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A Comprehensive Reliability Assessment of the Kumaraswamy Inverse Weibull Distribution with Application to Electromigration Failure-Time Data

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The Kumaraswamy Inverse Weibull (Kw–IW) distribution is a flexible lifetime model capable of representing highly skewed data, heavy-tailed behaviour, and complex hazard-rate patterns encountered in reliability and failure-time analysis. This paper presents a comprehensive assessment of the Kw–IW distribution, focusing on its structural properties, inferential characteristics, and empirical performance in modelling electromigration failure times. Maximum likelihood estimation is employed to estimate the model parameters, while its goodness of fit is evaluated and compared with several competing inverse-Weibull-type distributions using likelihood-based criteria, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Hannan–Quinn Information Criterion (HQIC), together with the Kolmogorov–Smirnov (KS), Cramér–von Mises (CvM), and Anderson–Darling (AD) goodness-of-fit statistics. The Kw–IW distribution is applied to a classical dataset comprising 59 failure times from an accelerated life test of conductors in microcircuits, where failures are associated with electromigration. The results demonstrate the suitability of the Kw–IW distribution for describing the observed failure-time behaviour and indicate its competitive performance relative to the alternative models considered. Graphical analyses further support the model's ability to capture the distributional characteristics and tail behaviour of the electromigration failure times. These findings demonstrate the potential of the Kw–IW distribution as a flexible and useful model for reliability analysis of electronic components and microcircuits subject to electromigration-related failures.

Keywords: Kumaraswamy inverse Weibull distribution; Reliability modelling; Lifetime analysis; Maximum likelihood estimation; Information criteria; Goodness-of-fit; Electromigration failure times; Heavy-tailed distributions.

1. Introduction

Lifetime and reliability data encountered in engineering, materials science, and industrial applications frequently exhibit strong right-skewness, heavy tails, and non-monotonic hazard-rate patterns. Classical lifetime models such as the inverse Weibull (IW) distribution have been widely used to analyse such data due to their ability to capture decreasing and unimodal hazard shapes. However, the two-parameter IW distribution often lacks the flexibility required to model complex empirical behaviours observed in modern reliability datasets, particularly those involving asymmetric degradation, extreme values, or multimodal hazard structures. To address these limitations, several extensions of the IW distribution have been proposed, including the generalised inverse Weibull, modified inverse Weibull, and transmuted inverse Weibull families. While these models offer greater flexibility, they still struggle to accommodate heavy tails, strong skewness, and diverse hazard-rate shapes simultaneously. The Kumaraswamy transformation provides a powerful mechanism for enhancing baseline distributions by introducing additional shape parameters while preserving analytical tractability. When applied to the IW distribution, it yields the Kumaraswamy inverse Weibull (Kw-IW) distribution, which offers substantially greater shape flexibility and improved control over tail behaviour. Despite its theoretical appeal, the Kw-IW distribution has not been comprehensively studied with respect to its structural properties, inferential behaviour, or practical performance in reliability modelling.

This study aims to investigate the Kumaraswamy–Inverse Weibull (Kw-IW) distribution as a flexible statistical model for lifetime and reliability data exhibiting skewness, heavy tails, and complex hazard rate structures. To achieve this aim, the study first establishes the theoretical properties of the Kw-IW distribution by deriving its fundamental functions, including the probability density, cumulative distribution, survival, hazard, and quantile functions. Subsequently, several important statistical properties, such as moments, probability weighted moments, incomplete moments, geometric and harmonic means, and entropy measures, are obtained. The study further develops a maximum likelihood estimation framework by deriving the corresponding score equations and observed information matrix for statistical inference. Finally, the practical utility and effectiveness of the Kw-IW distribution are demonstrated through an application to real reliability data involving reliability analysis of electronic components and microcircuits subject to electromigration-related failures, with comparative analyses conducted against

established competing models.

The inverse Weibull (IW) distribution has long been recognised as a flexible model for lifetime and reliability data, particularly in engineering and materials science. Originally introduced by Keller and Kamath (1982), the IW distribution has been applied to model failure times exhibiting decreasing or unimodal hazard rates, making it suitable for mechanical components, fatigue processes, and degradation phenomena. Its tractability and interpretability have motivated numerous methodological extensions aimed at improving its flexibility and tail behaviour.

Several authors have proposed generalisations of the IW distribution to address its limitations in modelling heavy-tailed or highly skewed data. Gusmão et al. (2011) introduced the generalized inverse Weibull (GIW) distribution, which incorporates an additional shape parameter to enhance tail flexibility. Khan and King (2012) proposed the modified inverse Weibull (MIW) distribution, offering improved modelling of asymmetric lifetime data. Khan, King, and Hudson (2014) further extended the IW family by introducing the transmuted inverse Weibull distribution, using a transmutation map to provide additional shape control. These models have been applied in reliability engineering, hydrology, and biomedical survival analysis, demonstrating the ongoing need for flexible IW-based distributions. Despite these developments, many IW extensions remain limited in their ability to simultaneously capture strong skewness, heavy tails, and complex hazard-rate shapes. Electromigration has become one of the most critical reliability challenges in modern electronic components and microcircuits, particularly as device geometries continue to shrink and current densities rise. Failures driven by electromigration typically exhibit strong right-skewness, early-life vulnerability, and heavy-tailed behaviour, making classical lifetime models such as the Weibull or lognormal distributions inadequate for accurately capturing the underlying failure dynamics. As a result, there is a growing need for flexible statistical models capable of representing complex hazard shapes and tail behaviours observed in electromigration-related failure data. The Kumaraswamy Inverse Weibull Distribution (KIWD) offers a promising alternative due to its ability to model non-monotonic hazard rates, upside-down bathtub failure patterns, and extreme skewness commonly present in microelectronic reliability datasets. Although KIWD has been introduced in the statistical literature, its practical value in engineering reliability applications, especially in the context of electromigration, has not been systematically explored.

This study is motivated by the need to evaluate whether KIWD can serve as a robust and flexible lifetime model for electronic components subjected to electromigration stress. By applying KIWD to real failure-time data from microcircuits, estimating its parameters via maximum likelihood, and comparing its performance against established reliability models, the paper aims to demonstrate that KIWD provides superior fit, more realistic hazard representation, and greater modelling flexibility for this important class of failures. The results highlight KIWD as a valuable tool for reliability engineers seeking accurate modelling of electromigration-induced degradation and failure mechanisms.

The Kumaraswamy distribution, originally proposed by Kumaraswamy (1980) for hydrological modelling, has emerged as a powerful generator for constructing flexible families of distributions. Its bounded support and tractable algebraic form make it particularly attractive for generating new lifetime models. Cordeiro and de Castro (2011) formalised the Kumaraswamy-G family, demonstrating that the transformation can significantly improve the flexibility of baseline distributions while preserving closed-form expressions for the density, distribution, and quantile functions. Since then, numerous Kumaraswamy-generated models have been proposed, including the Kumaraswamy Weibull, Kumaraswamy gamma, Kumaraswamy log-logistic, and Kumaraswamy Burr families. These models have been successfully applied in reliability analysis, environmental modelling, and actuarial science. Shahbaz et al. (2012) introduced the Kumaraswamy inverse Weibull (Kw-IW) distribution by applying the Kumaraswamy transformation to the IW baseline. Their work demonstrated that the Kw-IW distribution offers enhanced shape flexibility and improved modelling of heavy-tailed data. However, existing studies on the Kw-IW distribution have been limited to basic properties and simple applications. No comprehensive treatment of its statistical properties, inferential methods, or extensions has been provided in the literature.

