

On Multiplicative Y -index of Product Graphs and Polycyclic Compounds

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Abstract

Topological indices of graphs are widely used to analyse chemical significance of chemical compounds. In this work, a novel index named multiplicative Y-index (Π_Y) is defined and is computed for the tensor product of two graphs in terms of the multiplicative Y-index of the factor graphs. Additionally, its upper bounds for some other product graphs like Cartesian product, lexicographic product, strong product, disjunction and symmetric difference of two graphs in terms of other degree-based indices like the first Zagreb index, the forgotten index and the Y-index of the factor graphs are reported. Moreover, some graphs attaining these upper bounds are characterized. Furthermore, the index has been computed for polycyclic compounds like the polycyclic aromatic hydrocarbons, jagged rectangular benzenoid systems, polycyclic aromatic nanostar dendrimers and polycyclic metal organic network *TM-TCNB*.

Keywords: molecular graph, multiplicative Y-index, topological index, degree, product graphs

1 Introduction

A graph Γ consists of a nonempty set V , whose elements are called vertices and a collection E of unordered pairs of distinct elements from V , whose elements are called edges. The number of edges incident to a vertex v is said to be its degree and is denoted by $d(v)$. For other basic definitions and graph related results one can refer to [Harary \(1969\)](#). In graph theory, product graphs are graphs constructed by combining two or more base graphs, according to some adjacency rules. Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ and $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Formally, the product graph of Γ_1 and Γ_2 is a graph H , with vertex set as the Cartesian product of $V(\Gamma_1)$ and $V(\Gamma_2)$ in which adjacency is defined using some rules. Various product graphs are available in literature, a few of which are listed in Table 1. Two adjacent vertices u_1 and u_2 in Γ are denoted by $u_1 \sim u_2$. The base graphs also known as factor graphs serve as the building blocks of the product graphs. Structural features of the product graphs like their connectivity, distance between the

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Table 1 Adjacency rules and degree of every vertex (u_1, u_2) in some graph products.

Product graph H	Notation	Adjacency rule	Degree of $(u_1, u_2) \in V(H)$
Tensor product	$\Gamma_1 \otimes \Gamma_2$	$(u_1, v_1) \sim (u_2, v_2)$ iff $u_1 u_2 \in E(\Gamma_1)$ and $v_1 v_2 \in E(\Gamma_2)$.	$d_{(\Gamma_1 \otimes \Gamma_2)}(u_1, u_2) = d_{\Gamma_1}(u_1) d_{\Gamma_2}(u_2)$
Cartesian product	$\Gamma_1 \square \Gamma_2$	$(u_1, u_2) \sim (v_1, v_2)$ iff $u_1 v_1 \in E(\Gamma_1)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2 v_2 \in E(\Gamma_2)$.	$d_{(\Gamma_1 \square \Gamma_2)}(u_1, u_2) = d_{\Gamma_1}(u_1) + d_{\Gamma_2}(u_2)$
Lexicographic product or composition	$\Gamma_1[\Gamma_2]$	$(u_1, u_2) \sim (v_1, v_2)$ iff $u_1 v_1 \in E(\Gamma_1)$ or $u_1 = v_1$ and $u_2 v_2 \in E(\Gamma_2)$.	$d_{(\Gamma_1[\Gamma_2])}(u_1, u_2) = n_2 d_{\Gamma_1}(u_1) + d_{\Gamma_2}(u_2)$
Strong product	$\Gamma_1 * \Gamma_2$	$(u_1, u_2) \sim (v_1, v_2)$ iff $u_1 v_1 \in E(\Gamma_1)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2 v_2 \in E(\Gamma_2)$ or $u_1 v_1 \in E(\Gamma_1)$ and $u_2 v_2 \in E(\Gamma_2)$.	$d_{(\Gamma_1 * \Gamma_2)}(u_1, u_2) = d_{\Gamma_1}(u_1) + d_{\Gamma_2}(u_2) + d_{\Gamma_1}(u_1) d_{\Gamma_2}(u_2)$
Disjunction	$\Gamma_1 \vee \Gamma_2$	$(u_1, u_2) \sim (v_1, v_2)$ iff $u_1 v_1 \in E(\Gamma_1)$ or $u_2 v_2 \in E(\Gamma_2)$.	$d_{(\Gamma_1 \vee \Gamma_2)}(u_1, u_2) = n_2 d_{\Gamma_1}(u_1) + n_1 d_{\Gamma_2}(u_2) - d_{\Gamma_1}(u_1) d_{\Gamma_2}(u_2)$
Symmetric difference	$\Gamma_1 \oplus \Gamma_2$	$(u_1, u_2) \sim (v_1, v_2)$ iff $u_1 v_1 \in E(\Gamma_1)$ or $u_2 v_2 \in E(\Gamma_2)$ but not both.	$d_{(\Gamma_1 \oplus \Gamma_2)}(u_1, u_2) = n_2 d_{\Gamma_1}(u_1) + n_1 d_{\Gamma_2}(u_2) - 2 d_{\Gamma_1}(u_1) d_{\Gamma_2}(u_2)$

