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Madjid Eshaghi Gordji

Semnan University

Mohamadali Berahman

`mohamadali_berahman@semnan.ac.ir`

Semnan University

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Temporal robustness hierarchies distinguish structural and quantum-feasible traversals in living networks

Madjid Eshaghi Gordji¹ Mohamadali Berahman^{1,*}

¹Faculty of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran

*Corresponding author: mohamadali_berahman@semnan.ac.ir

Madjid Eshaghi Gordji: meshaghi@semnan.ac.ir

Abstract

Dynamic quantum networks require two questions to be separated: whether a time window contains a spanning graph-theoretic certificate, and whether that certificate can be executed chronologically while meeting quantum-quality and resource constraints. We formulate a quantum living temporal graph with atomic time cells, state-dependent transmit, receive, and memory capabilities, and discrete or positive-duration link opportunities. Hamiltonian and Eulerian robustness are classified as Weak, Local, Strong, or Interval-Strong. The hierarchy is strict in general. A channel-composition witness is required for the final quantum Eulerian separation; static identity channels alone do not establish it. Hamiltonian coverage collapses under fixed or nested vertex sets, while snapshot-additive occurrence aggregation with consecutive nonzero-degree vertex overlap yields a structural Eulerian collapse. The additivity condition is essential when positive-duration opportunities can span more than one atomic cell. Neither conclusion automatically extends to the physical layer. An analytical homogeneous teleportation-channel illustration shows how certificate length and memory waiting reduce fidelity; for initial link fidelity 0.95 and threshold 0.85, at most three zero-wait links are admissible under the declared model. The framework provides auditable single-payload robustness certificates without treating topology as a substitute for physical feasibility.

Keywords: temporal graphs; quantum networks; Hamiltonian cycle; Eulerian tour; entanglement fidelity; network robustness

1 Introduction

Quantum networks seek to distribute quantum states and entanglement among remote processors, sensors, and communication nodes, thereby enabling tasks that have no classical analogue^{1,2}. Long-distance operation is constrained by attenuation and imperfect local operations, which motivated quantum repeaters based on elementary entanglement generation, quantum memories, purification, and entanglement swapping^{3–5}. Repeater architectures now span memory-based and all-photonic approaches, and their resource requirements depend strongly on the physical error model and architecture^{6,7}. These facts make time and device state part of the network model rather than secondary implementation details.

A scalable quantum network also requires networking abstractions above the physical link. Link-layer proposals turn heralded entanglement generation into a service with explicit scheduling and quality semantics, while network-architecture studies emphasize that control, routing, and resource management must coexist with probabilistic quantum operations^{8,9}. Discrete-event simulation has consequently become important for resolving timing dependencies that are difficult to capture analytically¹⁰. At the routing level, work has incorporated link success probabilities, memory limitations, fidelity, swapping, purification, and resource allocation^{11–13}. Recent studies sharpen

this physical perspective by treating coherence and fidelity jointly and by quantifying how local errors change end-to-end resource requirements^{14;15}. Waiting-time calculations in repeater chains likewise show that temporal delay and physical quality are coupled¹⁶.

The quality criterion used in such a model must itself be stated precisely. Quantum teleportation and purification provide the operational background for converting shared entanglement into an effective payload channel¹⁷. Bell-diagonal or isotropic resources lead to standard channel-fidelity relations, but those relations depend on the declared state family and protocol rather than on graph connectivity alone^{18;19}. Entanglement fidelity supplies a channel-level comparison with an ideal target, while realistic entanglement swapping introduces additional conditional operations and imperfections^{20;21}. We therefore formulate physical acceptance as an explicit predicate over a chronologically ordered composite channel, rather than assigning a quality score directly to an aggregate graph; the channel notation follows standard quantum-information conventions²².

Temporal graph theory addresses the complementary question of whether interactions can be used in an admissible order. Reviews and time-varying-graph formalisms distinguish a temporal network from its static union and formalize journeys or paths whose edge times are ordered^{23–25}. Stream graphs further allow node presence to vary with time²⁶. The distinction is algorithmically substantive: reachability in a static aggregate need not imply temporal reachability, and temporal path problems can change complexity when waiting is constrained^{27–29}. Related exploration problems require a temporal walk to cover vertices or edges, and Eulerian temporal-walk formulations can be computationally hard even though Eulericity of a static directed multigraph is easy to test^{30;31}.

Quantum dynamics on changing graphs has also been studied from other perspectives. For example, quantum spatial search on random temporal networks shows that temporal variation can alter quantum-walk behaviour³². The living-temporal-graph proposal introduces explicit node states together with discrete and continuous interactions³³. We use that stateful motivation but do not rely on it as a theorem source: all definitions, witnesses, and proofs needed here are given in this article and its Supplementary Information.

The unresolved issue addressed here is narrower than general quantum routing. Existing temporal-network formalisms provide chronology and changing node/link presence but do not intrinsically impose a quantum channel-quality, heralding, and resource predicate. Quantum-routing models provide such physical constraints but do not ordinarily classify vertex-covering and occurrence-covering closed certificates over every contiguous time window. Classical Hamiltonian and Eulerian theory supplies natural vertex- and edge-coverage certificates, with sharply different computational behaviour^{34;35}; however, applying those static certificates to a living quantum network requires explicit semantics for aggregation, chronology, and physical admissibility. The gap is therefore a modelling and verification gap, not a claim that Hamiltonian or Eulerian traversals are universally optimal quantum routes.