Many new families of lifetime distributions have been proposed by modifying the inverse Weibull distribution for lifetime analysis. Let X be a positive continuous random variable. The classical inverse Weibull (IW) distribution, defined by a shape parameter $\alpha > 0$ and a scale parameter $\beta > 0$, has a probability density function

$$g(x; \alpha, \beta) = \alpha \beta^\alpha x^{-(\alpha+1)} \exp \left\{ - \left(\frac{\beta}{x} \right)^\alpha \right\}, \quad (x > 0) \quad (1)$$

The cumulative distribution function corresponding to (1) is given by

$$G(x; \alpha, \beta) = \exp \left\{ - \left(\frac{\beta}{x} \right)^\alpha \right\}. \quad (2)$$

Here $\alpha, \beta > 0$ are the shape and scale parameters of the inverse Weibull distribution. The main aim of this paper is to examine the potential usefulness of the Kumaraswamy inverse Weibull distribution with an application involving reliability analysis of electronic components and microcircuits subject to electromigration-related failures. The Kumaraswamy transformation offers a versatile mechanism for enhancing baseline distributions by introducing two additional shape parameters $a > 0$ and $b > 0$. When this generator is applied to the IW distribution, it produces the Kumaraswamy inverse Weibull (Kw-IW) distribution, first introduced by Shahbaz et al. (2012). This extended family inherits the heavy-tailed behaviour of the IW distribution while gaining additional flexibility through the parameters a and b , making it suitable for modelling diverse lifetime and reliability data.

The remainder of this paper is organised as follows. Section 2 outlines the Kumaraswamy Inverse Weibull (Kw-IW) distribution and summarises its sub-models, illustrated in Figure 1. Section 3 develops a range of new structural properties for the Kw-IW distribution, including closed-form expressions for its moments, probability-weighted moments, and incomplete moments. Section 4 derives additional characterisations through the geometric and harmonic means, while Section 5 formulates the Rényi and q -entropies. Section 6 presents maximum likelihood estimation for the model parameters. Section 7 demonstrates the practical utility of the Kw-IW distribution through a reliability analysis of electronic components and microcircuits subject to an electromigration-related failures dataset, followed by concluding remarks.

2. Kumaraswamy Inverse Weibull distribution

A positive continuous random variable X is said to follow the Kumaraswamy inverse Weibull (Kw-IW) distribution with parameters $a > 0$, $b > 0$, $\alpha > 0$, and $\beta > 0$, if its probability density function is

$$f_{KwIW}(x; a, b, \alpha, \beta) = ab\alpha \beta^\alpha x^{-(\alpha+1)} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{b-1}, \quad (3)$$

The corresponding cumulative distribution function is

$$F(x; a, b, \alpha, \beta) = 1 - \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^b. \quad (4)$$

Here α, b control the shape of the distribution, a and β control the scale of the distribution. If X is a random variable having the Kw inverse Weibull distribution, we can write

$X \sim KwIW(x; a, b, \alpha, \beta)$. This four-parameter family provides a substantial extension of the classical IW distribution, enabling a wide variety of density shapes and hazard-rate patterns. The reliability function (RF) corresponding to (5) is given by

$$R(x; a, b, \alpha, \beta) = 1 - \left[1 - \left\{ 1 - \exp \left\{ -a \left(\frac{\beta}{x} \right)^\alpha \right\} \right\}^b \right], \quad (5)$$

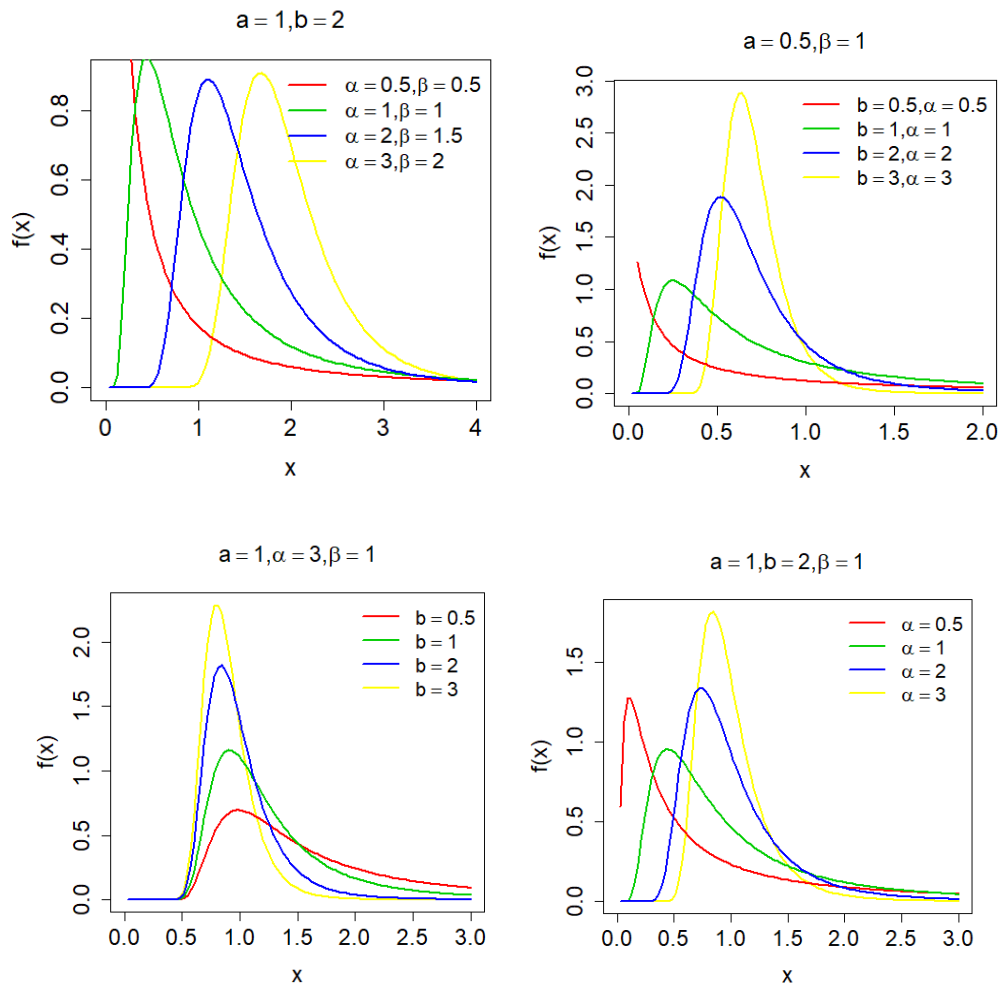


Figure 1: Plots of the *Kw* Inverse Weibull PDF for some selected parameter values.

Figure 1 presents the probability density function (PDF) of the Inverse Weibull distribution for several selected combinations of its shape and scale parameters. The plotted curves highlight the distribution's pronounced right-skewness and its sensitivity to parameter variation. Across all parameter settings, the density exhibits a sharp rise near the origin followed by a rapid decline, producing a long right tail whose thickness depends on the specific parameter values. The figure demonstrates that the shape parameter primarily governs the steepness of the initial peak and the rate of decay,

whereas the scale parameter influences the horizontal stretching of the distribution. As these parameters vary, the PDF transitions between highly concentrated forms characterised by a dominant peak near zero, and more dispersed shapes with heavier tails. Figure 2 displays the hazard function (HF) of the Inverse Weibull distribution for several combinations of its shape and scale parameters. The plotted curves reveal the characteristic non-monotonic failure behaviour associated with this distribution. Specifically, the hazard rate initially increases sharply, peaks, and then decreases over time.

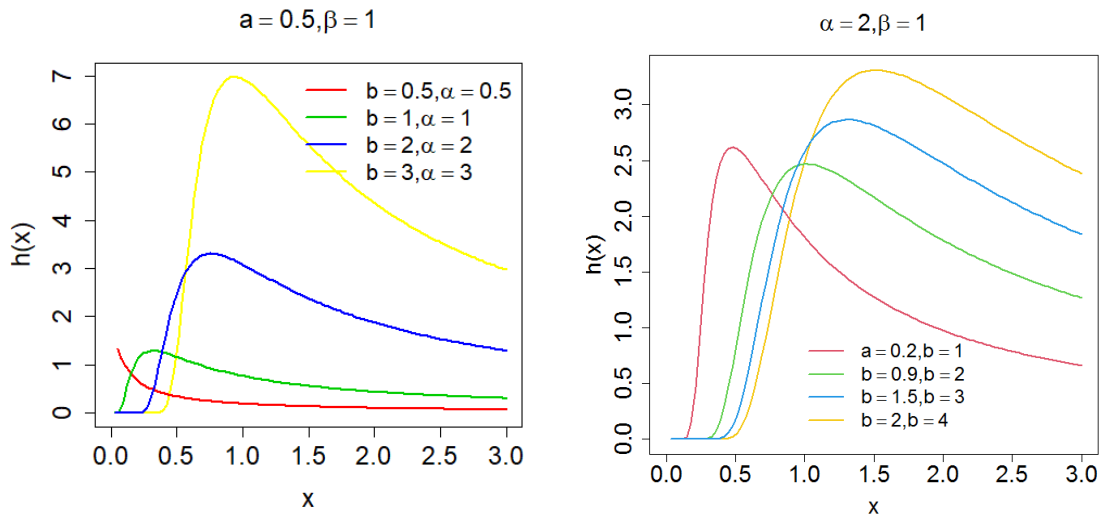


Figure 2: Plots of the Kw Inverse Weibull HF for some selected parameter values.

