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Analysis of Artificial Intelligence in Education Systems Based on an Advanced Fuzzy Decision-Making Model

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Abstract: By providing creative ways to improve learning outcomes, expedite administrative processes, and give pupils individualized instruction, artificial intelligence tools have the potential to completely transform the educational system. Although there are many educational benefits to artificial intelligence, it's important to deal with privacy concerns, and ethical issues, and make sure the technology is used to enhance rather than replace human talents. To fully utilize artificial intelligence in the educational system, researchers must continue their work, and educators and technologists must collaborate. The purpose of this study is to enhance decision-making in situations involving uncertainty, particularly within group decision-making processes. Two extensions of the fuzzy set to deal with uncertain information in actual decision-making situations are the complex fuzzy set and the interval-valued q -rung orthopair fuzzy set. We propose novel aggregation operators, namely, the complex interval-valued q -rung orthopair fuzzy weighted and ordered weighted arithmetic mean aggregation operators, as well as complex interval valued q -rung orthopair fuzzy weighted/ordered weighted geometric mean aggregation operators. These operators are developed within the framework of a new class of algebraic complex interval valued operational laws. We also examine the mathematical properties of these operators, such as boundedness, idempotency, and monotonicity, to establish their theoretical robustness. Furthermore, we introduce an advanced multi-attribute group decision-making approach that integrates these operators. This approach is applied to a practical problem in

the education sector, where artificial intelligence tools are evaluated for their effectiveness in enhancing educational systems. Comparative analyses and experimental results demonstrate that the proposed method consistently produces reliable rankings and accurately identifies the optimal alternative. The method's superior performance, as compared to existing techniques, confirms its applicability and effectiveness in handling complex decision-making scenarios.

Keywords: Algebraic complex interval valued operational laws, complex interval valued q -ROF weighted mean aggregation, Cq -ROFS.

1. INTRODUCTION

A technique called multi-attribute decision-making (MADM) is used to assess and prioritize options when there are conflicting goals or when judgments must be made quickly (Jafar et al. 2021; Khan et al. 2021; Riaz et al. 2021, 2022; Rong et al. 2022). Decisions implemented in the past frequently produced insufficient results because numerical data sets were insufficient to handle operational scenarios found in everyday circumstances. Decision-makers find it harder and harder to deal with the ambiguities in the data by applying conventional procedures as systems get more complicated over time. In order to communicate the knowledge they needed for their investigation, the scholars used fuzzy sets (FS) (Zadeh (1965)), a kind of mathematical model where items have a degree of membership in a set rather than a strict binary distinction between belonging or not belonging. Occasionally, even though someone is unsure, they may believe that an element is part of a FS. In these situations, the individual may be unsure or apprehensive about whether the element actually falls within the FS or not. Although conventional FSs only use one membership degree, they are unable to capture this kind of hesitancy or ambiguity. In order to overcome the problem that FS solely uses η MD for describing fuzzy knowledge, non-membership degree ψ (NMD) and hesitation degree (HD) are added to the FS structure to develop the intuitionistic fuzzy set (IFS) (Atanassov, 1989) theory, which is designed to provide better representations of uncertain data. At the moment, intuitionistic fuzziness studies have yielded valuable findings and are widely applied across various fields (Nguyen, 2019; Garg & Rani, 2019; Deveci et al. 2020; Jia & Wang, 2022). IFSs are not capable of representing conflicting and inconsistent data, although a great deal of research and answers to many doubtful choice and evaluation situations. Yager (2013) proposed Pythagorean fuzzy sets (PFS) as a solution to the problem. η and ψ are the characteristics of PFS, and the square sum of η and ψ is below or equal to 1. The application of PFSs across multiple sectors attracted scholars from many disciplines (Xue et al. 2018; Kumar & Chen, 2023). However, there might be some real-world situations when these requirements are insufficient to explain the scenario. On the other hand, if a DM indicates that their support for η is 0.8 and their support against ψ is 0.75, it may be determined that the total of the

squares of η and ψ is greater than one. Therefore, the ordered pair $(0.8, 0.75)$ is not permitted for a PFS. In order to resolve this issue, Yager (2016) developed the q -rung orthopair fuzzy set (q -ROFS), in which η and ψ fulfill $\eta^q + \psi^q \leq 1$, $q \geq 1$. For $q = 3$, the result for q -ROFS is $(0.8)^3 + (0.75)^3 = 0.5120 + 0.42188 = 0.93388 < 1$, which is the optimum value to address the aforementioned circumstance. Fermatean fuzzy sets were first proposed by Senapati and Yager (2019), who also expanded the model to include weighted averaging and geometric operators. This gave decision-makers and ambient intelligence environments a flexible way to express uncertainty. As a result, q -ROFS gives more opportunities for communicating preferences while also successfully addressing current problems (Xu et al., 2018; Wei et al., 2018; Yager et al., 2018). The scope of q -ROFSs was expanded by Yang and Pang (2019) by adding partitioned Bonferroni mean operators and how they are utilized. In a q -ROF environment, Ye et al. (2019) explored the characteristics of single variable differential calculus and its potential applications. The concept of neutrality aggregation operators under q -ROFSs has been proposed, and its applications have been explored by Garg and Chen (2020). q -ROFS has been applied in various disciplines, attracting the interest of scholars from various fields (Liu and Wang, 2017; Jana et al., 2019). In a q -ROF environment, Liu and Wang (2020) introduced a method to address multiple attribute decision making problems based on Maclaurin symmetric mean operators. Under a q -ROF environment, Dogu (2021) presents the concept of a TOPSIS-based decision-making mechanism. For the q -ROFSs, Khan et al. (2021) introduced the Knowledge measure. The idea of handling problems related to the selection of renewable energy sources using q -ROF environment is presented by Krishankumar et al. (2021). In a q -ROFSs, Pen et al. (2021) provide an approach for decision-making about the integration of preference mobile edge caching schemes. The domain of q -ROFSs was expanded by Li et al. (2020), who also developed correlation coefficients and discussed how to apply them to clustering analysis. q -ROF information is applied to patients' medical state, as discussed by Ejegwa and Davvaz (2022). All of these developments provide useful tools for simulation, decision-making, environmental sustainability and the advancement of the theoretical underpinnings and real-world implementations of intelligent systems in educational contexts. Fuzzy techniques to additive manufacturing, prequalification evaluation, and sustainability are among the critical applications that this study tackles.

1.1. Literature review. In real-world settings, decision-makers (DMs) face more obstacles in selecting the best option from a variety of workable possibilities as systems get more complicated. While it presents a significant obstacle, achieving a single goal is not insurmountable. Many companies have struggled with the complexities of setting goals, forming viewpoints, and inspiring employees. Decisions made by committees or people within an organization therefore

take into account several simultaneous goals. Each DM is compelled to develop the best answer for each realistic implications in real-world problems by using criteria that permit optional answers. As a result, DMs are becoming more committed to creating more dependable and practical techniques for identifying the best solutions. As per the research mentioned before, comparable methods have limitations and cannot capture the degree of uncertainty in the information and its variations throughout a specific time frame. To address this, complex FS (CFS) was developed by Ramot et al. (2002). The mathematical framework of complex IFS (CIFS), which was introduced by Alkouri and Salleh (2012), further expanded the domain of real numbers to complex numbers with a unit disc for the supporting grade and the supporting disregarding grade. Furthermore, the CIFS can capture two-dimensional data. Ullah et al. (2020) developed the complex PFS (CPFS), which is helpful for effectively characterizing ambiguous and untrustworthy knowledge, utilizing the CIFS as a basis. It should be noted that the sum of squares of the real and imaginary parts of the complex membership and non-membership grades cannot exceed $[0, 1]$ is a feature of CPFS. But when the sum of the squares for both the real and imaginary parts of the knowledge was greater than $[0, 1]$, which was beyond the capability of the CIFS and CPFS. Incorporating complex Pythagorean fuzzy knowledge, Ma et al. (2021) proposed an approach for collective decision-making. For instance, when analysts are requested to indicate which degree of an alternative G_i they preferred on a criterion C_j , the complex-valued membership grade could be $0.95e^{2i\pi(0.83)}$, while the non-membership grade may be $0.88e^{2i\pi(0.86)}$. In order to better understand this kind of data, Liu et al. (2020) put forth the complex q -ROFS (Cq -ROFS) theory, which states that the sum of q -power of the real and imaginary parts of the complex membership and non-membership grades cannot exceed $[0, 1]$. Subsequently, the Cq -ROFS is a modified version of the CIFS and CPFS to address real-world issues, it also serves as an effective tool for resolving ambiguous issues. Analysis of the aggregation operators under Cq -ROFSs was conducted by Garg et al. (2021). After that, significantly higher levels of ambiguity are often handled by the Cq -ROFS. When describing uncertainty in real-decision operations, the Cq -ROFS is applied more precisely than the CIFS and CPFS. After the invention of the Cq -ROFS, experts and other investigators have focused a lot of emphasis on decision-making issues involving numerous attributes in Cq -ROF situations (Liu et al. 2020). Fuzzy 2-tuple linguistic Maclaurin symmetric mean operators have been developed and some of their potential uses were explored by Rong et al. (2020) employing Cq -ROFS. Further, under the environment of Cq -ROF measurements, Ali and Mahmood (2020) developed a theoretical framework of Maclaurin symmetric mean operators. Further, the concepts of Schweizer-Sklar Muirhead mean aggregation operators under Cq -ROF environments were emphasized by Liu et al. (2021), along with their practical implications. Using Cq -ROF knowledge, Mahmood & Ali (2021) introduced the notion of the entropy measure and TOPSIS

approach utilizing the correlation coefficient and how it can be applied to MADM. Yang et al. (2021) addressed the implementation of partitioned Bonferroni mean operators in antiviral mask selection under complex q -rung orthopair uncertain linguistic circumstances. Further, the concept of Aczel Alsina aggregation operators was introduced by Ali and Naeem (2022) in the context of Cq -ROF, and how it can be applied for MAGDM with unknown weight information was explored. In a Cq -ROF environment, Ali & Mahmood (2022) demonstrate some Dombi aggregation operations and how they can be applied in multi-attribute decision making. In a Cq -ROF setting, Du et al. (2022) introduced the idea of Frank aggregation operators and how to use them for multi-attribute decision mapping. Muirhead mean operators were presented and their applications were explored by Akram et al. (2023) within the context of the Cq -ROF 2-tuple linguistic framework. As a whole, these new works demonstrate a variety of potential uses in fuzzy systems and intelligent decision support, contributing to the advancement of sustainable structure advancement and decision-making procedures.

1.2. Research gap and the main motivations. As a generalization of FS, Zadeh (1975) introduced the novel concept of interval-valued fuzzy set (IVFS), which is distinguished by truth degree as some closed subinterval of $[0, 1]$ rather than some value in $[0, 1]$. The concept of interval-valued intuitive fuzzy set (IVIFS), a generalization of FS and IVFS, was first put forth by Atanassov and Gargov in (1989). The invention of IVIFS defined a truth degree and a falsehood degree as closed subintervals of $[0, 1]$ subject to the requirement that the total of the upper parts of the degrees be less than or equal to one. An innovative accuracy function for tackling multicriteria decision-making problems was introduced by Garg (2016) in an interval-valued Pythagorean fuzzy framework. A approach for addressing multiple criterion group choice analysis was described by Liang et al. (2015) and involved maximizing deviation in the context of an interval-valued Pythagorean fuzzy weighted aggregating operator. In the context of interval-valued Pythagorean fuzzy sets, Garg (2017) introduced the TOPSIS technique and unveiled a revolutionary ranking score function. The idea of interval valued q -rung orthopair fuzzy sets was introduced by Joshi et al. (2018), who also covered some of its characteristics for the construction of this framework. Under interval-valued q -rung orthopair fuzzy environment and group decision making approach, Wan and Lu (2022) introduced the concept of power average operators. A generalization of IVFS and FS is the concept of interval-valued complex fuzzy set (IVCFS), which was introduced by Selvachandran et al. (2018). Having both the upper and lower complex value functions (UCVF and LCVF), as well as meeting the requirement that $UCVF \geq LCVF$, the IVC number is an element of a subset of the unit disc. The concept of generalized complex intuitionistic fuzzy aggregation operators

was emphasized by Garg and Rani (2019), along with their utilization in multicriteria decision-making processes. Ali et al. (2021) developed the innovative idea of complex interval-valued Pythagorean fuzzy based Einstein geometric aggregation operators and how it can be utilized in green supplier chain management. There are specific situations in which the CIVFS and CIVPFS fail. For example, if a decision maker gives $[0.7, 0.8]$, $[0.8, 0.9]$ for interval-valued complex MD (IVCMD) and $[0.6, 0.8]$, $[0.8, 0.8]$ for interval-valued complex NMD (IVCNMD), then the IVCIFS and IVC PFS are unable to handle the data appropriately. This results in $0.8+0.8 = 1.6 > 1$, $0.9+0.8 = 1.7 > 1$, and $(0.8)^2 + (0.8)^2 = 1.28 > 1$, $(0.9)^2 + (0.8)^2 = 1.45 > 1$. In order to address this shortcoming, Garg et al. (2021) developed the useful notion of CIVq-ROFS, which combines two-dimensional data into a single element, i.e. $(0.8)^5 + (0.8)^5 = 0.65536 < 1$, $(0.9)^5 + (0.8)^5 = 0.91817 < 1$. In the framework of CIVq-ROF measurements, Garg et al. (2021c) established a theoretical framework of the Muirhead mean operator based ranking methodology for MAGDM. The discipline has benefited from the proposal of new fuzzy hamy mean operators by Ali et al. (2022) and their implementation in decision-making strategies according to CIVq-ROF environments. Table 1 illustrates the detail limitations of previous studies in comparison to the suggested approach.

Table 1. Comparison of proposed model with extant models in literature

	MD	NMD	Indeterminacy	CIVMD	CIVNMD	Parameter q	Parameter λ
FS	Yes	No	No	No	No	No	No
CFS	Yes	No	No	No	No	No	No
IFS	Yes	Yes	Yes	No	No	No	No
CIFS	Yes	Yes	Yes	No	No	No	No
IVIFS	Yes	Yes	Yes	No	No	No	No
CIVIFS	Yes	Yes	Yes	Yes	Yes	No	No
PFS	Yes	Yes	Yes	No	No	No	No
IVPFS	Yes	Yes	Yes	No	No	No	No
CIVPFS	Yes	Yes	Yes	Yes	Yes	No	No
q -ROFS	Yes	Yes	Yes	No	No	Yes	No
IVq-ROF	Yes	Yes	Yes	No	No	Yes	No
CIVq-ROF	Yes	Yes	Yes	Yes	Yes	Yes	No
$\left(\begin{array}{c} \text{Proposed} \\ \text{CIVq-ROF} \end{array} \right)$	Yes	Yes	Yes	Yes	Yes	Yes	Yes

We found from the aforementioned investigation that the following are the main issues that all experts have:

- (1) On the basis of CIVq-ROFSs, how should novel operational laws be drafted?

- (2) On the basis of novel operational laws, how may new operators be developed?
- (3) How can all actions be ranked according to the developed operators?

1.3. The key objectives. The following are the primary objectives as stated in the study proposal:

- (i) To propose new algebraic operational laws that define how $CIVq$ -ROFSs can be combined and manipulated mathematically.
- (ii) In light of the benefits of the Aczel-Alsina operational laws, efforts are made to discover the $CIVq$ -ROFWAM operator and $CIVq$ -ROFWGM operator.
- (iii) To develop a new decision-making framework based on complex interval-valued q -rung orthopair fuzzy sets ($CIVq$ -ROFSs).
- (iv) To apply the proposed framework in a real-world multi-attribute decision-making problem related to AI tools in education.
- (v) To validate the method's performance through comparative analysis with existing techniques.

1.4. The main contributions. This study aims to create a logical and analytical approach for decision support so that the best option can be chosen from a variety of alternative sources. The incorporation of algebraic complex interval-valued operational rules into the $CIVq$ -ROF environment will enable the use of the $CIVq$ -ROF arithmetic and geometric mean aggregation operators, which will guarantee the efficiency of the conceptual framework.

In order to resolve the aforementioned issues, the main goals and contributions of this article are given below:

- (i) Being a modified version of CIVIFS and CIVPFS, $CIVq$ -ROFS is more capable, comprehensive, and dependable than current concepts like CIVIFS and CIVPFS when it comes to handling uncertain data during the decision-making process. Because the abstinence value is incorporated, the $CIVq$ -ROFS minimizes the loss of knowledge as compared to both CIVIFS and CIVPFS. Furthermore, the coupling of $CIVq$ -ROF circumstances with arithmetic and geometric mean aggregation operators using algebraic complex interval valued operational laws have not been studied before. Therefore, enhancing the geometric and arithmetic mean aggregation operators utilizing A -complex interval valued operational principles is crucial for solving MCDM problems in $CIVq$ -ROF situations.
- (ii) An insightful idea for handling two-dimensional data in a single set is the arithmetic and geometric mean aggregation operators pursuant to $CIVq$ -ROF settings, which employ

A -complex interval valued operational rules. Therefore, the purpose of this study is to offer Complex interval valued q -ROF weighted arithmetic mean aggregation operator ($CIVq$ -ROFWAM), Complex interval valued q -ROF ordered weighted arithmetic mean aggregation operator ($CIVq$ -ROFOWAM), Complex interval valued q -ROF weighted geometric mean aggregation operator ($CIVq$ -ROFWGM) and Complex interval valued q -ROF ordered weighted geometric mean aggregation operator ($CIVq$ -ROFOWGM).

(iii) The connections between these operators are highlighted in order to address certain of their characteristics, including their boundedness, idempotency, and monotonicity.

(iv) To create two distinct, cutting-edge methods based on the $CIVq$ -ROFWAM and $CIVq$ -ROFWGM.

(v) We give a graphical depiction of the proposed approach to further enhance the clarity and understandability of the created procedure. Using a flowchart to graphically depict the intended process, the recommended modelling technique is clearly communicated.

(vi) To provide application that demonstrate the viability and dependability of the suggested techniques. Additionally, by contrasting the suggested techniques with the current approaches, we will demonstrate that the suggested techniques are better and that the procedure for aggregation is more adaptable when using the arithmetic and geometric mean aggregation operators in accordance with $CIVq$ -ROF environments and A -complex interval valued operational rules.