vertices and other invariants are influenced by their factor graphs. Due to their rich structural features and numerous applications in domains like network architecture, parallel computing, and chemistry, these constructs are extensively explored. Complex

structures with interesting combinatorial properties result from product graphs, which preserve features of their factor graphs while adding new ones [Hammack et al. \(2011\)](#). Hence, the relationships between the topological indices of a product graph and that of its building blocks can be explored. Furthermore, some chemically significant graphs are obtained by means of different graph operations. For instance, a carbon nanotube and nanotorus are Cartesian product of path and a cycle and Cartesian product of two cycles, respectively.

In chemical graph theory, by a molecular graph of a chemical compound, we mean a graphical representation of the molecular structure of the compound where the atoms are represented by vertices and bonds are represented by edges. Analysis of chemical compounds from their molecular graphs plays a vital role in distinguishing structural isomers (compounds with the same molecular formula but different arrangement of atoms) and also simplifies the process of characterization of complicated structures. One of the means of characterization of different structures or networks is done with the help of topological indices. The indices, as the name suggests are numerical parameters, which are highly correlated with the topology (i.e. adjacency of the vertices and their branching) of a graph, and give an insight into its properties. The initiation of the concept of topological indices dates back to 1947 by Harold Wiener for the purpose of comparing the boiling points of alkanes. He introduced a parameter, the path number based on distance between the atoms, which was renamed to Wiener index after him [Wiener \(1947\)](#). One of the oldest pair of degree-based topological indices, $M_1(\Gamma)$ and $M_2(\Gamma)$, known as the first and second Zagreb indices was introduced by Gutman and Trinajstić in 1972 [Gutman and Trinajstić \(1972\)](#). These indices are indicative of the dependence of the total π -electron energy on molecular structure [Fath-Tabar \(2011\)](#). Over the years, the study of degree-based topological indices has garnered a lot of attention of researchers. Some of these indices are given below. The generalized Zagreb index is defined as

$$M_1^{\alpha+1}(\Gamma) = \sum_{u \in V(\Gamma)} d(u)^{\alpha+1},$$

where α is a real number. Depending on the values of α different degree-based indices are obtained. When $\alpha = 1$, we get the first Zagreb index ($M_1(\Gamma)$), which is defined as

$$M_1(\Gamma) = \sum_{u \in V(\Gamma)} d(u)^2.$$

The first and second Zagreb indices of some graph operations, have been studied and the results have been applied to obtain the indices of the C_4 tube, C_4 torus and q-multi-walled polyhex nanotorus [Khalifeh et al. \(2009\)](#). The first multiplicative topological index was introduced by Narumi and Katayama [Narumi and Katayama \(1984\)](#) and is defined as

$$NK(\Gamma) = \prod_{u \in V(\Gamma)} d(u).$$

The multiplicative version of the Zagreb indices [Das et al. \(2013\)](#) (multiplicative first Zagreb-index and multiplicative second Zagreb-index) are defined as

