

Supplementary Information

Memory in higher-order moments enables a quantum magic Mpemba inversion

Shuai Zeng

Chongqing University of Posts and Telecommunications, Chongqing, China

Purpose and scope. This Supplementary Information provides the complete derivations and numerical details supporting the manuscript. It establishes the exact separation between low-order ensemble moments and a fourth-order stabilizer-magic memory, derives the common-target Mpemba inversion, and reports finite-size overlapping-circuit calculations that characterize attenuation under spatial remixing. An accompanying Supplementary Code and Data archive provides the source data and plotting script for the main-text figure; the numerical values from the auxiliary checks are reported in the tables below.

Contents

1	Supplementary Note 1: Moment hierarchy and design-controlled relaxation	2
1.1	Ensemble moment operators	2
1.2	Moment channels of random-unitary ensembles	2
1.3	Specialization to a logical one-qubit Clifford active sector	3
2	Supplementary Note 2: Exact number-conserving construction and low-order matching	4
2.1	Three-site single-particle sector	4
2.2	Number-conserving random gate	4
2.3	Phase-twirl selection rule	5
3	Supplementary Note 3: Fourth-moment magic and the exact Mpemba solution	5
3.1	Stabilizer purity and stabilizer Rényi entropy	5
3.2	Closed form for the Clifford and Haar averages	6
3.3	Explicit common-target Mpemba pair	7
3.4	Persistent inversion under Clifford–Haar interpolation	8
3.5	Exact Clifford trajectory distributions and quenched inversion	8
4	Supplementary Note 4: Numerical checks and robustness	9
4.1	Deterministic Haar quadrature	9
4.2	Open neighborhood of Mpemba pairs	9
4.3	Robustness to gate-set perturbations	10
5	Supplementary Note 5: Attenuation under overlapping scrambling	12
5.1	Overlapping number-conserving circuit	12
5.2	L=6 high-statistics calculation	12
5.3	L=9 auxiliary consistency check	12
5.4	Finite-size interpretation	13

1 Supplementary Note 1: Moment hierarchy and design-controlled relaxation

1.1 Ensemble moment operators

Let $\mathcal{H}$ denote the system Hilbert space, and let an ensemble $\mathcal{E}$ produce a density matrix ρ_λ on $\mathcal{H}$ with probability measure $d\nu(\lambda)$. Its k th moment operator is

$$\Omega_k[\mathcal{E}] = \mathbb{E}_{\lambda \sim \nu} [\rho_\lambda^{\otimes k}]. \quad (1)$$

For any POVM element $0 \leq E_y \leq I$ on $\mathcal{H}^{\otimes k}$, the ensemble-averaged probability of outcome y in a k -copy test of the same random realization is

$$\bar{p}(y|\mathcal{E}) = \text{Tr}[E_y \Omega_k[\mathcal{E}]]. \quad (2)$$

Consequently, if $\Omega_k[\mathcal{E}_A] = \Omega_k[\mathcal{E}_B]$, all such k -copy linear tests have identical ensemble-averaged statistics.

This statement concerns equality of ensemble moment tensors. Repeated lower-copy experiments combined with nonlinear classical post-processing can access related higher-order information; accordingly, “moment-selective” refers here to tensor order rather than to a minimal simultaneous-copy experimental requirement.

1.2 Moment channels of random-unitary ensembles

For a probability measure μ on a compact unitary group, define the k th moment channel

$$\mathcal{T}_k^\mu(X) = \int d\mu(U) U^{\otimes k} X (U^\dagger)^{\otimes k}. \quad (3)$$

Let G be a compact subgroup of an enclosing compact group K , with normalized Haar measures μ_G and μ_K . Consider

$$\mu_p = p\mu_G + (1-p)\mu_K, \quad 0 \leq p \leq 1. \quad (4)$$

Supplementary Proposition 1 (Exact convolution law). *For $t \geq 1$ independent steps drawn from μ_p ,*

$$\mu_p^{*t} = p^t \mu_G + (1-p^t) \mu_K. \quad (5)$$

Proof. Normalized Haar measure on a compact subgroup is idempotent under convolution, so $\mu_G * \mu_G = \mu_G$. Haar measure on the enclosing group is absorbing under left or right convolution by a probability measure supported on the group: $\mu_G * \mu_K = \mu_K * \mu_G = \mu_K$ and $\mu_K * \mu_K = \mu_K$. Thus