(vii) The geometrical representation of the proposed works is shown in Fig. 1. $CIVq$ -ROF arithmetic and geometric mean aggregation operators based on algebraic complex interval-valued operational rules offer a high degree of flexibility when aggregating confusing data. By adjusting the parameters, one can change the focus placed on specific fuzzy IVMD, IVNMD, and their related complex interval values. These operators have been effectively applied in domains such as expert systems, data fusion, pattern recognition, and decision-making. Their ability to handle complicated and unpredictable information makes them suitable for managing ambiguous and inaccurate real-world situations. Their unique approach to data aggregation yields findings that are more reliable and accurate, assisting decision-makers in a range of applications in making better decisions.

1.5. The structure of this paper. According to the following, this article is organized: The main topic of section 2 is the construct of A -complex operational laws based Complex interval valued q -ROF weighted arithmetic mean aggregation operator ($CIVq$ -ROFWAM), Complex interval valued q -ROF ordered weighted arithmetic mean aggregation operator ($CIVq$ -ROFOWAM). The concept of Complex interval valued q -ROF weighted geometric mean aggregation operator ($CIVq$ -ROFWGM) and Complex interval valued q -ROF ordered weighted geometric mean aggregation operator ($CIVq$ -ROFOWGM) within the context of Cq -ROF value

sets and their characteristics are covered in Section 3. Section 4 suggests a novel approach of decision-making using the cutting-edge methods based on the $CIVq$ -ROFWAM and $CIVq$ -ROFWGM. Additionally, Section 5 provides an example to demonstrate the value of the suggested strategy for making decisions. Section 6 characterizes the comparability and sensitivity analysis that demonstrate the logic and stability of the suggested technique. Section 7 provides a conclusion to the article.

2. COMPLEX INTERVAL VALUED q -RUNG ORTHOPAIR FUZZY WEIGHTED ARITHMETIC MEAN AGGREGATION OPERATORS

We provide definitions of A -complex interval valued operational laws for $CIVq$ -ROFNs in the following section. Following them, several aggregation (complex interval valued q -ROF weighted arithmetic mean aggregation operator ($CIVq$ -ROFWAM), complex interval valued q -ROF ordered weighted arithmetic mean aggregation operator ($CIVq$ -ROFOWAM)) operators based on A -complex interval valued operational laws will be created.

Definition 1. (Liu et al. 2020; 2021) Cq -ROFS is defined by

$$C = \{(\chi_C(u), \xi_C(u)) : u \in X\}$$

where $\chi_C : X \rightarrow \{a_1 + ib_1 : a_1 + ib_1 \in C, |a_1 + ib_1| \leq 1\}$ and

$\xi_C : X \rightarrow \{a_2 + ib_2 : a_2 + ib_2 \in C, |a_2 + ib_2| \leq 1\}$ such that $\chi_C(u) = a_1 + ib_1$ and $\xi_C(u) = a_2 + ib_2$, provided that $|a_1 + ib_1|^q + |a_2 + ib_2|^q \leq 1$ or $\chi_C(u) = \eta_C(u) e^{i2\pi(\eta_C^{im}(u))}$ and $\xi_C(u) = \psi_C(u) e^{i2\pi(\psi_C^{im}(u))}$, where $(\eta_C(u))^q + (\psi_C(u))^q \leq 1$ and $(\eta_C^{im}(u))^q + (\psi_C^{im}(u))^q \leq 1$. The Cq -ROFN is defined by $P = (\eta_C e^{i2\pi(\eta_C^{im})}, \psi_C e^{i2\pi(\psi_C^{im})})$.

Definition 2. (Liu et al. 2020; 2021) For any Cq -ROFN $P_1 = (\eta_{C_1} e^{i2\pi(\eta_{C_1}^{im})}, \psi_{C_1} e^{i2\pi(\psi_{C_1}^{im})})$, the score and accuracy functions are defined by

$$SC(P_1) = \frac{1}{2} \left\{ (\eta_{C_1})^q - (\psi_{C_1})^q + (\eta_{C_1}^{im})^q - (\psi_{C_1}^{im})^q \right\}$$

and

$$AC(P_1) = \frac{1}{2} \left\{ (\eta_{C_1})^q + (\psi_{C_1})^q + (\eta_{C_1}^{im})^q + (\psi_{C_1}^{im})^q \right\}$$

where $SC(P_1) \in [-1, 1]$ and $AC(P_1) \in [0, 1]$.

Definition 3. (Liu et al. 2020; 2021) Let $P_1 = (\eta_{C_1} e^{i2\pi(\eta_{C_1}^{im})}, \psi_{C_1} e^{i2\pi(\psi_{C_1}^{im})})$,

$P_2 = (\eta_{C_2} e^{i2\pi(\eta_{C_2}^{im})}, \psi_{C_2} e^{i2\pi(\psi_{C_2}^{im})})$ be two Cq -ROFNs. Then

(1) if $SC(P_1) > SC(P_2)$, then $P_1 > P_2$,

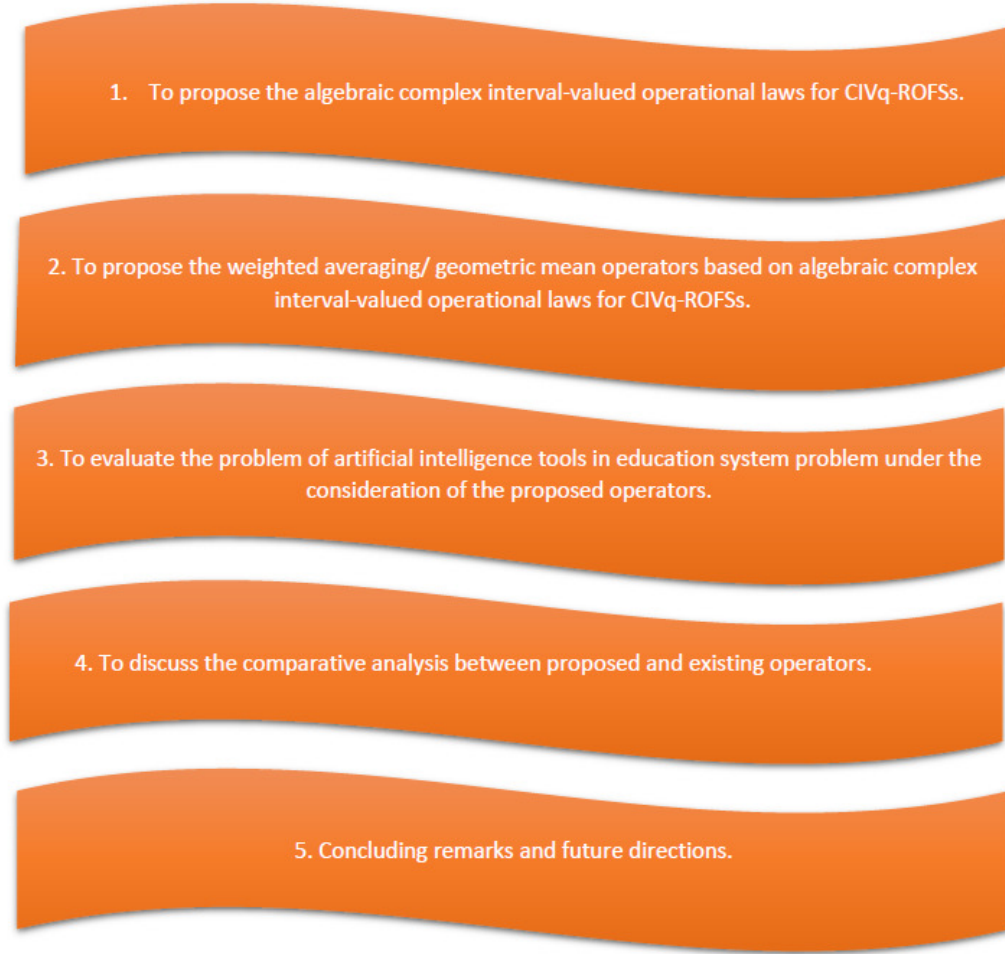


FIGURE 1. Geometricalr epresentation of the proposed works

- (2) if $SC(P_1) = SC(P_2)$ then
- (i) if $AC(P_1) > AC(P_2)$, then $P_1 > P_2$,
 - (ii) if $AC(P_1) = AC(P_2)$, then $P_1 = P_2$.

Definition 4. (Garg et al. 2021) CIVq-ROFS is defined by

$$C = \{([\chi_C^-(u), \chi_C^+(u)], [\xi_C^-(u), \xi_C^+(u)]) : u \in X\}$$

where $\chi_C^-, \chi_C^+ : X \rightarrow \{a_1 + ib_1 : a_1 + ib_1 \in C, |a_1^- + ib_1^-| \leq |a_1^+ + ib_1^+| \leq 1\}$ and

$\xi_C^-, \xi_C^+ : X \rightarrow \{a_2 + ib_2 : a_2 + ib_2 \in C, |a_2^- + ib_2^-| \leq |a_2^+ + ib_2^+| \leq 1\}$ such that $\chi_C^-(u) = a_1^- + ib_1^-$, $\chi_C^+(u) = a_1^+ + ib_1^+$ and $\xi_C^-(u) = a_2^- + ib_2^-$, $\xi_C^+(u) = a_2^+ + ib_2^+$ provided that $|a_1^- + ib_1^-|^q + |a_2^- + ib_2^-|^q \leq 1$, $|a_1^+ + ib_1^+|^q + |a_2^+ + ib_2^+|^q \leq 1$ or $\chi_C^-(u) = \eta_C^-(u) e^{i2\pi(\eta_C^{-im}(u))}$, $\chi_C^+(u) = \eta_C^+(u) e^{i2\pi(\eta_C^{+im}(u))}$ and $\xi_C^-(u) = \psi_C^-(u) e^{i2\pi(\psi_C^{-im}(u))}$, $\xi_C^+(u) = \psi_C^+(u) e^{i2\pi(\psi_C^{+im}(u))}$ where $(\eta_C^+(u))^q + (\psi_C^+(u))^q \leq 1$ and $(\eta_C^{+im}(u))^q + (\psi_C^{+im}(u))^q \leq 1$. The CIVq-ROFN is defined by $P_1 = \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]} \right)$.

2.1. Algebraic complex interval valued operational laws. The new direct sum, direct product, and scalar multiplication operations are defined for CIVq-ROFNs in this subsection.

Definition 5. Let $P_1 = \left([\eta_{C_1}^-, \eta_{C_1}^+] e^{i2\pi[\eta_{C_1}^{-im}, \eta_{C_1}^{+im}]}, [\psi_{C_1}^-, \psi_{C_1}^+] e^{i2\pi[\psi_{C_1}^{-im}, \psi_{C_1}^{+im}]} \right)$, $P_2 = \left([\eta_{C_2}^-, \eta_{C_2}^+] e^{i2\pi[\eta_{C_2}^{-im}, \eta_{C_2}^{+im}]}, [\psi_{C_2}^-, \psi_{C_2}^+] e^{i2\pi[\psi_{C_2}^{-im}, \psi_{C_2}^{+im}]} \right)$ be two CIVq-ROFNs. Then we define the following algebraic (A)-complex operational laws:

$$\begin{aligned}
 \text{(i) } P_1 \oplus P_2 &= \left(\begin{aligned} &\left[\left(\frac{1}{2} \left((\eta_{C_1}^-)^q + (\eta_{C_2}^-)^q \right) \right)^{\frac{1}{q}}, \left(\frac{1}{2} \left((\eta_{C_1}^+)^q + (\eta_{C_2}^+)^q \right) \right)^{\frac{1}{q}} \right] \\ &e^{i2\pi \left[\left(\frac{1}{2} \left((\eta_{C_1}^{-im})^q + (\eta_{C_2}^{-im})^q \right) \right)^{\frac{1}{q}}, \left(\frac{1}{2} \left((\eta_{C_1}^{+im})^q + (\eta_{C_2}^{+im})^q \right) \right)^{\frac{1}{q}} \right]} \\ &\left[\left(1 - \frac{1}{2} \left((\psi_{C_1}^+)^q + (\psi_{C_2}^+)^q \right) \right)^{\frac{1}{q}}, \left(1 - \frac{1}{2} \left((\psi_{C_1}^-)^q + (\psi_{C_2}^-)^q \right) \right)^{\frac{1}{q}} \right] \\ &e^{i2\pi \left[\left(1 - \frac{1}{2} \left((\psi_{C_1}^{+im}(u))^q + (\psi_{C_2}^{+im}(u))^q \right) \right)^{\frac{1}{q}}, \left(1 - \frac{1}{2} \left((\psi_{C_1}^{-im}(u))^q + (\psi_{C_2}^{-im}(u))^q \right) \right)^{\frac{1}{q}} \right]} \end{aligned} \right); \\
 \text{(ii) } P_1 \otimes P_2 &= \left(\begin{aligned} &\left[\left(1 - \frac{1}{2} \left((\psi_{C_1}^+)^q + (\psi_{C_2}^+)^q \right) \right)^{\frac{1}{q}}, \left(1 - \frac{1}{2} \left((\psi_{C_1}^-)^q + (\psi_{C_2}^-)^q \right) \right)^{\frac{1}{q}} \right] \\ &e^{i2\pi \left[\left(1 - \frac{1}{2} \left((\psi_{C_1}^{+im}(u))^q + (\psi_{C_2}^{+im}(u))^q \right) \right)^{\frac{1}{q}}, \left(1 - \frac{1}{2} \left((\psi_{C_1}^{-im}(u))^q + (\psi_{C_2}^{-im}(u))^q \right) \right)^{\frac{1}{q}} \right]} \\ &\left[\left(\frac{1}{2} \left((\eta_{C_1}^-)^q + (\eta_{C_2}^-)^q \right) \right)^{\frac{1}{q}}, \left(\frac{1}{2} \left((\eta_{C_1}^+)^q + (\eta_{C_2}^+)^q \right) \right)^{\frac{1}{q}} \right] \\ &e^{i2\pi \left[\left(\frac{1}{2} \left((\eta_{C_1}^{-im})^q + (\eta_{C_2}^{-im})^q \right) \right)^{\frac{1}{q}}, \left(\frac{1}{2} \left((\eta_{C_1}^{+im})^q + (\eta_{C_2}^{+im})^q \right) \right)^{\frac{1}{q}} \right]} \end{aligned} \right); \\
 \text{(iii) } \alpha P &= \left(\begin{aligned} &[(\alpha)^{\frac{1}{q}} \eta_C^-, (\alpha)^{\frac{1}{q}} \eta_C^+] e^{i2\pi[(\alpha)^{\frac{1}{q}} \eta_C^{-im}, (\alpha)^{\frac{1}{q}} \eta_C^{+im}]}, \\ &\left[(\alpha (1 - (\psi_C^+)^q))^{\frac{1}{q}}, (\alpha (1 - (\psi_C^-)^q))^{\frac{1}{q}} \right] e^{i2\pi[(\alpha (1 - (\psi_C^{+im})^q))^{\frac{1}{q}}, (\alpha (1 - (\psi_C^{-im})^q))^{\frac{1}{q}}]} \end{aligned} \right), \\
 &\text{where } 0 \leq \alpha \leq 1;
 \end{aligned}$$

$$(iv) P^\lambda = \left(\begin{array}{c} \left[r^{\frac{1}{q}} (\eta_C^-)^\lambda, r^{\frac{1}{q}} (\eta_C^+)^\lambda \right] e^{i2\pi \left[r^{\frac{1}{q}} (\eta_C^{-im})^\lambda, r^{\frac{1}{q}} (\eta_C^{+im})^\lambda \right]}, \\ \left[r^{\frac{1}{q}} (\psi_C^-)^\lambda, r^{\frac{1}{q}} (\psi_C^+)^\lambda \right] e^{i2\pi \left[r^{\frac{1}{q}} (\psi_C^{-im})^\lambda, r^{\frac{1}{q}} (\psi_C^{+im})^\lambda \right]} \end{array} \right), \text{ where } r \text{ is the total number}$$

of CIVq-ROFNs that are a part of the procedure;

$$(v) P^{\odot \alpha} = \left(\begin{array}{c} \left[(\alpha (1 - (\eta_C^+)^q))^{\frac{1}{q}}, (\alpha (1 - (\eta_C^-)^q))^{\frac{1}{q}} \right] e^{i2\pi \left[(\alpha (1 - (\eta_C^{+im})^q))^{\frac{1}{q}}, (\alpha (1 - (\eta_C^{-im})^q))^{\frac{1}{q}} \right]}, \\ \left[(\alpha)^{\frac{1}{q}} \psi_C^-, (\alpha)^{\frac{1}{q}} \psi_C^+ \right] e^{i2\pi \left[(\alpha)^{\frac{1}{q}} \psi_C^{-im}, (\alpha)^{\frac{1}{q}} \psi_C^{+im} \right]} \end{array} \right),$$

where $0 \leq \alpha \leq 1$.

Definition 6. Let $\{P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi [\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi [\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}) : r = 1, 2, \dots, m\}$ be the collection of CIVq-ROF values and let CIVq-ROFWAM : $\Omega^m \rightarrow \Omega$. If

CIVq-ROFWAM_E ($P_1, P_2, P_3, \dots, P_m$) = $\left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus \dots \oplus (\alpha_m P_m)^\lambda \right)^{\frac{1}{\lambda}}$ then CIVq-ROFWAM is called a complex interval valued q-rung orthopair fuzzy weighted averaging mean operator of dimension n , where Ω is the set of all CIVq-ROF values, $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$, $\sum_{r=1}^m \alpha_r = 1$, where $r = 1, 2, \dots, m$.

Theorem 1. Let $\{P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi [\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi [\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}) : r = 1, 2, \dots, m\}$ be the collection of CIVq-ROF values. Then by using the CIVq-ROFWAM_E operator their aggregated value is also a CIVq-ROF value and

$$\begin{aligned} & \text{CIVq-ROFWAM}_E (P_1, P_2, P_3, \dots, P_m) \\ &= \left(\begin{array}{c} \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}, \\ \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q)^\lambda) \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^+)^q)^\lambda) \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-im})^q)^\lambda) \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{+im})^q)^\lambda) \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{array} \right). \end{aligned}$$

$E = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^n \alpha_r = 1$, $r = 1, 2, \dots, m$.