The hazard function and cumulative hazard function corresponding to (5) are given by

$$h(x; a, b, \alpha, \beta) = \frac{ab\alpha \beta^\alpha x^{-(\alpha+1)} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{b-1}}{1 - \left[1 - \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^b\right]}, \quad (6)$$

The q th quantile x_q of the Kw inverse Weibull random variable given by

$$x_q = \beta \left\{ \frac{a}{-\ln\left(1 - (1-q)^{\frac{1}{b}}\right)} \right\}^{\frac{1}{\alpha}}, \quad 0 < q < 1 \quad (7)$$

This pattern corresponds to an upside-down bathtub-shaped hazard, which is commonly used to model early-life failures followed by a period of decreasing risk. The figure further illustrates how the parameters influence the hazard structure. Variations in the shape parameter modify the steepness of the initial rise and the peak location, while the scale parameter affects the overall spread of the hazard curve along the time axis. As these parameters change, the hazard function

transitions between sharply peaked profiles indicating strong early-life failure intensity and more gradually declining forms that reflect slower deterioration.

Table 1: The Quartiles of the *Kw* Inverse Weibull distribution for selected parameter values

a	b	α	β	Quantiles					
				Q_1	Q_2	Q_3	$I.Q.R$	$Q.D$	$C.Q.D$
1	0.5	1	0.5	0.6048	1.7380	7.7473	7.1425	3.5712	0.8552
1	0.5	1	0.5	0.6048	1.7380	7.7473	7.1425	3.5712	0.8552
1	0.5	1	0.5	0.6048	1.7380	7.7473	7.1425	3.5712	0.8552
1	0.5	1	0.5	0.6048	1.7380	7.7473	7.1425	3.5712	0.8552
1	1	2	0.5	0.4247	0.6006	0.9322	0.5075	0.2538	0.3741
1	1	2	0.5	0.4247	0.6006	0.9322	0.5075	0.2538	0.3741
1	1	2	0.5	0.4247	0.6006	0.9322	0.5075	0.2538	0.3741
1	1	2	0.5	0.4247	0.6006	0.9322	0.5075	0.2538	0.3741
2	2	3	1	1.5847	1.8677	2.2599	0.6752	0.3376	0.1756
2	2	3	1	1.5847	1.8677	2.2599	0.6752	0.3376	0.1756
2	2	3	1	1.5847	1.8677	2.2599	0.6752	0.3376	0.1756
2	2	3	1	1.5847	1.8677	2.2599	0.6752	0.3376	0.1756
2	3	3.5	1	1.5589	1.7555	2.0034	0.4445	0.2222	0.1248
2	3	3.5	1	1.5589	1.7555	2.0034	0.4445	0.2222	0.1248
2	3	3.5	1	1.5589	1.7555	2.0034	0.4445	0.2222	0.1248
2	3	3.5	1	1.5589	1.7555	2.0034	0.4445	0.2222	0.1248

I.Q.R: Inter Quartile Range; *Q.D*: Quartile Deviation; *C.Q.D*: Coefficient of Quartile Deviation

Table 1 reports the lower quartile Q_1 , median Q_2 , and upper quartile Q_3 of the Kumaraswamy Inverse Weibull distribution for several selected combinations of the shape and scale parameters. From these quartile values, we compute three additional dispersion measures: the interquartile range (I.Q.R.), quartile deviation (Q.D.), and coefficient of quartile deviation (C.Q.D.). Together, these statistics provide insight into the distribution's spread and skewness under different parameter settings. Across all parameter combinations, the quartile values exhibit a strong right-skewed pattern, with Q_3 substantially larger than both Q_1 and the median. This behaviour is consistent with the theoretical properties of the Inverse Weibull distribution, which places considerable probability mass near zero while allowing for a long right tail. The I.Q.R values are relatively large for parameter sets associated with heavier tails, indicating greater dispersion. Correspondingly, the Q.D and C.Q.D values decrease as the parameters increase, reflecting reduced spread and a more concentrated distribution.

3. Statistical Properties

This section develops key structural properties of the Kumaraswamy inverse Weibull

(Kw–IW) distribution, including raw moments, probability-weighted moments, and incomplete moments. These results provide the foundation for deriving measures such as mean, variance, skewness, kurtosis, and other functionals of interest.

Theorem 1: If X has the Kw Inverse Weibull distribution, then the k^{th} moment of X say μ_k is given as follows

$$\mu_k = \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{b\beta^k a^{\frac{k}{\alpha}} (-1)^i}{(i+1)^{1-\frac{k}{\alpha}}} \Gamma\left(1 - \frac{k}{\alpha}\right).$$

Proof: By definition

$$\mu_k = \int_0^{\infty} ab\alpha\beta^\alpha x^{k-(\alpha+1)} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{b-1} dx,$$

The above integral reduces to

$$\mu_k = ab\alpha\beta^\alpha \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \int_0^{\infty} x^{k-\alpha-1} \exp\left\{-(i+1)a\left(\frac{\beta}{x}\right)^\alpha\right\} dx,$$

Let $w = (i+1)a\left(\frac{\beta}{x}\right)^\alpha$, then

$$\mu_k = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i b\beta^k \int_0^{\infty} \left(\frac{w}{(i+1)a}\right)^{\frac{k}{\alpha}} \exp(-w) \frac{dw}{(i+1)},$$

Hence, the k^{th} moment of the Kw Inverse Weibull distribution can be obtained as

$$\mu_k = \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{b\beta^k a^{\frac{k}{\alpha}} (-1)^i}{(i+1)^{1-\frac{k}{\alpha}}} \Gamma\left(1 - \frac{k}{\alpha}\right). \quad (8)$$

The important statistical properties of the Kw Inverse Weibull distribution are obtained through moments such as mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis are displayed in Table 2.

Theorem 2: If X has the Kw Inverse Weibull distribution, then the probability weighted moment of X , $\xi_{(k,m)}(x)$ is given as follows

$$\xi_{(k,m)}(x) = \sum_{i,j=0}^{\infty} \binom{m}{i} \binom{bi+b-1}{j} \frac{b\beta^k a^{\frac{k}{\alpha}} (-1)^{i+j}}{(j+1)^{1-\frac{k}{\alpha}}} \Gamma\left(1 - \frac{k}{\alpha}\right)$$

Proof: By definition

$$\xi_{(k,m)}(x) = ab\alpha\beta^\alpha$$

$$\int_0^\infty x^{k-(\alpha+1)} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{b-1} \left(1 - \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^b\right)^m dx$$

We can obtain the expansions for cdf from the above integral reduces to

$$\left(1 - \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^b\right)^m = \sum_{i=0}^{\infty} \binom{m}{i} (-1)^i \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{bi}.$$

$$\xi_{(k,m)}(x) =$$

$$ab\alpha\beta^\alpha \sum_{i=0}^{\infty} \binom{m}{i} (-1)^i \int_0^\infty x^{k-\alpha-1} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{b(i+1)-1} dx$$

and then

$$\xi_{(k,m)}(x) = ab\alpha\beta^\alpha \sum_{i,j=0}^{\infty} \binom{m}{i} \binom{bi+b-1}{j} (-1)^{i+j} \int_0^\infty x^{k-\alpha-1} \exp\left\{-a(j+1)\left(\frac{\beta}{x}\right)^\alpha\right\} dx$$

