high- κ competitor study exhibits the projection gap predicted by Lemma 3.3. The κ sweep uses rescaled jumps and therefore does not establish physical κ -uniformity; the nonlinear-reaction sweep similarly probes the nonlinear algebra with consistently remanufactured sources. The matching-versus-nonmatching table additionally exposes common-refinement cell counts and uses a warm-up/median/IQR timing protocol on the recorded reference environment. The kinked and intersecting tests verify only the geometric coupling layer. We do not claim a full PDE convergence theorem or a numerical validation for arbitrary three-dimensional fracture networks, curved-surface intersections, or higher-order elements in the present paper.

8.9 Threats to validity

The principal verification benchmark is deliberately smooth and has matrix fields linear in the normal coordinate, constant diffusion coefficients, constant κ , and a constant exact Robin jump. It therefore does not exercise the full coefficient generality of Assumption 2.1; Table 8 quantifies how benign the baseline exchange contribution is. The high- κ and fixed-ratio tests mitigate, but do not eliminate, this limitation. Timing values depend on Python/SciPy, hardware and sparse ordering and are not performance benchmarks. The proof assumes $\beta^s \cdot n^s = 0$ on γ and the error theorem additionally assumes the uniform concentration range in Assumption 2.3. The full-PDE code uses one straight fracture and uniform parent partitions; the kinked/X tests audit only the intersection bookkeeping and do not impose a network-junction PDE law. Community fractured-flow benchmarks such as [27] are relevant external tests but solve a different single-phase flow problem; adapting the present reactive-transport operator to such benchmark geometries is future validation work.

9 Relation to recent DG fracture methods

It is useful to state precisely how the present method differs from closely related work.

- Chen and Jin [15] analyze DG plus backward Euler for hybrid-dimensional advection–diffusion and obtain optimal discrete H^1 estimates. Their matrix meshes coincide on the fracture to induce a unique fracture partition. Here the central new object is instead the conservative primal exchange operator on the common refinement of independently generated trace and fracture partitions; the monotone reaction is incorporated without changing that ratio-uniform coupling mechanism.

- Chen [14] treats hybrid-dimensional two-phase flow with DG and proves stability and convergence for a different coupled flow/saturation system.
- Leng and Chen [13] develop adaptive interior-penalty hybridized DG for fractured Darcy flow. Their primary unknowns and governing equations differ from the reactive transport system considered here.
- Antonietti et al. [3, 4] develop polytopic DG formulations for fractured porous flow, including fracture networks. Those works are central precedents for DG on complex fracture geometry, whereas our emphasis is the primal transport/reaction coupling of independently partitioned matrix traces and lower-dimensional fracture meshes.
- Gander and Japhet [26] give a linear-complexity algorithm for nonmatching-grid projections; this supplies a direct algorithmic precedent for the ordered two-pointer sweep used in Section 7.3. Jiao and Heath [19] use a geometric common refinement for accurate conservative transfer between nonmatching surface meshes in multiphysics simulations. Their setting is a general data-transfer problem between interface meshes. Here the common refinement is used directly to assemble the Robin exchange bilinear form, with no separate projection or transfer variable.
- Fumagalli, Keilegavlen and Scialò [18] provide a useful taxonomy of conforming, non-conforming and non-matching fracture couplings. The present method belongs to the nonmatching family, but its specific object is the primal reactive-transport exchange operator and the associated mesh-ratio-uniform energy estimate.
- Fu and Yang [20] develop an unfitted HDG method for Darcy flow with conductive and blocking fractures. Their geometry treatment is more general than the fitted-interface setting used in our full PDE experiments. Our claim is therefore not greater geometric generality, but a different conservative coupling analysis for hybrid-dimensional nonlinear transport/reaction.
- Mortar methods [22, 23] and INTERNODES [24] are established alternatives for nonconforming interfaces. In particular, dual mortar bases can retain local support, so our “mortar-free” terminology denotes the absence of an auxiliary multiplier/projection in this Robin exchange form rather than a blanket efficiency claim.
- The mixed-dimensional flux-coupling framework of Nordbotten et al. [28] provides a broader mortar-based view in which lower-dimensional and matrix discretizations are coupled abstractly. Hybrid mixed DG methods for convection–diffusion provide another relevant precedent [29]. Liu and Xu [16] are particularly close in motivation because their interior-penalty interface method avoids additional interface unknowns; their operator and interface model differ from the Robin reactive-transport law considered here, so the comparison is one of coupling architecture rather than identical PDEs.