Our contribution is a finite audit framework that separates three questions: whether a window contains a structural certificate, whether that certificate has a time-respecting closed realization, and whether the resulting single-payload channel meets a fixed physical specification. Within that scope we (i) define atomic time-cell, state-capability, support, occurrence, scheduling, and channel-admissibility semantics; (ii) classify Hamiltonian and Eulerian robustness as Weak, Local, Strong, or Interval-Strong; (iii) prove that the hierarchy is strict using structural witnesses and, where topology alone is insufficient, an explicit channel-composition witness; (iv) identify sufficient collapse conditions, including the snapshot-additivity requirement needed by the Eulerian result when duration-spanning opportunities are otherwise allowed; and (v) exhibit a homogeneous teleportation-channel example in which structural Eulerian collapse does not imply quantum-admissible collapse. The claims concern one logical payload and do not extend without additional state variables to simultaneous multipair storage, purification queues, or arbitrary repeater protocols.

Table 1 makes this positioning explicit. The comparison concerns the formal ingredients used in

this article, not a claim that every paper in each broad field has identical scope.

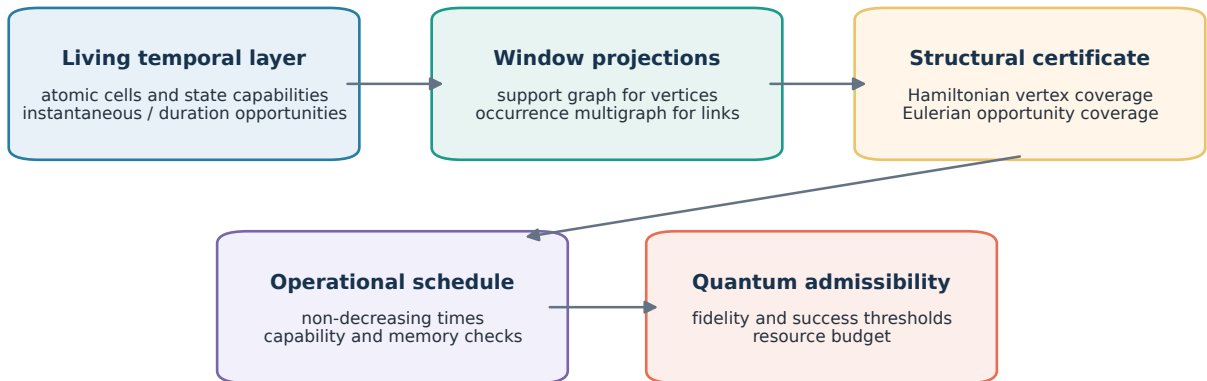
Table 1. Scope relative to the closest modelling traditions. “Not intrinsic” means that the ingredient can be added to a particular model but is not part of the cited tradition’s defining certificate.

Tradition	Core ingredient retained	Ingredient not intrinsic to that certificate
Temporal paths and stream graphs	time-respecting order, time-varying links, and node presence	channel fidelity, heralded success, and a resource budget
Temporal exploration and Euler walks	vertex or edge-occurrence coverage under temporal information	a state-dependent quantum channel admissibility test
Quantum entanglement routing Present framework	link quality, memories, success, swapping, and protocol resources atomic living states, support/occurrence projections, chronological schedules, and a fixed physical predicate	a four-level certificate over every contiguous window general multipair protocols; the proved scope is one logical payload

Hamiltonian and Eulerian certificates answer different engineering questions. A Hamiltonian cycle is a sparse cyclic backbone visiting each available node once. It may support a round-robin state-transfer or control schedule. An Eulerian tour covers every retained link opportunity and can represent a complete calibration, audit, or activation schedule. These interpretations are deliberately limited: neither certificate is claimed to be universally optimal for entanglement routing. Their value is that they test vertex coverage and opportunity coverage under the same temporal and physical rules.

The central distinction of this paper is between a *structural certificate* and a *physically admissible schedule*. A support graph may contain a Hamiltonian cycle, and an occurrence multigraph may be degree balanced, without either certificate being executable in time. Conversely, an executable traversal can still fail its quantum fidelity, success-probability, or resource thresholds. Figure 1 summarizes the resulting audit.

From changing topology to a physically executable quantum certificate



The robustness level is reported only after the structural and operational layers have both been evaluated.

Figure 1. Audit from topology to quantum feasibility. Node lifetimes, capabilities, and link opportunities first determine a window-specific support graph and occurrence multigraph. Hamiltonian or Eulerian structure is then only a candidate certificate. A separate scheduler enforces non-decreasing times, endpoint capabilities, and memory availability; the resulting composite operation is finally tested against fidelity, success-probability, and resource thresholds. This ordering prevents an aggregate graph from being interpreted as an executable quantum protocol without further evidence.

Our contributions are fourfold. First, we give a finite, reproducible single-payload model with atomic time cells, state-dependent capabilities, and an explicitly ordered composite channel. Second,

we retain the four robustness levels—Weak, Local, Strong, and Interval-Strong—while stating whether support or occurrence aggregation is used. Third, we prove strict separations with distinct structural and quantum witnesses, give corrected collapse conditions, and show why a topological Eulerian collapse need not be a quantum collapse. Fourth, we identify a careful complexity boundary: snapshot and window projections allow polynomial Euler tests, whereas time-respecting Eulerian walk and trail variants are NP-complete in general³¹.

2 Results

We first state the model and then use it to determine how the four robustness levels relate, when local checks collapse to window-wise guarantees, and where topology ceases to be a reliable proxy for physical feasibility.

