

# Supplementary Information

## Temporal robustness hierarchies distinguish structural and quantum-feasible traversals in living networks

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### S1 Conventions and relation to the main text

The horizon  $\mathcal{T} = \{1, \dots, \tau\}$  indexes the atomic cells defined in the main text: all cells are left-closed and right-open except the final cell, which is closed at the horizon endpoint. Every physical time therefore has a unique owning cell. Every interval is non-empty and contiguous, and its physical span is the union of its atomic cells. An instantaneous opportunity is included in a window only when its owning cell lies in the window; a positive-duration opportunity is included only when its closed start-to-completion interval lies in the window span. All state-dependent capability conditions must also hold. All graphs are finite and directed. We require at least two live vertices for a Hamiltonian certificate. A directed occurrence multigraph is Eulerian only when it is non-empty, its nonzero-degree vertices form one strongly connected component, and each has equal in- and out-degree<sup>1</sup>.

For a window  $I$ ,  $S[I]$  collapses repeated ordered endpoint pairs and  $O[I]$  retains each opportunity occurrence. Structural Hamiltonicity is evaluated on  $S[I]$ , and structural Eulericity on  $O[I]$ . A quantum-admissible certificate must additionally admit a closed chronological schedule  $W$  satisfying the fixed target, fidelity, success, and resource specification. The four levels are

$$\begin{aligned}\text{Weak}(P) &\iff \exists t \in \mathcal{T} : P(\{t\}), \\ \text{Local}(P) &\iff \forall t \in \mathcal{T} : P(\{t\}), \\ \text{Strong}(P) &\iff \text{Local}(P) \text{ and } P(\mathcal{T}), \\ \text{IntervalStrong}(P) &\iff \forall \text{ non-empty contiguous } I \subseteq \mathcal{T} : P(I).\end{aligned}$$

There are  $\tau(\tau + 1)/2$  intervals. Non-decreasing time is used throughout. The finite witnesses below use instantaneous opportunities, so their temporal order is unambiguous. Identity channels are used only where they genuinely preserve a chronological positive certificate.

### S2 Strict Hamiltonian hierarchy

**Theorem 1.** *For structural and quantum-admissible Hamiltonicity,*

$$\text{IntervalStrong} \implies \text{Strong} \implies \text{Local} \implies \text{Weak},$$

*and every converse fails.*

*Proof.* The forward implications follow directly from the definitions. For the converse failures, use the following finite instances.

*Weak does not imply Local.* Let  $\mathcal{T} = \{1, 2\}$ , with  $V_1 = V_2 = \{A, B, C\}$ . At time 1 take

$$E_1 = \{A \rightarrow B, B \rightarrow C, C \rightarrow A\},$$

and at time 2 take  $E_2 = \{A \rightarrow B\}$ . The first snapshot is Hamiltonian, whereas the second is not.

*Local does not imply Strong.* Let

$$\begin{aligned} V_1 &= \{A, B, C\}, & E_1 &= \{A \rightarrow B, B \rightarrow C, C \rightarrow A\}, \\ V_2 &= \{C, D, E\}, & E_2 &= \{C \rightarrow D, D \rightarrow E, E \rightarrow C\}. \end{aligned}$$

Each snapshot is Hamiltonian. In their support union, a cycle leaving  $A$  must follow  $A \rightarrow B \rightarrow C$ . Closing  $C \rightarrow A$  omits  $D, E$ , while continuing  $C \rightarrow D \rightarrow E \rightarrow C$  repeats  $C$  instead of returning to  $A$ . Hence the aggregate is not Hamiltonian.

*Strong does not imply Interval-Strong.* Retain the preceding snapshots and add

$$V_3 = \{A, D, E\}, \quad E_3 = \{A \rightarrow D, D \rightarrow E, E \rightarrow A\}.$$

All snapshots are Hamiltonian. The whole support contains  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ , but the window  $[1, 2]$  is the preceding non-Hamiltonian aggregate.