$$\Pi_1(\Gamma) = \prod_{u \in V(\Gamma)} d(u)^2,$$

and

$$\Pi_2(\Gamma) = \prod_{uv \in E(\Gamma)} d(u)d(v).$$

In the same article, the upper bounds on the multiplicative Zagreb indices of the join, Cartesian product, corona product, composition and disjunction of graphs are derived and the indices are evaluated for some well-known graphs. In 2015, Furtula and Gutman observed that the generalized Zagreb index possesses strong correlation properties when $\alpha = 2$, and shown that in combination with the first Zagreb index it can significantly enhance the physico-chemical applicability of the first Zagreb index. It was left unstudied for a long time since its inception in 1972, thus they named it the forgotten index [Furtula and Gutman \(2015\)](#). It is defined as

$$F(\Gamma) = \sum_{u \in V(\Gamma)} d(u)^3.$$

In [De et al. \(2016\)](#), the F-index of various graph products have been obtained. The multiplicative version of forgotten index [Yousefi et al. \(2019\)](#) (multiplicative F -index) is defined as follows

$$\Pi_F(\Gamma) = \prod_{u \in V(\Gamma)} d(u)^3.$$

In [Mondal et al. \(2020\)](#), the explicit formulas for another variant of the Zagreb index known as the neighbourhood Zagreb index of product graphs has been presented. While studying the F -index and F -coindex of some derived graphs Nilanjan De, introduced a novel index called the modern index for $\alpha = 3$ [De et al. \(2016\)](#). It is defined as

$$Y(\Gamma) = \sum_{u \in V(\Gamma)} d(u)^4.$$

This index was named as the Yemen index (Y -index) in 2020 [Alameria et al. \(2020\)](#), where, the Y -index of some graph operations has been determined. Also, a strong correlation coefficient between Y -index and some physico-chemical properties as acentric factor, entropy, enthalpy of vaporization ($HVAP$) and standard enthalpy of vaporisation ($DHVAP$) has been shown [Alameria et al. \(2020\)](#). In a recent study, a new invariant the Y -coindex is discussed [Alameri et al. \(2020\)](#). Here, their basic mathematical properties are explored and explicit formulae for this co-index under several graph operations have been presented. Apart from investigating the chemical applicability of the indices, a substantial amount of work has been done in exploring their basic mathematical properties. The determination of bounds of indices and characterization of extremal graphs with respect to the indices has been an active area of

research ever since their inception [Che and Chen \(2016\)](#); [Imran and Jamil \(2020\)](#). Furthermore, establishing relations among different topological indices is well studied in literature [Das et al. \(2015\)](#); [Xu \(2012\)](#). Therefore, in this article the work done on Y -index is extended to multiplicative version of Y -index (called multiplicative Y -index). It is defined as follows

$$\Pi_Y(\Gamma) = \prod_{u \in V(\Gamma)} d(u)^4.$$

Also, the expression of the multiplicative Y -index for tensor product of two graphs in terms of the multiplicative Y -index of the factor graphs is determined. Furthermore, the upper bounds for the multiplicative Y -index is computed for some graph products namely the Cartesian product, lexicographic product, strong product, disjunction and symmetric difference in terms of the orders, sizes and other degree-based indices, like the first and second Zagreb index, forgotten index, and Y -index of the factor graphs. Also, the extremal graphs with respect to the indices have been characterized. Lastly, the expressions of this index for some polycyclic aromatic compounds i.e., polycyclic aromatic hydrocarbons, jagged rectangular benzenoid systems, polycyclic nanostar dendrimers and a polycyclic metal organic network have been reported.

2 Main Results

In this section, our aim is to compute some exact expressions and bounds of multiplicative Y -index for some graph operations. We start by stating the following lemma.

Lemma 1 Let $\{x_i \in \mathbb{R}^+ | i = 1, 2, \dots, n\}$. Then, by AM-GM inequality,

$$\frac{\sum_{i=1}^n x_i}{n} \geq \sqrt[n]{\prod_{i=1}^n x_i},$$

where the equality holds iff all x_i 's are equal.