$$\mu_p * \mu_p = p^2 \mu_G + p(1-p)(\mu_K + \mu_K) + (1-p)^2 \mu_K \quad (6)$$

$$= p^2 \mu_G + (1-p^2) \mu_K. \quad (7)$$

Induction on t gives the claim. □

By linearity,

$$\mathcal{T}_k(t) = p^t \mathcal{T}_k^G + (1-p^t) \mathcal{T}_k^K. \quad (8)$$

We say that G is an exact unitary r -design relative to K in the represented Hilbert space when

$$\mathcal{T}_k^G = \mathcal{T}_k^K, \quad k = 1, \dots, r. \quad (9)$$

If this condition holds, then

$$\mathcal{T}_k(t) = \mathcal{T}_k^K, \quad k \leq r, \quad t \geq 1, \quad (10)$$

whereas at the next order

$$\mathcal{T}_{r+1}(t) - \mathcal{T}_{r+1}^K = p^t (\mathcal{T}_{r+1}^G - \mathcal{T}_{r+1}^K). \quad (11)$$

Thus low-order moments can coincide exactly with their Haar-twirled values after one update while the next-order defect retains exponentially decaying memory.

1.3 Specialization to a logical one-qubit Clifford active sector

The single-qubit Clifford group is an exact unitary 3-design but not a 4-design [1, 2, 3]. This design hierarchy supplies the exact mechanism that separates moment orders in the construction.

For the number-conserving construction below, the compact enclosing group is represented block-diagonally as $K = U(2) \times U(1)$ on a three-dimensional single-particle sector $\mathcal{H}_{ab} \oplus \mathcal{H}_c$, where $\mathcal{H}_{ab}$ is the two-dimensional active subspace and $\mathcal{H}_c$ the one-dimensional spectator subspace. Let $\tilde{\mathcal{C}}_1 \subset U(2)$ denote a phase-complete finite single-qubit Clifford group and let $C_8 = \{e^{i\pi m/4}\}_{m=0}^7$. The discrete branch is the finite subgroup

$$G = \{C \oplus \omega : C \in \tilde{\mathcal{C}}_1, \omega \in C_8\} \subset K. \quad (12)$$

The implementation code uses 24 projective Clifford representatives together with an independent C_8 spectator phase. This is the pushforward of uniform measure on $\tilde{\mathcal{C}}_1 \times C_8$ after quotienting the common global phase of the full three-dimensional block: if $C = \zeta C_0$ with C_0 a chosen projective representative and $\zeta \in C_8$, then

$$(\zeta C_0) \oplus \omega = \zeta (C_0 \oplus (\omega/\zeta)), \quad (13)$$

and the common prefactor cancels from density-matrix conjugation while ω/ζ remains uniform in C_8 . Hence the 192-trajectory projective enumeration implements the same moment channels as normalized Haar measure on G .

Supplementary Lemma 1 (Relative 3-design of the block ensemble). *In the representation above,*

$$\mathcal{T}_k^G = \mathcal{T}_k^K, \quad k = 1, 2, 3, \quad (14)$$

but

$$\mathcal{T}_4^G \neq \mathcal{T}_4^K. \quad (15)$$

Thus G is an exact unitary 3-design relative to K in this representation, but not a 4-design.

Proof. Consider a matrix-unit term in the operator space of $(\mathcal{H}_{ab} \oplus \mathcal{H}_c)^{\otimes k}$, and let j and j' be the numbers of active-sector factors on its ket and bra sides. Its relative active-spectator phase harmonic is determined by $\Delta = j - j'$, with $|\Delta| \leq k$. Averaging the relative phase over C_8 gives

$$\frac{1}{8} \sum_{m=0}^7 e^{i\pi m \Delta/4} = \delta_{\Delta,0} \quad (|\Delta| \leq 4), \quad (16)$$

which is exactly the same selection rule as the continuous $U(1)$ phase average through fourth order. Hence, for $k \leq 4$, both the discrete and continuous block twirls retain only sectors with $j = j'$.

On a surviving sector with $j = j'$, the active part of the discrete twirl is the j th moment channel of the single-qubit Clifford group, while the continuous twirl is the corresponding Haar- $U(2)$ channel.