Setting $u = a(j+1)\left(\frac{\beta}{x}\right)^\alpha$, we can write the above equation as

$$\xi_{(k,m)}(x) = b\beta^k \sum_{i,j=0}^{\infty} \binom{m}{i} \binom{bi+b-1}{j} (-1)^{i+j} \int_0^\infty \left(\frac{w}{(j+1)a}\right)^{-\frac{k}{\alpha}} \exp(-w) \frac{dw}{(j+1)}.$$

Hence, the $\xi_{(k,m)}(x)$ can be finally obtained as

$$\xi_{(k,m)}(x) = \sum_{i,j=0}^{\infty} \binom{m}{i} \binom{bi+b-1}{j} \frac{b\beta^k a^{\frac{k}{\alpha}} (-1)^{i+j}}{(j+1)^{1-\frac{k}{\alpha}}} \Gamma\left(1 - \frac{k}{\alpha}\right). \quad (9)$$

Table 2 presents numerical values of the first four moments of the Kumaraswamy Inverse Weibull (Kw-IW) distribution for selected values of the parameters $\alpha = 2$, $\beta = 0.5$, and $a = 1.5$, under varying choices of the shape parameter b . These moment estimates provide insight into the central tendency, dispersion, and tail behaviour of the distribution. The results show that the mean increases steadily as b increases, indicating that larger values of the Kumaraswamy shape parameter shift the distribution toward higher values of the random variable. The variance also increases with b , reflecting greater spread and heavier tail behaviour. The third and fourth moments grow more rapidly, highlighting the strong right-skewness and pronounced kurtosis characteristic of the Kw-IW distribution. Table 2 demonstrates how the Kumaraswamy generator enhances the flexibility of the baseline Inverse Weibull distribution. As the parameter b increases, the distribution becomes more dispersed and more heavily skewed, with higher-order moments amplifying these effects. These findings reinforce the suitability of the Kw-IW model for applications requiring a distribution capable of capturing substantial asymmetry and heavy-tailed behaviour.

Table 2: Moments for the Kw Inverse Weibull distribution for some selected values of Parameters $\alpha = 2$, $\beta = 0.5$ and $a = 1.5$

μ_k	Parameter					
	$b = 2$	$b = 2.5$	$b = 3$	$b = 4$	$b = 5$	$b = 6$
μ_1	0.6358	0.5719	0.5312	0.4807	0.4498	0.4284
μ_2	0.5198	0.3901	0.3236	0.2548	0.2187	0.1960
μ_3	0.6744	0.3428	0.2354	0.1514	0.1159	0.0963
μ_4	10.332	0.4788	0.2207	0.1032	0.0677	0.0511
Var	0.1156	0.0630	0.0414	0.0237	0.0164	0.0125
SD	0.3399	0.2511	0.2035	0.1540	0.1280	0.1117
CV	53.466	43.899	38.316	32.044	28.454	26.070
CS	0.1970	0.0476	0.0195	0.0061	0.0028	0.0016
CK	9.3874	0.1392	0.0295	0.0052	0.0019	0.0009

Theorem 3: If X has the Kw Inverse Weibull distribution, then the k^{th} incomplete moment of X say $\psi_k(q)$ is given as follows

$$\psi_k(q) = \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{b\beta^k a^{\frac{k}{\alpha}} (-1)^i}{(i+1)^{1-\frac{k}{\alpha}}} \gamma\left(1 - \frac{k}{\alpha}, (i+1)a\left(\frac{\beta}{q}\right)^\alpha\right).$$

Proof: By definition

$$\psi_k(q) = \int_0^q ab\alpha\beta^\alpha x^{k-(\alpha+1)} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{b-1} dx,$$

The above integral reduces to

$$\psi_k(q) = ab\alpha\beta^\alpha \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \int_0^q x^{k-\alpha-1} \exp\left\{-(i+1)a\left(\frac{\beta}{x}\right)^\alpha\right\} dx,$$

Let $w = (i+1)a\left(\frac{\beta}{x}\right)^\alpha$, then

$$\psi_k(q) = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i b\beta^k \int_0^{(i+1)a\left(\frac{\beta}{q}\right)^\alpha} \left(\frac{w}{(i+1)a}\right)^{-\frac{k}{\alpha}} \exp(-w) \frac{dw}{(i+1)},$$

Hence, the k^{th} incomplete moment of the kw Inverse Weibull distribution can be obtained as

$$\psi_k(q) = \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{b\beta^k a^{\frac{k}{\alpha}} (-1)^i}{(i+1)^{1-\frac{k}{\alpha}}} \gamma\left(1 - \frac{k}{\alpha}, (i+1)a\left(\frac{\beta}{q}\right)^\alpha\right). \quad (10)$$

4. Geometric mean and harmonic mean

A random variable X with probability density function from the Kw Inverse Weibull distribution then the geometric mean is defined as

$$G = \int_0^{\infty} \log(x) a b \alpha \beta^{\alpha} x^{-(\alpha+1)} \exp \left\{ -a \left(\frac{\beta}{x} \right)^{\alpha} \right\} \left\{ 1 - \exp \left\{ -a \left(\frac{\beta}{x} \right)^{\alpha} \right\} \right\}^{b-1} dx,$$

Solving the above integral, we obtain

$$G = a b \alpha \beta^{\alpha} \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \int_0^{\infty} \log(x) x^{-(\alpha+1)} \exp \left\{ -(i+1) a \left(\frac{\beta}{x} \right)^{\alpha} \right\} dx,$$

By using the n th derivatives of the gamma function, we obtain

$$\log G = b \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{(-1)^i}{(i+1)} \left\{ \log(\beta) - \Gamma_{(1)}^{\frac{1}{\alpha}} + \log(a(i+1)) \right\},$$

Hence the geometric mean of the Kw Inverse Weibull distribution follows that

$$G = \exp \left\{ b \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{(-1)^i}{(i+1)} \left\{ \log(\beta) - \Gamma_{(1)}^{\frac{1}{\alpha}} + \log(a(i+1)) \right\} \right\}, \quad (11)$$

For a random variable X , the harmonic mean for the Kw Inverse Weibull distribution is defined as

$$\frac{1}{H} = \int_0^{\infty} a b \alpha \beta^{\alpha} x^{-1-(\alpha+1)} \exp \left\{ -a \left(\frac{\beta}{x} \right)^{\alpha} \right\} \left\{ 1 - \exp \left\{ -a \left(\frac{\beta}{x} \right)^{\alpha} \right\} \right\}^{b-1} dx,$$

Solving the above integral, we obtain the harmonic mean of the Kw Inverse Weibull distribution as

$$\frac{1}{H} = \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{b a^{\frac{-1}{\alpha}} (-1)^i}{(i+1)^{1+\frac{1}{\alpha}}} \Gamma \left(1 + \frac{1}{\alpha} \right). \quad (12)$$

Table 3 reports the geometric mean (GM) and harmonic mean (HM) of the Kumaraswamy Inverse Weibull (Kw-IW) distribution for selected values of the parameter b , while keeping $\alpha = 2$, $\beta = 0.5$, and $a = 1.5$ fixed. These two means provide additional insight into the distribution's central tendency, particularly for highly skewed lifetime data where traditional arithmetic moments may be less informative. The results show that both GM and HM increase systematically as the parameter b increases. This behaviour indicates that larger values of the Kumaraswamy shape parameter shift the distribution toward higher values of the random variable, consistent with the trends observed in the raw moments. The harmonic mean remains consistently smaller than the

geometric mean across all parameter settings, reflecting the strong right-skewness of the Kw–IW distribution and the influence of small-valued observations on the HM. Table 3 demonstrates that the geometric and harmonic means respond predictably to changes in the Kumaraswamy parameter b . As b increases, both measures rise, indicating reduced concentration near zero and a heavier contribution from larger values. These findings reinforce the flexibility of the Kw–IW distribution in modelling asymmetric lifetime data and highlight the usefulness of GM and HM as complementary descriptors of its central tendency.