2.1 A model with structural, temporal, and quantum layers

Let $\mathcal{T} = \{1, \dots, \tau\}$ index atomic time cells determined by physical boundaries $\zeta_0 < \zeta_1 < \dots < \zeta_\tau$. We define

$$C_t = \begin{cases} [\zeta_{t-1}, \zeta_t), & t < \tau, \\ [\zeta_{\tau-1}, \zeta_\tau], & t = \tau, \end{cases}$$

so every physical time in the declared horizon belongs to exactly one cell. For a non-empty contiguous index window $I = [i, j] \subseteq \mathcal{T}$, its physical span is the union of its cells,

$$H(I) = \bigcup_{t=i}^j C_t = \begin{cases} [\zeta_{i-1}, \zeta_j), & j < \tau, \\ [\zeta_{i-1}, \zeta_\tau], & j = \tau. \end{cases} \quad (1)$$

This convention is important at cell boundaries: an instantaneous event at ζ_t belongs to cell $t + 1$, not to both adjacent cells. A positive-duration opportunity that completes exactly at ζ_t therefore requires the window to include cell $t + 1$ under the completion-within-window convention used below.

Let V be a finite universe of nodes. A node v has a lifetime $L_v \subseteq \mathcal{T}$ and a state

$$s_v(t) \in \{\text{Active}, \text{Sleep}, \text{RxOff}, \text{TxFOff}, \text{Offline}\}.$$

We impose $s_v(t) = \text{Offline}$ whenever $t \notin L_v$. The state determines transmit, receive, and memory indicators by the convention in Table 2. This is a modelling policy, not a universal hardware claim; another device can be represented by changing the capability map before any robustness certificate is evaluated. We write $V_t = \{v \in V : t \in L_v, s_v(t) \neq \text{Offline}\}$.

Table 2. State-to-capability convention. Sleep preserves the stored single payload while disabling communication; direction-specific off states retain the other communication direction and memory. Outside a node's lifetime the effective state is Offline.

State	x_v^{tx}	x_v^{rx}	x_v^{mem}
Active	1	1	1
Sleep	0	0	1
RxOff	1	0	1
TxFOff	0	1	1
Offline	0	0	0

A directed link opportunity $a \in \mathcal{A}$ has tail $u(a)$, head $v(a)$, physical start $b(a)$, completion $d(a) \geq b(a)$, and type $k(a) \in \{D, C\}$. Type D is instantaneous, with $b(a) = d(a)$, and requires transmit

capability at its tail and receive capability at its head in the unique cell containing that time. Type C has $b(a) < d(a)$. Its occupied-cell set is

$$\Gamma(a) = \{t \in \mathcal{T} : C_t \cap [b(a), d(a)] \neq \emptyset\},$$

and, under the conservative convention used here, tail transmission and head reception must be available in every $t \in \Gamma(a)$. Thus a completion exactly on a cell boundary is checked in the cell that owns that boundary. These requirements make the living state part of feasibility rather than a descriptive label.

All traversals carry one logical payload on a common d -dimensional Hilbert space $\mathcal{H}_Q$. The model therefore describes a sequential state-transfer or teleportation chain; it does not represent a general protocol with several Bell-pair halves stored simultaneously at different nodes. Conditional on a heralded success, opportunity a implements a completely positive trace-preserving channel $\mathcal{N}_a : \mathcal{D}(\mathcal{H}_Q) \rightarrow \mathcal{D}(\mathcal{H}_Q)$, with success probability $p_a \in [0, 1]$ and non-negative resource cost r_a . Memory at v for duration $w \geq 0$ is represented by $\mathcal{M}_{v,w}$, with non-negative declared cost $\mu_v(w)$. A common input-output space makes channel composition and comparison with an ideal target channel well defined^{20;22}. Protocol-specific swapping can be represented by a conditional operation and a separately tracked heralding probability²¹.

Definition 1 (Time-respecting schedule). A non-empty schedule $W = (a_1, \dots, a_k)$ is time respecting if $v(a_i) = u(a_{i+1})$ and $d(a_i) \leq b(a_{i+1})$ for every $i < k$. Every opportunity must satisfy its state-dependent capability conditions. During a positive wait $\Delta_i = b(a_{i+1}) - d(a_i) > 0$, the payload is stored at $v(a_i)$, where memory capability must remain one in every cell with positive-duration overlap with the open waiting interval $(d(a_i), b(a_{i+1}))$. Non-decreasing time is used; no claim is made for a strictly increasing convention. A schedule is *closed* when $v(a_k) = u(a_1)$.

For $k \geq 2$, the ordered conditional payload channel is

$$\oplus_W = \mathcal{N}_{a_k} \circ \mathcal{M}_{v(a_{k-1}), \Delta_{k-1}} \circ \mathcal{N}_{a_{k-1}} \circ \dots \circ \mathcal{M}_{v(a_1), \Delta_1} \circ \mathcal{N}_{a_1}, \quad (2)$$

with $\oplus_W = \mathcal{N}_{a_1}$ when $k = 1$. The resource accounting used below is additive,

$$R(W) = \sum_{i=1}^k r_{a_i} + \sum_{i=1}^{k-1} \mu_{v(a_i)}(\Delta_i).$$

The quantity $p(W)$ denotes the joint success probability of the declared execution model. Only when conditional independence has been justified may it be replaced by the product of link and memory success probabilities.

Given a fixed ideal unitary channel $\mathcal{U}$, the entanglement fidelity of $\oplus_W$ relative to $\mathcal{U}$ is

$$F_e(W; \mathcal{U}) = \langle \Omega_d | [\mathcal{I} \otimes (\mathcal{U}^\dagger \circ \oplus_W)] (|\Omega_d\rangle \langle \Omega_d|) | \Omega_d \rangle, \quad (3)$$

where $|\Omega_d\rangle = d^{-1/2} \sum_{j=1}^d |j\rangle |j\rangle$. The target $\mathcal{U}$, fidelity threshold $\theta \in [0, 1]$, success threshold $\pi \in [0, 1]$, and resource budget $B \geq 0$ are fixed before robustness is compared across windows. A schedule is quantum-admissible when

$$\text{Adm}(W) = 1 \iff F_e(W; \mathcal{U}) \geq \theta, \quad p(W) \geq \pi, \quad R(W) \leq B, \quad (4)$$

for the declared specification.