For $k \leq 3$, one has $j \leq 3$, and the Clifford 3-design property makes these active channels identical. Therefore $\mathcal{T}_k^G = \mathcal{T}_k^K$ for $k = 1, 2, 3$.

At $k = 4$, the C_8 phase average still agrees exactly with the continuous phase average, including the $\Delta = \pm 4$ harmonics, which vanish in both cases. Restricting the input operator to $\mathcal{H}_{ab}^{\otimes 4}$ removes the spectator sector entirely, so the two block twirls reduce respectively to the fourth Clifford and Haar- $U(2)$ moment channels. These channels are unequal because the single-qubit Clifford group is not a unitary 4-design. Hence $\mathcal{T}_4^G \neq \mathcal{T}_4^K$. The fourth-order defect therefore originates in the active Clifford sector rather than in discretization of the spectator phase. $\square$

2 Supplementary Note 2: Exact number-conserving construction and low-order matching

2.1 Three-site single-particle sector

Consider three qubits with total occupation operator

$$Q = n_1 + n_2 + n_3, \quad (17)$$

where n_i is the occupation-number operator of site i . We work in the $Q = 1$ sector

$$\mathcal{H}_1 = \text{span}\{|a\rangle, |b\rangle, |c\rangle\}, \quad (18)$$

where

$$|a\rangle = |100\rangle, \quad |b\rangle = |010\rangle, \quad |c\rangle = |001\rangle. \quad (19)$$

For fixed active weight $q \in [0, 1]$, let θ and ϕ denote the polar and azimuthal angles of the active Bloch sphere and let χ denote the relative active-spectator phase. Define

$$|\psi(q, v, \chi)\rangle = \sqrt{q}|v\rangle + \sqrt{1-q}e^{i\chi}|c\rangle, \quad (20)$$

with

$$|v\rangle = \cos \frac{\theta}{2}|a\rangle + e^{i\phi} \sin \frac{\theta}{2}|b\rangle. \quad (21)$$

The associated active Bloch vector is

$$\mathbf{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (22)$$

2.2 Number-conserving random gate

On the $Q = 1$ sector define

$$U_{\text{eff}} = U_{ab} \oplus e^{i\varphi}, \quad (23)$$

where $U_{ab} \in U(2)$ acts on $\text{span}\{|a\rangle, |b\rangle\}$ and the spectator state receives a phase. Extending U_{eff} block-diagonally to all occupation sectors gives a three-qubit unitary satisfying

$$[U, Q] = 0. \quad (24)$$

In the discrete branch U_{ab} is sampled uniformly from the logical single-qubit Clifford group on $\text{span}\{|a\rangle, |b\rangle\}$ and $e^{i\varphi}$ uniformly from $C_8 = \{e^{i\pi m/4}\}_{m=0}^7$. This encoded Clifford action is distinct from the physical three-qubit Clifford group; stabilizer magic throughout is defined with the physical three-qubit Pauli group. In the continuous branch, U_{ab} is Haar distributed on $U(2)$ and φ is uniform on $[0, 2\pi)$.

2.3 Phase-twirl selection rule

Write $\rho = |\psi\rangle\langle\psi|$ as active, spectator and cross blocks. In an expansion of $\rho^{\otimes k}$, a term acquires a phase factor $e^{i\Delta\varphi}$ where Δ equals the difference between spectator ket and bra occurrences. The C_8 average gives

$$\frac{1}{8} \sum_{m=0}^7 e^{i\pi m \Delta/4} = \begin{cases} 1, & \Delta \equiv 0 \pmod{8}, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

For $k \leq 4$, $|\Delta| \leq 4$, so the only surviving terms have $\Delta = 0$. Thus the discrete C_8 phase average reproduces the continuous $U(1)$ selection rule through fourth order, so the fourth-order separation below is attributable to the active Clifford sector rather than spectator-phase discretization.

Supplementary Theorem 1 (Exact matching through third order). *Let A and B be two initial states $|\psi(q, v_X, \chi_X)\rangle$ with the same active weight q . After one Clifford- C_8 twirl,*

$$\Omega_k^A = \Omega_k^B, \quad k = 1, 2, 3. \quad (26)$$

If the same random quantum channel is subsequently applied to both ensembles, the equality remains true at all later times for the corresponding k th moment operators.

