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The choice of probability mapping in ESG questionnaire scoring: a structural risk-metric perspective for venture-capital and private-equity due diligence, with a dynamical-systems foundation

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Abstract

Quantitative ESG -scoring frameworks for venture-capital (VC) and private-equity (PE) due diligence map questionnaire answers into an additive score and then, crucially, into a normalised ESG -alignment probability. This second step, the choice of the *probability map*, has so far been treated as a technical afterthought, yet it materially changes where rating boundaries fall on the score axis and therefore how comparable, auditable and defensible the resulting ESG -rating is. Building on the author's earlier additive scoring model, this paper (a) states the normalisation problem as an axiomatic choice among admissible probability maps; (b) provides a structural-risk reading of the problem, drawing an explicit analogy with distance-to-default credit-risk models, and shows, with a calibrated numerical example, that two economically reasonable maps (the logistic sigmoid and the arctangent) can assign the same questionnaire outcome to rating classes that differ by several notches; (c) derives a general, falsifiable criterion, grounded in a one-dimensional conservative-dynamics analogy, for selecting admissible maps, relegating the underlying classical -mechanics apparatus to a self-contained appendix so that the main text remains readable to a business and economics audience; and (iv) discusses the governance implications for ESG -rating design, comparability across questionnaires, and model risk. Two new schematic figures and one new sensitivity figure, together with a fully reproducible numerical illustration, are provided in support of the argument.

Keywords: ESG scoring, probability calibration, sigmoid function, structural credit risk, distance to default, rating comparability, venture capital, due diligence, dynamical systems

1 Introduction

1.1 Motivation: ESG questionnaires as due-diligence instruments

Venture-capital and private-equity investors routinely need to assess the environmental, social and governance (ESG) profile of small and medium-sized targets, typically start-ups, for which no historical market data, credit history or third-party ESG rating exists [14, 23]. In this data-poor regime, a detailed due-diligence-like Y-N questionnaire, answered by the target company itself, is often the only feasible instrument to obtain a structured, comparable ESG assessment [28]. The present paper is a direct methodological extension of [14], where an additive scoring model was introduced that turns the questionnaire answers into a real-valued score m and, subsequently, into an ESG-alignment probability p and its complement, the ESG-risk (default-like) probability $\psi = 1 - p$. That earlier work left one modelling choice explicitly open: *which* normalised probability map should be used to translate the score into the probability and on what grounds.

1.2 The comparability problem and rating disagreement

The choice matters economically because it is precisely the ingredient that makes scores *comparable* across different questionnaires, scales and methodologies [7]. It is well documented that commercial ESG rating agencies disagree substantially with one another even when scoring the same companies [1, 7], and that this divergence is partly attributable to differences in aggregation and normalisation choices rather than to disagreement on the underlying facts. The rapid growth of ESG-measurement methodologies, ranging from criteria-integration frameworks [5], to automated scoring engines [21], to systematic reviews of financial-sustainability reporting standards [15] and multi-objective portfolio-construction tools for socially responsible investment [31] – only sharpens the need for an explicit, disclosed normalisation step, since heterogeneous metrics can be compared only once they are expressed on a common probability scale. The regulatory relevance of a defensible ESG-risk metric has also grown, as ESG considerations are increasingly embedded in the finalised Basel III/IV prudential framework and in the due-diligence expectations placed on funds and their counterparties [23]. A poorly justified probability map is therefore not a purely academic concern: it can shift a target across a full rating notch, with consequences for capital allocation, covenants, and investment committee decisions.

1.3 Contribution and relation to prior work

This paper makes four contributions. First, it restates the score-to-probability normalisation problem in axiomatic form and recalls a general existence criterion for admissible maps (Section 3). Second – and this is the paper’s main novel contribution relative to both [14] and the author’s earlier dynamical-systems draft – it reframes the problem from a *structural credit-risk* perspective, drawing an explicit analogy with distance-to-default models [9, 24], gives an economic interpretation of the calibration parameters, and quantifies, with a reproducible numerical example, how much the

choice of map can move rating boundaries and individual company ratings (Section 4). Third, it shows that the same admissibility criterion can be obtained from a one-dimensional conservative-dynamics analogy, in which the probability map is read as the trajectory of a point particle under a conservative force; because this apparatus is more naturally the domain of mathematical physics than of business and economics, the full classical-mechanics exposition has been moved to Appendix A, and only the results needed for the economic argument are kept in the main text (Section 5). Fourth, it discusses the governance implications of the analysis for ESG -rating design (Section 6).

Beyond the immediate ESG application, the underlying question – how should an additive, questionnaire-based score be normalised into a probability that is comparable across instruments – is a special case of a much older measurement-theoretic problem [3, 22], and is closely related to the general theory of proper scoring rules [16] and to probability calibration methods used in credit scoring [30] and machine learning [26]. The sigmoid and arctangent maps studied here are two convenient, tractable representatives of a much larger family of admissible *saturating* functions, whose approximation and modelling properties have been studied extensively in statistics and machine learning [8, 10, 11, 18, 20, 25, 32], in dynamical-systems theory [17], and in constrained optimisation and robust statistics, where closely related log-sigmoid and concentrated-Cauchy densities arise naturally as admissible members of the same family [6, 27]. The analysis is intentionally kept general enough to extend to any of these alternatives via Theorem 1.

1.4 Structure of the paper

Section 2 recalls, in condensed form and using the author’s own notation, the questionnaire scoring framework of [14]. Section 3 states the normalisation axioms and a general admissibility criterion. Section 4 contains the paper’s main new material: the structural-risk analogy, the economic reading of the calibration parameters, the comparative-statics analysis of rating bands, an elasticity (marginal sensitivity) analysis, and a worked numerical illustration. Section 5 summarises the conservative-dynamics foundation for the admissibility criterion, with full derivations deferred to Appendix A–C. Section 6 discusses practical guidance, robustness and limitations, and Section 7 concludes.

2 The ESG questionnaire scoring framework

This section recalls, in condensed form and using the author’s own notation, the additive scoring model of [14]; it is not a verbatim reproduction of that paper but a self-contained restatement sufficient to make the present paper readable on its own.