Theorem 2 Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ and $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Then

$$\Pi_Y(\Gamma_1 \otimes \Gamma_2) = [\Pi_Y(\Gamma_1)]^{n_2} [\Pi_Y(\Gamma_2)]^{n_1}.$$

Proof Following the definition of multiplicative Y -index and Table 1 we have

$$\begin{aligned} \Pi_Y(\Gamma_1 \otimes \Gamma_2) &= \prod_{(u_i, v_j) \in V(\Gamma_1 \otimes \Gamma_2)} d(u_i, v_j)^4 \\ &= \prod_{u_i \in V(\Gamma_1)} \prod_{v_j \in V(\Gamma_2)} (d(u_i) d(v_j))^4 \\ &= \left[\prod_{u_i \in V(\Gamma_1)} d(u_i)^4 \right]^{n_2} \left[\prod_{v_j \in V(\Gamma_2)} d(v_j)^4 \right]^{n_1} \end{aligned}$$

$$= [\Pi_Y(\Gamma_1)]^{n_2} [\Pi_Y(\Gamma_2)]^{n_1}.$$

□

The following theorem gives an upper bound for the multiplicative Y -index of the Cartesian product of two graphs.

Theorem 3 *Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Then*

$$\Pi_Y(\Gamma_1 \square \Gamma_2) \leq \frac{1}{(n_1 n_2)^{n_1 n_2}} [n_2 Y(\Gamma_1) + 6M_1(\Gamma_1)M_1(\Gamma_2) + 8m_2 F(\Gamma_1) + 8m_1 F(\Gamma_2) + n_1 Y(\Gamma_2)]^{n_1 n_2}.$$

Proof Following the definition of multiplicative Y -index and Table 1 we have

$$\begin{aligned} \Pi_Y(\Gamma_1 \square \Gamma_2) &= \prod_{(u_i, v_j) \in V(\Gamma_1 \square \Gamma_2)} d(u_i, v_j)^4 \\ &= \prod_{u_i \in V(\Gamma_1)} \prod_{v_j \in V(\Gamma_2)} (d(u_i) + d(v_j))^4 \\ &= \prod_{u_i \in V(\Gamma_1)} \prod_{v_j \in V(\Gamma_2)} (d(u_i)^4 + 6d(u_i)^2 d(v_j)^2 \\ &\quad + 4d(u_i)^3 d(v_j) + 4d(v_j)^3 d(u_i) + d(v_j)^4). \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} &\Pi_Y(\Gamma_1 \square \Gamma_2) \\ &\leq \left[\frac{\sum_{u_i \in V(\Gamma_1)} \sum_{v_j \in V(\Gamma_2)} (d(u_i)^4 + 6d(u_i)^2 d(v_j)^2 + 4d(u_i)^3 d(v_j) + 4d(v_j)^3 d(u_i) + d(v_j)^4)}{n_1 n_2} \right]^{n_1 n_2} \\ &= \frac{1}{(n_1 n_2)^{n_1 n_2}} [n_2 Y(\Gamma_1) + 6M_1(\Gamma_1)M_1(\Gamma_2) + 8m_2 F(\Gamma_1) + 8m_1 F(\Gamma_2) + n_1 Y(\Gamma_2)]^{n_1 n_2}. \end{aligned}$$

Also, the equality holds iff $(d(u_i) + d(v_j))^4$ are equal for all pairs of vertices (u_i, v_j) in $V(\Gamma_1 \square \Gamma_2)$. This implies $d(u_i) = d(u_j)$ for all pairs of vertices u_i and u_j in Γ_1 and $d(v_j) = d(v_k)$ for all pairs of vertices v_j and v_k in Γ_2 i.e., when both Γ_1 and Γ_2 are regular graphs. □

The proofs of the Theorem 4-7 follow similarly to that of Theorem 2 and 3 using the definition of multiplicative Y -index and Table 1.

An upper bound for the multiplicative Y -index of the lexicographic product of two graphs is given in the following theorem.

Theorem 4 Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ and $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Then

$$\Pi_Y(\Gamma_1[\Gamma_2]) \leq \frac{1}{(n_1 n_2)^{n_1 n_2}} [n_2^5 Y(\Gamma_1) + 6n_2^2 M_1(\Gamma_1) M_1(\Gamma_2) + 8n_2^3 m_2 F(\Gamma_1) + 8n_2 m_1 F(\Gamma_2) + n_1 Y(\Gamma_2)]^{n_1 n_2}.$$

The equality holds iff both Γ_1 and Γ_2 are regular graphs.