Proof. After the phase twirl, every surviving term in $\rho^{\otimes k}$ can be grouped by the number j of active density-matrix factors. For $k \leq 3$, one has $j \leq 3$. Its active part is an average of the form

$$\mathbb{E}_C \left[(C|v\rangle\langle v|C^\dagger)^{\otimes j} \right]. \quad (27)$$

Because the single-qubit Clifford group is an exact unitary 3-design, this equals the Haar average for every $j \leq 3$ and is independent of the input Bloch direction v . All remaining coefficients depend only on q . Hence states A and B with the same q produce identical k th moment operators for $k \leq 3$. A later common random channel acts as the same linear map on both k th moment operators, preserving equality. $\square$

For the explicit states used in the manuscript, direct finite enumeration gives the numerical check in Supplementary Table 1.

Supplementary Table 1. Numerical verification of copy-moment matching for the explicit input pair.

k	$\ \Omega_k^A - \Omega_k^B\ _F$	interpretation
1	7.28×10^{-16}	roundoff
2	9.49×10^{-16}	roundoff
3	7.77×10^{-16}	roundoff
4	$5.6350057356 \times 10^{-3}$	finite separation

Here $\|\cdot\|_F$ denotes the Frobenius norm. The first three lines provide finite-precision checks of the exact theorem.

3 Supplementary Note 3: Fourth-moment magic and the exact Mpemba solution

3.1 Stabilizer purity and stabilizer Rényi entropy

For an N -qubit pure state, we use the second stabilizer Rényi entropy introduced in Ref. [4],

$$M_2(|\psi\rangle) = -\log_2 Z_2(|\psi\rangle), \quad (28)$$

with

$$Z_2(|\psi\rangle) = 2^{-N} \sum_{P \in \mathcal{P}_N} \langle \psi | P | \psi \rangle^4. \quad (29)$$

Here $\mathcal{P}_N = \{I, X, Y, Z\}^{\otimes N}$ is the phase-free N -qubit Pauli set. Equivalently, with $\rho_\psi = |\psi\rangle\langle\psi|$,

$$Z_2(\psi) = \text{Tr} \left[Q_4 \rho_\psi^{\otimes 4} \right], \quad Q_4 = 2^{-N} \sum_{P \in \mathcal{P}_N} P^{\otimes 4}. \quad (30)$$

Thus the stabilizer purity is linear in the fourth moment operator even though M_2 is nonlinear in the state.

We distinguish two ensemble averages:

$$M_2^{\text{ann}} = -\log_2 \mathbb{E}[Z_2], \quad M_2^{\text{que}} = \mathbb{E}[-\log_2 Z_2]. \quad (31)$$

3.2 Closed form for the Clifford and Haar averages

Define the fourth-order octahedral invariant

$$s_4(\mathbf{r}) = r_x^4 + r_y^4 + r_z^4. \quad (32)$$

After spectator-phase averaging, the Clifford-averaged stabilizer purity is an octahedrally invariant polynomial in $\mathbf{r}$ of degree at most four. On the unit sphere, the independent invariants at this degree reduce to a constant and s_4 , so

$$\overline{Z}_2^{\text{C}}(q, \mathbf{r}) = A(q) + B(q)s_4(\mathbf{r}). \quad (33)$$

It suffices to evaluate two representative directions. For an axis direction, $s_4 = 1$ and direct Pauli summation gives

$$F_{\text{ax}}(q) = 10q^4 - 20q^3 + 14q^2 - 4q + 1. \quad (34)$$

For the body diagonal $\mathbf{r} = (1, 1, 1)/\sqrt{3}$, $s_4 = 1/3$ and

$$F_{\text{diag}}(q) = \frac{29q^4 - 60q^3 + 42q^2 - 12q + 3}{3}. \quad (35)$$

Solving for A and B yields

$$\overline{Z}_2^{\text{C}}(q, \mathbf{r}) = \frac{19q^4 - 40q^3 + 28q^2 - 8q + 2}{2} + \frac{q^4}{2}s_4(\mathbf{r}). \quad (36)$$

For a Haar-random qubit Bloch vector, rotational symmetry gives

$$\mathbb{E}_{\text{H}}[s_4] = 3\mathbb{E}[r_z^4] = \frac{3}{5}, \quad (37)$$

so

$$\overline{Z}_2^{\text{H}}(q) = \frac{49q^4 - 100q^3 + 70q^2 - 20q + 5}{5}. \quad (38)$$

Subtracting Eqs. (38) from (36) gives the central fourth-moment memory:

$$\overline{Z}_2^{\text{C}} - \overline{Z}_2^{\text{H}} = \frac{q^4}{2} \left(s_4 - \frac{3}{5} \right). \quad (39)$$

The active direction is therefore erased from moments through third order by Theorem 1, while it survives in the fourth-order stabilizer sector through the single invariant s_4 .