Table 3: GM and HM for the *Kw* Inverse Weibull distribution for some selected values of Parameters $\alpha = 2$, $\beta = 0.5$ and $a = 1.5$

	Parameter					
	$b = 2$	$b = 2.5$	$b = 3$	$b = 4$	$b = 5$	$b = 6$
GM	0.5779	0.5318	0.5005	0.4597	0.4337	0.4152
HM	1.8711	2.0024	2.1072	2.2674	2.3874	2.4826

5. Entropies

Entropy measures quantify the uncertainty or information content associated with a probability distribution. For the Kumaraswamy inverse Weibull (Kw–IW) distribution, both Rényi and Havrda–Charvat (q -) entropies admit tractable analytical expressions. These measures provide insight into the tail behaviour and concentration properties of the distribution.

The entropy of a random variable X is a measure of the uncertainty for the *Kw* Inverse Weibull distribution. The Rényi (1961) entropy is defined as

$$I_R(\rho) = \frac{1}{1-\rho} \log\{\int f(x)^\rho dx\}, \quad (13)$$

where $\rho > 0$ and $\rho \neq 1$. The integral in $I_R(\rho)$ for the *Kw* Inverse Weibull distribution can be defined as

$$\int_0^\infty f(x)^\rho dx = (ab\alpha)^\rho \int_0^\infty \beta^{\rho\alpha} x^{-\rho(\alpha+1)} \exp\left\{-\rho a \left(\frac{\beta}{x}\right)^\alpha\right\} \left\{1 - \exp\left\{-a \left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{\rho(b-1)} dx,$$

We can obtain the expansions for the pdf from the above integral, which reduces to

$$\left\{1 - \exp\left\{-a \left(\frac{\beta}{x}\right)^\alpha\right\}\right\}^{\rho(b-1)} = \sum_{i=0}^{\infty} \binom{\rho(b-1)}{i} (-1)^i \exp\left\{-ai \left(\frac{\beta}{x}\right)^\alpha\right\}.$$

By applying binomial expansions, one can rewrite the integral as

$$\int_0^\infty f(x)^\rho dx = (ab\alpha)^\rho \beta^{\rho\alpha} \sum_{i=0}^{\infty} \binom{\rho(b-1)}{i} (-1)^i \int_0^\infty x^{-\rho(\alpha+1)} \exp\left(-(\rho+i)a \left(\frac{\beta}{x}\right)^\alpha\right) dx,$$

The above integral reduces to

$$\int_0^\infty f(x)^\rho dx =$$

$$(ab\alpha)^\rho \frac{\beta^{\alpha(\rho-1)}}{\alpha} \sum_{i=0}^{\infty} \binom{\rho(b-1)}{i} \frac{(-1)^i \beta^{-\frac{(\alpha+1)(1-\rho)}{\alpha}}}{\{a(\rho+i)\}^{\frac{(\alpha+1)(1-\rho)}{\alpha}+1}} \Gamma\left(\frac{(\alpha+1)(1-\rho)}{\alpha} + 1\right), \quad (14)$$

After some manipulations, the Rényi entropy is obtained as

$$I_R(\rho) = \frac{\rho}{1-\rho} \log(a) + \frac{\rho}{1-\rho} \log(b) - \log(\alpha) - \alpha \log(\beta)$$

$$+ \frac{1}{1-\rho} \log \left\{ \sum_{i=0}^{\infty} \binom{\rho(b-1)}{i} \frac{(-1)^i \beta^{-\frac{(\alpha+1)(1-\rho)}{\alpha}}}{\{a(\rho+i)\}^{\frac{(\alpha+1)(1-\rho)}{\alpha}+1}} \Gamma\left(\frac{(\alpha+1)(1-\rho)}{\alpha} + 1\right) \right\}. \quad (15)$$

The q -entropy was presented by Havrda and Charvat (1967). Ullah (1996) quantified that q -entropy measures are the monotonic functions of the Rényi entropy and is defined as

$$I_H(q) = \frac{1}{q-1} \left\{ 1 - \int_0^\infty f(x)^q dx \right\}, \quad (16)$$

where $q > 0$ and $q \neq 1$. The integral in $I_H(q)$ of the KIW distribution can be defined as

$$\int_0^\infty f(x)^q dx = (ab\alpha)^q \int_0^\infty \beta^{qa} x^{-q(\alpha+1)} \exp \left\{ -qa \left(\frac{\beta}{x} \right)^\alpha \right\} \left\{ 1 - \exp \left\{ -a \left(\frac{\beta}{x} \right)^\alpha \right\} \right\}^{q(b-1)} dx,$$

Using (14) and (16), we obtain the expression of the q -entropy as

$$I_H(q) = \frac{1}{q-1} \left\{ 1 - (ab\alpha)^q \frac{\beta^{\alpha(q-1)}}{\alpha} \sum_{i=0}^{\infty} \binom{q(b-1)}{i} \right.$$

$$\left. \times \frac{(-1)^i \beta^{-\frac{(\alpha+1)(1-q)}{\alpha}}}{\{a(q+i)\}^{\frac{(\alpha+1)(1-q)}{\alpha}+1}} \Gamma\left(\frac{(\alpha+1)(1-q)}{\alpha} + 1\right) \right\}. \quad (17)$$

6. Parameter Estimation

This section proposes the maximum likelihood estimators (MLEs) of the unknown parameters from the Kw Inverse Weibull distribution. Consider the random samples $x_1, x_2, \dots, x_n$ consisting of n observations from the $KwIW(x; a, b, \alpha, \beta)$ distribution based on complete samples. The log-likelihood function $\mathcal{L} = \ln L$ of (3) is given by

$$\mathcal{L} = n \ln a + n \ln b + n \ln \alpha + n \alpha \ln \beta - (\alpha + 1) \sum_{i=1}^n \ln x_i - a \sum_{i=1}^n \left(\frac{\beta}{x_i} \right)^\alpha$$

$$+ (b-1) \sum_{i=1}^n \ln \left\{ 1 - \exp \left\{ -a \left(\frac{\beta}{x_i} \right)^\alpha \right\} \right\}. \quad (18)$$

By differentiating (18) with respect to a, b, α and β then equating it to zero, we obtain the estimating equations are

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{n}{a} - \sum_{i=1}^n \left(\frac{\beta}{x_i}\right)^\alpha + (b-1) \sum_{i=1}^n \frac{\exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\} \left(\frac{\beta}{x_i}\right)^\alpha}{1 - \exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\}},$$

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \ln\left\{1 - \exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\}\right\},$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{n}{\alpha} + n \ln \beta - \sum_{i=1}^n \ln x_i - a \sum_{i=1}^n \left(\frac{\beta}{x_i}\right)^\alpha \ln\left(\frac{\beta}{x_i}\right) + (b-1) \sum_{i=1}^n \frac{\exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\} a \left(\frac{\beta}{x_i}\right)^\alpha \ln\left(\frac{\beta}{x_i}\right)}{1 - \exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\}}$$

and

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{n\alpha}{\beta} - \frac{a\alpha}{\beta} \sum_{i=1}^n \left(\frac{\beta}{x_i}\right)^{\alpha-1} \left(\frac{1}{x_i}\right) + (b-1) \sum_{i=1}^n \frac{\exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\} \frac{a\alpha}{\beta} \left(\frac{\beta}{x_i}\right)^{\alpha-1} \left(\frac{1}{x_i}\right)}{1 - \exp\left\{-a\left(\frac{\beta}{x_i}\right)^\alpha\right\}}$$

Because the score equations do not admit closed-form solutions, numerical optimisation is required. In practice, good starting values can be obtained from the *kw* inverse Weibull MLEs, the log-likelihood is smooth and well-behaved, and optimisation algorithms converge reliably for moderate sample sizes. The resulting MLEs $(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})$ can be used for inference, model comparison, and reliability analysis. Let $\theta = (a, b, \alpha, \beta)^\top$. The observed information matrix is defined as

$$\mathcal{I}(\theta) = -\frac{\partial^2 \ell}{\partial \theta \partial \theta^\top},$$

a 4×4 matrix whose entries are obtained by differentiating the score functions of equation (18). Because the second-order derivatives do not yield closed-form expressions, $\mathcal{I}(\theta)$ is evaluated numerically at the MLEs. Approximate standard errors for the parameter estimates follow from

$$\widehat{\text{Var}}(\hat{\theta}) = \mathcal{I}(\hat{\theta})^{-1}.$$

This matrix plays a central role in inference, model comparison, and constructing confidence intervals. Under standard regularity conditions, the MLEs are asymptotically multivariate normal:

$$\hat{\theta} \sim N_4(\theta, \mathcal{I}(\theta)^{-1}).$$

Thus, a $100(1 - \gamma)\%$ confidence interval for the j -th parameter θ_j is

$$\hat{\theta}_j \pm z_{1-\gamma/2} \sqrt{[\mathcal{I}(\hat{\theta})^{-1}]_{jj}},$$

where $z_{1-\gamma/2}$ is the upper quantile of the standard normal distribution. These intervals are used to assess uncertainty in the parameter estimates and to compare competing lifetime models for the cement-sheath reliability data.