2.1 Questionnaire, answers and score

A **Y-N** questionnaire $\mathbf{q} = (q_1, \dots, q_n)$ collects n questions, grouped into an *Overall* block plus the three ESG pillars (**E**nvironmental, **S**ocial, **G**overnance); each question q_i is answered by $a_i \in \{Y, N, \text{blank}\}$ and carries a *rank* $r_i \in \{1, 2, 3\}$ expressing

its materiality (low, medium, high impact on the final judgement) and a signum $\zeta(q_i) \in \{\pm 1, \mp 1\}$ recording whether a ‘Yes’ answer is ESG -positive or ESG -negative for that particular question. Calling $\varrho = (\mathbf{q}, \mathbf{a})$ the answered questionnaire, the score is additive,

$$m(\varrho) = \sum_{i=1}^n \delta_i(a_i), \quad \delta_i = \kappa_i e^{r_i}, \quad (1)$$

where $\kappa_i \in \{+1, -1\}$ (the $(-1, 0, 1)$ -walk model) or $\kappa_i \in \{0, 1\}$ (the $(0, 1)$ -walk model) is the signed outcome of question i after applying $\zeta(q_i)$, and un-answered questions receive $\delta_i = 0$. The exponential weight e^{r_i} encodes the idea that materially higher-ranked questions should carry a disproportionately larger *jump* in the score, akin to a random walk with rank-dependent step size. Two variants are used in [14]: the signed $(-1, 0, 1)$ walk of Eq. (1), floored at zero, and a purely positive $(0, 1)$ walk in which the score is instead the Euclidean length of the vector of per-rank positive contributions,

$$m(\mathbf{a}) = \sqrt{(a e^1)^2 + (b e^2)^2 + (c e^3)^2}, \quad (2)$$

a, b, c being the number of positively-answered questions of rank 1, 2, 3 respectively. Both variants share the same rank-induced coordinate system and therefore the same normalisation problem addressed below.

2.2 From score to probability: the open problem

The realised score $m \geq 0$ has no intrinsic upper bound that is common across questionnaires of different length or design, so it cannot be compared directly across companies unless it is first mapped into a normalised, dimensionless quantity. [14] therefore introduces a probability $p \equiv p(m) \in (0, 1)$ – the ESG -alignment (sustainability) probability – and its complement $\psi = 1 - p$, the ESG -risk (default-like) probability, and couples ψ to a letter rating scale $(AAA, \dots, D)$ via fixed cut-offs, reported for reference in Table 1. That paper adopted, as a working choice, the logistic sigmoid $p(m) = (1 + \beta e^{-\alpha m})^{-1}$, calibrated by two boundary conditions ($p(0)$ and $p(m_{\max})$), but explicitly left open *why* the sigmoid, rather than some other saturating function, is the appropriate choice, and what would change if a different, equally reasonable map were used instead. Answering that question – with an explicit economic motivation and a general, falsifiable criterion – is the object of the remainder of this paper.

3 Admissible probability maps: axioms and a general criterion

3.1 Normalisation axioms

We seek a family of maps $\pi(s) : \mathbb{R}^+ \rightarrow (0, 1)$, applicable to any non-negative additive score s (not only the ESG score m), that satisfy three natural requirements: (i) *monotonicity* – a higher score must never correspond to a lower alignment probability; (ii) *boundedness* – $\pi(s) \in (0, 1)$ for every admissible s , so that π can be read as a genuine probability; and (iii) *normalisability* – π must be expressible as the integral of

Table 1 Rating scale and ESG -risk probability
 $\psi = 1 - p(m)$, reproduced from [14] for reference
(sigmoid calibration, $p(0) = \frac{1}{2}$, $p(m_{\max}=580) = 0.998$).

Rating class	Score range m	$\psi = 1 - p$ (%)	p (%)
AAA	580–400	0.200–1.355	99.800
AA	400–340	1.355–2.547	98.645
A	340–290	2.547–4.275	97.453
BBB	290–250	4.275–6.417	95.725
BB	250–215	6.417–9.074	93.583
B	215–165	9.074–14.570	90.926
CCC	165–80	14.570–29.785	85.430
CC	80–28	29.785–42.552	70.215
C	28–0	42.552–50.000	57.448
D	≤ 0	50.000	50.000

a non-negative, integrable “density” function, so that the map is pinned down, up to boundary conditions, by a single functional form. Formally, we restrict attention to maps of the form

$$\pi(s) = \frac{1}{M} \int_0^s f(\xi) d\xi, \quad M = \int_{\mathbb{R}^+} f(\xi) d\xi < +\infty, \quad (3)$$

for some non-negative f . Condition (3) automatically grants monotonicity ($\pi' = f/M \geq 0$) and boundedness (by construction $\pi(s) \rightarrow 1$ as $s \rightarrow \infty$ and $\pi(0) = 0$ before the additive shift used below), while the finiteness of M is what makes the normalisation constant well defined.

3.2 A general admissibility criterion

The following criterion, proved independently in Section 5 via a conservative-dynamics argument, and hence stated here without proof, characterises the entire admissible family in a form that is convenient for applied use.

Theorem 1 (General criterion for admissible probability maps) *A scoring function $s : X \rightarrow \mathbb{R}^+$ admits a normalised probability map $\pi(s)$ satisfying axioms (i) to (iii) above if and only if there exists a non-negative function f such that*

$$\pi' = f(\pi), \quad \int_{\mathbb{R}^+} f(\xi) d\xi = M < +\infty, \quad (4)$$

in which case

$$\pi(s) = \frac{1}{M} \int_0^s f(\xi) d\xi. \quad (5)$$

Two features of Theorem 1 deserve comment. First, condition (4) is an *autonomous first-order differential equation* for π as a function of s : the admissible maps are precisely the solutions of $\pi' = f(\pi)$ for some integrable f . Second, this makes the criterion directly testable: given any candidate map π , one only needs to check that π' can be

written as a function of π alone (not explicitly of s) and that the corresponding f integrates to a finite constant on $\mathbb{R}^+$.

3.3 Two canonical admissible families

Two convenient and economically meaningful choices satisfy Theorem 1, and are compared throughout the remainder of the paper.

The logistic sigmoid.

$\pi' = \pi(1 - \pi)$, giving

$$\pi_{\alpha,\beta}(s) = \frac{1}{1 + \beta e^{-\alpha s}}, \quad \alpha, \beta > 0, \quad (6)$$

with $M = 1/2$ under the natural normalisation $f = \pi(1 - \pi)$ (Table 2). This is the map used in [14] and in most credit-scoring practice [30].

The (rescaled) arctangent.

$f(\xi) = 1/(a^2 + \xi^2)$, giving

$$\pi_a(s) = \frac{1}{2} + \frac{1}{\pi} \arctan(as), \quad a > 0, \quad (7)$$

with $M = \pi/2$ (up to the additive shift that centres the map at $s = 0$). Both maps are monotone, bounded in $(0, 1)$, symmetric around their respective midpoints, and calibrated by exactly two free parameters, which makes them directly comparable.

Table 2 Two admissible probability-map families satisfying Theorem 1.

Family	$f(\xi)$	M	normalised map	note
Sigmoid	$\xi(1 - \xi)$	$1/2$	$(1 + \beta e^{-\alpha s})^{-1}$	solves $\pi' = \pi(1 - \pi)$
Arctangent	$\frac{1}{a^2 + \xi^2}$	$\pi/2$	$\frac{1}{2} + \frac{1}{\pi} \arctan(as)$	primitive is $\arctan$

Both families are admissible under Theorem 1: neither axiom (i)–(iii) discriminates between them. Yet, as Section 4 shows quantitatively, they are far from interchangeable in practice. This is precisely the comparability problem that motivates the structural-risk reading developed next.