3.3 Explicit common-target Mpemba pair

Fix $q = 2/3$ and choose

$$A : (\theta_A, \phi_A, \chi_A) = \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4} \right), \quad (40)$$

$$B : (\theta_B, \phi_B, \chi_B) = \left(\frac{3\pi}{4}, \frac{7\pi}{4}, 0 \right). \quad (41)$$

Their active invariants are

$$s_{4,A} = \frac{1}{2}, \quad s_{4,B} = \frac{3}{8}. \quad (42)$$

Direct evaluation of the 64 Pauli terms gives

$$Z_{2,A}(0) = \frac{37}{81}, \quad Z_{2,B}(0) = \frac{4(12 - \sqrt{2})}{81}. \quad (43)$$

The Clifford means follow from Eq. (36),

$$\overline{Z}_{2,A}^C = \frac{5}{9}, \quad \overline{Z}_{2,B}^C = \frac{44}{81}, \quad (44)$$

and the common constrained-Haar mean is

$$\overline{Z}_2^H = \frac{229}{405}. \quad (45)$$

The exact inequalities are

$$\frac{37}{81} < \frac{4(12 - \sqrt{2})}{81} < \frac{229}{405}, \quad (46)$$

so initially

$$M_{2,A}(0) > M_{2,B}(0) > M_{2,H}^{\text{ann}}. \quad (47)$$

After the Clifford update,

$$\frac{44}{81} < \frac{5}{9} = \frac{225}{405} < \frac{229}{405}, \quad (48)$$

which reverses the distance ordering while keeping both states on the same side of the common target:

$$M_{2,B}^{\text{ann},C} > M_{2,A}^{\text{ann},C} > M_{2,H}^{\text{ann}}. \quad (49)$$

The corresponding values are summarized in Supplementary Table 2.

Supplementary Table 2. Exact magic values and annealed target distances for the explicit Mpemba pair.

	initial M_2	Clifford M_2^{ann}	distance to annealed Haar target
A	1.1303966373	0.8479969066	0.3078223276 $\rightarrow$ 0.0254225969
B	0.9357934542	0.8804183842	0.1132191445 $\rightarrow$ 0.0578440746

Thus A starts farther from the same magic target but becomes closer after one update.

3.4 Persistent inversion under Clifford–Haar interpolation

Let μ_C and μ_H denote the discrete Clifford–phase measure and the constrained-Haar measure, respectively, and use the per-step mixture $\mu_p = p\mu_C + (1-p)\mu_H$. Proposition 1 gives

$$\bar{Z}_{2,X}(t) = Z_{2,H} + p^t (Z_{2,X}^C - Z_{2,H}), \quad X = A, B, \quad (50)$$

for all integer $t \geq 1$. Since

$$Z_{2,A}^C - Z_{2,B}^C = \frac{1}{81}, \quad (51)$$

we obtain

$$\bar{Z}_{2,A}(t) - \bar{Z}_{2,B}(t) = \frac{p^t}{81} > 0 \quad (52)$$

for $p > 0$. Because $-\log_2 x$ is decreasing,

$$M_{2,A}^{\text{ann}}(t) < M_{2,B}^{\text{ann}}(t), \quad t \geq 1. \quad (53)$$

For a genuine relaxation process we take $0 < p < 1$: both post-update trajectories then converge monotonically to the same Haar target from the same side. The inversion is generated at the first update, after which the exact construction gives both inputs a common geometric decay factor and hence a common long-time exponent. At $p = 0$ both inputs reach the Haar target immediately, while at $p = 1$ the discrete-branch residual remains fixed under this interpolation.