Proof. See Appendix 1. □

Proposition 1. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right) : r = 1, 2, \dots, m\right\}$ be the collection of CIVq-ROF values. Let $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. If $\left([\eta_{C_1}^-, \eta_{C_1}^+] e^{i2\pi[\eta_{C_1}^{-im}, \eta_{C_1}^{+im}]}, [\psi_{C_1}^-, \psi_{C_1}^+] e^{i2\pi[\psi_{C_1}^{-im}, \psi_{C_1}^{+im}]}\right) = \left([\eta_{C_2}^-, \eta_{C_2}^+] e^{i2\pi[\eta_{C_2}^{-im}, \eta_{C_2}^{+im}]}, [\psi_{C_2}^-, \psi_{C_2}^+] e^{i2\pi[\psi_{C_2}^{-im}, \psi_{C_2}^{+im}]}\right) = \dots = \left([\eta_{C_m}^-, \eta_{C_m}^+] e^{i2\pi[\eta_{C_m}^{-im}, \eta_{C_m}^{+im}]}, [\psi_{C_m}^-, \psi_{C_m}^+] e^{i2\pi[\psi_{C_m}^{-im}, \psi_{C_m}^{+im}]}\right) = \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]}\right)$ and $\lambda = 1$, then CIVq-ROFWAM $_E(P_1, P_2, P_3, \dots, P_m) = \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]}\right)$.

Proof. See Appendix 2. □

Theorem 2. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right) : r = 1, 2, \dots, m\right\}$ be the collection of CIVq-ROF values and $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$, where $r = 1, 2, \dots, m$. Let

$$P^- = \left(\left[\min_{1 \leq r \leq m} \eta_{C_r}^-, \min_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \eta_{C_r}^{-im}, \min_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\max_{1 \leq r \leq m} \psi_{C_r}^-, \max_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \psi_{C_r}^{-im}, \max_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right)$$

$$P^+ = \left(\left[\max_{1 \leq r \leq m} \eta_{C_r}^-, \max_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\min_{1 \leq r \leq m} \psi_{C_r}^-, \min_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_{C_r}^{-im}, \min_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right),$$

where and $\lambda = 1$. Then $P^- \leq \text{CIVq-ROFWAM}_E(P_1, P_2, P_3, \dots, P_m) \leq P^+$.

Proof. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right) : r = 1, 2, \dots, m\right\}$ be the collection of CIVq-ROF values and $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r , where $r = 1, 2, \dots, m$ with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. Let

$$P^- = \left(\left[\min_{1 \leq r \leq m} \eta_{C_r}^-, \min_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \eta_{C_r}^{-im}, \min_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\max_{1 \leq r \leq m} \psi_{C_r}^-, \max_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \psi_{C_r}^{-im}, \max_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right)$$

$$P^+ = \left(\left[\max_{1 \leq r \leq m} \eta_{C_r}^-, \max_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\min_{1 \leq r \leq m} \psi_{C_r}^-, \min_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_{C_r}^{-im}, \min_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right),$$

where and $\lambda = 1$, we get $(\eta_{C_t}^-)^q \leq \left(\max_{1 \leq r \leq m} \eta_{C_r}^- \right)^q$ and $(\eta_{C_t}^+)^q \leq \left(\max_{1 \leq r \leq m} \eta_{C_r}^+ \right)^q$, where $t = 1, 2, \dots, m$. Then we have $\alpha_r (\eta_{C_t}^-)^q \leq \alpha_r \left(\max_{1 \leq r \leq m} \eta_{C_r}^- \right)^q$ and $\alpha_r (\eta_{C_t}^+)^q \leq \alpha_r \left(\max_{1 \leq r \leq m} \eta_{C_r}^+ \right)^q$. This implies that $\sum_{r=1}^m \alpha_r (\eta_{C_t}^-)^q \leq \sum_{r=1}^m \alpha_r \left(\max_{1 \leq r \leq m} \eta_{C_r}^- \right)^q$ and $\sum_{r=1}^m \alpha_r (\eta_{C_t}^+)^q \leq \sum_{r=1}^m \alpha_r \left(\max_{1 \leq r \leq m} \eta_{C_r}^+ \right)^q$ then we have, $\sum_{r=1}^m \alpha_r (\eta_{C_t}^-)^q \leq \left(\max_{1 \leq r \leq m} \eta_{C_r}^- \right)^q \sum_{r=1}^m \alpha_r$ and $\sum_{r=1}^m \alpha_r (\eta_{C_t}^+)^q \leq \left(\max_{1 \leq r \leq m} \eta_{C_r}^+ \right)^q \sum_{r=1}^m \alpha_r$. This

implies that $\sum_{r=1}^m \alpha_r (\eta_{C_t}^-)^q \leq \left(\max_{1 \leq r \leq m} \eta_{C_r}^- \right)^q$ and $\sum_{r=1}^m \alpha_r (\eta_{C_t}^+)^q \leq \left(\max_{1 \leq r \leq m} \eta_{C_r}^+ \right)^q$ and this implies that $\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^-)^q \right)^{\frac{1}{q}} \leq \max_{1 \leq r \leq m} \eta_{C_r}^-$ and $\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^+)^q \right)^{\frac{1}{q}} \leq \max_{1 \leq r \leq m} \eta_{C_r}^+$. Hence

$$\left[\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^-)^q \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^+)^q \right)^{\frac{1}{q}} \right] \leq \left[\max_{1 \leq r \leq m} \eta_{C_r}^-, \max_{1 \leq r \leq m} \eta_{C_r}^+ \right].$$

Also

$$\left[\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{-im})^q \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{+im})^q \right)^{\frac{1}{q}} \right] \leq \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right].$$

This implies that

$$e^{i2\pi \left[\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{-im})^q \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{+im})^q \right)^{\frac{1}{q}} \right]} \leq e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}.$$

Thus

$$\begin{aligned} & \left[\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{-im})^q \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{+im})^q \right)^{\frac{1}{q}} \right] e^{i2\pi \left[\left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{-im})^q \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m \alpha_r (\eta_{C_t}^{+im})^q \right)^{\frac{1}{q}} \right]} \\ & \leq \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}. \end{aligned}$$

On the other hand, we get $(\psi_{C_r}^-)^q \geq \left(\min_{1 \leq r \leq m} \psi_C^- \right)^q$ and $(\psi_{C_r}^+)^q \geq \left(\min_{1 \leq r \leq m} \psi_C^+ \right)^q$. This implies that $-(\psi_{C_r}^-)^q \leq -\left(\min_{1 \leq r \leq m} \psi_C^- \right)^q$ and $-(\psi_{C_r}^+)^q \leq -\left(\min_{1 \leq r \leq m} \psi_C^+ \right)^q$, also $1 - (\psi_{C_r}^-)^q \leq 1 - \left(\min_{1 \leq r \leq m} \psi_C^- \right)^q$ and $1 - (\psi_{C_r}^+)^q \leq 1 - \left(\min_{1 \leq r \leq m} \psi_C^+ \right)^q$, where, $r = 1, 2, \dots, m$. Then we have $\alpha_r (1 - (\psi_{C_r}^-(u))^q) \leq \alpha_r (1 - (\psi_C^-(u))^q)$ and $\alpha_r (1 - (\psi_{C_r}^+(u))^q) \leq \alpha_r (1 - (\psi_C^+(u))^q)$.

$\alpha_r (1 - (\psi_{C_r}^+)^q) \leq \alpha_r \left(1 - \left(\min_{1 \leq r \leq m} \psi_C^+\right)^q\right)$ then we get,

$$\begin{aligned}
\alpha_r (1 - (\psi_{C_r}^-)^q) &\leq \alpha_r \left(1 - \left(\min_{1 \leq r \leq m} \psi_C^-\right)^q\right) \\
&\Rightarrow \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q) \leq \sum_{r=1}^m \alpha_r \left(1 - \left(\min_{1 \leq r \leq m} \psi_C^-\right)^q\right) \\
&\Rightarrow \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q) \leq \left(1 - \left(\min_{1 \leq r \leq m} \psi_C^-\right)^q\right) \sum_{r=1}^m \alpha_r \\
&\Rightarrow -\sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q) \geq -\left(1 - \left(\min_{1 \leq r \leq m} \psi_C^-\right)^q\right) \\
&\Rightarrow 1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q) \geq 1 - \left(1 - \left(\min_{1 \leq r \leq m} \psi_C^-\right)^q\right) \\
&\Rightarrow 1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q) \geq \left(\min_{1 \leq r \leq m} \psi_C^-\right)^q \\
&\Rightarrow \left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q)\right)^{\frac{1}{q}} \geq \min_{1 \leq r \leq m} \psi_C^-.
\end{aligned}$$

Also $\left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^+)^q)\right)^{\frac{1}{q}} \geq \min_{1 \leq r \leq m} \psi_C^+$. This implies that

$$\begin{aligned}
&\left[\left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q)\right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^+)^q)\right)^{\frac{1}{q}}\right] \geq \left[\min_{1 \leq r \leq m} \psi_C^-, \min_{1 \leq r \leq m} \psi_C^+\right]. \text{ Also} \\
&\left[\left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^{-im})^q)\right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^{+im})^q)\right)^{\frac{1}{q}}\right] \geq \left[\min_{1 \leq r \leq m} \psi_C^{-im}, \min_{1 \leq r \leq m} \psi_C^{+im}\right].
\end{aligned}$$

This implies that

$$e^{i2\pi \left[\left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^{-im})^q)\right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^{+im})^q)\right)^{\frac{1}{q}}\right]} \geq e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_C^{-im}, \min_{1 \leq r \leq m} \psi_C^{+im}\right]}. \text{ Thus}$$

$$\begin{aligned}
&\left[\left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^-)^q)\right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^+)^q)\right)^{\frac{1}{q}}\right] \\
&e^{i2\pi \left[\left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^{-im})^q)\right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m \alpha_r (1 - (\psi_{C_r}^{+im})^q)\right)^{\frac{1}{q}}\right]} \\
&\geq \left[\min_{1 \leq r \leq m} \psi_C^-, \min_{1 \leq r \leq m} \psi_C^+\right] e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_C^{-im}, \min_{1 \leq r \leq m} \psi_C^{+im}\right]}.
\end{aligned}$$

Based on Definition 3 and Definition 6, we get $CIVq\text{-ROFWAM}_E(P_1, P_2, P_3, \dots, P_m) \leq P^+$. Similarly, we can have $CIVq\text{-ROFWAM}_E(P_1, P_2, P_3, \dots, P_m) \geq P^-$. Finally we get,

$$P^+ \geq CIVq\text{-ROFWAM}_E(P_1, P_2, P_3, \dots, P_m) \geq P^-.$$

□

Theorem 3. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right) : r = 1, 2, \dots, m\right\}$ and $\left\{P_r^* = \left([\eta_{C_r}^{*-}, \eta_{C_r}^{+*}] e^{i2\pi[\eta_{C_r}^{-im*}, \eta_{C_r}^{+im*}]}, [\psi_{C_r}^{*-}, \psi_{C_r}^{+*}] e^{i2\pi[\psi_{C_r}^{-im*}, \psi_{C_r}^{+im*}]}\right) : r = 1, 2, \dots, m\right\}$ are two collections of CIVq-ROF values. If $\eta_{C_r}^- \leq \eta_{C_r}^{*-}$, $\eta_{C_r}^+ \leq \eta_{C_r}^{+*}$, $\eta_{C_r}^{-im} \leq \eta_{C_r}^{-im*}$, $\eta_{C_r}^{+im} \leq \eta_{C_r}^{+im*}$, $\psi_{C_r}^- \geq \psi_{C_r}^{*-}$, $\psi_{C_r}^+ \geq \psi_{C_r}^{+*}$, $\psi_{C_r}^{-im} \geq \psi_{C_r}^{-im*}$ and $\psi_{C_r}^{+im} \geq \psi_{C_r}^{+im*}$ where $r = 1, 2, \dots, m$, then $CIVq\text{-ROFWAM}_E(P_1, P_2, P_3, \dots, P_m) \leq CIVq\text{-ROFWAM}_E(P_1^*, P_2^*, P_3^*, \dots, P_m^*)$.

Proof. Given that $P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right)$ and $P_r^* = \left([\eta_{C_r}^{*-}, \eta_{C_r}^{+*}] e^{i2\pi[\eta_{C_r}^{-im*}, \eta_{C_r}^{+im*}]}, [\psi_{C_r}^{*-}, \psi_{C_r}^{+*}] e^{i2\pi[\psi_{C_r}^{-im*}, \psi_{C_r}^{+im*}]}\right)$ are two collections of CIVq-ROF values, where $r = 1, 2, \dots, m$. If $\eta_{C_r}^- \leq \eta_{C_r}^{*-}$, $\eta_{C_r}^+ \leq \eta_{C_r}^{+*}$, $\eta_{C_r}^{-im} \leq \eta_{C_r}^{-im*}$, $\eta_{C_r}^{+im} \leq \eta_{C_r}^{+im*}$, $\psi_{C_r}^- \geq \psi_{C_r}^{*-}$, $\psi_{C_r}^+ \geq \psi_{C_r}^{+*}$, $\psi_{C_r}^{-im} \geq \psi_{C_r}^{-im*}$ and $\psi_{C_r}^{+im} \geq \psi_{C_r}^{+im*}$, then $(\eta_{C_r}^-)^q \leq (\eta_{C_r}^{*-})^q$ and $(\eta_{C_r}^+)^q \leq (\eta_{C_r}^{+*})^q$. As we have

$$\begin{aligned} (\eta_{C_r}^-)^q &\leq (\eta_{C_r}^{*-})^q \\ &\Rightarrow \alpha_r (\eta_{C_r}^-)^q \leq \alpha_r (\eta_{C_r}^{*-})^q \\ &\Rightarrow (\alpha_r (\eta_{C_r}^-)^q)^\lambda \leq (\alpha_r (\eta_{C_r}^{*-})^q)^\lambda \\ &\Rightarrow \sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda \leq (\alpha_r (\eta_{C_r}^{*-})^q)^\lambda \\ &\Rightarrow \left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{*-})^q)^\lambda\right)^{\frac{1}{\lambda}} \\ &\Rightarrow \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \leq \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{*-})^q)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}. \end{aligned}$$

Also

$$\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^+)^q)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \leq \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+*})^q)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}.$$

Thus

$$\begin{aligned} & \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & \leq \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{*-})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{*+})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]. \quad (i) \end{aligned}$$

Further,

$$\begin{aligned} (\eta_{C_r}^{-im}(u))^q & \leq (\eta_{C_r}^{-im*}(u))^q \leq \\ & \Rightarrow \alpha_r (\eta_{C_r}^{-im}(u))^q \leq \alpha_r (\eta_{C_r}^{-im*}(u))^q \\ & \Rightarrow (\alpha_r (\eta_{C_r}^{-im}(u))^q)^\lambda \leq (\alpha_r (\eta_{C_r}^{-im*}(u))^q)^\lambda \\ & \Rightarrow \sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im}(u))^q)^\lambda \leq (\alpha_r (\eta_{C_r}^{-im*}(u))^q)^\lambda \\ & \Rightarrow \left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \\ & \Rightarrow \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \leq \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}. \end{aligned}$$

Also

$$\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \leq \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}.$$

Thus

$$\begin{aligned} & \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & \leq \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Hence

$$\begin{aligned}
& e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \\
& \leq e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}. \tag{ii}
\end{aligned}$$

From (i) and (ii)

$$\begin{aligned}
& \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\
& e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \\
& \leq \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\
& e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}. \tag{A}
\end{aligned}$$

Now

$$\begin{aligned}
(\psi_{C_r}^-)^q & \geq (\psi_{C_r}^{-*})^q \\
& \Rightarrow -(\psi_{C_r}^-)^q \leq -(\psi_{C_r}^{-*})^q \\
& \Rightarrow (1 - (\psi_{C_r}^-)^q) \leq (1 - (\psi_{C_r}^{-*})^q) \\
& \Rightarrow \alpha_r (1 - (\psi_{C_r}^-)^q) \leq \alpha_r (1 - (\psi_{C_r}^{-*})^q) \\
& \Rightarrow (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \leq (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \\
& \leq \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda \\
& \Rightarrow -\sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \geq -\sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda \\
& \Rightarrow 1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \geq 1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda \\
& \Rightarrow \left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda\right)^{\frac{1}{\lambda}} \geq \left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda\right)^{\frac{1}{\lambda}} \\
& \Rightarrow \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \geq \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}.
\end{aligned}$$

Similarly

$$\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \geq \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{+*})^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}.$$

Thus

$$\begin{aligned}
& \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \right] \\
& \geq \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{+*})^q))^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \right]. \quad (\text{iii})
\end{aligned}$$

Also

$$\begin{aligned}
& \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \\
& \leq \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im*})^q \right) \right)^\lambda \\
& \Rightarrow - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \geq - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im*})^q \right) \right)^\lambda \\
& \Rightarrow 1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \geq 1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im*})^q \right) \right)^\lambda \\
& \Rightarrow \left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \geq \left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \\
& \Rightarrow \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \geq \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}.
\end{aligned}$$

Further

$$\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \geq \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}.$$

Furthermore,

$$\begin{aligned}
& \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\
& \geq \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

This implies that

$$\begin{aligned}
& e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \\
& \geq e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}. \quad (\text{iv})
\end{aligned}$$

From (iii) and (iv), we get

$$\begin{aligned}
& \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\
& e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \\
& \geq \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\
& e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^{+im*})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}. \quad (B).
\end{aligned}$$

Thus, from (A) and (B), we have

$$\left(\begin{array}{c} \left[\left(\left(\sum_{r=1}^m \left(\alpha_r \left(\eta_{C_r}^- \right)^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m \left(\alpha_r \left(\eta_{C_r}^+ \right)^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(\sum_{r=1}^m \left(\alpha_r \left(\eta_{C_r}^- \right)^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m \left(\alpha_r \left(\eta_{C_r}^+ \right)^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}, \\ \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^-)^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^+)^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^-)^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\psi_{C_r}^+)^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{array} \right)$$

$$\leq \left(\begin{array}{c} \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im*})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \\ \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-*})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{+*})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-im*})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{+im*})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{array} \right).$$

Hence $CIVq\text{-ROFWAM}_E(P_1, P_2, P_3, \dots, P_m) \leq CIVq\text{-ROFWAM}_E(P_1^*, P_2^*, P_3^*, \dots, P_m^*)$. $\square$

Definition 7. Let $\{P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}) : r = 1, 2, \dots, m\}$ be the collection of $CIVq\text{-ROF}$ values and let $CIVq\text{-ROFOWAM} : \Omega^m \rightarrow \Omega$, if $CIVq\text{-ROFOWAM}_E(P_1, P_2, P_3, \dots, P_m) = ((\alpha_1 P_{\delta(1)})^\lambda \oplus (\alpha_2 P_{\delta(1)})^\lambda \oplus (\alpha_3 P_{\delta(3)})^\lambda \oplus \dots \oplus (\alpha_m P_{\delta(m)})^\lambda)^{\frac{1}{\lambda}}$ then $CIVq\text{-ROFOWAM}$ is called a $CIVq\text{-rung orthopair fuzzy ordered weighted averaging mean operator of dimension } n$, where $(\delta(1), \delta(2), \dots, \delta(m))$ is a permutation of $(1, 2, \dots, m)$ such that $P_{\delta(r-1)} \geq P_{\delta(r)}$ for all r , Ω is the set of all $CIVq\text{-ROF}$ values, $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$, where $r = 1, 2, \dots, m$.