4 A structural risk-management perspective

This section contains the paper’s principal new contribution. It reframes the abstract normalisation problem of Section 3 as a special case of a well-known class of models in quantitative finance – structural (distance-to-default) credit-risk models – and uses

that analogy to make the practical consequences of the choice of map concrete and quantitative.

4.1 The distance-to-default analogy

Corporate credit-risk modelling has long relied on mapping an aggregated, partially-informative indicator into a probability of financial distress: the earliest and best known example is Altman’s accounting-ratio-based Z -score for bankruptcy prediction [2], itself an additive scoring model conceptually close to Eq. (1). The structural family of models refines this idea by grounding the state variable in an explicit economic mechanism. In the classical structural model of corporate default [24], later industrialised as the Moody’s KMV framework [9], a firm defaults when its asset value falls below its liabilities. The key state variable is the *distance to default* d_t , a standardised measure of how many standard deviations of asset-value uncertainty separate the firm from the default boundary; the probability of default over a horizon T is then obtained by pushing d_t through a saturating map – the standard normal cumulative distribution function $\Phi(\cdot)$ – so that $PD = \Phi(-d_t) \in (0, 1)$.

The parallel with the ESG questionnaire framework of Section 2 is direct and, we argue, illuminating rather than merely decorative (Figure 1). The additive score $m(\varrho)$ plays the role of a (reversed and rescaled) distance-to-default measure: it aggregates many partially informative signals (questionnaire answers; asset-value drivers) into a single real-valued state variable. The probability map $\pi(\cdot)$ plays the role of $\Phi(\cdot)$: an essentially arbitrary, but conventionally agreed, saturating transformation that converts the state variable into a probability. And the ESG -risk probability $\psi = 1 - \pi(m)$ plays the role of PD . The analogy immediately suggests two lessons well known in the credit-risk literature that had not previously been transferred to ESG scoring. First, the map is a *modelling convention*, not a physical law: the normal-CDF choice in Merton-type models is itself a simplification (asset returns are not exactly Gaussian), yet the industry standardised on it because it is analytically tractable and its parameters (drift, volatility) have a clear economic meaning. The sigmoid and arctangent maps studied here play the analogous tractability role for ESG scores. Second, and more consequentially, *changing the map without recalibrating the rating boundaries silently reshuffles the rating scale* – exactly as would happen if a bank switched from a Gaussian to a fat-tailed default-probability model without re-fitting cut-offs. Section 4.3 quantifies this effect for the ESG case.

4.2 Economic interpretation of the calibration parameters

Writing the sigmoid map as $\pi_{\alpha,\beta}(m) = (1 + \beta e^{-\alpha m})^{-1}$, the two parameters admit a direct economic reading that mirrors the role of drift and volatility in the structural analogy of Section 4.1. The parameter β is pinned down by the *base rate* $p(0)$: an unanswered or minimally-informative questionnaire ($m = 0$) is conventionally treated as a coin flip, $p(0) = \frac{1}{2} \Rightarrow \beta = 1$, mirroring the assumption in structural models that a firm exactly at the default boundary ($d_t = 0$) has $PD = 50\%$. The parameter α controls the *steepness* of the map, i.e. how quickly an improving score is translated into a lower ESG -risk probability; it is fixed by a single additional calibration point

Structural credit-risk model

ESG questionnaire scoring

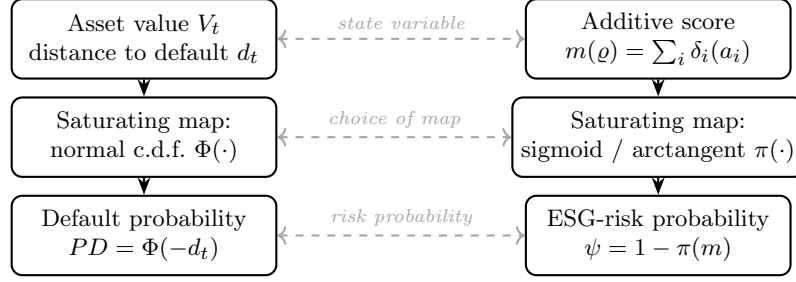


Figure 1 The distance-to-default analogy. In both frameworks a partially-informative state variable (distance to default; ESG score) is pushed through a conventional, calibrated saturating map to obtain a probability. The economic content of the model lies as much in the choice and calibration of the map as in the construction of the state variable.

$(\bar{m}, \bar{p})$, typically the maximum attainable score of the questionnaire family, exactly as the credit-risk analogue would be fixed by requiring a plausible PD at a chosen distance-to-default benchmark. Economically, α is therefore best read as an (inverse) *volatility* or *risk-aversion* parameter of the rating design: a large α produces a scale on which small score improvements translate into large rating upgrades (a “nervous” scale), while a small α produces a more forgiving scale in which large score changes are needed to move the rating. The arctangent map admits the analogous reading with its single scale parameter a playing the role of α (its β -equivalent is fixed automatically by the symmetry of $\arctan$ around zero).

4.3 Comparative statics: how the map reshapes the rating scale

To make this concrete we calibrate both maps under identical, minimally informative boundary conditions – $p(0) = \frac{1}{2}$ for both, and $p(m_{\max}=580) = 99.8\%$ for both, i.e. the same two conditions used in [14] to calibrate the sigmoid (Table 1). Solving Eq. (7) for a under the same condition gives $a \approx 0.2744$ (versus $\alpha \approx 0.01071$, $\beta = 1$ for the sigmoid; full calibration formulas are given in Appendix D). We then invert both maps at each of the ESG -risk cut-offs of Table 1 to obtain the score m at which each rating boundary falls under each map. Table 3 and Figure 2 report the result.

Under the sigmoid map the AA/D boundary spans essentially the whole score range (about 69% of $m_{\max}$ separates AA from D), matching the intuitive idea that a well-designed rating scale should spread companies across the full range of realised scores. Under the arctangent map, calibrated with exactly the same two boundary conditions, more than 85% of the entire rating scale (from AA down to D) is compressed into the bottom 15% of the score axis: a company scoring $m = 100$ would already be rated AA under the arctangent map, while it would be rated only CC under the sigmoid map. This is not a pathology of the arctangent function – it is an admissible map under Theorem 1 and is used routinely elsewhere in applied probability – but a direct illustration of why the choice of map cannot be treated as a detail: two equally

Table 3 Score thresholds implied by the sigmoid and the arctangent probability maps under *identical* boundary conditions $p(0) = \frac{1}{2}$, $p(m_{\max}=580) = 99.8\%$. Despite agreeing exactly at the two calibration points (by construction), the two maps disagree sharply on where every intermediate rating boundary falls.