For the quantum versions, assign the identity channel, unit success, and zero opportunity cost to every opportunity; take every live node to be Active and every memory channel used during an inter-cell wait to be the identity with zero memory cost. The positive singleton certificates are chronological. In the final witness, the whole-horizon cycle uses opportunity times 1, 1, 2, 2, 3, so it too is chronological, and the waits do not alter the payload channel. The structural failures remain failures. Thus the same three instances establish the quantum Hamiltonian separations.  $\square$

Table 1: Hamiltonian separation instances. A semicolon separates snapshots.

| Separation                      | Snapshot arc sets                    |
|---------------------------------|--------------------------------------|
| <b>Weak / Local</b>             | $AB, BC, CA; AB$                     |
| <b>Local / Strong</b>           | $AB, BC, CA; CD, DE, EC$             |
| <b>Strong / Interval-Strong</b> | $AB, BC, CA; CD, DE, EC; AD, DE, EA$ |

### S3 Strict Eulerian hierarchy

**Theorem 2.** *Under occurrence aggregation, both the structural and quantum-admissible Eulerian hierarchies are strict. The structural Strong/Interval-Strong witness and the corresponding quantum witness are different constructions.*

*Proof.* The forward implications are definitional.

*Weak does not imply Local.* At time 1 use the directed cycle  $A \rightarrow B \rightarrow C \rightarrow A$ . At time 2 retain the three vertices but only  $A \rightarrow B$ . The first snapshot is Eulerian and the second is degree imbalanced. Identity opportunity channels, Active live nodes, and identity memory during any required wait lift the positive snapshot to the quantum property.

*Local does not imply Strong.* At time 1 use the directed 3-cycle on  $\{A, B, C\}$ , and at time 2 the directed 3-cycle on the disjoint set  $\{D, E, F\}$ . Both snapshots are Eulerian, but the occurrence aggregate is disconnected. Identity opportunity channels, Active live nodes, and identity memory during any required wait lift the local positive certificates, while no whole-horizon structural or quantum Euler certificate exists.

*Structural Strong does not imply structural Interval-Strong.* Retain the two disjoint cycles and add at time 3

$$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow A.$$

The whole occurrence aggregate is strongly connected and each vertex has in- and out-degree two. The interval  $[1, 2]$  remains the disjoint union of two cycles. This construction is a structural witness only: its first two cycles cannot be joined by a chronological tour before time 3.

*Quantum Strong does not imply quantum Interval-Strong.* Let  $\mathcal{T} = \{1, 2, 3\}$  and  $V_t = \{A, B\}$  for all  $t$ . In each cell place one occurrence of  $A \rightarrow B$  with identity channel  $\mathcal{I}$  and one occurrence of  $B \rightarrow A$  with Pauli channel  $\mathcal{X}(\rho) = X\rho X$ , both at the same interior instant of that cell. All opportunities have unit success and zero resource cost. Take both nodes Active throughout and set every memory channel used between cells to the identity with zero memory cost. Fix the target channel  $\mathcal{U} = \mathcal{X}$ ,  $\theta = 1$ ,  $\pi = 1$ , and  $B = 0$ .

Because the two opportunities are simultaneous under the non-decreasing-time convention and  $\mathcal{I}$  commutes with  $\mathcal{X}$ , each cell-cycle has composite channel  $\mathcal{X}$ , regardless of whether its closed tour is written from  $A$  or  $B$ . Hence every singleton is quantum-admissible. A chronological whole-horizon Euler tour concatenates three such cycles; the positive inter-cell waits contribute only identity memory channels, so the resulting channel is  $\mathcal{X}^3 = \mathcal{X}$  and the instance is Strong. Every chronological tour of  $[1, 2]$  must consume the two occurrences in each cell before moving forward in time and therefore implements  $\mathcal{X}^2 = \mathcal{I}$ . For qubits,

$$F_e(\mathcal{I}; \mathcal{X}) = \frac{|\text{Tr}(X^\dagger \mathcal{I})|^2}{4} = 0 < \theta.$$

Thus  $[1, 2]$  fails and the instance is not Interval-Strong. □

Table 2: Channel products for the quantum Eulerian Strong/Interval-Strong witness.

| Window                | Composite channel | $F_e(\cdot; \mathcal{X})$ | Outcome |
|-----------------------|-------------------|---------------------------|---------|
| $\{1\}, \{2\}, \{3\}$ | $\mathcal{X}$     | 1                         | Pass    |
| $[1, 2]$              | $\mathcal{I}$     | 0                         | Fail    |
| $[2, 3]$              | $\mathcal{I}$     | 0                         | Fail    |
| $[1, 3]$              | $\mathcal{X}$     | 1                         | Pass    |

Table 3: Complete summary of the strictness witnesses. “Identity lift” means that every positive structural certificate has an explicitly chronological realization with identity opportunity channels, identity memory during required waits, unit success, and zero cost.