For a non-empty contiguous window I , let $V_I = \bigcup_{t \in I} V_t$. An instantaneous opportunity belongs to I when its unique owning cell lies in I ; a positive-duration opportunity belongs to I when $[b(a), d(a)] \subseteq H(I)$. In both cases all capability conditions must hold. The *support graph* $S[I]$ contains one arc (u, v) whenever at least one such opportunity goes from u to v . The *occurrence*

Table 3. Objects used at each audit layer. The purpose column states why the object is introduced and prevents one layer from silently replacing another.

Object	What it retains	Purpose
Snapshot $G[t]$	nodes and operations contained in atomic cell C_t	tests cell-local structural availability
Support graph $S[I]$	one arc per ordered endpoint pair	tests whether a window contains a sparse vertex-covering backbone
Occurrence multigraph $O[I]$	every time-labelled opportunity	tests whether every retained opportunity can be covered
Time-respecting schedule W	adjacency, start/completion times, and waits	rejects paths that exist only after erasing time order
Admissibility predicate $\text{Adm}(W)$	channel fidelity, success, and resources	rejects executable schedules that are physically below specification

multigraph $O[I]$ retains every such opportunity as a distinct arc. We use $S[I]$ for Hamiltonian vertex coverage and $O[I]$ for Eulerian opportunity coverage.

The capability convention has observable consequences. Suppose a discrete opportunity carries the payload from A to B in C_1 , the next opportunity leaves B for C in C_3 , and the payload waits at B during C_2 . The schedule remains feasible if B is Sleep in C_2 , because memory is retained, but fails if B is Offline. It also fails if B is RxOff when the first opportunity completes or TxOff when the second begins. Likewise, an instantaneous type-D opportunity in the interior of C_1 belongs to $\{1\}$, while an instantaneous opportunity exactly at ζ_1 belongs only to $\{2\}$. A type-C opportunity starting in C_1 and completing in the interior of C_2 belongs to $[1, 2]$ but neither singleton. These examples explain why state, duration, and boundary ownership change the existence and scope of a certificate.

We use non-degenerate cycle conventions throughout. A structural Hamiltonian certificate in I exists only when $|V_I| \geq 2$ and is a directed cycle that visits every vertex in V_I exactly once before returning to its start. A structural Eulerian certificate is a directed Eulerian circuit in a *non-empty* $O[I]$, using every retained occurrence exactly once; isolated live vertices are irrelevant to occurrence coverage. A quantum Hamiltonian or Eulerian certificate additionally requires a *closed*, time-respecting realization W satisfying $\text{Adm}(W) = 1$. The Hamiltonian schedule selects one opportunity for each cycle arc; the Eulerian schedule uses every occurrence exactly once.

2.2 The robustness hierarchy is strict

For any one of the four window properties just defined—structural Hamiltonian, structural Eulerian, quantum Hamiltonian, or quantum Eulerian—write $P(I)$ for its truth value in window I .

Definition 2 (Four robustness levels).

$$\begin{aligned}
\text{Weak}(P) &: \iff \exists t \in \mathcal{T} : P(\{t\}), \\
\text{Local}(P) &: \iff \forall t \in \mathcal{T} : P(\{t\}), \\
\text{Strong}(P) &: \iff \text{Local}(P) \text{ and } P(\mathcal{T}), \\
\text{IntervalStrong}(P) &: \iff \forall \text{ non-empty contiguous } I \subseteq \mathcal{T} : P(I).
\end{aligned}$$

Theorem 3 (Hierarchy and strict separation). *For each property P above,*

$$\text{IntervalStrong}(P) \implies \text{Strong}(P) \implies \text{Local}(P) \implies \text{Weak}(P).$$

Every converse fails in general for both structural and quantum-admissible Hamiltonian and Eulerian properties. Identity channels suffice for all Hamiltonian separations and for the first two Eulerian separations. Quantum Eulerian Strong does not imply Interval-Strong by a separate channel-composition witness.

The implications are definitional, but strictness is not. Figure 2 displays the three Hamiltonian separations. Identity opportunity channels together with identity memory during required waits lift these Hamiltonian witnesses and the first two Eulerian separations to the quantum setting. They do *not* lift the structural Eulerian Strong/Interval-Strong witness, because a static Euler circuit need not have a chronological realization.

The missing quantum separation is supplied by channel composition. Let $\mathcal{T} = \{1, 2, 3\}$, $V = \{A, B\}$, and place the directed 2-cycle $A \rightarrow B \rightarrow A$ in every cell. The two opportunities in each cell occur at the same interior instant, which is permitted by the declared non-decreasing-time convention. Give $A \rightarrow B$ the identity channel and $B \rightarrow A$ the Pauli channel $\mathcal{X}(\rho) = X\rho X$. Set every node-memory channel needed between cells to the identity, with zero memory cost, and take joint success probability one. Either cyclic starting point therefore yields the cell product $\mathcal{X}$, while the inter-cell waits do not alter the payload channel. Fix the target to $\mathcal{U} = \mathcal{X}$, take $B = 0$, and set $\theta = \pi = 1$. Every singleton passes and the three-cell tour passes because $X^3 = X$, whereas the window $[1, 2]$ fails because $X^2 = I$ and

$$F_e(I; \mathcal{X}) = \frac{|\text{Tr}(X^\dagger I)|^2}{4} = 0.$$

Thus quantum Eulericity is Strong but not Interval-Strong in this instance. The structural Euler witness and this channel witness are deliberately separate. Complete edge lists, schedule checks, and proofs are in Supplementary Notes S2–S3.