3.5 Exact Clifford trajectory distributions and quenched inversion

Complete enumeration of the 24 projective Clifford representatives and eight spectator phases gives 192 trajectories per input state. For A ,

$$\Pr\left(Z_2 = \frac{37}{81}\right) = \frac{1}{3}, \quad \Pr\left(Z_2 = \frac{49}{81}\right) = \frac{2}{3}, \quad (54)$$

so

$$M_{2,A}^{\text{que},C} = -\frac{1}{3} \log_2 \frac{37}{81} - \frac{2}{3} \log_2 \frac{49}{81} = 0.8602256516. \quad (55)$$

For B ,

$$\Pr\left(Z_2 = \frac{2(21 - \sqrt{2})}{81}\right) = \frac{1}{3}, \quad (56)$$

$$\Pr\left(Z_2 = \frac{4(12 - \sqrt{2})}{81}\right) = \frac{1}{6}, \quad (57)$$

$$\Pr\left(Z_2 = \frac{2(21 + \sqrt{2})}{81}\right) = \frac{1}{3}, \quad (58)$$

$$\Pr\left(Z_2 = \frac{4(12 + \sqrt{2})}{81}\right) = \frac{1}{6}, \quad (59)$$

which gives

$$M_{2,B}^{\text{que},C} = 0.8888664398. \quad (60)$$

The deterministic Haar integral below gives the common quenched target

$$M_{2,H}^{\text{que}} = 0.8434111519421856. \quad (61)$$

Hence

$$M_{2,A}(0) > M_{2,B}(0) > M_{2,H}^{\text{que}}, \quad (62)$$

while

$$M_{2,B}^{\text{que},C} > M_{2,A}^{\text{que},C} > M_{2,H}^{\text{que}}. \quad (63)$$

The same common-target Mpemba inversion therefore holds under quenched averaging. Since the accumulated distribution is a convex mixture of the discrete and Haar distributions,

$$M_{2,X}^{\text{que}}(t) = M_{2,H}^{\text{que}} + p^t \left(M_{2,X}^{\text{que},C} - M_{2,H}^{\text{que}} \right), \quad (64)$$

and

$$M_{2,B}^{\text{que}}(t) - M_{2,A}^{\text{que}}(t) = 0.0286407882 p^t > 0. \quad (65)$$

4 Supplementary Note 4: Numerical checks and robustness

4.1 Deterministic Haar quadrature

The deterministic quadrature used for the quenched Haar target is specified in the Methods section of the main manuscript. Writing $u = \cos \theta$ for the polar coordinate of the active Bloch sphere, let n_u be the Gauss–Legendre order in u and let n_ϕ and n_χ be the uniform angular quadrature orders for the active azimuth and spectator phase. Supplementary Table 3 reports convergence through $(n_u, n_\phi, n_\chi) = (48, 96, 96)$.

Supplementary Table 3. Deterministic convergence of the quenched Haar target. The final row is used as the reference value.

(n_u, n_ϕ, n_χ)	$M_{2,H}^{\text{que}}$	absolute error	$\mathbb{E}[Z_2]$
(8, 16, 16)	0.843416953825	5.80×10^{-6}	0.565432098765
(12, 24, 24)	0.843411164614	1.27×10^{-8}	0.565432098765
(16, 32, 32)	0.843411151976	3.34×10^{-11}	0.565432098765
(24, 48, 48)	0.843411151942	2.11×10^{-15}	0.565432098765
(32, 64, 64)	0.843411151942	1.33×10^{-15}	0.565432098765
(48, 96, 96)	0.843411151942	0	0.565432098765

The annealed Haar mean equals 229/405 to numerical precision at every listed order.

The reported value differs from the highest-order reference by only 3.34×10^{-11} in the quenched target, far below the precision relevant to the manuscript.

4.2 Open neighborhood of Mpemba pairs

At the central pair, all inequalities required for the common-target inversion are strict. Initial stabilizer purity, the Clifford averages, and the Haar target are continuous functions of the state parameters. Therefore an open neighborhood of the central pair within the fixed- q angular manifold inherits the same ordering by continuity.