Rating class	ψ cut-off (%)	m (sigmoid)	m (arctangent)
AAA	0.200	580.0	580.0
AA	1.355	400.3	85.6
A	2.547	340.2	45.4
BBB	4.275	290.2	27.0
BB	6.417	250.2	17.8
B	9.074	215.2	12.4
CCC	14.570	165.1	7.4
CC	29.785	80.1	2.7
C	42.552	28.0	0.9
D	50.000	0.0	0.0

defensible, equally “normalised” probability maps, calibrated on the same two anchor points, produce materially different rating architectures.

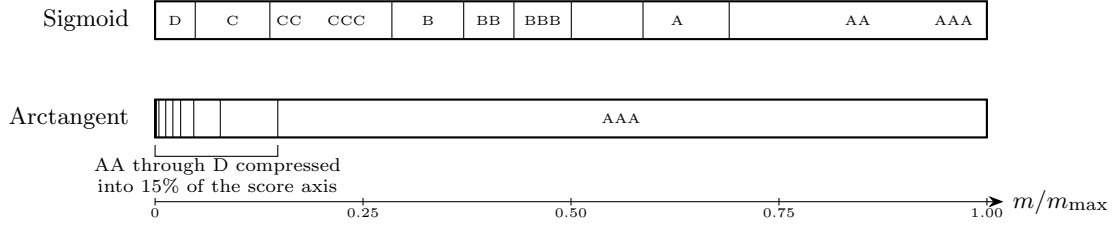


Figure 2 Rating bands induced by the sigmoid and the arctangent probability maps on the same relative score axis $m/m_{\max}$, calibrated with identical boundary conditions $p(0) = \frac{1}{2}$ and $p(m_{\max}) = 99.8\%$. The arctangent map compresses almost the whole rating scale into a narrow region near the origin, while the sigmoid spreads ratings across the full score range.

4.4 Marginal risk sensitivity: the elasticity of the alignment probability

A second, complementary way to compare the two maps is through their *marginal sensitivity* dp/dm : how much an incremental improvement of the score translates into an incremental change of the ESG -alignment probability. For the sigmoid,

$$\frac{dp_{\alpha,\beta}}{dm} = \frac{\alpha\beta e^{-\alpha m}}{(1 + \beta e^{-\alpha m})^2}, \quad (8)$$

while for the arctangent map,

$$\frac{dp_a}{dm} = \frac{1}{\pi} \frac{a}{1 + (a m)^2}. \quad (9)$$

Figure 3 plots both quantities under the calibration of Section 4.3. The arctangent map exhibits a very sharp sensitivity peak near $m = 0$ (where a small number of positive answers has a disproportionate effect on the assigned probability) that decays quickly, whereas the sigmoid’s sensitivity is more gradual and remains materially positive over a much wider range of scores. In economic terms, the arctangent calibration used here rewards the *first* favourable answers very heavily and is almost insensitive to further improvements beyond a modest score, which is a questionable property for a due-diligence instrument meant to discriminate consistently across the full range of realistic ESG profiles; the sigmoid, by contrast, remains informative (non-negligible marginal sensitivity) over a much larger portion of the questionnaire’s attainable score range, which is why it was adopted as the working choice in [14] and is retained here as the reference map, while the arctangent is kept throughout as an explicit benchmark for the comparability argument.

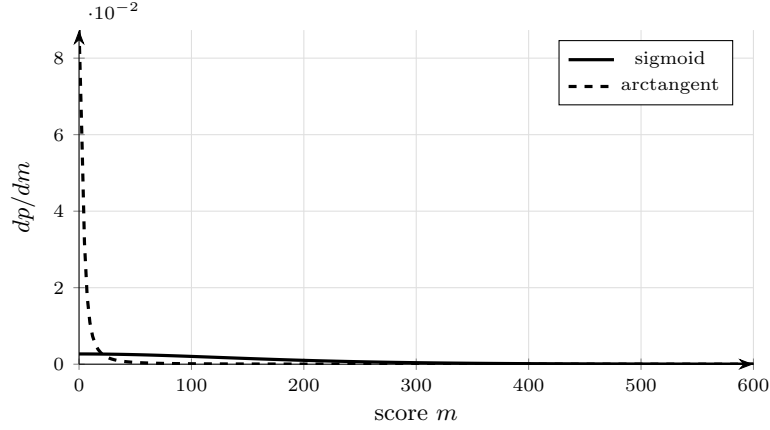


Figure 3 Marginal sensitivity dp/dm of the ESG-alignment probability to the score, for the sigmoid and arctangent maps calibrated under identical boundary conditions ($p(0)=\frac{1}{2}$, $p(580)=99.8\%$). The arctangent map is far more sensitive near $m = 0$ and far less sensitive thereafter.

4.5 A worked numerical illustration

Table 4 applies both calibrated maps to five synthetic companies spanning the plausible score range of a ~ 100 -question overall questionnaire. The disagreement is largest, and most consequential, for low-to-medium scores: a company scoring $m = 50$ would be rated C (compromised, remediation required) under the sigmoid map but A (no remediation needed) under the arctangent map – a seven-notch difference on a ten-notch scale, obtained from the very same questionnaire answers. The two maps converge only

once the score approaches the upper calibration anchor ($m = 550$), which is mechanically guaranteed by the shared boundary condition rather than being informative of any deeper agreement between the models.

Table 4 Worked numerical illustration: five synthetic companies, same questionnaire answers, two admissible probability maps. Ratings use the cut-offs of Table 1.

score m	Sigmoid			Arctangent		
	p (%)	ψ (%)	Rating	p (%)	ψ (%)	Rating
50	63.08	36.92	C	97.68	2.32	A
150	83.30	16.70	CC	99.23	0.77	AA
300	96.13	3.87	BBB	99.61	0.39	AA
450	99.20	0.80	AA	99.74	0.26	AA
550	99.72	0.28	AA	99.79	0.21	AA

This example illustrates, quantitatively, the comparability failure motivating Section 3: without an explicit, economically motivated criterion for selecting (and disclosing) the probability map, two analysts applying two equally “correct” normalisations to the same questionnaire outcome can reach opposite conclusions about whether a target requires remediation before an investment closes.

5 A dynamical-systems foundation for the admissibility criterion

Section 4 motivated the choice of map on economic grounds. This section shows that Theorem 1 also follows from a purely mathematical analogy: reading the probability map $\pi(s)$ as the trajectory $x(t)$ of a one-dimensional conservative dynamical system, with the score s playing the role of time t . This analogy is not needed to use the results of Section 4, and readers mainly interested in the economic argument may skip directly to Section 6; it is retained here, in condensed form, because it independently derives the same admissibility criterion, gives an alternative and complementary reading of the parameters α, β in terms of energy conservation, and connects the present analysis to a classical body of work on ordinary differential equations. All the underlying classical-mechanics material (Newton’s equations, phase portraits, the harmonic oscillator and pendulum, and the Lagrangian/Hamiltonian formalism) has been moved to Appendix A, so as to keep the main exposition focused on its economic content; only the results needed here are stated.