7. Application

The IW and its generalisations have been widely applied in modelling fatigue life, mechanical failure, and material degradation. The electromigration failure-time data reported in Lawless (2003) were used for the analysis. The study concerns electromigration, a failure mechanism caused by the movement of atoms within the conductors of an electrical circuit. The observed failure times are measured in hours, and the dataset contains no censored observations, meaning that the failure time is observed for every conductor. For the 59 conductors, the observed failure times range from 2.997 to 11.038 hours, with a mean failure time of approximately 6.980 hours and a median of 6.923 hours. The sample standard deviation is approximately 1.614 hours, indicating moderate variability in the observed lifetimes. The data provide a useful basis for investigating the distributional characteristics of conductor failure times and for assessing the suitability of alternative lifetime and reliability models. The dataset can therefore be used to demonstrate the application of lifetime distribution models and statistical inference for reliability analysis, particularly in modelling failure times associated with electromigration in microelectronic components.

(1) Generalized inverse Weibull (GIW) distribution

$$f(x) = a\alpha \beta^\alpha x^{-(\alpha+1)} \exp\left\{-a\left(\frac{\beta}{x}\right)^\alpha\right\}, \quad x > 0,$$

where $\alpha > 0$ is the shape parameter and $a, \beta > 0$ are the scale parameters of the GIW distribution proposed by Gusmao et al. (2011).

(2) Modified inverse Weibull (MIW) distribution

$$f(x) = \left\{a + b\alpha \left(\frac{1}{x}\right)^{b-1}\right\} \left(\frac{1}{x}\right)^2 \exp\left\{-\frac{a}{x} - \alpha \left(\frac{1}{x}\right)^b\right\}, \quad x > 0,$$

where $\alpha, \beta > 0$ are the shape parameters and $b > 0$ is the shape parameter of the MIW distribution proposed by Khan and King (2012).

(3) Inverse Weibull (IW) distribution

$$f(x) = \alpha \beta^\alpha x^{-(\alpha+1)} \exp\left\{-\left(\frac{\beta}{x}\right)^\alpha\right\}, \quad x > 0,$$

where $\alpha > 0$ is the shape parameter and $\theta > 0$ is the scale parameter of the IW distribution proposed by Keller and Kamath, (1982).

Table 4: MLEs of the Parameters for electromigration failure-time data with their corresponding SEs in parentheses, AIC, BIC and HQIC goodness-of-fit tests

Model	Parameter Estimates				AIC	BIC	HQIC
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$			
<i>kwIW</i>	4.8487 (2.8101)	132.7577 (245.8133)	0.9159 (0.3216)	7.5045 (3.7030)	231.11	239.42	234.35
<i>GIW</i>	3.1339 (17.3534)	-	3.5111 (0.3008)	4.3147 (6.8064)	254.69	260.93	257.12
<i>MIW</i>	6.5812 (0.8568)	8.6083 (5.4112)	0.5588 (41.4704)	-	353.64	359.87	356.07
<i>IW</i>	-	-	3.5110 (0.3008)	5.9744 (0.2356)	252.69	256.85	254.31

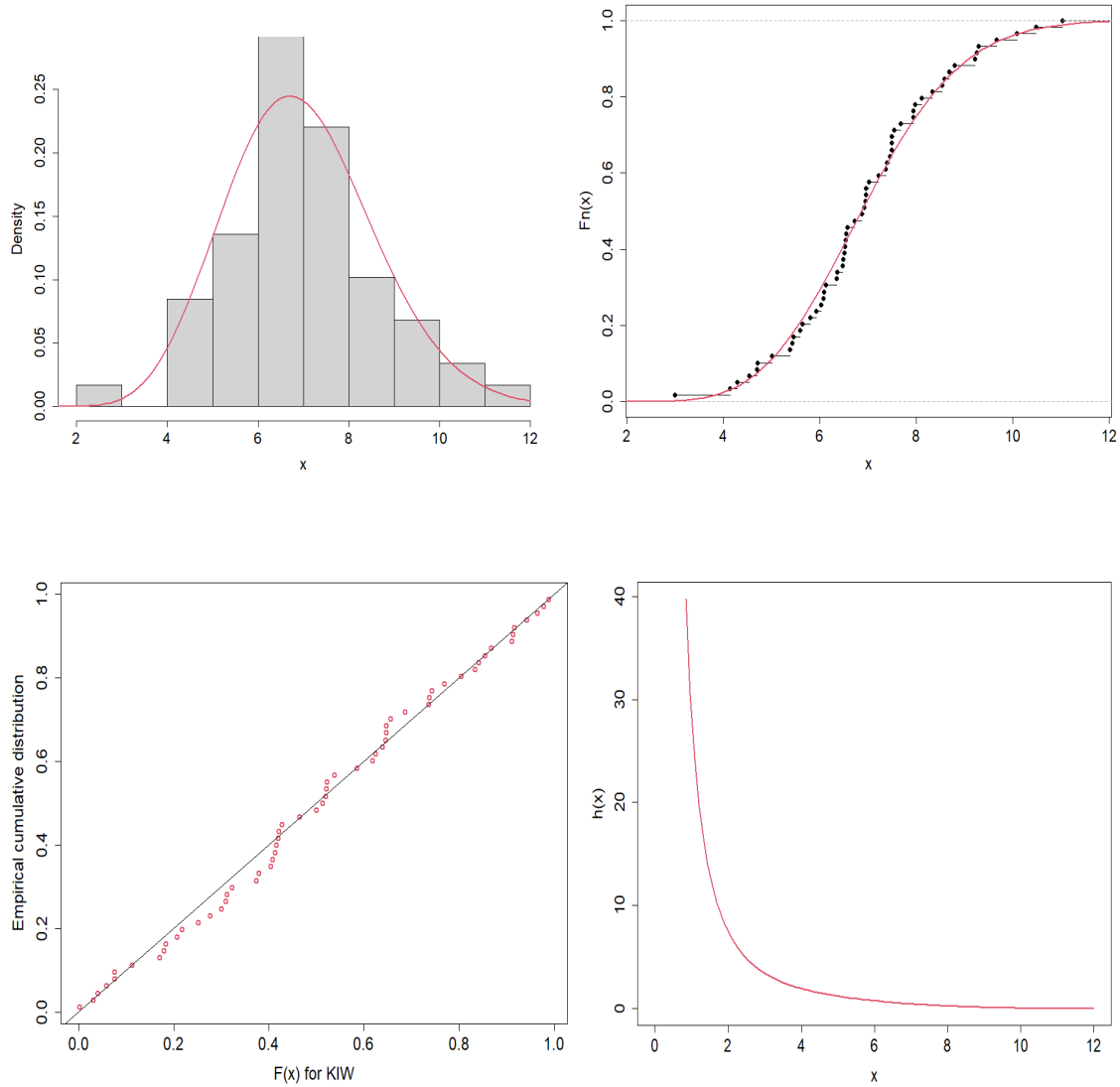


Figure 3: Plots of fitted *kw* Inverse Weibull model for the reliability of electromigration failure-time data.