2.3 Correct collapse conditions and their physical limit

The term *hierarchy collapse* must identify exactly which levels and which aggregation semantics are involved. The following theorem replaces the earlier, over-broad collapse claims.

Theorem 4 (Window collapse conditions). *The following statements hold.*

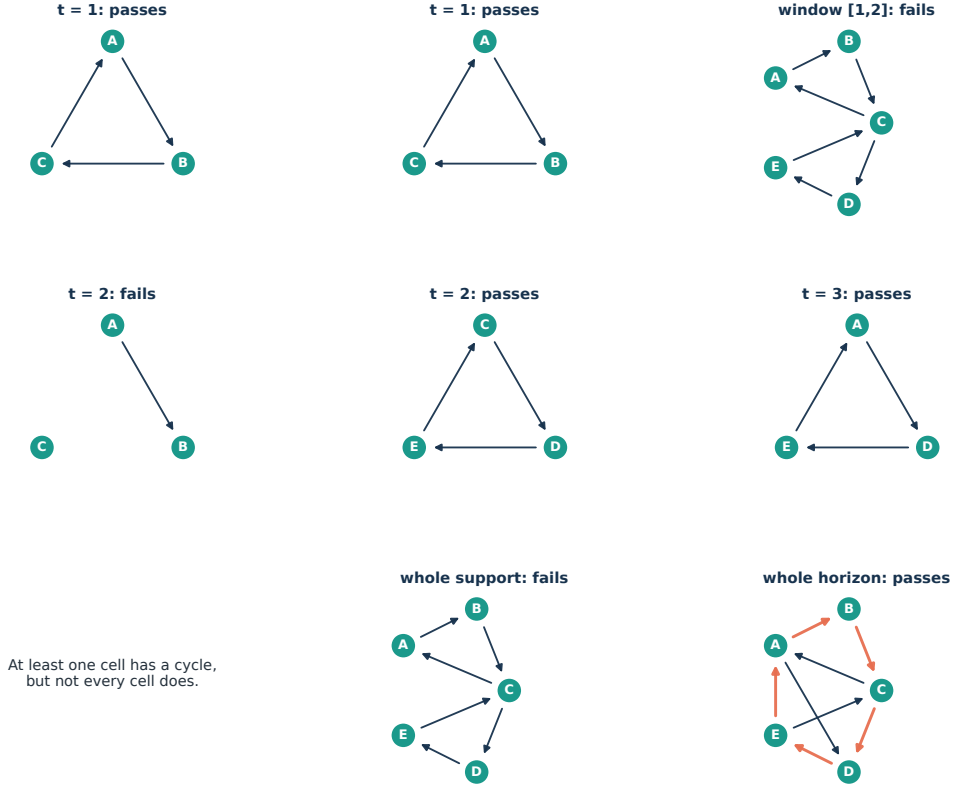
- (a) *If $V_t = V$ for every t , Local Hamiltonicity implies Interval-Strong Hamiltonicity under support aggregation, both structurally and quantum-admissibly.*
- (b) *More generally, if $V_t \subseteq V_{t+1}$ for every t , Local Hamiltonicity implies Interval-Strong Hamiltonicity under support aggregation, both structurally and quantum-admissibly. Monotone edge addition is not required.*
- (c) *Let $U_t \subseteq V_t$ be the vertices with non-zero degree in the snapshot occurrence multigraph. Assume additionally that occurrence aggregation is snapshot-additive,*

$$O[I] = \biguplus_{t \in I} O[\{t\}]$$

for every non-empty contiguous interval I . This holds, for example, when every retained opportunity is contained in one atomic cell. If every snapshot occurrence multigraph is non-empty and directed Eulerian, and $U_t \cap U_{t+1} \neq \emptyset$ for consecutive times, then every non-empty interval occurrence aggregate is structurally Eulerian. Hence the instance is Interval-Strong structurally under this occurrence aggregation.

For Hamiltonicity, any interval contains a snapshot whose live vertex set equals the interval union: every snapshot works in (a), and the last snapshot works in (b). Its cycle is therefore a valid witness for the entire interval. For Eulericity, the stated snapshot-additivity assumption makes multiset degree balance additive across snapshots, while overlap at a vertex incident to an opportunity links their strongly connected components. Without snapshot additivity, a positive-duration opportunity spanning several cells can appear in $O[I]$ while appearing in no singleton occurrence multigraph, so the displayed degree sum need not hold and the collapse conclusion can fail. Requiring only $V_t \cap V_{t+1} \neq \emptyset$ would also be insufficient because the shared vertex could be isolated. These arguments do not require a claim that channel fidelity improves under concatenation.

Formal proofs of all three collapse statements are given in Supplementary Note S4; living-state and duration boundary cases are proved in Supplementary Note S5.

Weak does not imply Local**Local does not imply Strong****Strong does not imply Interval-Strong**

Directed Hamiltonian constructions. Positive cycles are chronological; identity channels preserve their quantum admissibility.

Figure 2. Finite witnesses for strict Hamiltonian separation. Left: the $t = 1$ cycle establishes Weak Hamiltonicity, while the $t = 2$ single arc prevents Local Hamiltonicity. Centre: the cycles at $t = 1$ and $t = 2$ are each Hamiltonian on their live vertex sets, but their support union has no cycle through all five vertices. Right: the first two-cell window still fails, whereas the added $t = 3$ cycle creates the highlighted whole-horizon cycle $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$. Identity channels make these Hamiltonian positive certificates quantum-admissible; the Eulerian quantum separation requires the distinct Pauli-channel construction described in the text.

Occurrence preservation is essential. Under set aggregation, repeated arcs are collapsed and can change degree balance. Figure 3 uses two individually Eulerian snapshots. After duplicates are removed, node A has out-degree one and in-degree two, so the set aggregate is not Eulerian. The full second snapshot and degree calculation appear in Supplementary Note S4.