5.1 Scores as trajectories

Recall from Appendix A that a one-dimensional conservative dynamical system is described by

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial U}{\partial x}, \quad (10)$$

where $U = U(x)$ is the potential energy, $p = m\dot{x}$ the momentum, and the total mechanical energy $E = T + U$, with kinetic energy $T = p^2/2m$, is conserved along trajectories (Theorem 4 in Appendix A). The position is recovered from the momentum by $x(t) - x_0 = \int_0^t p(\xi)/m d\xi$ (Eq. (A5)).

Comparing this with the definition of the probability map, $\pi(s) = \frac{1}{M} \int_0^s f(\xi) d\xi$ (Eq. (3)), the two expressions have the same structure once we identify $s \equiv t$ (the score plays the role of time), $\pi \equiv x$ (the probability plays the role of position), and $f/M \equiv p/m$ (the normalised density plays the role of momentum per unit mass). This identification yields the following result.

Theorem 2 (Kinetic energy of a questionnaire) *Given a questionnaire ϱ with score $s(\varrho)$ and probability map $\pi(s) = \frac{1}{M} \int_0^s f(\xi) d\xi$, the associated kinetic energy is*

$$T_\varrho = \frac{1}{2M} f(s(\varrho))^2. \quad (11)$$

Proof Immediate from the structural identification above and $T = p^2/(2m)$: setting $p/m \equiv f/M$ gives $T = \frac{1}{2M} f^2$ after matching units so that M plays the role of the mass m . ■ □

Interpreting $s \equiv t$ as “time” and $\pi(s)$ as the “position” of the questionnaire dynamic, a conservative potential $V(\pi)$ acts on the system, and by the energy-conservation theorem (Appendix A), $V(\pi) = E - T(\pi)$ for every admissible map. This is precisely the content of Theorem 1: the condition $\pi' = f(\pi)$ (i.e., the momentum depends on π alone, not explicitly on s) is what makes the underlying force conservative and hence guarantees the existence of a well-defined potential $V(\pi) = E - T(\pi)$, while $\int_{\mathbb{R}^+} f(\xi) d\xi < +\infty$ guarantees that the trajectory is confined to $(0, 1)$, exactly as a bound orbit is confined between its turning points $E = U(x)$ (Appendix A).

5.2 Conservative potentials for the two canonical maps

Appendices B and C carry out, in full detail, the calculation of the potential V implied by each of the two maps of Table 2. The results are summarised here.

For the sigmoid, $\dot{\pi} = \pi(1 - \pi)$ implies (Appendix B)

$$V(\pi) = E - \frac{1}{2M} \pi^2(1 - \pi)^2, \quad (12)$$

a quartic double-well-like potential (Figure B2 in Appendix B) that is attractive towards $\pi = \frac{1}{2}$ from above and flattens out only asymptotically as $\pi \rightarrow 1$. Economically, this reproduces the observation of Section 4.4: it is comparatively easy to move the alignment probability away from the uninformative prior $\pi = \frac{1}{2}$, but increasingly costly (in score terms) to approach full alignment $\pi \rightarrow 1$ – the sigmoid map treats ESG alignment as subject to *diminishing marginal returns* to the score.

For the arctangent map, the Lorentzian momentum profile $p(t) = 1/(1+t^2)$ implies, after eliminating t in favour of x (Appendix C),

$$V(x) = -\frac{1}{2m} \cos^4[m(x - x_0)] + C, \quad (13)$$

a single potential well bounded by two symmetric, comparatively soft barriers at $x = x_0 \pm \pi/(2m)$. The barrier is reached, in finite score terms, faster than for the sigmoid – which is the dynamical-systems counterpart of the sharp early sensitivity documented numerically in Section 4.4 and Figure 3.

5.3 The general criterion, revisited

Collecting the above, Theorem 1 can equivalently be restated, and proved, in dynamical language.

Theorem 3 (Class of admissible probability maps – dynamical form) *A scoring function $s : X \rightarrow \mathbb{R}^+$ admits a probability map $\pi(s)$ associated with a conservative potential $V(\pi) = E - T(\pi)$ if and only if $\pi' = f(\pi)$ for some f with $\int_{\mathbb{R}^+} f(\xi) d\xi = M < +\infty$, in which case $\pi(s) = \frac{1}{M} \int_0^s f(\xi) d\xi$.*

Proof The condition $\pi' = f(\pi)$ ensures the momentum $p(s) \equiv f(\pi(s))$ depends on π only, so $T = p^2/(2M)$ is a function of π alone and the associated force $-\partial V/\partial \pi$ is conservative (Appendix A); energy conservation then gives $V(\pi) = E - T(\pi)$. The integrability condition $\int_{\mathbb{R}^+} f d\xi = M < +\infty$ guarantees $\pi(s) \rightarrow 1$ as $s \rightarrow \infty$ so that $\pi(s) : \mathbb{R}^+ \rightarrow (0, 1)$ is a valid probability map. The sigmoid satisfies the hypothesis with $f(\pi) = \pi(1 - \pi)$ and $\int_{\mathbb{R}^+} \pi(1 - \pi) d\pi < +\infty$. ■

Theorems 1 and 3 are two proofs of the same statement, obtained from unrelated arguments (a direct calculus argument versus a mechanical-energy argument); their agreement is, we believe, itself evidence that the admissibility criterion is a structural property of the normalisation problem rather than an artefact of either derivation.

6 Discussion: implications for ESG rating design and governance

6.1 Practical guidance for choosing and disclosing a probability map

Three recommendations follow directly from Sections 4 and 5. First, *the probability map is a disclosable modelling choice*, not a computational detail: two ESG -rating providers using an identical questionnaire and an identical additive score, but different admissible maps, can and will produce materially different ratings, as Table 4 shows starkly. Any ESG -metric methodology should therefore state explicitly which map is used and how it was calibrated, in the same way that a structural credit model discloses its assumed asset-return distribution. Second, *calibration anchors should be chosen for*

their economic content, not merely their mathematical convenience: the two boundary conditions used throughout this paper ($p(0) = \frac{1}{2}$, coin-flip at zero information, and $p(m_{\max})$ tied to a target top-rating default probability) are natural because they map onto observable, interpretable quantities; arbitrary alternative anchors would produce yet another rating architecture. Third, *the marginal- sensitivity diagnostic of Section 4.4 should be part of routine model validation:* a map whose sensitivity collapses too quickly (as the arctangent does here) effectively stops discriminating between companies over most of the questionnaire’s practical range, undermining the metric’s use for tracking remediation progress over time (cf. the re-submission logic of [14]).