An upper bound for the multiplicative Y -index of strong product of two graphs is obtained in the following theorem.

Theorem 5 Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ and $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Then

$$\begin{aligned} \Pi_Y(\Gamma_1 * \Gamma_2) \leq & \frac{1}{(n_1 n_2)^{n_1 n_2}} [n_2 Y(\Gamma_1) + n_1 Y(\Gamma_2) + Y(\Gamma_1) Y(\Gamma_2) + \\ & 6M_1(\Gamma_1) M_1(\Gamma_2) + 6M_1(\Gamma_1) Y(\Gamma_2) + 6Y(\Gamma_1) M_1(\Gamma_2) + 8m_2 F(\Gamma_1) + \\ & 8m_1 F(\Gamma_2) + 8m_1 Y(\Gamma_2) + 4F(\Gamma_1) Y(\Gamma_2) + 8m_2 Y(\Gamma_1) + 4Y(\Gamma_1) F(\Gamma_2) \\ & + 12F(\Gamma_1) F(\Gamma_2) + 12M(\Gamma_1) F(\Gamma_2) + 12F(\Gamma_1) M_1(\Gamma_2)]^{n_1 n_2}. \end{aligned}$$

The equality holds iff both Γ_1 and Γ_2 are regular graphs.

An upper bound for the multiplicative Y -index of the disjunction of two graphs is obtained in the following theorem.

Theorem 6 Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ and $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Then

$$\begin{aligned} \Pi_Y(\Gamma_1 \vee \Gamma_2) \leq & \frac{1}{(n_1 n_2)^{n_1 n_2}} [n_2^5 Y(\Gamma_1) + n_1^5 Y(\Gamma_2) + Y(\Gamma_1) Y(\Gamma_2) \\ & + 6n_1^2 n_2^2 M_1(\Gamma_1) M_1(\Gamma_2) + 6n_1^2 M_1(\Gamma_1) Y(\Gamma_2) + 6n_2^2 Y(\Gamma_1) M_1(\Gamma_2) \\ & + 8n_1 n_2^3 m_2 F(\Gamma_1) + 8n_2 n_1^3 m_1 F(\Gamma_2) - 8n_1^3 m_1 Y(\Gamma_2) \\ & - 4n_1 F(\Gamma_1) Y(\Gamma_2) - 8m_2 n_2^3 Y(\Gamma_1) - 4n_2 Y(\Gamma_1) F(\Gamma_2) \\ & - 12n_1 n_2^2 F(\Gamma_1) M_1(\Gamma_2) - 12n_2 n_1^2 M_1(\Gamma_1) F(\Gamma_2) + 12n_1 n_2 F(\Gamma_1) F(\Gamma_2)]^{n_1 n_2}. \end{aligned}$$

The equality holds iff both Γ_1 and Γ_2 are regular graphs.

An upper bound for the multiplicative Y -index of symmetric difference of two graphs is deduced in the following theorem.

Theorem 7 Let Γ_1 and Γ_2 be two graphs with $V(\Gamma_1) = \{u_1, \dots, u_n\}$ and $V(\Gamma_2) = \{v_1, \dots, v_n\}$ and sizes m_1 and m_2 , respectively. Then

$$\begin{aligned} \Pi_Y(\Gamma_1 \oplus \Gamma_2) \leq & \frac{1}{(n_1 n_2)^{n_1 n_2}} [n_2^5 Y(\Gamma_1) + n_1^5 Y(\Gamma_2) + 16Y(\Gamma_1) Y(\Gamma_2) \\ & + 6n_1^2 n_2^2 M_1(\Gamma_1) M_1(\Gamma_2) + 24n_1^2 M_1(\Gamma_1) Y(\Gamma_2) + 24n_2^2 Y(\Gamma_1) M_1(\Gamma_2) \\ & + 8n_1 n_2^3 m_2 F(\Gamma_1) + 8n_2 n_1^3 m_1 F(\Gamma_2) - 16n_1^3 m_1 Y(\Gamma_2) \\ & - 32n_1 F(\Gamma_1) Y(\Gamma_2) - 16m_2 n_2^3 Y(\Gamma_1) - 32n_2 Y(\Gamma_1) F(\Gamma_2) \\ & - 24n_1 n_2^2 F(\Gamma_1) M_1(\Gamma_2) - 24n_2 n_1^2 M_1(\Gamma_1) F(\Gamma_2) \\ & + 48n_1 n_2 F(\Gamma_1) F(\Gamma_2)]^{n_1 n_2}. \end{aligned}$$

The equality holds iff both Γ_1 and Γ_2 are regular graphs.