The physical conclusion is different. Consider two identical snapshots on $\{A, B\}$, each containing one occurrence of $A \rightarrow B$ and one of $B \rightarrow A$, with all four opportunities contained in their respective atomic cells. Each snapshot is Eulerian, consecutive nonzero-degree vertex sets overlap, and the snapshot-additive occurrence aggregate is Eulerian. Suppose each directed opportunity implements standard teleportation using the two-qubit Bell-diagonal isotropic state with fully entangled fraction $F_0 = 0.95$ specified in Supplementary Note S6. Under the standard teleportation-channel correspondence, its conditional payload channel is depolarizing with visibility $q_0 = (4F_0 - 1)/3$ ¹⁹. Assume the memory channel during the unavoidable inter-cell wait is the identity with zero memory cost, and take joint success probability one. Composing m independent identical opportunity channels then gives

$$F(m) = \frac{1}{4} + \frac{3}{4} \left(\frac{4F_0 - 1}{3} \right)^m, \quad (5)$$

for m sequential traversals under this declared independent-channel, ideal-memory model. We fix the ideal round-trip target to $\mathcal{U} = \mathcal{I}$; on a closed Eulerian route, Eq. (5) is then the entanglement fidelity of the payload channel relative to that target. Thus $F(2) = 0.903$, whereas $F(4) = 0.819$. At threshold $\theta = 0.86$, both snapshots pass but the four-occurrence aggregate fails. Local structural Eulericity has collapsed to Interval-Strong structural Eulericity, while quantum admissibility has not.

Supplementary Note S6 gives the exact rational calculation and states the independence and ideal-memory assumptions used in this physical non-collapse example.

Static aggregation and chronological channel composition are different tests

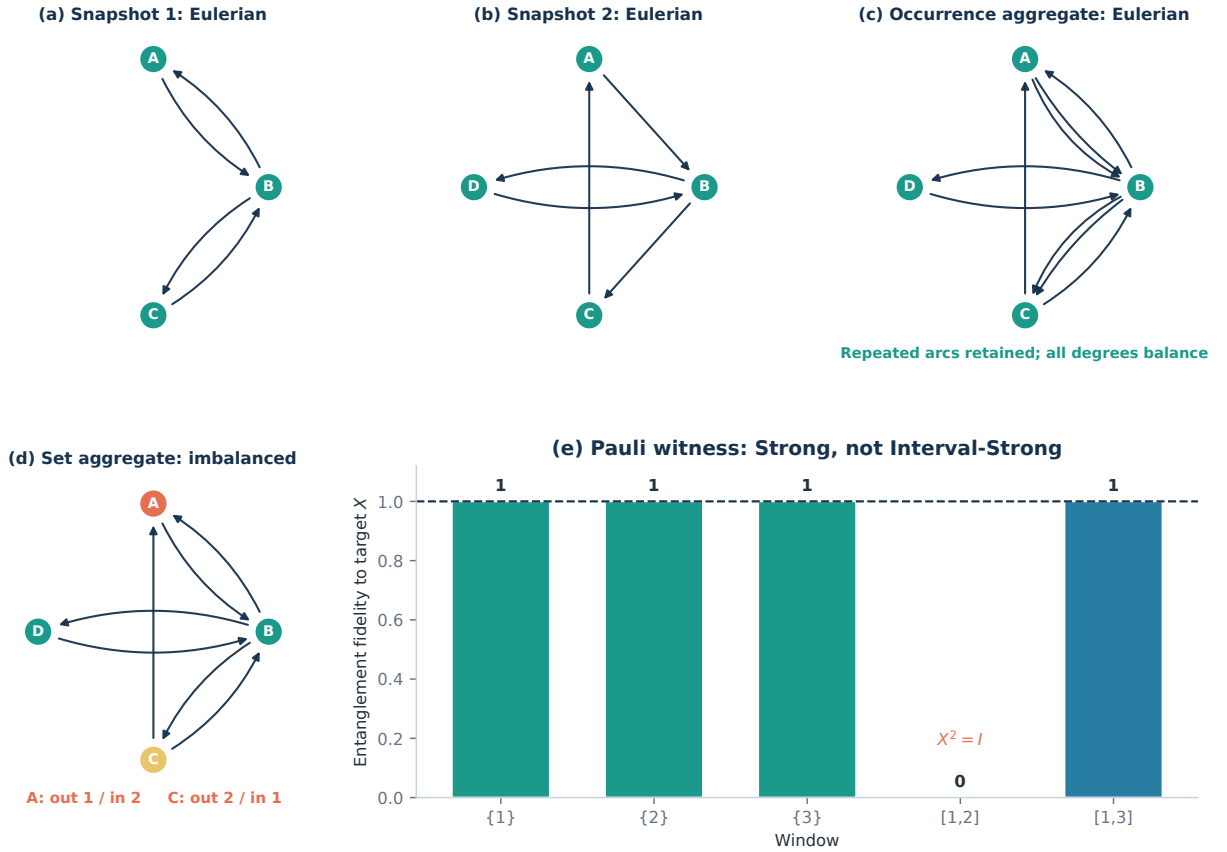


Figure 3. Aggregation and channel composition require separate checks. Panels (a) and (b) show two individually Eulerian snapshots. Panel (c) preserves repeated occurrences, so degrees remain balanced and the aggregate is Eulerian. Panel (d) retains only one copy of each ordered pair; this set aggregate is imbalanced at A and C. Panel (e) reports the Pauli-channel witness: singleton and three-cell tours implement X and pass, whereas the two-cell window implements I and has zero entanglement fidelity relative to the fixed target X. All panels are authors' constructions based on the definitions introduced in this study.