6.2 Robustness and limitations

The present analysis inherits the limitations already acknowledged in [14]: the empirical base is still small (four questionnaires scored to date), the rank structure is intentionally simple (three materiality levels), and the qualitative, non-Y-N part of due-diligence questionnaires is deliberately excluded from the quantitative routine. The structural-risk analogy of Section 4.1 is, by design, qualitative and illustrative rather than a formal isomorphism: distance-to-default models are grounded in an explicit stochastic process for the firm’s asset value, whereas the ESG score is a deterministic, rank-weighted aggregation of discrete answers. The value of the analogy lies in transferring two well-tested risk-management disciplines – explicit disclosure of the calibration map, and routine sensitivity diagnostics – to a setting where they had not previously been applied, not in claiming a literal equivalence between the two model classes. The comparison in Section 4.3 is likewise a *comparative-statics* exercise under matched boundary conditions, and is not meant to argue that the arctangent map is unsuitable in general; rather, it demonstrates that admissibility under Theorem 1 alone is an insufficient criterion for practical rating design, and that the additional, economically motivated marginal-sensitivity diagnostic of Section 4.4 is needed to discriminate between otherwise admissible candidates.

6.3 Future work

Three extensions appear promising. First, the criterion of Theorem 1 could be used to search systematically over a wider family of admissible maps (e.g., generalised logistic maps [32]), ranked by the marginal-sensitivity diagnostic of Section 4.4 against an explicit loss function reflecting the investor’s rating-stability preferences. Second, the time dependence of the score, $m(a) \rightarrow m(a')$ between two questionnaire submissions separated by a remediation horizon τ , together with the vector interpretation of the **E,S,G** sub-scores already outlined in [14], could be studied with the same conservative-dynamics apparatus used here, this time tracking genuine time evolution rather than a score-as-time reparametrisation. Third, as the empirical base of scored questionnaires grows, the boundary-condition calibration of Section 4.3 could be replaced by a data-driven estimation of α, β (or a), moving the framework from a purely axiomatic to a partly empirical calibration, in the spirit of standard practice in credit-scoring model validation [30].

7 Conclusion

This paper closed an explicitly open modelling choice in the author’s earlier additive ESG -scoring framework [14]: which probability map should be used to normalise a questionnaire score into an ESG -alignment probability, and why. We showed that the normalisation problem admits a general, falsifiable admissibility criterion (Theorem 1), obtainable independently from a direct calculus argument and from a one-dimensional conservative-dynamics analogy (Theorem 3), and that this criterion alone is not sufficient to select a single map for practical use: two admissible maps, calibrated on identical boundary conditions, were shown, both analytically and through a reproducible numerical illustration, to imply materially different rating architectures and to differ sharply in their marginal sensitivity to score improvements. Reading the normalisation problem through the lens of structural credit-risk models clarifies what is at stake economically in the choice of map, and suggests three concrete governance recommendations – disclosure of the map, economically motivated calibration anchors, and routine marginal-sensitivity diagnostics – for ESG -rating design in venture-capital and private-equity due diligence. The classical-mechanics apparatus that grounds the dynamical form of the admissibility criterion has been confined to Appendix A–C, so that the paper’s economic argument can be read, and used, independently of it.

Appendix A Classical-mechanics background

The idea of transferring a mathematical apparatus from one field to an apparently unrelated one, in order to expose a common underlying structure, has a long and distinguished history, from Einstein’s own reflections on the relationship between geometry and physical experience [12, 13] to Kolmogorov’s axiomatic unification of probability theory [19] and modern historical-epistemological accounts of how such unifications are achieved [29]; the same drive toward quantification, tempered by an awareness of its limits, motivates classical treatments of the methodology of the social sciences [33]. In that spirit, this appendix collects, in self-contained form, the elements of classical one-dimensional dynamics used analogically in Section 5. It follows standard textbook treatments, in particular the notation of [4], and is included so that a reader unfamiliar with Hamiltonian mechanics can verify every step of the analogy without consulting external references; a reader satisfied with the economic argument of Sections 4–6 does not need this appendix at all.

A.1 Newton’s equation and conserved energy

A one-dimensional dynamical system for a point mass P is governed by the first-order system

$$\begin{cases} m\dot{x} = p \\ \dot{p} = \Phi(x, \dot{x}, t) \\ x(0) = x_0, \quad p(0) = p_0, \end{cases} \quad (\text{A1})$$

where $p = m\dot{x}$ is the momentum. If the force Φ does not depend explicitly on time (a Galilean, time-translation-invariant system), the system reduces to the autonomous

form

$$\begin{cases} m\dot{x} = p \\ \dot{p} = \Phi(x, p) \\ x(0) = x_0, \quad p(0) = p_0. \end{cases} \quad (\text{A2})$$

When Φ depends only on position, $\Phi = \Phi(x)$, the force is called *conservative* and admits a potential energy $U(x) = -\int_0^x \Phi(\xi) d\xi$, so that $\Phi = -\partial U/\partial x$ and Eq. (A2) becomes

$$\begin{cases} \dot{x} = p/m \\ \dot{p} = -\frac{\partial U}{\partial x}. \end{cases} \quad (\text{A3})$$

Defining the kinetic energy $T = \frac{1}{2}m\dot{x}^2 = p^2/(2m)$ (Eq. (A4)), the total mechanical energy $E = T + U$ is conserved.

$$T = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m}. \quad (\text{A4})$$

Theorem 4 (Energy conservation) *For a conservative force, $E = T + U$ satisfies $\dot{E} = 0$.*

Proof By the chain rule, $\dot{U} = \frac{\partial U}{\partial x}\dot{x}$, so

$$\dot{E} = \dot{T} + \dot{U} = \frac{p\dot{p}}{m} + \frac{\partial U}{\partial x}\dot{x} = \dot{x}\left(\dot{p} + \frac{\partial U}{\partial x}\right) = 0,$$

using $\dot{p} = -\partial U/\partial x$ from Eq. (A3). ■

□

Theorem 5 (Conservation of kinetic energy in the absence of forces) *If $\Phi \equiv 0$, then $U \equiv 0$ and $\dot{T} = 0$.*

Proof Immediate from Theorem 4 with $U = 0$, or directly: $\dot{T} = \frac{1}{m}p\dot{p}$ and $\Phi = 0 \Rightarrow \dot{p} = 0$. ■

□

A trajectory under a conservative force is confined to the region $U(x) \leq E$, the time to move between two positions x_1, x_2 being $\Delta t = \int \sqrt{m/[2(E - U)]} dx$; equivalently, the position is recovered from the momentum by

$$x(t) - x_0 = \int_0^t \frac{p(\xi)}{m} d\xi. \quad (\text{A5})$$

This relation is the one exploited, by analogy, in Theorem 2: whenever $\int_{\mathbb{R}^+} p(\xi)/m d\xi < \infty$, the trajectory is confined and bounded, exactly as an admissible probability map is confined to $(0, 1)$.