3 Multiplicative Y -index of Polycyclic Compounds

Polycyclic Aromatic Hydrocarbons (PAH_n)

During the incomplete combustion of gas, coal, oil and other organic materials, a group of more than 100 chemicals are formed which are known as polycyclic aromatic compounds (PAH_n). These compounds are also manufactured and exist as pale yellow-green, colorless, and white solids. The PAH_n considered in this paper is a family of hydrocarbons containing benzene rings on the circumference [Sardar et al. \(2017\)](#).

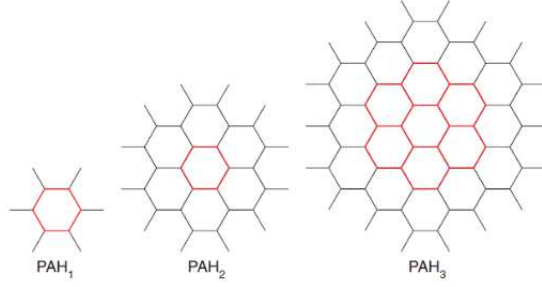


Fig. 1 First three members of polycyclic aromatic hydrocarbons PAH_n .

Let $\Gamma = PAH_n$ and $V_1 = \{i : d_i = 1\}$ and $V_3 = \{i : d_i = 3\}$ be the vertex partition of $V(\Gamma)$. Then $|V_1| = 6n$ and $|V_3| = 6n^2$.

Jagged Rectangle Benzenoid System ($B_{m,n}$)

A jagged rectangular benzenoid system $B_{m,n}$ consists of a family of hexagonal cells, arranged in a rectangular shape [Kulli \(2018\)](#). The number of cells in every chain alternate between $m-1$ and m with $4mn + 4m + 2n - 2$ vertices of degree either 2 or 3.

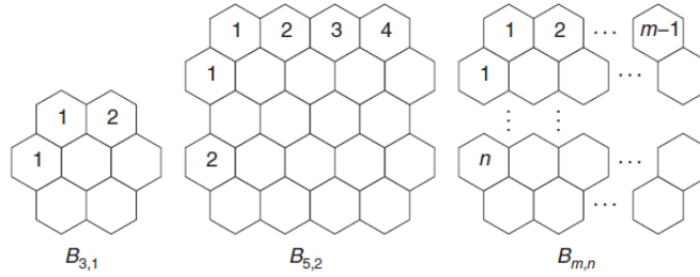


Fig. 2 Jagged rectangle benzenoid system $B_{m,n}$.

Let $\Gamma = B_{m,n}$ and $V_2 = \{i : d_i = 2\}$ and $V_3 = \{i : d_i = 3\}$ be the vertex partition of $V(\Gamma)$. Then $|V_2| = 2m + 4n + 2$ and $|V_3| = 4mn + 2m - 2n - 4$.

Polycyclic Aromatic Nanostar Dendrimers ($D_1[n]$)

A star-shaped macromolecule with a nanometer scale, built up of branched units called monomers is known as a dendrimer. These are artificially synthesized and have an intricate chemical structure and three distinct components namely the core, branches and the end groups. These are the principal components of nanotechnology owing to their energy collecting, photophysical and photochemical properties [Chu et al. \(2022\)](#).

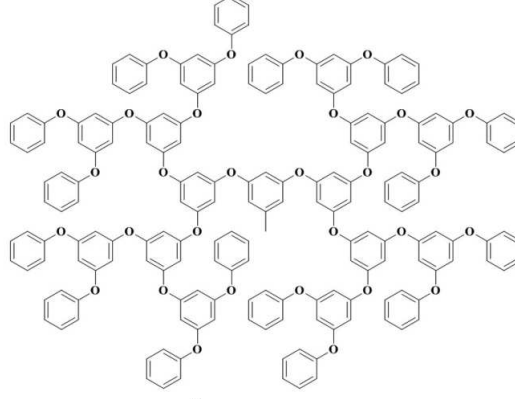


Fig. 3 Polycyclic aromatic nanostar dendrimer $D_1[n]$.