The robustness scan defined in the Methods section of the main manuscript uses seed 20260803 and independently perturbs the six angles of the two states uniformly within ± 0.08 rad at fixed

$q = 2/3$. For each sampled pair we define

$$\Delta_0 = M_{2,A}(0) - M_{2,B}(0), \quad (66)$$

$$\Delta_{\text{ann}} = M_{2,B}^{\text{ann},C} - M_{2,A}^{\text{ann},C}, \quad (67)$$

$$\Delta_{\text{que}} = M_{2,B}^{\text{que},C} - M_{2,A}^{\text{que},C}, \quad (68)$$

and use the same-side margin

$$\Delta_{\text{side}} = \min\left\{M_{2,B}(0) - M_{2,H}^{\text{que}}, M_{2,A}^{\text{ann},C} - M_{2,H}^{\text{ann}}, M_{2,A}^{\text{que},C} - M_{2,H}^{\text{que}}\right\}. \quad (69)$$

The first term uses the larger quenched target, so positivity also implies that both initial states lie above the annealed target; positivity of the crossing margins then places both post-update states on the same respective side. All 200 samples retain $\Delta_0 > 0$, $\Delta_{\text{ann}} > 0$, $\Delta_{\text{que}} > 0$, and $\Delta_{\text{side}} > 0$. The minimum observed margins are

$$\Delta_{0,\min} = 0.1254074, \quad (70)$$

$$\Delta_{\text{ann},\min} = 0.0190518, \quad (71)$$

$$\Delta_{\text{que},\min} = 0.0184625, \quad (72)$$

$$\Delta_{\text{side},\min} = 0.0117649. \quad (73)$$

The open-region existence follows from continuity, while the scan illustrates its finite width.

4.3 Robustness to gate-set perturbations

We perturb the implemented active gates themselves to quantify stability away from the exact Clifford-design point. For each calibration pattern and each of the 24 fixed Clifford representatives C_j , draw an independent isotropic Bloch-sphere axis $\mathbf{n}_j$ and replace

$$C_j \mapsto R_j(\epsilon)C_j, \quad R_j(\epsilon) = \exp\left[-\frac{i\epsilon}{2}\mathbf{n}_j \cdot \boldsymbol{\sigma}\right]. \quad (74)$$

The axes are held fixed within a calibration pattern, while the Clifford label and the independent C_8 spectator phase are sampled as in the exact construction. We use 200 patterns with seeds 20260900–20261099. The ordered Clifford list is generated deterministically from I by breadth-first closure under (HC, SC, CH, CS) , where H and S are the Hadamard and phase gates, with each matrix canonicalized by removing the phase of its first nonzero entry and duplicates skipped in order of first appearance. For each calibration seed, NumPy’s `default_rng(seed)` generates a $(24, 3)$ array of independent standard-normal vectors, which are normalized row by row and paired indexwise with the ordered Clifford list. This rule, together with the stated seed, fixes every perturbation axis and gate used in the scan. Gate dependence of R_j generically removes both the exact 3-design identity and idempotence of the discrete measure.

For each pattern we evaluate the exact 192-output one-step averages and require positive annealed and quenched crossing margins together with positive same-side margins. The results are reported in Supplementary Table 4.

Supplementary Table 4. One-step robustness to coherent gate-set perturbations across 200 calibration patterns.

ϵ (rad)	successful patterns	min. Δ_{ann}	min. Δ_{que}	min. same-side margin
0.02	200/200	0.02486	0.02137	0.01311
0.05	200/200	0.01334	0.01030	0.00753
0.08	199/200	0.00163	-0.00093	0.00193
0.10	195/200	-0.00625	-0.00847	-0.00180

Thus the exact-design point has a finite numerical neighborhood in gate space for the tested perturbation model. At $\epsilon = 0.05$ rad, the low-order moment equality is already visibly broken: across the 200 patterns the median Frobenius residuals

$$\|\Omega_k^A - \Omega_k^B\|_F, \quad k = 1, 2, 3, \quad (75)$$

are respectively 5.00×10^{-3} , 6.49×10^{-3} , and 7.49×10^{-3} ; the corresponding maxima are 1.25×10^{-2} , 1.26×10^{-2} , and 1.34×10^{-2} . Thus the tested inversion persists in a regime where low-order moment residuals are finite, quantifying tolerance to small low-order contamination around the exact design point.