Figure 3 presents the fitted *kw* Inverse Weibull (IW) model applied to the reliability of electromigration failure-time data. The plot displays the empirical reliability curve derived from the observed electromigration failure-time data alongside the theoretical reliability function

estimated via maximum likelihood. We fitted four distributions, namely: Kumaraswamy inverse Weibull distribution (Shahbaz, et al., 2012), Generalized inverse Weibull distribution proposed by Gusmao et al., (2011), modified inverse Weibull distribution (Khan and King, 2012) and inverse Weibull distribution (Keller and Kamath, 1982) to the reliability of electromigration failure-time data by the method of maximum likelihood. Table 4 presents the maximum likelihood estimates (MLEs) of the parameters for four lifetime distributions fitted to electromigration failure-time data, together with their corresponding standard errors (SEs) and three model-selection criteria: AIC, BIC, and HQIC. The Kumaraswamy Inverse Weibull Distribution (KIWD) was fitted to the electromigration failure-time data using maximum likelihood estimation, and the optimization procedure converged successfully, indicating a well-behaved likelihood surface and stable parameter estimates. The estimated KIWD parameters were $\hat{a} = 4.8487$, $\hat{b} = 132.7577$, $\hat{\alpha} = 0.9159$, and $\hat{\beta} = 7.5045$, with moderate standard errors that confirm numerical reliability. These values reflect a sharply increasing cumulative failure pattern driven by the large Kumaraswamy shape parameter b , combined with a heavy-tailed inverse-Weibull component that aligns with the physics of electromigration-induced degradation. Goodness-of-fit statistics strongly support the adequacy of KIWD: the Cramér–von Mises statistic was extremely small ($W = 0.0379$), the Anderson–Darling statistic was similarly low ($A = 0.2205$), and the Kolmogorov–Smirnov test yielded a very small KS statistic ($D = 0.0689$) with a high p-value (0.9235), indicating that the empirical distribution is statistically indistinguishable from the fitted KIWD model. Information criteria further reinforced KIWD’s superiority, with $AIC = 231.109$, $CAIC = 231.850$, $BIC = 239.419$, and $HQIC = 234.353$ —each substantially lower than those of the competing models. To contextualize KIWD’s performance, three alternative lifetime models commonly used in reliability analysis were also fitted: the Inverse Weibull (IW), Generalized Inverse Weibull (GIW), and Modified Inverse Weibull (MIW) distributions. The IW model provided a moderate fit, with $W = 0.3176$, $A = 1.8883$, and KS p-value = 0.1613, indicating acceptable but noticeably weaker agreement with the empirical data. Its information criteria ($AIC = 252.693$, $BIC = 256.848$) were considerably higher than those of KIWD, confirming reduced parsimony and explanatory power. In contrast, the GIW and MIW models performed extremely poorly. Both models produced very large goodness-of-fit statistics (GIW: $W = 17.1656$, $A = 114.0687$; MIW: $W = 18.5780$, $A = 116.8053$) and catastrophic KS results ($D \approx 1.0$, $p \approx 7.77 \times 10^{-16}$), indicating severe misfit across the entire distribution. Their information criteria were also substantially worse (GIW $AIC = 254.693$; MIW $AIC = 353.638$), and both models exhibited numerical instability, including large or undefined standard errors and failed convergence. Table 5 reports three widely used goodness-of-fit statistics, the Kolmogorov–Smirnov (K-S) test, the Cramér–von Mises (CvM) statistic, and the Anderson–Darling (A-D) statistic, for four lifetime distributions fitted to the electromigration failure-time data. These statistics evaluate how closely each fitted model aligns with the empirical distribution of the observed reliability data.

Lower values indicate a better fit. The K-S statistic measures the maximum deviation between the empirical and fitted cumulative distribution functions. The Kumaraswamy Inverse Weibull (KW-IW) model achieves the smallest K-S value (0.0689), indicating the closest agreement with the empirical distribution. Competing models such as GIW, MIW, and IW show substantially larger deviations, confirming weaker fit. The CvM statistic evaluates the overall squared difference between the empirical and fitted CDFs. Again, the KW-IW model produces the lowest CvM value (0.0379), demonstrating superior global fit. The GIW and IW models show moderate discrepancies, while MIW exhibits the largest CvM value, indicating poor alignment with the data. The A-D statistic places greater weight on tail behaviour, making it particularly relevant for reliability data with heavy tails. The KW-IW model attains the smallest A-D statistic, confirming that it captures both central and tail behaviour more accurately than the alternative models. The MIW and IW models show much larger A-D values, indicating substantial tail misfit. Across all three goodness-of-fit measures, the Kumaraswamy Inverse Weibull distribution consistently outperforms the GIW, MIW, and IW models. Its uniformly smallest K-S, CvM, and A-D statistics demonstrate that it provides the closest approximation to the empirical reliability behaviour of the electromigration failure-time data. These results corroborate the findings from Table 4 and Figure 3, where the KW-IW model also exhibited superior likelihood-based fit and graphical agreement. Collectively, the evidence establishes the KW-IW distribution as the most appropriate and robust model for this dataset.

Table 5: The K-S test, Cramér-von Mises and Anderson-Darling goodness of-fit tests

Model	K-S test	$\mathcal{W}$	$\mathcal{A}$
<i>kwIW</i>	0.0689	0.0379	0.2205
GIW	0.9999	17.1656	114.0687
MIW	0.8887	18.578	116.8053
IW	0.1431	0.3176	1.8883

Generally, the comparative analysis demonstrates that KIWD provides a markedly superior fit to electromigration failure-time data compared with IW, GIW, and MIW. KIWD not only achieves the best goodness-of-fit performance but also exhibits the lowest information criteria and the most stable parameter estimation. Its flexible structure allows it to capture the complex hazard shapes and heavy-tailed behaviour characteristic of electromigration failures, outperforming traditional and generalized inverse-Weibull-type models. These results establish KIWD as the most appropriate and reliable model for characterising electromigration-induced failures in microelectronic components and microcircuits.

7.1 Survival Behaviour and Model Fit

The reliability behaviour of the electromigration failure-time data was examined by comparing the empirical survival function with the fitted Kumaraswamy Inverse Weibull Distribution

(KIWD). Figure 4 illustrates the close agreement between the empirical step-wise survival curve and the smooth KIWD-based parametric estimate. The empirical curve shows a rapid decline in survival probability during the early stages, reflecting the microstructural degradation typical of electromigration. The KIWD curve successfully captures this behaviour, closely following the empirical trajectory throughout the entire time range. The strong alignment between the two curves indicates that the KIWD model provides an accurate representation of both the central tendency and tail behaviour of the failure-time distribution. This agreement validates the suitability of KIWD for modelling electromigration reliability, supporting its use in quantile estimation, mission-time prediction, and accelerated test analysis. The model's ability to reproduce early-life failures and the gradual decline in survival probability demonstrates its effectiveness in characterising the underlying physical degradation mechanisms.

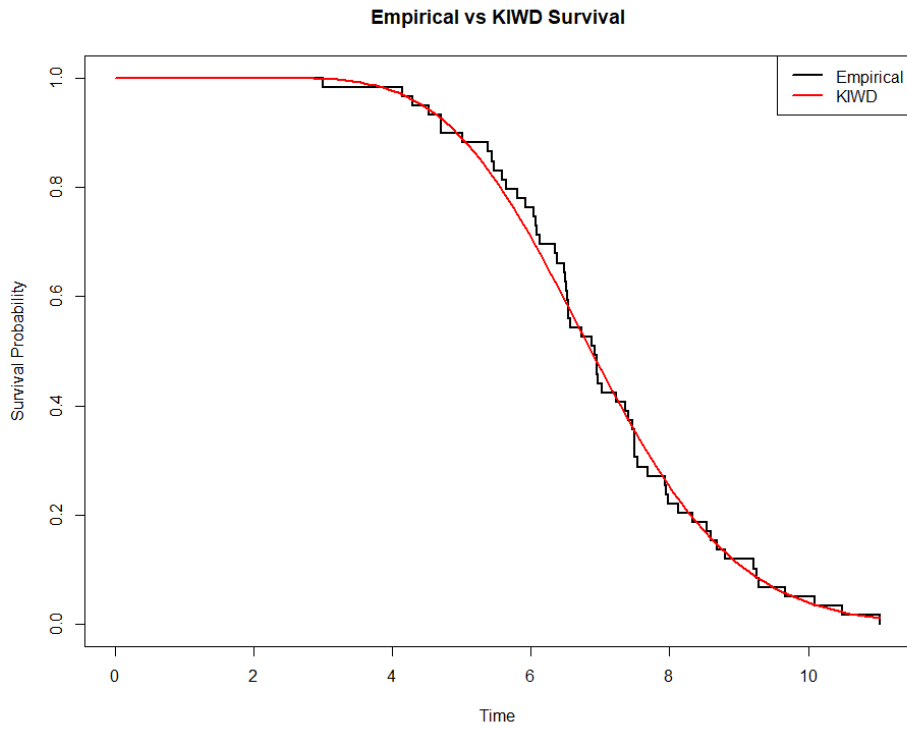


Figure 4: Empirical vs. KIWD Survival Function for Electromigration Failure-Time Data