The polycyclic aromatic nanostar dendrimer $D_1[n]$ consists of $2^{n+4} - 9$ vertices of degree 1, 2 or 3.

Let $\Gamma = D_1[n]$ and $V_1 = \{i : d_i = 1\}$, $V_2 = \{i : d_i = 2\}$ and $V_3 = \{i : d_i = 3\}$ be the vertex partition of $V(\Gamma)$. Then $|V_1| = 1$, $|V_2| = 12 \times 2^n - 7$ and $|V_3| = 4 \times 2^n - 3$.

Polycyclic Metal Organic Network (TM - $TCNB$)

The polycyclic metal organic network like the transition metal (TM) of the three-dimensional series such as: $Ti, V, Cr, \dots$, or Z_n , phthalocyanine, and tetracyanobenzene ($TCNB$) are commonly used in alloy synthesis due to its high temperature stability. These have high melting point and boiling point and have immense applications in catalysis, biosensing and photodynamic treatment [Zhao et al. \(2022\)](#). The polycyclic metal organic network consists of vertices of degree 1, 2 or 3 and 4.

Let $\Gamma = TM$ - $TCNB$ and $V_1 = \{i : d_i = 1\}$, $V_2 = \{i : d_i = 2\}$ and $V_3 = \{i : d_i = 3\}$ be the vertex partition of $V(\Gamma)$. Then $|V_1| = 12n - 4$, $|V_2| = 24n + 8$, $|V_3| = 76n - 20$, and $|V_4| = 4n$.

The following result follows from the above definitions.

Proposition 8 The multiplicative Y -index of PAH_n , $B_{m,n}$, $D_1[n]$ and TM - $TCNB$ is given by 81^{6n^2} , $16^{2m+4n+2} \times 81^{4mn+2m-2n-4}$, $16^{12 \times 2^n - 7} \times 81^{4 \times 2^n - 3}$, and $16^{24n+8} \times 81^{76n-20} \times 256^{4n}$, respectively.

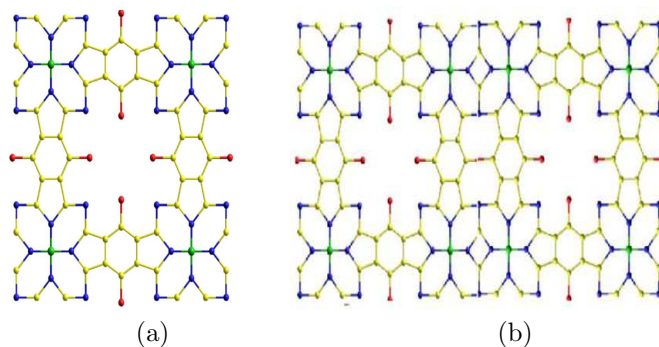


Fig. 4 (a) Unit and (b) Generalized structures of polycyclic metal organic network *TM-TCNB*.

4 Conclusion

*In this study, a new variant of Y -index has been defined and determined for tensor product of two graphs. Also, the upper bounds for other product graphs, in terms of parameters like number of vertices and edges and other topological indices like the first Zagreb index, forgotten index and Y -index has been computed. The expressions of the graph operations so derived help to understand how their topological indices can relate to the values of the corresponding topological indices of their components. Not only from the application point of view, but the computation of new variants of topological indices of different product graphs from a mathematical perspective can be explored for future references. Moreover, the index has been determined for some polycyclic compounds like the polycyclic aromatic hydrocarbons, jagged rectangular benzenoid system, polycyclic aromatic nanostar dendrimers and polycyclic metal organic network *TM-TCNB*. Along these lines, some possible future directions include the computation of lower bounds of the multiplicative Y -index, and hence, the characterization of minimal graphs with respect to the topological indices. Furthermore, the variants of topological indices, for important class of chemical compounds and nanostructures can be computed using the results so obtained.*

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