2.4 Complexity and analytical illustration

The structural and operational questions have different computational boundaries. Directed Eulericity of an explicitly given multigraph can be checked by degree balance and strong connectivity³⁵. Consequently, all four *structural* Euler robustness levels are polynomial-time decidable by checking at most $O(\tau^2)$ windows. Structural Hamiltonian variants are NP-complete: membership follows from one cycle certificate per relevant window, and hardness follows already at $\tau = 1$ from directed Hamiltonian cycle³⁴.

Time-respecting Euler coverage is not the same problem. Marino and Silva proved that undirected

temporal Eulerian walk and trail variants are NP-complete in general, including strong restrictions on lifetime and edge appearances³¹. This establishes a genuine complexity boundary for operational coverage, but it is not cited as a proof that every directed closed-tour variant is NP-hard. Each of the four quantum-constrained Hamiltonian robustness decisions is NP-complete when channel, success, and resource evaluators are polynomial-time computable: at $\tau = 1$, assign every directed opportunity the identity channel and the same interior time, so a directed Hamiltonian cycle has an immediate chronological realization; polynomial-size schedules establish membership in NP. Each of the four directed quantum-constrained Euler robustness decisions defined here is in NP under the same evaluator assumption, but no polynomial-time or NP-hardness conclusion follows merely from static Eulericity. Table 4 records only claims supported by these arguments.

Supplementary Note S7 supplies the corresponding certificate-size and reduction arguments.

Table 4. Verified computational boundary. Structural questions erase schedule order after constructing a window projection; operational questions require a non-decreasing temporal traversal.

Decision problem	Complexity	Reason
Structural Hamiltonian hierarchy	NP-complete	hardness at one time; polynomial-size cycle certificates for at most $O(\tau^2)$ windows
Structural Euler hierarchy	polynomial	directed balance and strong-connectivity test for each required occurrence multigraph
Temporal Euler walk/trail variants	NP-complete in general	established for the literature's undirected formulations; not an NP-hardness claim for every directed closed-tour variant
Quantum Hamiltonian hierarchy	NP-complete under polynomial evaluators	identity-channel reduction and verifiable schedules
Quantum Euler hierarchy	in NP under polynomial evaluators	schedule verification is polynomial; no general polynomial search claim is made

To show how physical parameters alter a structural certificate, we extend Eq. (5) by an explicitly declared depolarizing-memory and swap-visibility model:

$$F(m, w) = \frac{1}{4} + \frac{3}{4} q_0^m \exp\left(-\frac{w}{T_m}\right) s^{m-1}, \quad q_0 = \frac{4F_0 - 1}{3}, \quad (6)$$

where w is total waiting time, T_m is a memory time constant, and $s \in [0, 1]$ is an independently declared isotropic visibility factor for each of the $m - 1$ intermediate composition operations. Entanglement swapping is one possible interpretation when the protocol actually contains swaps; the factor is not intrinsic to every sequential payload traversal. Equation (6) is a transparent analytical model, not a universal law. More complete studies use density-matrix propagation, amplitude damping, purification, and measured device parameters^{14;16}.

Figure 4 evaluates Eq. (6) with $s = 1$. With threshold 0.85 and zero waiting, the maximum admissible link counts are 1, 3, and 8 for $F_0 = 0.90, 0.95$, and 0.98 , respectively. At $F_0 = 0.95$, even a modest normalized wait moves the admissibility boundary toward shorter certificates. The exact values are reported in Table 5; the script reproduces them without external data.

3 Discussion

The four-level hierarchy separates occasional availability from persistent and window-wise guarantees. Its value does not come from the forward implications, which follow from the definitions, but from the finite separations and from conditions under which repeated checks can be reduced. Fixed

Analytical fidelity landscape under the declared homogeneous teleportation-channel model

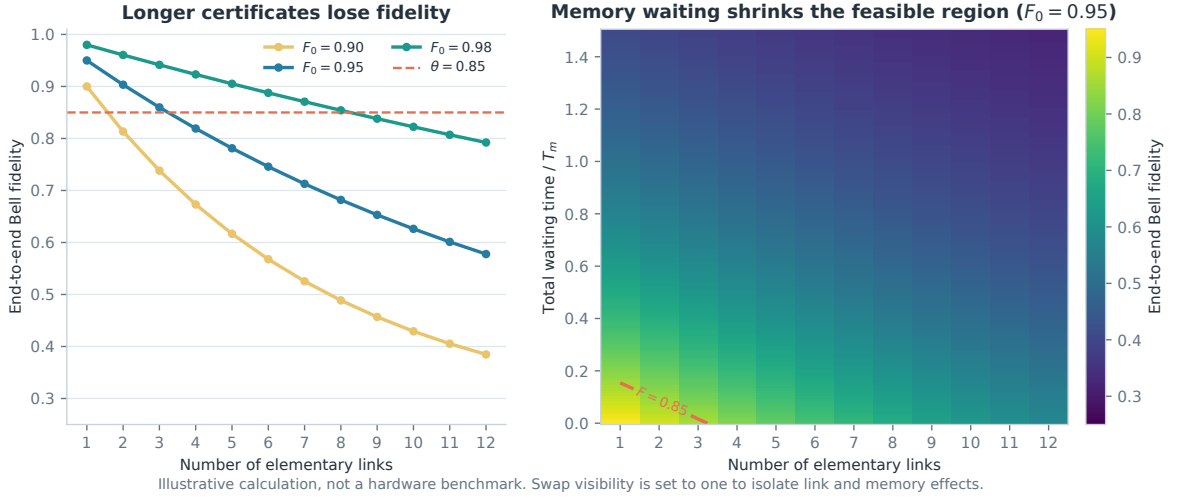


Figure 4. Certificate length and waiting time under the declared homogeneous teleportation-channel model. Left: end-to-end fidelity falls with the number of elementary links for three initial link fidelities. Right: for $F_0 = 0.95$, accumulated memory waiting further shrinks the region above $\theta = 0.85$. Perfect swap visibility ($s = 1$) is used to isolate link and memory effects. These curves are analytical illustrations and should not be read as device benchmarks.