Thus the contribution is not “DG applied once more to a fracture equation.” The distinguishing object is the nonmatching primal exchange form (6), together with an error analysis whose constants do not require a matching trace partition.

10 Conclusions

The main conclusion is structural: a Robin matrix–fracture law can be coupled on independently generated interface partitions by direct physical $L^2(\gamma)$ integration, without introducing a multiplier

inf-sup condition or a projection-stability constant. Explicit SIP thresholds give coercivity, the nonlinear operator is strongly monotone, and finite-dimensional existence and uniqueness follow. Under the additional mesh-ratio-uniform concentration-range Assumption 2.3, the quasi-optimal nonlinear energy estimate is obtained by recombining intersection cells before estimation, so no h/H or H/h factor enters its constant. Lemma 3.3 further orders direct and projected Robin couplings exactly: the direct form exceeds the projected one by the unresolved fracture component.

For the supplied two-dimensional Q_1/P_1 implementation, deliberately nonmatching sequences show first-order energy convergence and an observed matrix L^2 rate close to two. The latter is reported as numerical evidence rather than as a proved ratio-uniform nonlinear duality theorem. Fixed-ratio refinements at $H/h = 4, 1, 1/4$ retain first-order energy convergence. In the high-transfer regime the L^2 -projection competitor exhibits the predicted information-loss gap and stagnates under fracture refinement with a fixed matrix trace, while the direct coupling continues to improve. Geometry-only stress tests on kinked and intersecting fractures recover the direct interface integral to roundoff and give a near-roundoff balance residual from an independently assembled transfer block.

The scope is intentionally limited. The full PDE verification uses one straight fracture, a common matrix-side trace partition, and first-order approximation spaces. Extending the complete PDE study to mutually nonconforming matrix-side traces, intersecting networks, curved fractures, three-dimensional matrix/surface-fracture configurations, higher-order elements, and adaptive refinement is future work. The present paper isolates the common-refinement coupling mechanism and establishes its energy stability uniformly with respect to the matrix/fracture mesh ratio under fixed coefficient bounds; no uniformity in κ is claimed.

Data availability

All numerical data reported in the tables are included as CSV files in the accompanying reproducibility package. They can be regenerated from the supplied Python source without external proprietary software.

Code availability

The accompanying source file `hybrid_dg_2d.py` assembles and solves the complete two-dimensional matrix/one-dimensional fracture DG problem. The script `additional_tests.py` regenerates the matching/nonmatching, projection-gap, fixed-ratio, energy-budget, conditioning and geometry audits. The driver `reproduce_all.py`, supplied in the `code/` directory, orchestrates the complete regeneration.

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Author contributions

Alvaro H. Salas: Conceptualization, methodology, mathematical analysis, software verification, numerical investigation, writing, and revision.

Use of generative AI tools declaration

Generative AI tools were used as assistive tools for language editing and portions of code drafting during manuscript preparation. All mathematical arguments, numerical experiments, source files, references, and final statements were checked by the author, who takes responsibility for the submitted content. This declaration should be replaced by the journal-prescribed wording if a specific disclosure format is required.

Conflict of interest

The author declares no conflict of interest.