Theorem 4. Let $P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]})$ be the collection of $CIVq\text{-ROF}$ values, where $r = 1, 2, \dots, m$. Then by using the $CIVq\text{-ROFOWAM}_E$ operator their aggregated value is also a $CIVq\text{-ROF}$ value and

$$CIVq\text{-ROFOWAM}_E(P_1, P_2, P_3, \dots, P_m) = \left(\begin{array}{c} \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_{\delta(r)}}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_{\delta(r)}}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_{\delta(r)}}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_{\delta(r)}}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \\ \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_{\delta(r)}}^-)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_{\delta(r)}}^+)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_{\delta(r)}}^{-im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_{\delta(r)}}^{+im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{array} \right)$$

where $E = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^n \alpha_r = 1$, where

Proof. Similar to the proof of Theorem 1. $\square$

Theorem 5. Let $\{P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}) : r = 1, 2, \dots, m\}$ be the collection of CIVq-ROF values. Let $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. If $\left([\eta_{C_1}^-, \eta_{C_1}^+] e^{i2\pi[\eta_{C_1}^{-im}, \eta_{C_1}^{+im}]}, [\psi_{C_1}^-, \psi_{C_1}^+] e^{i2\pi[\psi_{C_1}^{-im}, \psi_{C_1}^{+im}]}\right) = \left([\eta_{C_2}^-, \eta_{C_2}^+] e^{i2\pi[\eta_{C_2}^{-im}, \eta_{C_2}^{+im}]}, [\psi_{C_2}^-, \psi_{C_2}^+] e^{i2\pi[\psi_{C_2}^{-im}, \psi_{C_2}^{+im}]}\right) = \dots = \left([\eta_{C_m}^-, \eta_{C_m}^+] e^{i2\pi[\eta_{C_m}^{-im}, \eta_{C_m}^{+im}]}, [\psi_{C_m}^-, \psi_{C_m}^+] e^{i2\pi[\psi_{C_m}^{-im}, \psi_{C_m}^{+im}]}\right)$ and $\lambda = 1$, then CIVq-ROFOWAM $_E(P_1, P_2, P_3, \dots, P_m) = ([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]})$.

Proof. Similar to the proof of Theorem 1. $\square$

Theorem 6. Let $\{P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}) : r = 1, 2, \dots, m\}$ be the collection of CIVq-ROF values and $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. Let

$$P^- = \left(\left[\min_{1 \leq r \leq m} \eta_{C_r}^-, \min_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \eta_{C_r}^{-im}, \min_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\max_{1 \leq r \leq m} \psi_{C_r}^-, \max_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \psi_{C_r}^{-im}, \max_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right)$$

and

$$P^+ = \left(\left[\max_{1 \leq r \leq m} \eta_{C_r}^-, \max_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\min_{1 \leq r \leq m} \psi_{C_r}^-, \min_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_{C_r}^{-im}, \min_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right),$$

where and $\lambda = 1$. Then $P^- \leq \text{CIVq-ROFOWAM}_E(P_1, P_2, P_3, \dots, P_m) \leq P^+$.

Proof. Similar to the proof of Theorem 2. $\square$

Theorem 7. Let $\{P_r = ([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}) : r = 1, 2, \dots, m\}$ and $\{P_r^* = ([\eta_{C_r}^{*-}, \eta_{C_r}^{+*}] e^{i2\pi[\eta_{C_r}^{-im*}, \eta_{C_r}^{+im*}]}, [\psi_{C_r}^{*-}, \psi_{C_r}^{+*}] e^{i2\pi[\psi_{C_r}^{-im*}, \psi_{C_r}^{+im*}]}) : r = 1, 2, \dots, m\}$ are two collections of CIVq-ROF values. If $\eta_{C_r}^- \leq \eta_{C_r}^{*-}, \eta_{C_r}^+ \leq \eta_{C_r}^{+*}, \eta_{C_r}^{-im} \leq \eta_{C_r}^{-im*}, \eta_{C_r}^{+im} \leq \eta_{C_r}^{+im*}, \psi_{C_r}^- \geq \psi_{C_r}^{*-}, \psi_{C_r}^+ \geq \psi_{C_r}^{+*}, \psi_{C_r}^{-im} \geq \psi_{C_r}^{-im*}$ and $\psi_{C_r}^{+im} \geq \psi_{C_r}^{+im*}$ where $r = 1, 2, \dots, m$, then CIVq-ROFOWAM $_E(P_1, P_2, P_3, \dots, P_m) \leq \text{CIVq-ROFOWAM}_E(P_1^*, P_2^*, P_3^*, \dots, P_m^*)$.

Proof. Similar to the proof of Theorem 3. $\square$

3. COMPLEX INTERVAL VALUED q -RUNG ORTHOPAIR FUZZY WEIGHTED GEOMETRIC MEAN AGGREGATION OPERATORS

Here, we provide some new geometric aggregation operators, complex interval valued q -ROF weighted geometric mean aggregation operator ($CIVq$ -ROFWGM) and complex interval valued q -ROF ordered weighted geometric mean aggregation operator ($CIVq$ -ROFOWGM), using the suggested operations.

Definition 8. Let $\left\{ P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]} , [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) : r = 1, 2, \dots, m \right\}$ be the collection of q -ROF values and let $CIVq$ -ROFOWGM : $\Omega^m \rightarrow \Omega$, if $CIVq$ -ROFWGM $_E(P_1, P_2, P_3, \dots, P_m) = \left((P_1^{\odot \alpha_1})^\lambda \otimes (P_2^{\odot \alpha_2})^\lambda \otimes (P_3^{\odot \alpha_3})^\lambda \otimes \dots \otimes (P_m^{\odot \alpha_m})^\lambda \right)^{\frac{1}{\lambda}}$ then $CIVq$ -ROFWGM is called a complex interval valued q -rung orthopair fuzzy weighted geometric mean operator of dimension n , where Ω is the set of all $CIVq$ -ROF values, $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$, where $r = 1, 2, \dots, m$.

Theorem 8. Let $\left\{ P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]} , [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) : r = 1, 2, \dots, m \right\}$ be the collection of $CIVq$ -ROF values. Then by using the $CIVq$ -ROFWGM $_E$ operator their aggregated value is also a $CIVq$ -ROF value and

$$CIVq\text{-ROFWGM}_E(P_1, P_2, P_3, \dots, P_m) = \left(\begin{aligned} & \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\eta_{C_r}^-)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\eta_{C_r}^+)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\eta_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\eta_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}, \\ & \left[\left(\left(\sum_{r=1}^m (\alpha_r (\psi_{C_r}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\psi_{C_r}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\psi_{C_r}^{-im}(u))^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\psi_{C_r}^{+im}(u))^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{aligned} \right),$$

where $E = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^n \alpha_r = 1$, where $r = 1, 2, \dots, m$.

Proof. Similar to the proof of Theorem 1. □

Proposition 2. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+]\right) e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right\} : r = 1, 2, \dots, m$ be the collection of CIVq-ROF values. Let $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. If $\left([\eta_{C_1}^-, \eta_{C_1}^+] e^{i2\pi[\eta_{C_1}^{-im}, \eta_{C_1}^{+im}]}, [\psi_{C_1}^-, \psi_{C_1}^+] e^{i2\pi[\psi_{C_1}^{-im}, \psi_{C_1}^{+im}]}\right) = \left([\eta_{C_2}^-, \eta_{C_2}^+] e^{i2\pi[\eta_{C_2}^{-im}, \eta_{C_2}^{+im}]}, [\psi_{C_2}^-, \psi_{C_2}^+] e^{i2\pi[\psi_{C_2}^{-im}, \psi_{C_2}^{+im}]}\right) = \dots = \left([\eta_{C_m}^-, \eta_{C_m}^+] e^{i2\pi[\eta_{C_m}^{-im}, \eta_{C_m}^{+im}]}, [\psi_{C_m}^-, \psi_{C_m}^+] e^{i2\pi[\psi_{C_m}^{-im}, \psi_{C_m}^{+im}]}\right) = \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]}\right)$ and $\lambda = 1$, then $CIVq-ROFWGM_E(P_1, P_2, P_3, \dots, P_m) = \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]}\right)$.

Proof. Similar to the proof of Theorem 1. $\square$

Theorem 9. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+]\right) e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right\} : r = 1, 2, \dots, m$ be the collection of CIVq-ROF values and $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. Let

$$P^- = \left(\left[\min_{1 \leq r \leq m} \eta_{C_r}^-, \min_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \eta_{C_r}^{-im}, \min_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\max_{1 \leq r \leq m} \psi_{C_r}^-, \max_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \psi_{C_r}^{-im}, \max_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right)$$

and

$$P^+ = \left(\left[\max_{1 \leq r \leq m} \eta_{C_r}^-, \max_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\min_{1 \leq r \leq m} \psi_{C_r}^-, \min_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_{C_r}^{-im}, \min_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right),$$

where and $\lambda = 1$. Then $P^- \leq CIVq-ROFWGM_E(P_1, P_2, P_3, \dots, P_m) \leq P^+$.

Proof. Similar to the proof of Theorem 2. $\square$

Theorem 10. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+]\right) e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right\} : r = 1, 2, \dots, m$ and $\left\{P_r^* = \left([\eta_{C_r}^{*-}, \eta_{C_r}^{*+}]\right) e^{i2\pi[\eta_{C_r}^{-im*}, \eta_{C_r}^{+im*}]}, [\psi_{C_r}^{*-}, \psi_{C_r}^{*+}] e^{i2\pi[\psi_{C_r}^{-im*}, \psi_{C_r}^{+im*}]}\right\} : r = 1, 2, \dots, m$ are two collections of CIVq-ROF values. If $\eta_{C_r}^- \leq \eta_{C_r}^{*-}, \eta_{C_r}^+ \leq \eta_{C_r}^{*+}, \eta_{C_r}^{-im} \leq \eta_{C_r}^{-im*}, \eta_{C_r}^{+im} \leq \eta_{C_r}^{+im*}, \psi_{C_r}^- \geq \psi_{C_r}^{*-}, \psi_{C_r}^+ \geq \psi_{C_r}^{*+}, \psi_{C_r}^{-im} \geq \psi_{C_r}^{-im*}$ and $\psi_{C_r}^{+im} \geq \psi_{C_r}^{+im*}$, where $r = 1, 2, \dots, m$, then $CIVq-ROFWGM_E(P_1, P_2, P_3, \dots, P_m) \leq CIVq-ROFWGM_E(P_1^*, P_2^*, P_3^*, \dots, P_m^*)$.

Proof. Similar to the proof of Theorem 3. $\square$

Definition 9. Let $\left\{P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+]\right) e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}\right\} : r = 1, 2, \dots, m$ be the collection of CIVq-ROF values and let $CIVq-ROFOWGM : \Omega^m \rightarrow \Omega$, if

$$CIVq-ROFOWGM_E(P_1, P_2, P_3, \dots, P_m) = \left(\left(P_{\delta(1)}^{\odot \alpha_1} \right)^\lambda \otimes \left(P_{\delta(2)}^{\odot \alpha_2} \right)^\lambda \otimes \left(P_{\delta(3)}^{\odot \alpha_3} \right)^\lambda \otimes \dots \otimes \left(P_{\delta(r)}^{\odot \alpha_m} \right)^\lambda \right)^{\frac{1}{\lambda}}$$

then $CIVq-ROFOWGM_E$ is called a complex interval valued q-rung orthopair fuzzy ordered weighted geometric operator of dimension n , where $(\delta(1), \delta(2), \dots, \delta(m))$ is a permutation

of $(1, 2, \dots, m)$ such that $P_{\delta(r-1)} \geq P_{\delta(r)}$ for all r , Ω is the set of all CIVq-ROF values, $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$.

Theorem 11. Let $\left\{ P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]} , [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) : r = 1, 2, \dots, m \right\}$ be the collection of CIVq-ROF values. Then by using the CIVq-ROFWGM_E operator their aggregated value is also a CIVq-ROF value and

$$\begin{aligned} & \text{CIVq-ROFWGM}_E(P_1, P_2, P_3, \dots, P_m) \\ &= e^{\left[\left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\eta_{C_{\delta(r)}}^-)^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} , \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\eta_{C_{\delta(r)}}^+)^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] } \right. \\ & \quad \left. i2\pi \left[\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\eta_{C_{\delta(r)}}^{-im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} , \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\eta_{C_{\delta(r)}}^{+im})^q \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \right. \\ & \quad \left. \left[\left(\left(\sum_{r=1}^m \left(\alpha_r (\psi_{C_{\delta(r)}}^-)^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} , \left(\left(\sum_{r=1}^m \left(\alpha_r (\psi_{C_{\delta(r)}}^+)^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \right. \\ & \quad \left. i2\pi \left[\left(\left(\sum_{r=1}^m \left(\alpha_r (\psi_{C_{\delta(r)}}^{-im}(u))^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} , \left(\left(\sum_{r=1}^m \left(\alpha_r (\psi_{C_{\delta(r)}}^{+im}(u))^q \right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \right] \right) \end{aligned}$$

where $E = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^n \alpha_r = 1$.

Proof. Similar to the proof of Theorem 1. $\square$

Proposition 3. Let $\left\{ P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]} , [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) : r = 1, 2, \dots, m \right\}$

be the collection of CIVq-ROF values. Let $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r

with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$. If $\left([\eta_{C_1}^-, \eta_{C_1}^+] e^{i2\pi[\eta_{C_1}^{-im}, \eta_{C_1}^{+im}]} , [\psi_{C_1}^-, \psi_{C_1}^+] e^{i2\pi[\psi_{C_1}^{-im}, \psi_{C_1}^{+im}]} \right)$

$$= \left([\eta_{C_2}^-, \eta_{C_2}^+] e^{i2\pi[\eta_{C_2}^{-im}, \eta_{C_2}^{+im}]} , [\psi_{C_2}^-, \psi_{C_2}^+] e^{i2\pi[\psi_{C_2}^{-im}, \psi_{C_2}^{+im}]} \right) = \dots$$

$$= \left([\eta_{C_m}^-, \eta_{C_m}^+] e^{i2\pi[\eta_{C_m}^{-im}, \eta_{C_m}^{+im}]} , [\psi_{C_m}^-, \psi_{C_m}^+] e^{i2\pi[\psi_{C_m}^{-im}, \psi_{C_m}^{+im}]} \right)$$

$$= \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]} , [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]} \right) \text{ and } \lambda = 1, \text{ then}$$

$$\text{CIVq-ROFWGM}_E(P_1, P_2, P_3, \dots, P_m) = \left([\eta_C^-, \eta_C^+] e^{i2\pi[\eta_C^{-im}, \eta_C^{+im}]} , [\psi_C^-, \psi_C^+] e^{i2\pi[\psi_C^{-im}, \psi_C^{+im}]} \right).$$

Proof. Similar to the proof of Theorem 1. $\square$

Theorem 12. Let $\left\{ P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi[\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]} , [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi[\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) : r = 1, 2, \dots, m \right\}$

be the collection of CIVq-ROF values and $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r

with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$, where $r = 1, 2, \dots, m$. Let

$$P^- = \left(\left[\min_{1 \leq r \leq m} \eta_{C_r}^-, \min_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \eta_{C_r}^{-im}, \min_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\max_{1 \leq r \leq m} \psi_{C_r}^-, \max_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \psi_{C_r}^{-im}, \max_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right)$$

and

$$P^+ = \left(\left[\max_{1 \leq r \leq m} \eta_{C_r}^-, \max_{1 \leq r \leq m} \eta_{C_r}^+ \right] e^{i2\pi \left[\max_{1 \leq r \leq m} \eta_{C_r}^{-im}, \max_{1 \leq r \leq m} \eta_{C_r}^{+im} \right]}, \left[\min_{1 \leq r \leq m} \psi_{C_r}^-, \min_{1 \leq r \leq m} \psi_{C_r}^+ \right] e^{i2\pi \left[\min_{1 \leq r \leq m} \psi_{C_r}^{-im}, \min_{1 \leq r \leq m} \psi_{C_r}^{+im} \right]} \right),$$

where and $\lambda = 1$. Then $P^- \leq CIVq\text{-ROFOWGM}_E(P_1, P_2, P_3, \dots, P_m) \leq P^+$.

Proof. Similar to the proof of Theorem 2. $\square$

Theorem 13. Let $\left\{ P_r = \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi [\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi [\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) : r = 1, 2, \dots, m \right\}$ and $\left\{ P_r^* = \left([\eta_{C_r}^{*-}, \eta_{C_r}^{*+}] e^{i2\pi [\eta_{C_r}^{*-im}, \eta_{C_r}^{*+im}]}, [\psi_{C_r}^{*-}, \psi_{C_r}^{*+}] e^{i2\pi [\psi_{C_r}^{*-im}, \psi_{C_r}^{*+im}]} \right) : r = 1, 2, \dots, m \right\}$ are two collections of CIVq-ROF values. If $\eta_{C_r}^- \leq \eta_{C_r}^{*-}, \eta_{C_r}^+ \leq \eta_{C_r}^{*+}, \eta_{C_r}^{-im} \leq \eta_{C_r}^{*-im}, \eta_{C_r}^{+im} \leq \eta_{C_r}^{*+im}, \psi_{C_r}^- \geq \psi_{C_r}^{*-}, \psi_{C_r}^+ \geq \psi_{C_r}^{*+}, \psi_{C_r}^{-im} \geq \psi_{C_r}^{*-im}$ and $\psi_{C_r}^{+im} \geq \psi_{C_r}^{*+im}$ where $r = 1, 2, \dots, m$, then $CIVq\text{-ROFOWGM}_E(P_1, P_2, P_3, \dots, P_m) \leq CIVq\text{-ROFOWGM}_E(P_1^*, P_2^*, P_3^*, \dots, P_m^*)$.