A representative multi-step calculation probes persistence of the perturbed inversion beyond the first update. Let ν_ϵ denote the perturbed discrete measure. With the same Haar branch as in the ideal interpolation,

$$[p\nu_\epsilon + (1-p)\mu_H]^{*t} = p^t\nu_\epsilon^{*t} + (1-p^t)\mu_H. \quad (76)$$

We use $\epsilon = 0.05$, calibration seed 20260911, $p = 0.8$, trajectory seed 20260999, and 10^5 paired trajectories to sample the conditional product ν_ϵ^{*t} . Because ν_ϵ is not idempotent, this conditional distribution changes with t ; the resource curves therefore combine the common Haar weight p^t with the evolving conditional distribution ν_ϵ^{*t} . The common-target distances are reported in Supplementary Tables 5 and 6.

Supplementary Table 5. Multistep annealed target distances under the representative coherent gate perturbation.

t	D_A^{ann}	D_B^{ann}	$D_B^{\text{ann}} - D_A^{\text{ann}}$	standard error (s.e.)
1	0.02021	0.04537	0.02516	0.00077
2	0.01639	0.03618	0.01978	0.00061
4	0.00960	0.02277	0.01317	0.00039
6	0.00641	0.01404	0.00762	0.00025
8	0.00408	0.00889	0.00481	0.00016

Supplementary Table 6. Multistep quenched target distances under the representative coherent gate perturbation.

t	D_A^{que}	D_B^{que}	$D_B^{\text{que}} - D_A^{\text{que}}$	s.e.
1	0.01333	0.03566	0.02233	0.00077
2	0.01107	0.02859	0.01751	0.00062
4	0.00623	0.01810	0.01187	0.00040
6	0.00431	0.01116	0.00684	0.00026
8	0.00277	0.00710	0.00434	0.00017

Here $D_X^{\text{ann}} = |M_{2,X}^{\text{ann}} - M_{2,H}^{\text{ann}}|$ and $D_X^{\text{que}} = |M_{2,X}^{\text{que}} - M_{2,H}^{\text{que}}|$. Both trajectories remain above their respective common targets throughout the tested window, and $D_A < D_B$ at every listed time. The smallest same-side distances occur at $t = 8$, where $D_A^{\text{ann}} = 0.00408$ and $D_A^{\text{que}} = 0.00277$. These calculations quantify multi-step robustness for the specified coherent-error model.

5 Supplementary Note 5: Attenuation under overlapping scrambling

5.1 Overlapping number-conserving circuit

The overlapping-circuit construction and paired-sampling protocol are specified in the Methods section of the main manuscript. Here we report the resulting finite-size calculations. The observable is the trajectory-wise difference

$$\Delta M_2 = M_{2,A} - M_{2,B}. \quad (77)$$

A negative value corresponds to the post-inversion ordering of the exact three-site construction.

5.2 L=6 high-statistics calculation

Two independent batches of 10,000 paired trajectories give the combined 20,000-trajectory estimate in Supplementary Table 7.

Supplementary Table 7. Overlapping-circuit ordering signal for $L = 6$ from 20,000 paired trajectories.

layer	$\mathbb{E}[\Delta M_2]$	standard error
1	-0.0426583	0.0032807
2	-0.0204867	0.0031773
3	-0.0087667	0.0035795

The full-cycle mean has only about 20% of the first-layer magnitude. The sign remains negative at $L = 6$, but the exact design-limited inversion is strongly attenuated by the two subsequent overlapping layers.

5.3 L=9 auxiliary consistency check

This lower-statistics calculation provides an auxiliary finite-size consistency check. Four independent batches of 100 paired trajectories give the results in Supplementary Table 8.

Supplementary Table 8. Auxiliary overlapping-circuit ordering signal for $L = 9$ from 400 paired trajectories.

layer	$\mathbb{E}[\Delta M_2]$	standard error
1	-0.0554146	0.0275722
2	-0.0249067	0.0279521
3	+0.0045984	0.0325510

The complete-cycle value is statistically consistent with zero. Together with the $L = 6$ calculation, this auxiliary sample characterizes attenuation of the post-inversion ordering over the tested finite sizes.

5.4 Finite-size interpretation

The exact analytical result is the moment-selective Mpemba inversion in the design-limited number-conserving construction. The overlapping-circuit data characterize its finite-size response to spatial remixing: the post-inversion trajectory-level ordering signal is strongly suppressed over the reported circuits and sizes.

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