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A Implementation details

A.1 Matrix element

The reference square is $[-1, 1]^2$ with the four bilinear basis functions

$$\begin{aligned}\phi_1 &= \frac{1}{4}(1 - \xi)(1 - \eta), & \phi_2 &= \frac{1}{4}(1 + \xi)(1 - \eta), \\ \phi_3 &= \frac{1}{4}(1 + \xi)(1 + \eta), & \phi_4 &= \frac{1}{4}(1 - \xi)(1 + \eta).\end{aligned}$$

Each physical rectangle carries its own four unknowns; no nodal continuity is imposed. Volume terms are evaluated by tensor Gauss quadrature. The production code uses four Gauss points per coordinate. The supplied `quadrature_check.py` repeats the representative 16×23 benchmark with six points and writes the resulting difference to `data/quadrature_check.csv`; the comparison verifies that quadrature is not limiting the reported convergence at that level.

A.2 Fracture element

Each fracture interval uses the local basis

$$\psi_1(s) = \frac{1-s}{2}, \quad \psi_2(s) = \frac{1+s}{2}, \quad -1 \leq s \leq 1.$$

Fracture unknowns are discontinuous at element endpoints. One-dimensional SIP terms weakly enforce interelement continuity in the diffusion operator, while upwind fluxes treat tangential advection.

A.3 Common-refinement quadrature

Suppose a matrix trace segment occupies $F = [a_0, a_1]$ in the fracture coordinate and a fracture element occupies $E = [b_0, b_1]$. The overlap is

$$I = [\max(a_0, b_0), \min(a_1, b_1)].$$

If $|I| > 0$, quadrature points $y_q \in I$ are mapped independently to the matrix-face reference coordinate and the fracture-element reference coordinate. If $\phi(y_q)$ and $\psi(y_q)$ denote their basis vectors, the local transfer matrix is

$$\kappa w_q \begin{bmatrix} \phi\phi^T & -\phi\psi^T \\ -\psi\phi^T & \psi\psi^T \end{bmatrix}.$$

This 2×2 block structure is positive semidefinite because it is the outer product of $(\phi, -\psi)$ with itself. Summing all intersections gives (6).

A.4 Boundary treatment

For a matrix Dirichlet face with datum g , the SIP/Nitsche contribution is

$$-\int_e D\partial_n u_h v_h - \int_e D\partial_n v_h u_h + \int_e \frac{\sigma D}{h_e} u_h v_h,$$

with right-hand side

$$-\int_e D\partial_n v_h g + \int_e \frac{\sigma D}{h_e} g v_h.$$

For advection, the exterior state equals g on inflow and the interior state is used on outflow. The fracture endpoints are treated identically in one dimension.

B Manufactured-source formulas

For the exact fields in Section 8.1,

$$c'_\gamma(y) = 0.2\pi \cos(\pi y), \quad c''_\gamma(y) = -0.2\pi^2 \sin(\pi y).$$

Because the matrix dependence in x is linear, $\partial_{xx}c^s = 0$ and $\partial_{yy}c^s = c''_\gamma$. Thus

$$f^s = -D_m c''_\gamma + \beta_{m,y} c'_\gamma + \lambda_m c^s + \mu_m (c^s)^3.$$

The fracture source is

$$f_\gamma = -D_\gamma c''_\gamma + \beta_\gamma c'_\gamma + \lambda_\gamma c_\gamma + \mu_\gamma c_\gamma^3 - \kappa(J_- + J_+).$$

These are evaluated at quadrature points by the reproducibility code rather than symbolically simplified into hard-coded polynomials.

C Reproducibility checklist

The submission package contains:

1. `main.tex` and the compiled `main.pdf`;
2. `references.bib`;
3. `code/hybrid_dg_2d.py` and `code/additional_tests.py`;
4. CSV files in `data/` for convergence, parameter sweeps, timing, quadrature, and geometry audits;

5. vector PDF figures and PNG previews;
6. the cover letter, highlights, and a short reproducibility README.

A clean regeneration requires the versions pinned in `requirements.txt`; `ENVIRONMENT.txt` records the reference machine and library versions. An MIT license is included for the supplied code, and `ZENODO_DEPOSIT_REQUIRED.txt` records the remaining DOI-deposit action without inventing a repository identifier. From the `code` directory, run

```
python reproduce_all.py --out ..
```

to regenerate the numerical tables and figures and write their hashes to the generated-file manifest.

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