Proof. Similar to the proof of Theorem 3. $\square$

4. MADM TECHNIQUE USING CIVq-RUNG ORTHOPAIR FUZZY WEIGHTED MEAN AGGREGATION OPERATORS

According to the judgments of certain specialists related to qualities, the MADM is a highly important technique for choosing an option within the collection of attractive alternatives. The decision-maker always pursues the best and most logical path of action beneficial when applying MADM, which is frequently utilized in real-world circumstances. This means that selecting the appropriate strategy for decision-making is essential, and several methods should be applied for different contexts. For this reason, in this section, we offer a complex interval valued theory based on the proposed q -rung orthopair to handle the MADM problem.

4.1. Decision-making steps. Let $G = \{G_k : k = 1, 2, 3, \dots, l\}$ be the set of experts, whose weight vector $w = (w_1, w_2, w_3, \dots, w_l)$. Let $\mathcal{U} = \{u_i : i = 1, 2, 3, \dots, m\}$ be the set of alternatives and $C = \{C_t : t = 1, 2, 3, \dots, n\}$ be the set of attributes. Let $Q^k = ((\eta_{it}, \psi_{it}))_{m \times n}$ be the decision matrix (Information system) for each expert G_k . In this section, we construct a MADM methodology, including the following key steps:

Step 1 : Two categories of attributes, such as cost and benefits, are included in this procedure. Normalized the decision matrix to convert costs into benefits based on the following formula:

$$P_{ij}^k = \begin{cases} \left([\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi [\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]}, [\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi [\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]} \right) & \text{for benefit attribute } C_r \\ \left([\psi_{C_r}^-, \psi_{C_r}^+] e^{i2\pi [\psi_{C_r}^{-im}, \psi_{C_r}^{+im}]}, [\eta_{C_r}^-, \eta_{C_r}^+] e^{i2\pi [\eta_{C_r}^{-im}, \eta_{C_r}^{+im}]} \right) & \text{for cost attribute } C_r \end{cases}$$

Step 2 : Using Definition 6 or Definition 8, to aggregate the decision matrix G_k , i.e.

$$P_i^k = CIVq-ROFWAM_E \left(P_{i1}^k, P_{i2}^k, P_{i3}^k, \dots, P_{im}^k \right)$$

or

$$P_i^k = CIVq-ROFWGM_E \left(P_{i1}^k, P_{i2}^k, P_{i3}^k, \dots, P_{im}^k \right)$$

where $k = 1, 2, 3, \dots, l$.

Step 3 : Aggregate the difference preferences $P_i^k, i = 1, 2, 3, \dots, m$ into P_i by applying Definition 6 or Definition 8, i.e.

$$P_i = CIVq-ROFWAM_E \left(P_i^1, P_i^2, P_i^3, \dots, P_i^l \right)$$

or

$$P_i = CIVq-ROFWGM_E \left(P_i^1, P_i^2, P_i^3, \dots, P_i^l \right).$$

Step 4 : Compute the score values of the aggregated values obtained in Step 3, where the score function is

$$\Psi(P_r) = \frac{1}{8} \left(1 - \left((\eta_{C_r}^-)^q + (\eta_{C_r}^+)^q + (\eta_{C_r}^{-im})^q + (\eta_{C_r}^{+im})^q + (\psi_{C_r}^-)^q + (\psi_{C_r}^+)^q + (\psi_{C_r}^{-im})^q + (\psi_{C_r}^{+im})^q \right) \right)$$

Step 5 : Determine which selection is better by ranking each one based on the Step 4 score values or accuracy values.

Further, Fig. 2 represent the graphical representation of proposed method. Furthermore, the above Step 4 can be easily repalced by Remark 1 with the involvement of a new parameter $\lambda \in [0, 1]$.

Remark 1. Determine the score values using the score function for the aggregated values acquired in Step 3. We may also utilize the new score formula that is based on CIVq-ROFWAM and CIVq-ROFWGM, that is to say

$$New \ score = \Omega(C_i) = \alpha.S(C_i)_{CIVq-ROFWAM} + (1 - \alpha).S(C_i)_{CIVq-ROFWGM}.$$

5. ANALYSIS OF ARTIFICIAL INTELLIGENCE IN EDUCATION SYSTEMS

In this section, we address the use of Artificial intelligence (AI) tools in education, drawing inspiration from the works of Fitria [47], Bagunaid et al. [50], and Harry [1].

Artificial intelligence tools hold great promise for revolutionizing and improving several facets of the education industry. These resources can be utilized to enhance instruction, customize student experiences, and give administrators and teachers insightful information. The following artificial tools are frequently employed in the field of education:

(1) *Adaptive learning platforms:*

Adaptive learning platforms are artificial technologies created especially for the educational industry. Examples of these kinds of systems are Knewton, DreamBox, and Smart Sparrow. Artificial learning platforms are systems that use cutting-edge technology, such as machine learning and artificial intelligence (AI), to give students individualized educational activities. Based on each student's skills, weaknesses, and preferred methods of learning, these platforms modify the pace, written material, and teaching strategies to suit their specific needs.

(2) *Educational chatbots:*

Artificial intelligence technologies known as educational chatbots are made to converse with users to reply to inquiries, give advice, and offer support about educational materials. These conversational agents utilize machine learning techniques and natural language processing (NLP) to comprehend and react to user inquiries. Prompt assistance, personalized learning support, help with language acquisition, feedback and evaluation, management assistance, student engagement, and adaptable learning paths are some of the main characteristics and uses of educational chatbots in the field of education. By giving teachers and students prompt, individualized support, educational chatbots improve the quality of the learning process. They are useful instruments that support efficiency, accessibility, and participation in learning environments.

(3) *Automated grading system:*

An artificial tool created to simplify and speed up the steps of analyzing and marking assignments, tests, and evaluations is known as an automated grading system. This tool makes use of technology to evaluate and grade student work automatically. It frequently incorporates machine learning and intelligence-based algorithms. Educational institutions can benefit from automated grading systems due to their effectiveness and time savings, uniformity and reliability, instantaneous response, flexibility, customization of marking rubrics, and compliance with rubric standards.

(4) *Language learning apps:*

Applications for language learning are made by humans to make learning new languages easier by providing interesting and dynamic experiences. By utilizing technology, these applications frequently include capabilities like artificial intelligence, natural language processing, and other aspects that improve language acquisition. Personalized learning pathways, adaptive learning algorithms, classroom interaction, entertainment components, vocabulary building, grammar training, and monitoring of progress are some of the salient characteristics and educational tools of language learning apps. By making language instruction available to a worldwide audience and providing a practical and efficient means for people to improve their language abilities, language learning apps help to democratize the field.

(5) *Intelligent tutoring systems:*

Intelligent Tutoring Systems (ITS) are artificial intelligence systems that are intended to give students individualized and flexible training. These systems mimic the job of an instructor by evaluating each student's unique needs, providing tailored feedback, and modifying the course material in response to achievement. They do this by utilizing artificial intelligence, machine learning, and data analytics. ITS are essential for addressing individual requirements, fostering successful learning outcomes, and giving learners focused tailored support. Their efficacy in diverse educational situations can be attributed to their versatility, data-driven insights, and emphasis on individualized instruction.

5.1. Importance of AI in educational sector. AI tools are essential in the education sector because they offer a number of advantages that lead to better learning outcomes for both students and teachers as well as increased efficiency and learning experiences. The following are some major points that emphasize the value of artificial tools in education:

- (i) Learning materials can be customized to meet the desires of each student by using artificial technologies like intelligent tutoring systems and adaptable learning platforms. With a deeper comprehension of the subject matter and the ability to learn at their speed, personalization helps students meet both of these goals.
- (ii) Learning management platforms, educational chatbots, and automated grading systems simplify handling paperwork so that teachers can concentrate more on instructing and giving learners individualized assistance. Reduced workloads and a more fruitful atmosphere for learning may result from this effectiveness.
- (iii) Huge amounts of information about student achievement, involvement, and educational patterns may be gathered and analyzed thanks to artificial tools. With the use of this information, administrators and teachers can improve educational results by identifying areas for enhancements, making well-informed decisions, and putting solutions into practice.
- (iv) Learning languages, pronouncing words correctly, and communicating effectively are all aided by AI-powered communication tools and applications. Students from various cultural backgrounds can benefit from these resources while they learn a new language.
- (v) Teaching with AI equips learners with an employment environment that is dominated by technology. Students who are exposed to robotics, coding, and other technology skills gain the competencies necessary for employment in a variety of professions.

In conclusion, artificial tools play a critical role in modernizing education and enabling it to better meet the changing demands of both educators and students. Although those applications have many benefits, it's important to use them carefully, taking into account privacy issues, ethical issues, and the necessity of striking a balance between technology and human contact in gaining knowledge.

When choosing artificial intelligence tools for learning institutions, things must be carefully considered to make sure that the tools selected are in line with the unique requirements and objectives of the educational institution. These are important criteria to take into account while choosing artificial tools for the education sector

(1) *Cost-effectiveness:*

Thoroughly weighing the advantages of artificial tools against their related expenses is necessary to ensure their cost-effectiveness in the education sector. The following factors are required to optimize the cost-effectiveness of using artificial intelligence in education: precise learning goals, return on investment analysis, total cost of ownership, open-source and free software, cooperative licensing and buying, continuing education expenditure, support and maintenance contracts, and cost-benefit evaluation. Through careful consideration of these variables, educational establishments can optimize the financial efficiency of artificial tools while guaranteeing that the tools are congruent with learning goals and enhance the overall learning surroundings.

(2) *Adaptability and compatibility:*

The efficiency of educational endeavors is greatly enhanced by artificial technologies in education that possess essential features like adaptability and interactivity. An examination of each idea about educational technology is provided below:

(a) *Adaptability:*

The tool must be flexible enough to accommodate different topic areas, levels of schooling, and learning styles. For instance, adaptive learning systems can customize information to meet the needs of each unique student. Adaptive assessments, learning routes, individualized instruction, and data-driven modifications are a few of the key features and components of educational adaptations in the field of education.

(b) *Compatibility:*

A key factor to examine when assessing an AI tool's suitability for the education industry is how effectively it integrates with the current educational technology, mechanisms, and processes. When evaluating compatibility, it's important to take into account the following factors; flexibility, adhering to educational requirements, safety and anonymity, interoperability with learning channels, connectivity with cloud services, cross-device compatibility, connectivity with web browsers, integration with educational administration mechanisms, and connectivity with student information systems. Through a systematic evaluation of these compatibility factors, educational establishments can facilitate the seamless incorporation of AI tools into their current framework, thereby fostering a harmonious and effective learning environment.

(3) *Integration with learning standards:*

AI technologies must be integrated with learning standards in the education sector to comply with existing educational frameworks and rules. The following factors are crucial to consider

when integrating AI tools with learning requirements: curriculum requirements alignment, educational guidelines connection, subject-specific standard adherence, accessibility requirements, interconnection with educational systems, uniformity with national or international standards, records of alignment, and customization flexibility. Educational organizations may improve the efficacy of AI tools for promoting curriculum goals, academic goals, and the general quality of student experiences by thoroughly assessing and making sure that these tools are aligned with educational requirements.

(4) *Feedback mechanism:*

To give teachers and students immediate and useful knowledge, AI technologies for educational institutions must include feedback mechanisms. Good feedback finds areas for growth, helps with personalized education, and enhances student achievement in general. Key elements of feedback processes seen in AI education systems include focused on objective feedback, adjustable feedback, formative evaluation feedback, summative evaluation feedback, multi-dimensional feedback, real-time feedback, progress monitoring, and constructive criticism. Efficient feedback systems in AI tools enable students to actively participate in their educational experience, foster a positive learning environment, and promote continuous enhancement. Specification, interaction, and the synchronization of feedback with learning objectives should be the main design principles for these systems.

(5) *Research-based design:*

Evidence-based concepts and development processes are integrated into the creation technique for AI solutions for educational institutions through research-based design. It seeks to guarantee that standard procedures, education research, and teaching concepts are taken into consideration during the development, deployment, and use of AI systems. The following are important components of research-based design for AI tools in education: educational analytics, data-driven decision-making, inspirational design, user-focused design, adaptive educational paths, cognitive load theory, flexible learning paths, evaluation and feedback techniques, pedagogical alignment, and user-centered architecture.

Let $E = \{G_1\}$ be the set of one expert. Let $\mathcal{U} = \{P_1, P_2, P_3, P_4, P_5\}$ be the set of (artificial tools) alternatives, where P_1 : adaptive learning platforms, P_2 : educational chatbots, P_3 : automated grading system, P_4 : language learning apps, P_5 : intelligent tutoring systems and $C = \{C_1, C_2, C_3, C_4, C_5\}$ be the set of attributes, where C_1 : cost-effectiveness, C_2 : adaptability and compatibility, C_3 : integration with learning standards, C_4 : feedback mechanism, C_5 : research-based design. Based on these attributes, we evaluated the best kind of alternatives among our five alternatives. For this, we consider the following weight vector for our considered attributes C_1, C_2, C_3, C_4 and C_5 such as (0.35, 0.19, 0.06, 0.17, 0.23). Thus,

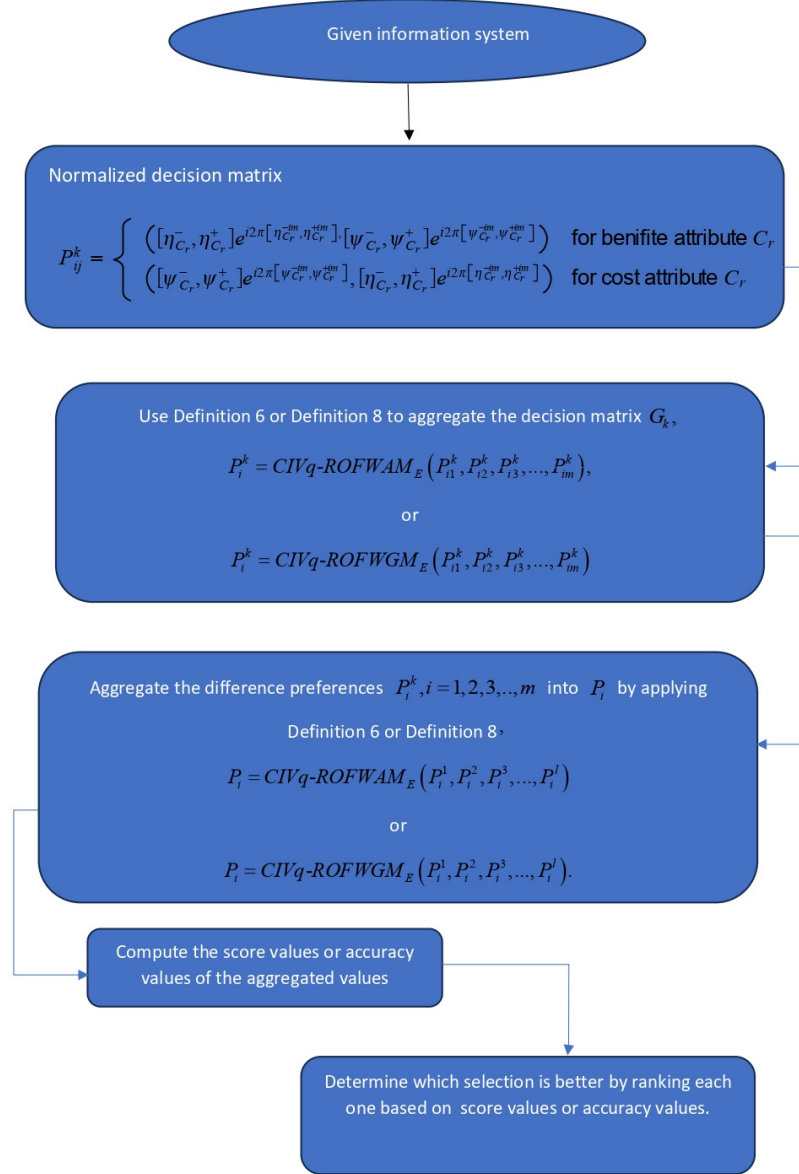


FIGURE 2. The graphical representaion of proposed model

we considered our procedure and tried to evaluate our targeted result. Table 2 represent the information table given by expert G_1 .