Table 5. Zero-wait admissible lengths at $\theta = 0.85$. Values are computed from Eq. (6) with $w = 0$ and $s = 1$.

F_0	maximum m	$F(m)$	$F(m + 1)$
0.90	1	0.9000	0.8133
0.95	3	0.8598	0.8191
0.98	8	0.8542	0.8381

or nested live vertex sets are sufficient for a Hamiltonian collapse because one snapshot witness already covers the vertices of its window. Consecutive overlap at nonzero-degree vertices is sufficient for a structural Eulerian collapse only under snapshot-additive occurrence aggregation, because only then is interval degree balance exactly the sum of the singleton balances. Duration-spanning opportunities that are absent from all singleton projections must therefore be handled separately.

The quantum layer changes the interpretation of these topological statements. Fidelity is generally not preserved by making a tour longer, stationary noise does not imply a minimum-of-components fidelity rule, and local success cannot be concatenated without accounting for resources and heralding. The two-snapshot homogeneous teleportation-channel example makes this failure explicit. It is intentionally simple: imperfect Bell measurements, heterogeneous channels, purification, correlated errors, and queueing would require a protocol-specific density-matrix or discrete-event analysis^{10;14;17}. The present theorems remain valid because physical assumptions are isolated in $\text{Adm}(W)$, while the numerical claims are restricted to Eq. (6).

The structural projections should therefore be used as screening certificates. A support Hamiltonian cycle says that a sparse cyclic backbone is present somewhere in a time window; it does not say that a quantum payload can traverse it. An occurrence Eulerian circuit says that degree balance and connectivity permit complete opportunity coverage after time labels are erased; it does not replace temporal scheduling. The time-expanded or direct schedule check must follow.

An urban emergency network provides a cautious illustration. Hospitals, laboratories, and distribution depots may be linked by continuously available fibre and intermittent free-space or mobile links. The quantum layer would protect or coordinate classical emergency operations through key material or distributed quantum services; medical supplies themselves remain classical. A Hamiltonian certificate can test whether all live coordination nodes participate in a cyclic backbone,

whereas an Eulerian certificate can test whether a maintenance schedule covers every retained link opportunity. To claim practical performance, a future study would need measured link loss, memory coherence, operation durations, success probabilities, and demand traces. No such empirical claim is made here.

Four limitations delimit the conclusions. First, the five-state capability map is a declared abstraction and must be replaced when a device has different sleep or memory behaviour. Second, the model follows one logical payload or teleportation chain; simultaneous storage of several entangled subsystems, swapping dependencies, and purification queues require a larger state space. Third, a single scalar fidelity cannot capture every useful quantum resource; coherence, entanglement rate, and application-specific utility may add independent constraints¹⁴. Fourth, Hamiltonian and Eulerian certificates are deliberately interpretable but may be too rigid for redundancy-oriented routing. Future work should study multipair state tracking, k -coverage, disjoint temporal routes, purification policies, and uncertainty in opportunity times. Experiments or high-fidelity simulations would be appropriate only after a hardware scenario and its atomic time scale are fixed.

4 Methods

4.1 Formal construction and proof strategy

The construction is explicit and finite. Rational cell boundaries, node states, endpoint labels, opportunity types, physical start and completion times, channel descriptions, and thresholds are given as input. State-dependent capabilities are derived from Table 2. For each index window, support aggregation removes duplicate endpoint pairs, whereas occurrence aggregation retains every opportunity identity. Directed structural Eulericity is tested on the nonzero-degree vertices by equality of in- and out-degree and strong connectivity in both the graph and its transpose.

The structural strictness witnesses were built as directed finite graphs. Positive Hamiltonian certificates and the first two Eulerian separations are lifted with identity opportunity channels and identity memory during required waits. Quantum Eulerian Strong/Interval-Strong strictness is checked separately by ordered Pauli-channel composition. Collapse proofs use either vertex-set containment or additivity of multigraph degrees. The physical non-collapse uses Eq. (5) with an explicit threshold separating two- and four-occurrence tours. Full proofs, edge lists, schedule arguments, and boundary cases are supplied in the Supplementary Information.

4.2 Reproducible analytical calculation

The accompanying Python code exhaustively checks the small Hamiltonian instances, structural Euler balance and connectivity, chronological closed Euler schedules, state-dependent opportunity feasibility, unique ownership of cell-boundary instants, positive-duration completion semantics, and Pauli-channel products. It also evaluates Eqs. (5)–(6) and regenerates all four figures. Automated tests require the Pauli witness to be Local and Strong but not Interval-Strong, and require the original static Euler witness to fail chronological quantum Strong. The code writes JSON and CSV reports; all inputs are authors' constructions or deterministic model parameters.

Data availability

No empirical datasets were generated or analysed in this study. All constructed instances and parameter choices used in the theoretical demonstrations are reported in the article and its Supplementary Information.

Code availability

The reproducibility code supporting the constructed examples, automated validation checks, and figures is supplied with the manuscript as Supplementary Software. No public archive DOI is asserted here because the repository record must be confirmed before submission.

Author contributions

M.E.G. and M.B. contributed to the conception, formal development, verification, and writing of the manuscript. Both authors reviewed and approved the final manuscript.

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Ethics statement

This theoretical study did not involve human participants, animals, human tissue, or identifiable personal data; ethics approval and informed consent were therefore not applicable.

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