A.2 Phase space and two worked examples

Writing $\dot{x} = p/m$, $\dot{p} = f(x, p)$, the pair $\mathbb{M} = (x, p)$ traces a curve in the *phase plane*, called a phase portrait; energy conservation restates the dynamics as motion along the level sets of $E(x, p) = T + U$. Figure A1 shows the two classical examples used for intuition throughout this appendix: the harmonic oscillator, $\Phi(x) = -kx$, $U = \frac{1}{2}kx^2$, whose phase portrait consists of closed orbits (ellipses) around the single equilibrium; and the simple pendulum, $\Phi(x) = -k \sin x$, $U = k(1 - \cos x)$, whose phase portrait exhibits both closed orbits (librations) around the stable equilibrium and open orbits (rotations) beyond a critical energy, separated by a separatrix through the unstable equilibria at $x = \pm\pi$.

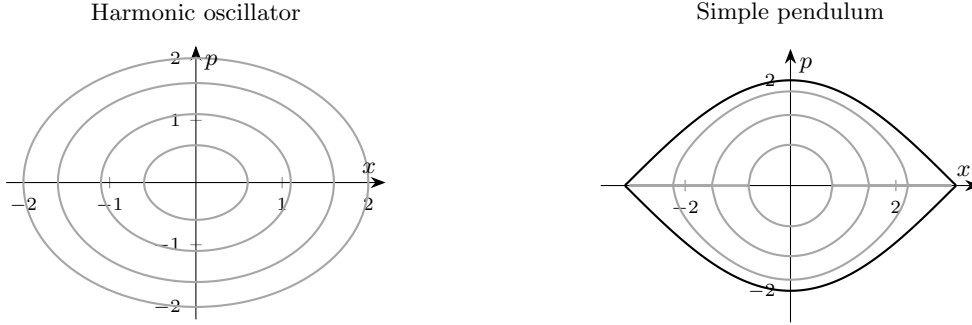


Figure A1 Phase portraits of the harmonic oscillator (closed elliptical orbits) and the simple pendulum (closed librations near the origin, separated by the black separatrix from open rotations at higher energy).

For the harmonic oscillator, the general solution of Eq. (A3) with $\Phi = -kx$, given $x(0) = x_0$, $\dot{x}(0) = v_0$, is $x(t) = x_0 \cos \omega t + (v_0/\omega) \sin \omega t$ with $\omega^2 = k/m$; using the energy-conservation theorem to eliminate time, $E = T + U$ with $U(x) = \frac{1}{2}kx^2$ recovers the same potential directly from the dynamics, which is exactly the inversion exploited in Section 5 to associate a potential with an admissible probability map: given the trajectory (the probability map $\pi(s)$) and the conserved energy, the potential $V(\pi)$ is obtained as $V = E - T$.

A.3 Lagrangian and Hamiltonian formulation: the pendulum

For systems subject to geometric constraints it is more economical to work with generalised coordinates. With kinetic energy $T(q, \dot{q})$ and potential $U(q)$, the Lagrangian is $L = T - U$ and the stationary-action principle gives the Euler–Lagrange equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$. Passing to the conjugate momenta $p_i = \partial L / \partial \dot{q}_i$, the Legendre transform of L gives the Hamiltonian $H(q, p) = \sum_i p_i \dot{q}_i - L$, and when the constraints are time-independent and forces derive from a potential, $H = T + U = E$ is a conserved first integral of the motion.

For the simple pendulum, a mass m on a rigid rod of length ℓ has one degree of freedom, the angle θ from the downward vertical, with $T = \frac{1}{2}m\ell^2\dot{\theta}^2$ and $U(\theta) =$

$mg\ell(1 - \cos \theta)$. The Euler–Lagrange equation gives $\ddot{\theta} + (g/\ell) \sin \theta = 0$, which reduces to the harmonic oscillator for small θ . The conjugate momentum is $p_\theta = m\ell^2 \dot{\theta}$, the Hamiltonian is $H(\theta, p_\theta) = p_\theta^2/(2m\ell^2) + mg\ell(1 - \cos \theta) = E$, and solving for $\dot{\theta}$ gives the first-order equation

$$\dot{\theta} = \pm \sqrt{\frac{2}{m\ell^2} [E - mg\ell(1 - \cos \theta)]}, \quad (\text{A6})$$

which contains the entire qualitative theory of the pendulum shown in Figure A1: motion confined to $E \geq U(\theta)$, turning points where $\dot{\theta} = 0$, and a separatrix at the critical energy $E_s = 2mg\ell$ separating librations from rotations.

A.4 One-dimensional gravitational dynamics and escape velocity

As a further worked example illustrating bounded versus unbounded motion under a conservative force, consider a point mass moving radially in the gravitational field of a central body of mass $M_\bullet$, with potential $U(r) = -GM_\bullet m/r$. Energy conservation gives the specific energy $\varepsilon = \frac{1}{2} \dot{r}^2 - GM_\bullet/r$, and the critical case of a particle reaching infinity with vanishing residual velocity corresponds to $\varepsilon = 0$, which at an initial radius r_0 gives the escape velocity

$$v_{\text{esc}} = \sqrt{\frac{2GM_\bullet}{r_0}}, \quad (\text{A7})$$

so that $v_0 < v_{\text{esc}}$ implies bounded motion, $v_0 = v_{\text{esc}}$ the critical escape trajectory, and $v_0 > v_{\text{esc}}$ unbounded motion. This is the mechanical counterpart of the integrability condition $\int_{\mathbb{R}^+} f(\xi) d\xi < +\infty$ in Theorem 1: exactly as a sub-escape-velocity trajectory remains gravitationally bound, an admissible probability map remains bound within $(0, 1)$.

Appendix B Derivation of the potential for the sigmoid map

Consider the trajectory $x(t) = 1/(1 + e^{-t})$, satisfying the first-order autonomous equation $\dot{x} = x(1 - x)$. With momentum $p = m\dot{x} = mx(1 - x)$, the force is

$$F(t) = \dot{p}(t) = m \dot{x}(1 - 2x) = mx(1 - x)(1 - 2x) = m(x - 3x^2 + 2x^3),$$

which depends on x only, so the dynamics is conservative and $F(x) = -dV/dx$ gives, upon integration,

$$V(x) = -m \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2} \right) + C. \quad (\text{B8})$$

Equation (B8) is the quartic potential quoted as Eq. (12) in the main text (up to the sign convention and normalisation constant M used there). Figure B2 plots the sigmoid trajectory together with the implied potential.