7.2 Characteristics of Electromigration Failure-Time Data

The electromigration failure-time data exhibit several defining characteristics that reflect both the physical degradation mechanism and the statistical behaviour captured by the fitted Kumaraswamy Inverse Weibull Distribution (KIWD). Electromigration is known for producing early-life failures followed by a tapering hazard rate, and this pattern is clearly visible in the distributional structure of the observed lifetimes. The quartile estimates derived from the KIWD model: $Q_1 = 5.0206$, $Q_2 = 5.9425$, and $Q_3 = 6.9300$, show that half of all failures occur within

a narrow interval of roughly two hours. This compact inter-quartile range ($IQR = 1.9095$) and corresponding quartile deviation (0.9547) indicate moderate dispersion, meaning most conductors fail within a predictable time window. The coefficient of quartile deviation (0.1599), a relative measure of spread used to compare variability across datasets with different units, further confirms that the central portion of the distribution is tightly clustered.

The mild positive skewness (0.034) suggests that while most failures occur early, a small number of conductors survive longer due to favourable microstructural conditions, consistent with the physics of electromigration where atomic flux divergence accelerates failure in weaker interconnects. The kurtosis value (≈ 2.87) indicates a slightly flatter-than-normal distribution, meaning the tails are not excessively heavy; however, the presence of longer-life components still contributes to tail behaviour relevant for reliability qualification. The 90th percentile lifetime ($p_{90} = 7.8737$) shows that only 10% of conductors survive beyond this point, reinforcing the steep reliability decline typical of electromigration-driven degradation. The KIWD-based statistical characteristics portray electromigration failure times as moderately dispersed, slightly right-skewed, and dominated by early-life failures with a small proportion of extended lifetimes. These features align closely with the known physical behaviour of electromigration, where microstructural weaknesses lead to rapid failure progression, and only a minority of interconnects exhibit prolonged resistance to atomic migration.

8. Concluding Remarks

This study revisited the Kumaraswamy Inverse Weibull (KW-IW) distribution, originally introduced by Shahbaz et al. (2012), and provided a comprehensive development of its statistical properties together with a rigorous reliability application. The distribution's structural behaviour was examined through its density and hazard functions, quartile-based measures, raw moments, geometric and harmonic means, and entropy measures. These analytical results demonstrate that the KW-IW distribution possesses substantial flexibility, allowing it to model a wide range of asymmetric and heavy-tailed lifetime patterns. This comparative study demonstrates that the Kumaraswamy Inverse Weibull Distribution (KIWD) provides a markedly superior statistical and practical framework for modelling electromigration-related failure-time data in microelectronic components. Among the four lifetime models evaluated KIWD, Inverse Weibull (IW), Generalized Inverse Weibull (GIW), and Modified Inverse Weibull (MIW)- KIWD consistently outperformed all alternatives across every goodness-of-fit metric, information criterion, and convergence diagnostic. The KIWD model achieved exceptionally small Cramér–von Mises and Anderson–Darling statistics, a highly favourable Kolmogorov–Smirnov p-value, and the lowest AIC, BIC, and HQIC values, confirming both excellent empirical fit and strong parsimony. In contrast, the IW model provided only moderate performance, while the GIW and MIW models

exhibited severe misfit, numerical instability, and failed convergence, rendering them unsuitable for this application. Beyond statistical superiority, KIWD offers meaningful engineering interpretability. Its flexible structure captures the heavy-tailed behaviour, early-life vulnerability, and complex hazard shapes characteristic of electromigration-induced degradation features that traditional inverse-Weibull-type models struggle to represent. The stability of the KIWD parameter estimates further reinforces its reliability for practical use in failure prediction, reliability qualification, and accelerated testing analysis. The findings establish KIWD as the most robust, accurate, and physically relevant model for characterising electromigration failure-time behaviour in microcircuits and electronic components. Its demonstrated advantages make it a valuable tool for reliability engineers and researchers seeking precise modelling of degradation mechanisms in advanced microelectronic systems.

References

1. Cordeiro, G. M., & de Castro, M. (2011). A new family of Generalized distributions. *Journal of Statistical Computation and Simulation*, Vol. 81 (7), 883–898.
2. Gusmao, F.R.S, Ortega E.M.M, and Cordeiro G.M. (2011). The Generalized inverse Weibull distribution, *Statistical Papers*. Vol. 52: 591–619.
3. Hussian M. A and Amin E. A. (2014). Estimation and prediction for the Kumaraswamy-inverse Rayleigh distribution based on records, *International Journal of Advanced Statistics and Probability*. Vol. 2(1): 21-27.
4. Havrda. J. and Charvat. F. Quantification method in classification processes: concept of structural α -entropy, *Kybernetika*. 1967; 3: 30–35.
5. Kumaraswamy, P. (1980). A generalized probability density function for double- bounded random processes. *Journal of Hydrology*, 46(1–2), 79–88.
6. Keller AZ, Kamath AR (1982). Reliability analysis of CNC machine tools. *Reliability Eng*, Vol. 3, pp. 449–473.
7. Keller, A. Z., & Kamath, A. R. (1982). Alternative reliability models for mechanical components. *Microelectronics Reliability*, 22(2), 213–227.
8. Khan M.S and King. R. (2012). Modified inverse Weibull distribution, *Journal of Statistics Applications and Probability*. Vol. 1(2): 115–132.
9. Khan M. S, King, R, and Hudson I. L. (2014). Characterizations of the transmuted Inverse Weibull distribution. *ANZIAM J*. 55 (EMAC2013): C197–C217.
10. Khan, M. S., & King, R. (2014). A New Class of Transmuted Inverse Weibull Distribution for Reliability Analysis. *American Journal of Mathematical and Management Sciences*, 33(4), 261–286. <https://doi.org/10.1080/01966324.2014.929989>
11. Khan M. S, King, R, and Hudson I. L. (2017). Transmuted Weibull distribution Properties and Estimation, *Communications in Statistics: theory and methods*. 46 (11): 5394-5418, DOI: 10.1080/03610926.2015.1100744.
12. Lawless, J. F. (2003). *Statistical models and methods for lifetime data* (2nd ed.). Wiley-Interscience.
13. Oguntunde. P. E, Adejumo A. O, and Owoloko E. A. (2017). Application of Kumaraswamy inverse exponential distribution to real lifetime data. *International Journal of Applied Mathematics and Statistics*. 2017; 56(5): 34–47.

14. Rényi, A. On measures of entropy and information. University of California Press, Berkeley, California. 1961; 547–561.
15. Shahbaz, M. Q., Shahbaz, S. and Butt, N. S. The Kumaraswamy–inverse Weibull distribution. Pakistan Journal of Statistics and Operation Research. 2012; 8(3): 479-489.
16. Ullah. A. Entropy, divergence and distance measures with econometric applications. Journal of Statistical Planning and Inference. 1996; 49: 137–162.
17. R Development Core Team. A Language and Environment for Statistical Computing, R Foundation for Statistical Computing. 2019; Vienna, Austria.
18. Jon Makar and Karen Luke., (2011). Thermal Stability of the Cement Sheath in Steam Treat Oil Wells. J. of the Amer. Ceramic Soc., 4463–4470.