Table 2. Information table given by expert G_1

	C_1	C_2	C_3	C_4	C_5
P_1	$\begin{pmatrix} [0.7, 0.8] \\ e^{i2\pi[0.5, 0.6]}, \\ [0.77, 0.9] \\ e^{i2\pi[0.66, 0.72]} \end{pmatrix}$	$\begin{pmatrix} [0.56, 0.84] \\ e^{i2\pi[0.44, 0.66]}, \\ [0.76, 0.85] \\ e^{i2\pi[0.44, 0.66]} \end{pmatrix}$	$\begin{pmatrix} [0.28, 0.80] \\ e^{i2\pi[0.7, 0.8]}, \\ [0.38, 0.7] \\ e^{i2\pi[0.8, 0.82]} \end{pmatrix}$	$\begin{pmatrix} [0.45, 0.84] \\ e^{i2\pi[0.67, 0.75]}, \\ [0.58, 0.78] \\ e^{i2\pi[0.45, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.9] \\ e^{i2\pi[0.6, 0.7]}, \\ [0.70, 0.8] \\ e^{i2\pi[0.6, 0.8]} \end{pmatrix}$
P_2	$\begin{pmatrix} [0.45, 0.79] \\ e^{i2\pi[0.61, 0.75]}, \\ [0.18, 0.91] \\ e^{i2\pi[0.36, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.82, 0.86] \\ e^{i2\pi[0.36, 0.65]}, \\ [0.83, 0.88] \\ e^{i2\pi[0.46, 0.8]} \end{pmatrix}$	$\begin{pmatrix} [0.58, 0.78] \\ e^{i2\pi[0.6, 0.71]}, \\ [0.83, 0.9] \\ e^{i2\pi[0.5, 0.55]} \end{pmatrix}$	$\begin{pmatrix} [0.46, 0.77] \\ e^{i2\pi[0.56, 0.8]}, \\ [0.55, 0.66] \\ e^{i2\pi[0.55, 0.79]} \end{pmatrix}$	$\begin{pmatrix} [0.18, 0.9] \\ e^{i2\pi[0.24, 0.85]}, \\ [0.28, 0.79] \\ e^{i2\pi[0.26, 0.75]} \end{pmatrix}$
P_3	$\begin{pmatrix} [0.58, 0.67] \\ e^{i2\pi[0.62, 0.65]}, \\ [0.68, 0.79] \\ e^{i2\pi[0.63, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.9] \\ e^{i2\pi[0.4, 0.5]}, \\ [0.6, 0.7] \\ e^{i2\pi[0.5, 0.6]} \end{pmatrix}$	$\begin{pmatrix} [0.38, 0.7] \\ e^{i2\pi[0.6, 0.66]}, \\ [0.39, 0.75] \\ e^{i2\pi[0.46, 0.76]} \end{pmatrix}$	$\begin{pmatrix} [0.48, 0.87] \\ e^{i2\pi[0.65, 0.76]}, \\ [0.82, 0.89] \\ e^{i2\pi[0.66, 0.75]} \end{pmatrix}$	$\begin{pmatrix} [0.18, 0.77] \\ e^{i2\pi[0.61, 0.65]}, \\ [0.28, 0.76] \\ e^{i2\pi[0.62, 0.85]} \end{pmatrix}$
P_4	$\begin{pmatrix} [0.48, 0.84] \\ e^{i2\pi[0.66, 0.76]}, \\ [0.82, 0.9] \\ e^{i2\pi[0.67, 0.88]} \end{pmatrix}$	$\begin{pmatrix} [0.83, 0.85] \\ e^{i2\pi[0.36, 0.56]}, \\ [0.81, 0.88] \\ e^{i2\pi[0.46, 0.58]} \end{pmatrix}$	$\begin{pmatrix} [0.58, 0.76] \\ e^{i2\pi[0.62, 0.75]}, \\ [0.48, 0.77] \\ e^{i2\pi[0.67, 0.7]} \end{pmatrix}$	$\begin{pmatrix} [0.75, 0.78] \\ e^{i2\pi[0.45, 0.78]}, \\ [0.68, 0.75] \\ e^{i2\pi[0.56, 0.79]} \end{pmatrix}$	$\begin{pmatrix} [0.28, 0.71] \\ e^{i2\pi[0.26, 0.65]}, \\ [0.38, 0.72] \\ e^{i2\pi[0.36, 0.75]} \end{pmatrix}$
P_5	$\begin{pmatrix} [0.58, 0.67] \\ e^{i2\pi[0.65, 0.86]}, \\ [0.77, 0.78] \\ e^{i2\pi[0.45, 0.78]} \end{pmatrix}$	$\begin{pmatrix} [0.56, 0.72] \\ e^{i2\pi[0.52, 0.54]}, \\ [0.63, 0.8] \\ e^{i2\pi[0.5, 0.58]} \end{pmatrix}$	$\begin{pmatrix} [0.6, 0.7] \\ e^{i2\pi[0.5, 0.6]}, \\ [0.8, 0.81] \\ e^{i2\pi[0.6, 0.8]} \end{pmatrix}$	$\begin{pmatrix} [0.7, 0.8] \\ e^{i2\pi[0.6, 0.7]}, \\ [0.81, 0.83] \\ e^{i2\pi[0.7, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.9] \\ e^{i2\pi[0.7, 0.8]}, \\ [0.85, 0.86] \\ e^{i2\pi[0.6, 0.82]} \end{pmatrix}$

We normalized the values in Table 1 because C_1 is a cost type, and the outcome is shown in Table 3.

Table 3. Normalized information table of expert G_1

	C_1	C_2	C_3	C_4	C_5
P_1	$\begin{pmatrix} [0.77, 0.9] \\ e^{i2\pi[0.66, 0.72]}, \\ [0.7, 0.8] \\ e^{i2\pi[0.5, 0.6]} \end{pmatrix}$	$\begin{pmatrix} [0.56, 0.84] \\ e^{i2\pi[0.44, 0.66]}, \\ [0.76, 0.85] \\ e^{i2\pi[0.44, 0.66]} \end{pmatrix}$	$\begin{pmatrix} [0.28, 0.80] \\ e^{i2\pi[0.7, 0.8]}, \\ [0.38, 0.7] \\ e^{i2\pi[0.8, 0.82]} \end{pmatrix}$	$\begin{pmatrix} [0.45, 0.84] \\ e^{i2\pi[0.67, 0.75]}, \\ [0.58, 0.78] \\ e^{i2\pi[0.45, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.9] \\ e^{i2\pi[0.6, 0.7]}, \\ [0.70, 0.8] \\ e^{i2\pi[0.6, 0.8]} \end{pmatrix}$
P_2	$\begin{pmatrix} [0.18, 0.91] \\ e^{i2\pi[0.36, 0.85]}, \\ [0.45, 0.79] \\ e^{i2\pi[0.61, 0.75]}, \end{pmatrix}$	$\begin{pmatrix} [0.82, 0.86] \\ e^{i2\pi[0.36, 0.65]}, \\ [0.83, 0.88] \\ e^{i2\pi[0.46, 0.8]} \end{pmatrix}$	$\begin{pmatrix} [0.58, 0.78] \\ e^{i2\pi[0.6, 0.71]}, \\ [0.83, 0.9] \\ e^{i2\pi[0.5, 0.55]} \end{pmatrix}$	$\begin{pmatrix} [0.46, 0.77] \\ e^{i2\pi[0.56, 0.8]}, \\ [0.55, 0.66] \\ e^{i2\pi[0.55, 0.79]} \end{pmatrix}$	$\begin{pmatrix} [0.18, 0.9] \\ e^{i2\pi[0.24, 0.85]}, \\ [0.28, 0.79] \\ e^{i2\pi[0.26, 0.75]} \end{pmatrix}$
P_3	$\begin{pmatrix} [0.58, 0.67] \\ e^{i2\pi[0.62, 0.65]}, \\ [0.68, 0.79] \\ e^{i2\pi[0.63, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.9] \\ e^{i2\pi[0.4, 0.5]}, \\ [0.6, 0.7] \\ e^{i2\pi[0.5, 0.6]} \end{pmatrix}$	$\begin{pmatrix} [0.38, 0.7] \\ e^{i2\pi[0.6, 0.66]}, \\ [0.39, 0.75] \\ e^{i2\pi[0.46, 0.76]} \end{pmatrix}$	$\begin{pmatrix} [0.48, 0.87] \\ e^{i2\pi[0.65, 0.76]}, \\ [0.82, 0.89] \\ e^{i2\pi[0.66, 0.75]} \end{pmatrix}$	$\begin{pmatrix} [0.18, 0.77] \\ e^{i2\pi[0.61, 0.65]}, \\ [0.28, 0.76] \\ e^{i2\pi[0.62, 0.85]} \end{pmatrix}$
P_4	$\begin{pmatrix} [0.82, 0.9] \\ e^{i2\pi[0.67, 0.88]}, \\ [0.48, 0.84] \\ e^{i2\pi[0.66, 0.76]} \end{pmatrix}$	$\begin{pmatrix} [0.83, 0.85] \\ e^{i2\pi[0.36, 0.56]}, \\ [0.81, 0.88] \\ e^{i2\pi[0.46, 0.58]} \end{pmatrix}$	$\begin{pmatrix} [0.58, 0.76] \\ e^{i2\pi[0.62, 0.75]}, \\ [0.48, 0.77] \\ e^{i2\pi[0.67, 0.7]} \end{pmatrix}$	$\begin{pmatrix} [0.75, 0.78] \\ e^{i2\pi[0.45, 0.78]}, \\ [0.68, 0.75] \\ e^{i2\pi[0.56, 0.79]} \end{pmatrix}$	$\begin{pmatrix} [0.28, 0.71] \\ e^{i2\pi[0.26, 0.65]}, \\ [0.38, 0.72] \\ e^{i2\pi[0.36, 0.75]} \end{pmatrix}$
u_5	$\begin{pmatrix} [0.77, 0.78] \\ e^{i2\pi[0.45, 0.78]}, \\ [0.58, 0.67] \\ e^{i2\pi[0.65, 0.86]}, \end{pmatrix}$	$\begin{pmatrix} [0.56, 0.72] \\ e^{i2\pi[0.52, 0.54]}, \\ [0.63, 0.8] \\ e^{i2\pi[0.5, 0.58]} \end{pmatrix}$	$\begin{pmatrix} [0.6, 0.7] \\ e^{i2\pi[0.5, 0.6]}, \\ [0.8, 0.81] \\ e^{i2\pi[0.6, 0.8]} \end{pmatrix}$	$\begin{pmatrix} [0.7, 0.8] \\ e^{i2\pi[0.6, 0.7]}, \\ [0.81, 0.83] \\ e^{i2\pi[0.7, 0.85]} \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.9] \\ e^{i2\pi[0.7, 0.8]}, \\ [0.85, 0.86] \\ e^{i2\pi[0.6, 0.82]} \end{pmatrix}$

Utilizing Step 2, the aggregated values obtain from expert G_1 by using $CIVq$ -ROFWAM operator with $q = 8$ and $\lambda = 1$. Table 4(a&b) represent aggregated values obtain from expert G_1 , we have Applying the score function in accordance with Step 4 to determine the score values of the aggregated values in Table 4(a&b) are

$$\begin{aligned} \Psi(P_1) &= 0.00553, \Psi(P_2) = 0.00494, \Psi(P_3) = 0.0373, \Psi(P_4) = 0.01116 \\ \text{and } \Psi(P_5) &= 0.01016. \end{aligned}$$

Table 4 (a). Aggregated values obtain from expert G_1

	P_1	P_2	P_3
	$\begin{pmatrix} [0.73369, 0.69868] \\ e^{i2\pi[0.63695, 0.61022]}, \\ [0.8767, 0.80471] \\ e^{i2\pi[0.7203, 0.80213]} \end{pmatrix}$	$\begin{pmatrix} [0.66863, 0.70103] \\ e^{i2\pi[0.47976, 0.553056]}, \\ [0.87692, 0.81161] \\ e^{i2\pi[0.81596, 0.76413]} \end{pmatrix}$	$\begin{pmatrix} [0.68484, 0.67376] \\ e^{i2\pi[0.61186, 0.61187]}, \\ [0.82877, 0.77009] \\ e^{i2\pi[0.77074, 0.75056]} \end{pmatrix}$

Table 4 (b). Aggregated values obtain from expert G_1

	P_4	P_5
	$\begin{pmatrix} [0.77647, 0.67720] e^{i2\pi[0.59575, 0.60147]}, \\ [0.83974, 0.81577] e^{i2\pi[0.79793, 0.74378]} \end{pmatrix}$	$\begin{pmatrix} [0.74459, 0.76181] e^{i2\pi[0.60502, 0.63539]}, \\ [0.81395, 0.79792] e^{i2\pi[0.74869, 0.82304]} \end{pmatrix}$

Using the score values acquired above, the alternatives are ranked based on Step 5, we get

$$P_3 \succ P_4 \succ P_5 \succ P_1 \succ P_2.$$

6. COMPARATIVE ANALYSIS FOR $CIVq$ -RUNG ORTHOPAIR FUZZY ENVIRONMENT

Based on the data in Tables 2, 3, 4(a & b), 5 and 6, we compare the suggested techniques ($CIVq$ -ROFWAM and $CIVq$ -ROFWGM) with a few current operators. When comparing the suggested techniques and the existing approach, the best and top choices are the same, demonstrating the reliability and practicability of the suggested work. We employ the existing operators listed below to complete this study: The concept of geometric aggregation was introduced by Xu and Chen (2007) in the context of interval-valued intuitionistic fuzzy knowledge, and its potential uses were explored. In the framework of interval-valued intuitionistic fuzzy sets, Nayagam et al. (2011) established the notion of the multicriteria decision-making technique. Interval-valued intuitionistic fuzzy sets were ranked using a method devised by Nayagam and Sivaraman (2011), who also talked about practical applications. An innovative accuracy function for tackling multicriteria decision-making problems was introduced by Garg (2016) in an interval-valued Pythagorean fuzzy framework. A approach for addressing multiple criterion group choice analysis was described by Liang et al. (2015) and involved maximizing deviation in the context of an interval-valued Pythagorean fuzzy weighted aggregating operator. In the context of interval-valued Pythagorean fuzzy sets, Garg (2017) introduced the TOPSIS technique and unveiled a revolutionary ranking score function. The idea of interval valued q -rung orthopair fuzzy sets was introduced by Joshi et al. (2018), who also covered some of its characteristics for the construction of this framework. The concept of generalized complex intuitionistic fuzzy aggregation operators was emphasized by Garg and Rani (2019), along with their utilization in multi criteria decision-making processes. Different methods are designed by

Garg et al. (2021) for AHP, TOPSIS, and decision-making employing complex interval-valued q -rung orthopair fuzzy environment. Then, using the data from Tables 2, 3, 4(a & b), 5 and 6, we perform a comparative analysis, which are detailed in Tables 5 and 6. The operators put out by Garg and Rani (2019) and Garg et al. (2021) as well as in this paper, can all get the same optimum choice except Xu and Chen (2007), Nayagam et al. (2011), Nayagam and Sivaraman (2011), Garg (2017), Joshi et al. (2018), Liang et al. (2015) and Garg et al. (2021) AHP. Our suggested technique has the same ranking as the operators defined by Garg and Rani (2019) and Garg et al. (2021). In comparison to other approaches already in use, our suggested alternative selection methodology based on the *CIV q -ROFWAM* and *CIV q -ROFWGM* techniques with entropy measure is more flexible and efficient. According to all procedures, P_3 is the best alternative, as indicated in Table 6 based on Table 5, however various computation stages are used by various techniques. For instance, the authors of the current approaches employed the *CIV q -rung orthopair fuzzy weighted arithmetic mean* and *CIV q -rung orthopair fuzzy weighted geometric mean* operators to aggregate the information, however in the suggested method, we used the *CIV q -ROFWAM* and *CIV q -ROFWGM* methodologies with weight to aggregate complex interval valued q -rung orthopair fuzzy information. Furthermore, we employed various λ , q values and ranked the P_i ($i = 1, 2, 3, 4, 5$) options according to the *CIV q -ROFWAM* and *CIV q -ROFWGM* operator in Table 7 and Table 8. Further, Tables 9, 10 and 11 shows the graphical representations of Tables 5, 7 and 8.

Table 5. Score functions of the alternatives using different methods

Measure	$\Psi (P_1)$	$\Psi (P_2)$	$\Psi (P_3)$	$\Psi (P_4)$	$\Psi (P_5)$
(Xu & Chen, 2007)	-	-	-	-	-
(Nayagam et al., 2011)	-	-	-	-	-
(Nayagam & Sivaraman, 2011)	-	-	-	-	-
(Garg, 2016)	-	-	-	-	-
(Joshi et al., 2018)	-	-	-	-	-
(Liang et al., 2015)	-	-	-	-	-
(Garg, 2017)	-	-	-	-	-
(Garg et al., 2021)AHP	0.190	0.400	0.160	0.220	0.033
(Garg et al., 2021)TOPSIS	0.4448	0.5476	0.4594	0.3384	0.5463
(Garg & Rani, 2019) <i>CIVIFWA</i> for $q = 1$	-0.745	-0.765	-0.710	-0.710	-0.770
(Garg & Rani, 2019) <i>CIVIFWG</i> for $q = 1$	-0.82	-0.95	-0.81	-0.825	-0.815
(Garg et al., 2021) $q = 2$	-0.568	-0.607	-0.510	-0.520	-0.590
(Garg et al., 2021) <i>CIV_q-ROFWA</i> for $q = 4$	-0.340	-0.400	-0.290	-0.310	-0.350
(Garg et al., 2021) $q = 8$	-0.150	-0.190	-0.110	-0.120	-0.130
(Garg et al., 2021) $q = 12$	-0.070	-0.090	-0.040	-0.052	-0.059
(Garg et al., 2021) $q = 15$	-0.040	-0.050	-0.022	-0.031	-0.025
Proposed <i>CIV_q-ROFWAM</i> for $q = 7, \lambda = 1$	-0.02507	-0.02354	0.00965	-0.01967	-0.0222
$q = 8, \lambda = 1$	0.00553	0.00494	0.0373	0.01116	0.01016
$q = 9, \lambda = 1$	0.02908	0.02720	0.05787	0.03491	0.03504
$q = 10, \lambda = 1$	0.04737	0.04482	0.07330	0.05336	0.05428
$q = 12, \lambda = 1$	0.07308	0.07022	0.09384	0.07917	0.08090
$q = 15, \lambda = 1$	0.09535	0.09306	0.10992	0.10101	0.10289
$q = 18, \lambda = 1$	0.10737	0.10583	0.11747	0.11220	0.11374
$q = 20, \lambda = 1$	0.11232	0.11120	0.12020	0.11651	0.11776
$q = 23, \lambda = 1$	0.11711	0.11641	0.12251	0.12037	0.12124
$q = 25, \lambda = 1$	0.11918	0.11869	0.12338	0.12189	0.12256

Table 6. Ranking of the alternatives using different methods

Measure	Ranking
(Xu & Chen, 2007)	No ranking
(Nayagam et al., 2011)	No ranking
(Nayagam & Sivaraman, 2011)	No ranking
(Garg, 2016)	No ranking
(Joshi et al., 2018)	No ranking
(Liang et al., 2015)	No ranking
(Garg, 2017)	No ranking
(Garg et al., 2021)AHP	$P_2 \succ P_4 \succ P_1 \succ P_3 \succ P_5$
(Garg et al., 2021)TOPSIS	$P_2 \succ P_5 \succ P_3 \succ P_1 \succ P_4$
(Garg & Rani, 2019) <i>CIVIFWA</i> for $q = 1$	$P_3 \succ P_4 \succ P_1 \succ P_2 \succ P_5$
(Garg & Rani, 2019) <i>CIVIFWG</i> for $q = 1$	$P_3 \succ P_5 \succ P_1 \succ P_4 \succ P_2$
(Garg et al., 2021) $q = 2$	$P_3 \succ P_4 \succ P_1 \succ P_5 \succ P_2$
(Garg et al., 2021) <i>CIV_q-ROFWA</i> for $q = 4$	$P_3 \succ P_4 \succ P_1 \succ P_5 \succ P_2$
(Garg et al., 2021) $q = 8$	$P_3 \succ P_4 \succ P_5 \succ P_1 \succ P_2$
(Garg et al., 2021) $q = 12$	$P_3 \succ P_4 \succ P_5 \succ P_1 \succ P_2$
(Garg et al., 2021) $q = 15$	$P_3 \succ P_5 \succ P_4 \succ P_2 \succ P_1$
Proposed <i>CIV_q-ROFWAM</i> for $q = 7, \lambda = 1$	$P_3 \succ P_4 \succ P_5 \succ P_1 \succ P_2$
$q = 8, \lambda = 1$	$P_3 \succ P_4 \succ P_5 \succ P_1 \succ P_2$
$q = 9, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 10, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 12, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 15, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 18, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 20, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 23, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$
$q = 25, \lambda = 1$	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$