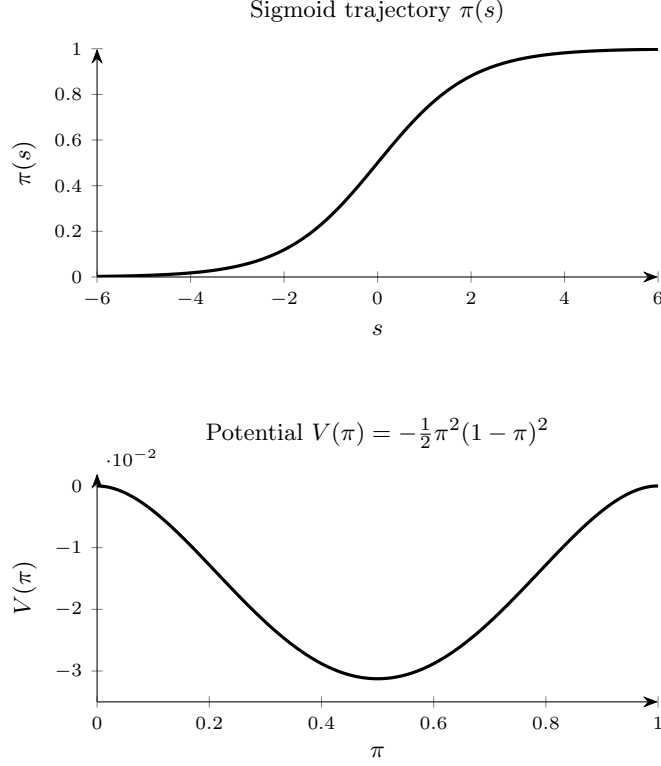


Figure B2 The sigmoid trajectory (top) and its associated conservative potential (bottom), attractive towards $\pi = \frac{1}{2}$ and flattening only asymptotically as $\pi \rightarrow 1$.

Appendix C Tangent and arctangent functions; potential of the arctangent map

Remark 1 (Tangent function) The equation $t' = 1 + t^2$ is solved by $t = \tan(x) + C$, $x \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$, since $\frac{d}{dx} \tan x = 1 + \tan^2 x$.

Remark 2 (Arctangent integral) $\int_0^\alpha \frac{1}{1+x^2} dx = \arctan \alpha$ and $\int_{\mathbb{R}} \frac{1}{1+x^2} dx = \pi$; more generally $\int_0^\alpha \frac{1}{a^2+x^2} dx = \arctan(\alpha/a)$ and $\int_{\mathbb{R}} \frac{1}{a^2+x^2} dx = \pi/a$ for $a \neq 0$, so $\frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{1+x^2} dx = 1$ is normalised.

Theorem 6 (Arctangent as a probability function) $P(\alpha) = \frac{1}{a} \frac{2}{\pi} \arctan(\alpha/a)$, $a > 0$, is a probability map of $\alpha \in \mathbb{R}^+$ with codomain $(0, 1)$.

C.1 Potential of the arctangent map

Consider a Lorentzian momentum profile $p(t) = 1/(1+t^2)$. From $p = m\dot{x}$, $\dot{x}(t) = 1/[m(1+t^2)]$, so $x(t) = \frac{1}{m} \arctan(t) + x_0$. Differentiating $p(t)$, $F(t) = \dot{p}(t) = -2t/(1+t^2)$

$t^2)^2$. Setting $\theta \equiv m(x - x_0)$ so that $\theta = \arctan(t)$, $t = \tan \theta$, $1 + t^2 = \sec^2 \theta$, gives F as a function of x alone,

$$F(x) = -2 \sin \theta \cos^3 \theta, \quad \theta = m(x - x_0),$$

which is conservative. Writing $F(x) = -dV/dx$, using $d\theta = m dx$ and $u = \cos \theta$,

$$V(x) = - \int F(x) dx = \frac{1}{m} \int (-2u^3) du = -\frac{1}{2m} u^4 + C = -\frac{1}{2m} \cos^4[m(x - x_0)] + C,$$

which is Eq. (13) in the main text. The trajectory is energetically closed: as $t \rightarrow \pm\infty$, $p(t) \rightarrow 0$ and $x \rightarrow x_0 \pm \pi/(2m)$, with $V(x) \rightarrow C$, i.e. the potential exhibits a single stable well bounded by two symmetric barriers, softer and reached at a finite value of the “time” t (the score s) compared with the sigmoid case of Appendix B – the dynamical counterpart of the sharper early marginal sensitivity documented in Figure 3.

Appendix D Calibration formulas used in Section 4

D.1 Sigmoid calibration

Writing $p_{\alpha,\beta}(m) = (1 + \beta e^{-\alpha m})^{-1}$, the base-rate condition $p(0) = p_0$ gives $\beta = (1 - p_0)/p_0$ (so $\beta = 1$ for the symmetric choice $p_0 = \frac{1}{2}$ used throughout this paper), and a second anchor $p(\bar{m}) = \bar{p}$ gives

$$\alpha = \frac{1}{\bar{m}} \ln \left(\frac{\beta \bar{p}}{1 - \bar{p}} \right). \quad (\text{D9})$$

With $p_0 = \frac{1}{2}$ ($\beta = 1$), $\bar{m} = 580$, $\bar{p} = 0.998$, this gives $\alpha \approx 0.0107114$, the value used throughout Section 4 and reported to six significant digits for reproducibility.

D.2 Arctangent calibration

Writing $p_a(m) = \frac{1}{2} + \frac{1}{\pi} \arctan(am)$, the base-rate condition $p(0) = \frac{1}{2}$ is satisfied automatically for every $a > 0$ (by symmetry of $\arctan$ around zero), so a single anchor $p(\bar{m}) = \bar{p}$ suffices to identify a :

$$a = \frac{1}{\bar{m}} \tan \left[\pi \left(\bar{p} - \frac{1}{2} \right) \right]. \quad (\text{D10})$$

With $\bar{m} = 580$, $\bar{p} = 0.998$, this gives $a \approx 0.2744015$, the value used throughout Section 4. Both calibrations, and the rating-threshold inversions of Table 3, were verified numerically; the inverse maps used to obtain Table 3 are $m_{\text{sigmoid}}(\psi) = \frac{1}{\alpha} \ln[\beta(1 - \psi)/\psi]$ and $m_{\text{arctan}}(\psi) = \frac{1}{a} \tan[\pi(\frac{1}{2} - \psi)]$.

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- **Conflict of interest / competing interests.** There are no competing interests. The work is of theoretical and methodological nature, carried out solely by the unaffiliated author, extending his own prior publication [14]; there are no co-authors or co-contributors.
- **Consent for publication.** The author consents to publication and is not aware of any impediment to it.
- **Data availability.** No new empirical data were used in this paper; the numerical illustrations of Section 4 are fully reproducible from the closed-form calibration formulas given in Appendix D. The empirical questionnaire data underlying Table 1 are as described in [14] and are available from the corresponding author on reasonable request.
- **Materials availability.** Material can be provided upon request.
- **Author contribution.** The author is the sole contributor to this article.

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