Table 7. Ranking order of alternatives for different λ using *CIVq-ROFWAM*

Measure	$\Psi(P_1)$	$\Psi(P_2)$	$\Psi(P_3)$	$\Psi(P_4)$	$\Psi(P_5)$	Ranking
$q = 7, \lambda = 2$	-0.37765	-0.38463	-0.36878	-0.39098	-0.37381	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 8, \lambda = 2$	-0.35912	-0.36533	-0.35001	-0.36961	-0.35578	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 9, \lambda = 2$	-0.34394	-0.34916	-0.33469	-0.35198	-0.34128	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 10, \lambda = 2$	-0.33118	-0.33533	-0.32182	-0.33715	-0.32928	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 12, \lambda = 2$	-0.31043	-0.31262	-0.30098	-0.31336	-0.31019	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 8, \lambda = 3$	-0.39860	-0.40624	-0.39445	-0.41243	-0.39473	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 9, \lambda = 3$	-0.38932	-0.39606	-0.38545	-0.40056	-0.38597	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 12, \lambda = 3$	-0.37241	-0.37596	-0.36854	-0.37779	-0.37013	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 15, \lambda = 3$	-0.36338	-0.36506	-0.35993	-0.36595	-0.36250	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 18, \lambda = 5$	-0.37600	-0.37696	-0.37499	-0.37735	-0.37505	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 20, \lambda = 5$	-0.37509	-0.37567	-0.37431	-0.37593	-0.37449	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 23, \lambda = 5$	-0.37428	-0.37459	-0.37373	-0.37472	-0.37401	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 25, \lambda = 5$	-0.37394	-0.37413	-0.37348	-0.37425	-0.37381	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 28, \lambda = 5$	-0.37379	-0.37368	-0.37320	-0.37379	-0.37357	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 25, \lambda = 7$	-0.37525	-0.37544	-0.37501	-0.37547	-0.37502	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 28, \lambda = 7$	-0.37505	-0.37515	-0.37491	-0.37517	-0.37494	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 33, \lambda = 7$	-0.37491	-0.37495	-0.37485	-0.37497	-0.37488	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 38, \lambda = 7$	-0.37486	-0.37487	-0.37481	-0.37488	-0.37485	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$

Table 8. Ranking order of alternatives for different λ using *CIVq-ROFWGM*

Measure	$\Psi(P_1)$	$\Psi(P_2)$	$\Psi(P_3)$	$\Psi(P_4)$	$\Psi(P_5)$	Ranking
$q = 12, \lambda = 2$	-0.31091	-0.31267	-0.30461	-0.31482	-0.31049	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 15, \lambda = 2$	-0.29063	-0.29289	-0.28389	-0.29509	-0.28688	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 18, \lambda = 2$	-0.27370	-0.27640	-0.26626	-0.27864	-0.26782	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 20, \lambda = 2$	-0.26336	-0.26635	-0.25538	-0.26859	-0.25644	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 23, \lambda = 2$	-0.24874	-0.25213	-0.23991	-0.25438	-0.24055	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 25, \lambda = 2$	-0.23943	-0.24308	-0.23003	-0.24532	-0.23053	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 28, \lambda = 2$	-0.22602	-0.23004	-0.21578	-0.23224	-0.21616	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 33, \lambda = 2$	-0.20493	-0.20954	-0.19337	-0.21159	-0.19365	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 18, \lambda = 3$	-0.35896	-0.35913	-0.35708	-0.36091	-0.35810	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 20, \lambda = 3$	-0.35674	-0.35699	-0.35475	-0.35887	-0.35542	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 23, \lambda = 3$	-0.35375	-0.35410	-0.35155	-0.35615	-0.35193	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 25, \lambda = 3$	-0.35186	-0.35227	-0.34945	-0.35444	-0.34980	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 28, \lambda = 3$	-0.34911	-0.34958	-0.34649	-0.35197	-0.34673	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$q = 33, \lambda = 3$	-0.34462	-0.34519	-0.34156	-0.34795	-0.34179	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$

6.1. Results and discussion. In addition, we observed some intriguing discoveries, which are explained in more detail below:

- (i) Nayagam et al. (2011), Joshi et al. (2018) and Liang et al. (2015) operators facilitate decision-making regarding parameter values, allowing for good measures. However, it should be noted that these operators only handle one-dimensional knowledge at the same time, leading to frequent destruction of information. It follows that the suggested work is better than Nayagam et al. (2011), JJoshi et al. (2018) and Liang et al. (2015) because of this shortcoming.
- (ii) About the option element in connection to optimism and pessimism, it is observed that Nayagam et al. (2011), JJoshi et al. (2018) and Liang et al. (2015) techniques offer freedom in selecting any one of the analytical structures, whereas the suggested technique enables the decision-maker to make selections about each of the aforementioned crucial factors.
- (iii) Compared with the current approaches in the interval-valued Pythagorean fuzzy environment, and interval-valued intuitionistic fuzzy environment, the presented method offers significantly more information for handling data ambiguities. It provides a more accurate and precise statement of the object-related knowledge. As such, it is an invaluable mechanism for handling vague and perplexing information when making decisions.

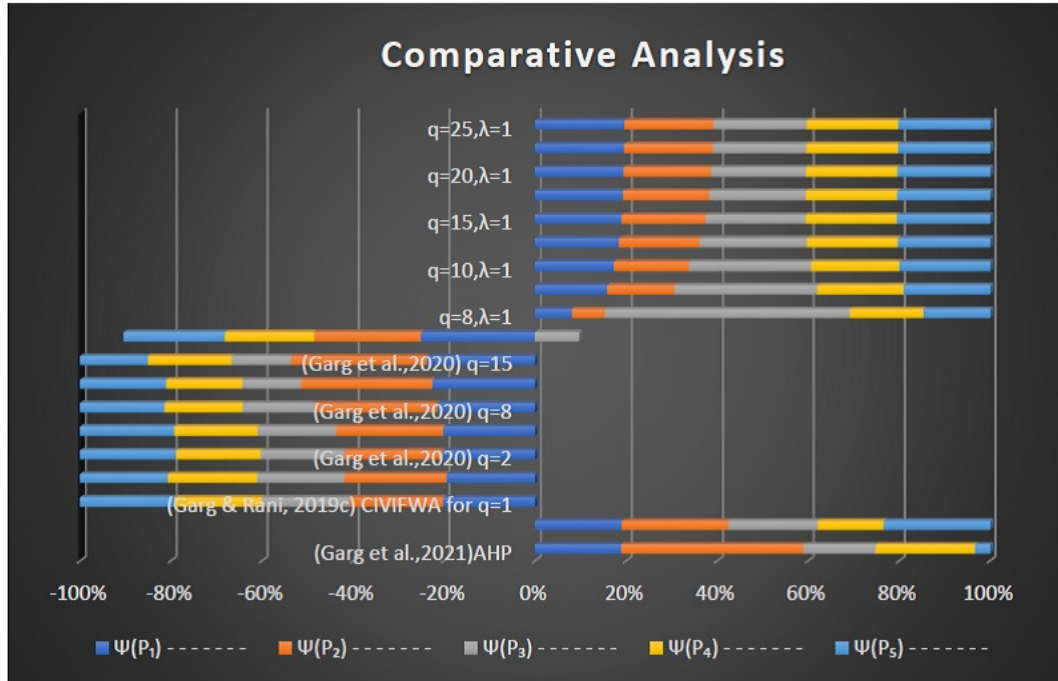


FIGURE 3. Graphical representation of Table 5

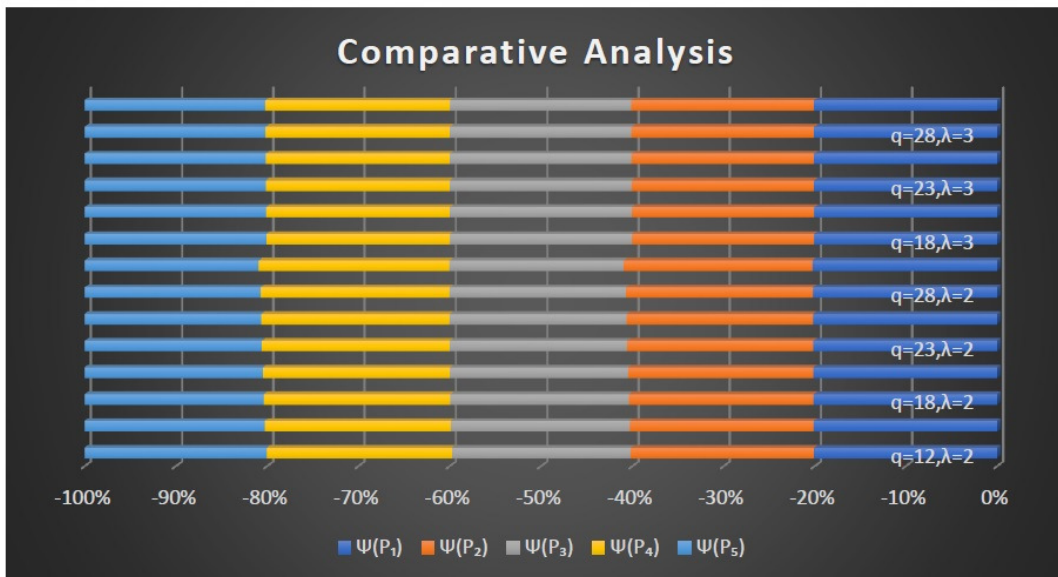


FIGURE 4. Graphical representation of Table 7

(iv) We have thoroughly compared the performance of these operators using a variety of metrics, including accuracy, precision, and remembering, in order to show the advantages of our suggested mean operators over current methods like Nayagam et al. (2011), JJoshi et al. (2018) and Liang et al. (2015). Our evaluation's findings demonstrate that our suggested mean operator continually performs better than the other strategies taken into account across all the measures examined, offering compelling evidence for its application in real-world decision-making situations. An overview of the ideal score values and the order in which the alternatives are ranked is provided in Tables 6, 7, 8, 9 and 10. Looking at these tables, we can see that the optimal choice agrees with the outcomes of our suggested strategy. This demonstrates the consistency and efficacy of our suggested strategy when compared to the marketplace's most advanced techniques. It's also critical to remember that our suggested approach's computing process is different from previous approaches used in different contexts. But when it comes to making decisions, the outcomes of our suggested strategy are more in line with actuality. This is essentially one of the key factors in making decisions that our method considers is the consistent degree of prioritization throughout the points of view in pairings. Our suggested method is more sensible and useful in real-world situations because of this aspect.

(v) Conclusively, the operators that have been suggested consider the decision maker's parameters, λ , α and q , so providing them with an increased range of choices. The reason for this is that the decision-maker can choose the option that most effectively fits their requirements and preferences due to the alternative's ratings vary depending on the parameterization values of λ , α and q . With the help of this characteristic of the suggested operators, the decision-maker has greater control and autonomy over the entire procedure. Tables 6, 7, 8, 9 and 10 shows that solutions (Nayagam et al. (2011), JJoshi et al. (2018) and Liang et al. (2015)) are insufficient to solve the problem since they don't meet the standards of the *CIVq*-ROFSs. Since IVIFSs and IVPFSs are unable to manage the data supplied in Section 5 in an efficient manner, we must investigate an alternate real-world decision-making challenge.

(vi) Tables 7, 8, 9 and 10 makes it clear that different alternatives receive different scores depending on the results of characteristics λ , α and q . The alternates are still ranked in the same order regardless these changes. Variations in parameters do not significantly affect the ranking order of the alternates, which is a strong indication of the robustness and stability of the suggested technique. This result reinforces the applicability of the suggested approach for real-world decision-making settings and boosts our confidence in its dependability.

Table 9. Ranking order of alternatives for different α using new score function with $q = 12$, $\lambda = 2$

Measure	$\Psi(P_1)$	$\Psi(P_2)$	$\Psi(P_3)$	$\Psi(P_4)$	$\Psi(P_5)$	Ranking
$\alpha = 0.1$	-0.310862	-0.312665	-0.304247	-0.314674	-0.31046	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.2$	-0.310814	-0.31266	-0.303884	-0.314528	-0.31043	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.3$	-0.310766	-0.312655	-0.303521	-0.314382	-0.3104	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.4$	-0.310718	-0.31265	-0.303158	-0.314236	-0.31037	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.5$	-0.31067	-0.312645	-0.302795	-0.31409	-0.31034	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.6$	-0.310622	-0.31264	-0.302432	-0.313944	-0.31031	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.7$	-0.310574	-0.312635	-0.302069	-0.313798	-0.31028	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.8$	-0.310526	-0.31263	-0.301706	-0.313652	-0.31025	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.9$	-0.310478	-0.312625	-0.301343	-0.313506	-0.31022	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$

Table 10. Ranking order of alternatives for different α using new score function with $q = 15, 18$, $\lambda = 3$

Measure	$\Psi(P_1)$	$\Psi(P_2)$	$\Psi(P_3)$	$\Psi(P_4)$	$\Psi(P_5)$	Ranking
$\alpha = 0.1$	-0.359402	-0.359723	-0.357365	-0.361414	-0.35854	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.2$	-0.359844	-0.360316	-0.35765	-0.361918	-0.35898	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.3$	-0.360286	-0.360909	-0.357935	-0.362422	-0.35942	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.4$	-0.360728	-0.361502	-0.35822	-0.362926	-0.35986	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.5$	-0.36117	-0.362095	-0.358505	-0.36343	-0.3603	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.6$	-0.361612	-0.362688	-0.35879	-0.363934	-0.36074	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.7$	-0.362054	-0.363281	-0.359075	-0.364438	-0.36118	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.8$	-0.362496	-0.363874	-0.35936	-0.364942	-0.36162	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$
$\alpha = 0.9$	-0.362938	-0.364467	-0.359645	-0.365446	-0.36206	$P_3 \succ P_5 \succ P_1 \succ P_2 \succ P_4$

6.2. Practical implications of the proposed model. (a) Financial Environments:

Our proposed model based on complex interval-valued q -rung orthopair fuzzy sets ($CIVq$ -ROFSs) is highly applicable to multi-attribute financial decision-making problems, where data is often imprecise, vague, or conflicting. Practical applications include:

- (i) **Credit risk assessment:** Financial institutions often deal with conflicting expert opinions and uncertain borrower data. Our model can aggregate expert judgments more accurately using $CIVq$ -ROFWAM and $CIVq$ -ROFWGM operators.
- (ii) **Investment portfolio selection:** Investor preferences, market volatility, and return-risk assessments are naturally vague. The proposed fuzzy environment allows more robust modeling of interval-valued and complex stakeholder opinions.

(iii) **Mergers and acquisitions analysis:** The model can integrate both qualitative (strategic fit, leadership) and quantitative (revenue, cost) factors from multiple decision-makers under uncertainty.

(b) **AI-Driven Education Systems:**

In AI-enhanced learning environments, decision-makers (e.g., school administrators, policy-makers) must evaluate and select among various AI tools and platforms based on multiple vague or partially known criteria. Practical applications include:

(i) **Evaluating AI-based tutoring systems:** Criteria like adaptability, learning analytics, scalability, and ethical compliance can be assessed using interval-valued opinions from teachers, technologists, and students.

(ii) **Curriculum recommendation systems:** Aggregating fuzzy feedback from students and educators on AI-generated course paths.

(iii) **EdTech vendor selection:** Comparing AI solutions based on uncertain or subjective factors like cost-effectiveness, data privacy compliance, and student engagement.

7. CONCLUSION

The purpose of this study, we discussed the notion of A -complex operational laws for $CIVq$ -ROFNs. Based on A -complex interval valued operational laws under $CIVq$ -ROFNs, we offered the idea of Complex interval valued q -ROF weighted arithmetic mean aggregation operator ($CIVq$ -ROFWAM), complex interval valued q -ROF ordered weighted arithmetic mean aggregation operator ($CIVq$ -ROFOWAM), complex interval valued q -ROF weighted geometric mean aggregation operator ($CIVq$ -ROFWGM) and complex q -ROF ordered weighted geometric mean aggregation operator ($CIVq$ -ROFOWGM). Additionally, we go into extensive detail on the newly suggested multi-attribute group decision process that is based on the q -rung orthopair fuzzy preference environment. The paper also looks into and analyses the different advantageous aspects of these operators. Lastly, we offer a framework that takes into account various permutations of the parameters λ and q in order to tackle decision-making problems. We used the artificial intelligence tools in education system problem to further verified the feasibility of the approach. The decision-maker has a variety of opportunities for decision evaluation using this technique, which also increases the adaptability of the suggested operators. while comparing the suggested operators and their related methodologies with various available operators, it becomes clear that the former provides the decision-maker with a more favorable, genuine, and consistent strategy while gathering data to make choices. Considering five important factors, the current investigation integrated feedback from specialists in five distinct choices. Future studies could broaden the scope by include more choices and qualities in order to confirm the findings of this one. According to the study, the options' ranking order

should stay the same for different values of λ and q . Furthermore, this technique is widely applicable and can be used to tackle a range of decision-making difficulties such as medical assessment, domestic aviation assessments, handling medical waste, and evaluation of medical waste treatment methods.

The major outcomes and findings of the study are :

- (i) The proposed CIVq-ROF aggregation operators effectively handle complex, interval-valued, and hesitant data in uncertain decision environments.
- (ii) The case study involving AI tool evaluation in education demonstrates that the method consistently provides accurate and discriminative rankings across multiple decision-makers and conflicting criteria.
- (iii) The algebraic operational laws support stable aggregation and preserve essential properties such as boundedness, monotonicity, and idempotency.
- (iv) Comparative results confirm that the model performs better than existing fuzzy and intuitionistic fuzzy frameworks in terms of flexibility, interpretability, and computational robustness.

These key findings now provide a concise and informative summary of the study's impact. Furthermore, we highlight some meaningful future study options below:

- (i) We will introduce a novel decision-making method that combines the TOPSIS method with our proposed models. We will applying this method to a real multiple conditional attributes in addition to multiple decision attributes problems of renewable energy source selection (Deveci et al. 2020; Jafer et al. 2021; Krishankumar et al. 2021), green supplier chain management (Ali et al. 2021), dynamic decision making (Rong et al. 2022), business and marketing management (Riaz et al. 2021), pattern recognition (Ullah et al. 2020), Assessment of clustering (Li et al. 2020) and medical treatment (Riaz et al. 2022; Ejegwa 2022). Through comparative analysis, parameter analysis, and experimental analysis, we will demonstrating the effectiveness and feasibility of our proposed methods.
- (ii) We will establish a three way decision model according to our novel approach of *CIVq-ROFWAM* and *CIVq-ROFWGM* from multisource fuzzy information and their applications to in selecting the best breed of Horsegram (Janani et al. 2022).
- (iii) We will apply the suggested methodology to large-scale group decision-making and base the attribute weights on user feedback gathered through social media.

Author Contributions

Noor Rehman: Methodology, Investigation, Writing–Review & Editing, Writing– original draft & Revision, Supervision. **Abbas Ali:** Data curation, software, formal analysis, and

writing original draft preparation, Conceptualization, methodology and validation. **Zabihullah Movaheedi:** Writing— original draft, review & editing, Validation, Project administration. **Mohsin Sajjad:** Writing original draft, validation, editing and methodology. **Kaleem Ullah:** Software, formal analysis, and writing original draft preparation, Conceptualization, methodology and validation.

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Data Availability Statement

The datasets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Declarations

Ethics approval and consent to participate

The study protocol was reviewed and approved by the relevant research ethics authority of Bacha Khan University, Charsadda, Pakistan, in accordance with the applicable institutional ethical guidelines and regulations. The expert involved in the study was provided with information regarding the purpose and nature of the study, and informed consent was obtained prior to the collection of the expert judgments. Participation was voluntary, and the confidentiality of the information provided by the expert was maintained throughout the study.

Consent for publication

Informed consent was obtained from the participating expert for the use of the provided judgments for research and publication purposes. No personally identifiable information is reported in this manuscript.

Competing interests

The authors declare no competing interests.

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Appendix 1.

Let

$$\alpha_1 P_1 = \left(\begin{array}{c} \left[(\alpha_1)^{\frac{1}{q}} \eta_{C_1}^-, (\alpha_1)^{\frac{1}{q}} \eta_{C_1}^+ \right] e^{i2\pi \left[(\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{-im}, (\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{+im} \right]}, \\ \left[\left(\alpha_1 \left(1 - (\psi_{C_1}^+)^q \right) \right)^{\frac{1}{q}}, \left(\alpha_1 \left(1 - (\psi_{C_1}^-)^q \right) \right)^{\frac{1}{q}} \right] e^{i2\pi \left[\left(\alpha_1 \left(1 - (\psi_{C_1}^+)^q \right) \right)^{\frac{1}{q}}, \left(\alpha_1 \left(1 - (\psi_{C_1}^-)^q \right) \right)^{\frac{1}{q}} \right]} \end{array} \right)$$

and

$$\alpha_2 P_2 = \left(\begin{array}{c} \left[(\alpha_2)^{\frac{1}{q}} \eta_{C_2}^-, (\alpha_2)^{\frac{1}{q}} \eta_{C_2}^+ \right] e^{i2\pi \left[(\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{-im}, (\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{+im} \right]}, \\ \left[\left(\alpha_2 \left(1 - (\psi_{C_2}^+)^q \right) \right)^{\frac{1}{q}}, \left(\alpha_2 \left(1 - (\psi_{C_2}^-)^q \right) \right)^{\frac{1}{q}} \right] e^{i2\pi \left[\left(\alpha_2 \left(1 - (\psi_{C_2}^+)^q \right) \right)^{\frac{1}{q}}, \left(\alpha_2 \left(1 - (\psi_{C_2}^-)^q \right) \right)^{\frac{1}{q}} \right]} \end{array} \right)$$

Then

$$(\alpha_1 P_1)^\lambda = \left(\begin{array}{c} \left[r^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^- \right)^\lambda, r^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^+ \right)^\lambda \right] e^{i2\pi \left[r^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{-im} \right)^\lambda, r^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{+im} \right)^\lambda \right]}, \\ \left[r^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^+)^q \right) \right)^{\frac{1}{q}} \right)^\lambda, r^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^-)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right] e^{i2\pi \left[r^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^+)^q \right) \right)^{\frac{1}{q}} \right)^\lambda, r^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^-)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right]} \end{array} \right)$$

and

$$(\alpha_2 P_2)^\lambda = \begin{pmatrix} \begin{bmatrix} r^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^- \right)^\lambda, r^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^+ \right)^\lambda \end{bmatrix} e^{i2\pi \left[r^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{-im} \right)^\lambda, r^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{+im} \right)^\lambda \right]}, \\ \begin{bmatrix} r^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^+)^q \right) \right)^{\frac{1}{q}} \right)^\lambda, r^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^-)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \end{bmatrix} \\ e^{i2\pi \left[r^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^{+im})^q \right) \right)^{\frac{1}{q}} \right)^\lambda, r^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^{-im})^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right]} \end{pmatrix}.$$

Now

$$\begin{aligned} & (\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \\ &= \begin{pmatrix} \begin{bmatrix} \left(\frac{1}{2} \left(\left(2^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^- \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^- \right)^\lambda \right)^q \right)^{\frac{1}{q}}, \right. \\ \left. \left(\frac{1}{2} \left(\left(2^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^+ \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^+ \right)^\lambda \right)^q \right)^{\frac{1}{q}} \right] \\ i2\pi \begin{bmatrix} \left(\frac{1}{2} \left(\left(2^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{-im} \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{-im} \right)^\lambda \right)^q \right)^{\frac{1}{q}}, \right. \\ \left. \left(\frac{1}{2} \left(\left(2^{\frac{1}{q}} \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{+im} \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{+im} \right)^\lambda \right)^q \right)^{\frac{1}{q}} \right] \\ e \end{bmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} \left(1 - \frac{1}{2} \left(\left(2^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^-)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^-)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}}, \right. \\ \left. \left(1 - \frac{1}{2} \left(\left(2^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^+)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^+)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}} \right] \\ i2\pi \begin{bmatrix} \left(1 - \frac{1}{2} \left(\left(2^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^{-im})^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^{-im})^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}}, \right. \\ \left. \left(1 - \frac{1}{2} \left(\left(2^{\frac{1}{q}} \left(\left(\alpha_1 \left(1 - (\psi_{C_1}^{+im})^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(2^{\frac{1}{q}} \left(\left(\alpha_2 \left(1 - (\psi_{C_2}^{+im})^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}} \right] \\ e \end{bmatrix} \end{pmatrix} \end{aligned}$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \left(\frac{1}{2} \left(2 \left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^- \right)^\lambda \right)^q + 2 \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^- \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}}, \\ \left(\frac{1}{2} \left(2 \left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^+ \right)^\lambda \right)^q + 2 \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^+ \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}} \end{array} \right] \\ e^{i2\pi} \left[\begin{array}{c} \left(\frac{1}{2} \left(2 \left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{-im} \right)^\lambda \right)^q + 2 \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{-im} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}}, \\ \left(\frac{1}{2} \left(2 \left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{+im} \right)^\lambda \right)^q + 2 \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{+im} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}} \end{array} \right] \\ \left[\begin{array}{c} \left(1 - \frac{1}{2} \left(2 \left(\left((\alpha_1 (1 - (\psi_{C_1}^-)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + 2 \left(\left((\alpha_2 (1 - (\psi_{C_2}^-)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}}, \\ \left(1 - \frac{1}{2} \left(2 \left(\left((\alpha_1 (1 - (\psi_{C_1}^+)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + 2 \left(\left((\alpha_2 (1 - (\psi_{C_2}^+)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}} \end{array} \right] \\ e^{i2\pi} \left[\begin{array}{c} \left(1 - \frac{1}{2} \left(2 \left(\left((\alpha_1 (1 - (\psi_{C_1}^{-im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + 2 \left(\left((\alpha_2 (1 - (\psi_{C_2}^{-im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}}, \\ \left(1 - \frac{1}{2} \left(2 \left(\left((\alpha_1 (1 - (\psi_{C_1}^{+im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + 2 \left(\left((\alpha_2 (1 - (\psi_{C_2}^{+im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}} \end{array} \right] \end{array} \right)$$

$$= \left(e^{i2\pi} \left[\begin{array}{c} \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^- \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^- \right)^\lambda \right)^q \right]^{\frac{1}{q}}, \\ \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^+ \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^+ \right)^\lambda \right)^q \right]^{\frac{1}{q}} \\ \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{-im} \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{-im} \right)^\lambda \right)^q \right]^{\frac{1}{q}}, \\ \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{+im} \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{+im} \right)^\lambda \right)^q \right]^{\frac{1}{q}} \end{array} \right] \right) \\ \\ = \left(e^{i2\pi} \left[\begin{array}{c} \left(1 - \left(\left(\left((\alpha_1 (1 - (\psi_{C_1}^-)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(\left((\alpha_2 (1 - (\psi_{C_2}^-)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}}, \\ \left(1 - \left(\left(\left((\alpha_1 (1 - (\psi_{C_1}^+)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(\left((\alpha_2 (1 - (\psi_{C_2}^+)^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}} \\ \left(1 - \left(\left(\left((\alpha_1 (1 - (\psi_{C_1}^{-im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(\left((\alpha_2 (1 - (\psi_{C_2}^{-im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}}, \\ \left(1 - \left(\left(\left((\alpha_1 (1 - (\psi_{C_1}^{+im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(\left((\alpha_2 (1 - (\psi_{C_2}^{+im})^q) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right)^{\frac{1}{q}} \end{array} \right] \right)$$

$$= \left(e^{i2\pi} \left[\begin{array}{c} \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^- \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^- \right)^\lambda \right)^q \right]^{\frac{1}{q}}, \\ \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^+ \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^+ \right)^\lambda \right)^q \right]^{\frac{1}{q}} \\ \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{-im} \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{-im} \right)^\lambda \right)^q \right]^{\frac{1}{q}}, \\ \left[\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{+im} \right)^\lambda \right)^q + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}^{+im} \right)^\lambda \right)^q \right]^{\frac{1}{q}} \end{array} \right] \right) \\ \\ = \left(e^{i2\pi} \left[\begin{array}{c} \left(1 - \left((\alpha_1 (1 - (\psi_{C_1}^-)^q) \right)^\lambda + (\alpha_2 (1 - (\psi_{C_2}^-)^q) \right)^\lambda \right)^{\frac{1}{q}}, \\ \left(1 - \left((\alpha_1 (1 - (\psi_{C_1}^+)^q) \right)^\lambda + (\alpha_2 (1 - (\psi_{C_2}^+)^q) \right)^\lambda \right)^{\frac{1}{q}} \\ \left(1 - \left((\alpha_1 (1 - (\psi_{C_1}^{-im})^q) \right)^\lambda + (\alpha_2 (1 - (\psi_{C_2}^{-im})^q) \right)^\lambda \right)^{\frac{1}{q}}, \\ \left(1 - \left((\alpha_1 (1 - (\psi_{C_1}^{+im})^q) \right)^\lambda + (\alpha_2 (1 - (\psi_{C_2}^{+im})^q) \right)^\lambda \right)^{\frac{1}{q}} \end{array} \right] \right)$$

$$= \left(\begin{array}{c} \left[\left(\sum_{r=1}^2 (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^2 (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\sum_{r=1}^2 (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^2 (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{q}} \right]}, \\ \left[\left(1 - \sum_{r=1}^2 (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^2 (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(1 - \sum_{r=1}^2 (\alpha_r (1 - (\psi_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^2 (\alpha_r (1 - (\psi_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{q}} \right]} \end{array} \right).$$

Let suppose that the result is true for $r = k$.

$$\left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus \dots \oplus (\alpha_k P_k)^\lambda \right) \\ = \left(\begin{array}{c} \left[\left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{q}} \right]}, \\ \left[\left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{q}} \right]} \end{array} \right).$$

We shows that the result is true for $r = k + 1$. So

$$\begin{aligned}
& \left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus \dots \oplus (\alpha_k P_k)^\lambda \right) \oplus (\alpha_{k+1} P_{k+1})^\lambda \\
&= \left(\begin{array}{c} \left[\left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^k (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{q}} \right]} \\ \left[\left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^k (\alpha_r (1 - (\psi_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{q}} \right]} \end{array} \right) \oplus \\
& \left(\begin{array}{c} \left[r^{\frac{1}{q}} \left((\alpha_{k+1})^{\frac{1}{q}} \eta_{C_{k+1}}^- \right)^\lambda, r^{\frac{1}{q}} \left((\alpha_{k+1})^{\frac{1}{q}} \eta_{C_{k+1}}^+ \right)^\lambda \right] e^{i2\pi \left[r^{\frac{1}{q}} \left((\alpha_{k+1})^{\frac{1}{q}} \eta_{C_{k+1}}^{-im} \right)^\lambda, r^{\frac{1}{q}} \left((\alpha_{k+1})^{\frac{1}{q}} \eta_{C_{k+1}}^{+im} \right)^\lambda \right]} \\ \left[r^{\frac{1}{q}} \left((\alpha_{k+1} (1 - (\psi_{C_{k+1}}^+)^q)^\lambda \right)^{\frac{1}{q}}, r^{\frac{1}{q}} \left((\alpha_{k+1} (1 - (\psi_{C_{k+1}}^-)^q)^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[r^{\frac{1}{q}} \left((\alpha_{k+1} (1 - (\psi_{C_{k+1}}^{+im})^q)^\lambda \right)^{\frac{1}{q}}, r^{\frac{1}{q}} \left((\alpha_{k+1} (1 - (\psi_{C_{k+1}}^{-im})^q)^\lambda \right)^{\frac{1}{q}} \right]} \end{array} \right) \\
&= \left(\begin{array}{c} \left[\left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{q}}, \left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{q}} \right]} \\ \left[\left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{q}} \right]} \end{array} \right).
\end{aligned}$$

Thus

$$\begin{aligned}
& \left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus \dots \oplus (\alpha_k P_k)^\lambda \right) \oplus (\alpha_{k+1} P_{k+1})^\lambda \Big)^{\frac{1}{\lambda}} \\
&= \left(\begin{aligned} & \left[\left(\left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & e^{i2\pi \left[\left(\left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^{k+1} (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}, \\ & \left[\left(\left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^{k+1} (\alpha_r (1 - (\psi_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{aligned} \right).
\end{aligned}$$

Appendix 2.

Let $\left(\left[\eta_{C_1}^-, \eta_{C_1}^+ \right] e^{i2\pi [\eta_{C_1}^{-im}, \eta_{C_1}^{+im}]}, [\psi_{C_1}^-, \psi_{C_1}^+] e^{i2\pi [\psi_{C_1}^{-im}, \psi_{C_1}^{+im}]} \right)$
 $= \left(\left[\eta_{C_2}^-, \eta_{C_2}^+ \right] e^{i2\pi [\eta_{C_2}^{-im}, \eta_{C_2}^{+im}]}, [\psi_{C_2}^-, \psi_{C_2}^+] e^{i2\pi [\psi_{C_2}^{-im}, \psi_{C_2}^{+im}]} \right) = \dots$
 $= \left(\left[\eta_{C_m}^-, \eta_{C_m}^+ \right] e^{i2\pi [\eta_{C_m}^{-im}, \eta_{C_m}^{+im}]}, [\psi_{C_m}^-, \psi_{C_m}^+] e^{i2\pi [\psi_{C_m}^{-im}, \psi_{C_m}^{+im}]} \right)$
 $= \left(\left[\eta_C^-, \eta_C^+ \right] e^{i2\pi [\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi [\psi_C^{-im}, \psi_C^{+im}]} \right)$ and $E = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\sum_{r=1}^m \alpha_r = 1$, where $r = 1, 2, \dots, m$. Based on Definition 6, we get

$$\begin{aligned}
& CIVq\text{-}ROFWAM_E(P_1, P_2, P_3, \dots, P_m) \\
&= \left(\begin{aligned} & \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^-)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^+)^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & e^{i2\pi \left[\left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{-im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(\sum_{r=1}^m (\alpha_r (\eta_{C_r}^{+im})^q)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]}, \\ & \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^-)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^+)^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right] \\ & e^{i2\pi \left[\left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{-im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}, \left(\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_{C_r}^{+im})^q))^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} \right]} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\begin{array}{c} \left[\left(\sum_{r=1}^m (\alpha_r (\eta_C^-)^q) \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m (\alpha_r (\eta_C^+)^q) \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(\sum_{r=1}^m (\alpha_r (\eta_C^{-im})^q) \right)^{\frac{1}{q}}, \left(\sum_{r=1}^m (\alpha_r (\eta_C^{+im})^q) \right)^{\frac{1}{q}} \right]} \\ \left[\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_C^-)^q)) \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_C^+)^q)) \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_C^{-im})^q)) \right)^{\frac{1}{q}}, \left(1 - \sum_{r=1}^m (\alpha_r (1 - (\psi_C^{+im})^q)) \right)^{\frac{1}{q}} \right]} \end{array} \right) \\
&= \left(\begin{array}{c} \left[\left((\eta_C^-)^q \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}}, \left((\eta_C^+)^q \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left((\eta_C^{-im})^q \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}}, \left((\eta_C^{+im})^q \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}} \right]} \\ \left[\left(1 - (1 - (\psi_C^-)^q) \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}}, \left(1 - (1 - (\psi_C^+)^q) \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}} \right] \\ e^{i2\pi \left[\left(1 - (1 - (\psi_C^{-im})^q) \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}}, \left(1 - (1 - (\psi_C^{+im})^q) \sum_{r=1}^m (\alpha_r) \right)^{\frac{1}{q}} \right]} \end{array} \right) \\
&= \left(\begin{array}{c} \left[((\eta_C^-)^q)^{\frac{1}{q}}, ((\eta_C^+)^q)^{\frac{1}{q}} \right] e^{i2\pi \left[((\eta_C^{-im})^q)^{\frac{1}{q}}, ((\eta_C^{+im})^q)^{\frac{1}{q}} \right]} \\ \left[(1 - (1 - (\psi_C^-)^q))^{\frac{1}{q}}, (1 - (1 - (\psi_C^+)^q))^{\frac{1}{q}} \right] \\ e^{i2\pi \left[(1 - (1 - (\psi_C^{-im})^q))^{\frac{1}{q}}, (1 - (1 - (\psi_C^{+im})^q))^{\frac{1}{q}} \right]} \end{array} \right) \\
&= \left([\eta_C^-, \eta_C^+] e^{i2\pi [\eta_C^{-im}, \eta_C^{+im}]}, [\psi_C^-, \psi_C^+] e^{i2\pi [\psi_C^{-im}, \psi_C^{+im}]} \right).
\end{aligned}$$

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