| Property                    | Separation        | Finite witness                                                                       | Exact reason for failure                                                                                       |
|-----------------------------|-------------------|--------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------|
| <b>Hamiltonian</b>          | Weak / Local      | 3-cycle; then one arc                                                                | second singleton has no spanning cycle; identity lift applies                                                  |
| <b>Hamiltonian</b>          | Local / Strong    | cycles on $ABC$ and $CDE$                                                            | whole support has no cycle through all five vertices; identity lift applies                                    |
| <b>Hamiltonian</b>          | Strong / Interval | add cycle on $ADE$                                                                   | whole support has a 5-cycle, but $[1, 2]$ does not; identity lift applies                                      |
| <b>Eulerian</b>             | Weak / Local      | directed 3-cycle; then one arc                                                       | second singleton is degree imbalanced; identity lift applies                                                   |
| <b>Eulerian</b>             | Local / Strong    | disjoint cycles on $ABC$ and $DEF$                                                   | whole occurrence aggregate is disconnected; identity lift applies                                              |
| <b>Eulerian, structural</b> | Strong / Interval | add a whole-set cycle at time 3                                                      | whole occurrence aggregate is balanced and connected, but $[1, 2]$ is disconnected; no quantum lift is claimed |
| <b>Eulerian, quantum</b>    | Strong / Interval | three $A \rightarrow B \rightarrow A$ cells with $\mathcal{I}, \mathcal{X}$ channels | $[1, 2]$ implements $\mathcal{I}$ and has $F_e = 0$ relative to target $\mathcal{X}$                           |

## S4 Collapse proofs and aggregation boundary

**Theorem 3** (Fixed or nested live vertices). *If  $V_t = V$  for every  $t$ , Local Hamiltonicity implies Interval-Strong Hamiltonicity under support aggregation. The same conclusion holds when  $V_t \subseteq V_{t+1}$  for all  $t$ . Both statements hold structurally and quantum-admissibly.*

*Proof.* Fix  $I = [a, b]$ . Under a fixed vertex set, any local cycle in the interval visits  $V = V_I$ , and all of its opportunities belong to the interval support. Under nested vertex sets,  $V_I = V_b$ ; the local cycle at  $b$  visits exactly this union and lies inside the window. For the quantum versions, use the same local chronological schedule. Enlarging the containing window does not alter its channel, success probability, or resource cost. Because the physical specification is fixed across windows, admissibility is unchanged.  $\square$

**Theorem 4** (Snapshot-additive structural Euler collapse). *For each  $t$ , let  $U_t$  contain exactly the vertices of nonzero degree in the snapshot occurrence multigraph. Suppose occurrence aggregation is snapshot-additive,  $O[I] = \uplus_{t \in I} O[\{t\}]$  for every non-empty contiguous interval  $I$ ; in particular, this condition holds when every retained opportunity is contained in one atomic cell. If every snapshot occurrence multigraph is non-empty and directed Eulerian and  $U_t \cap U_{t+1} \neq \emptyset$  for consecutive times, then every non-empty interval occurrence aggregate is structurally Eulerian.*

*Proof.* Fix  $I = [a, b]$ . By the snapshot-additivity assumption, occurrence degrees add:

$$\deg_{O[I]}^{\pm}(v) = \sum_{t=a}^b \deg_{O[\{t\}]}^{\pm}(v).$$

Each summand pair is balanced, so the aggregate is balanced. Every snapshot is strongly connected on  $U_t$ , and consecutive nonzero-degree sets share a vertex. These components therefore form an overlapping chain. Paths can be concatenated through the overlaps in both directions, so the union is strongly connected on all nonzero-degree vertices. It is consequently Eulerian.  $\square$

**Remark 5.** The use of nonzero-degree sets, rather than live vertex sets, is essential. A shared isolated live vertex cannot connect two Eulerian components. Snapshot additivity is also essential under the general duration semantics of the main text. For example, let both singleton snapshots contain the directed 2-cycle  $A \rightarrow B \rightarrow A$ , and add one positive-duration opportunity  $A \rightarrow B$  whose start lies in cell 1 and completion in cell 2. That opportunity belongs to the two-cell window but to neither singleton. Both singleton occurrence multigraphs are Eulerian, whereas the two-cell occurrence multigraph has one extra  $A \rightarrow B$  occurrence and is degree imbalanced. Thus the displayed degree sum and the collapse conclusion fail without the additivity assumption. The result is structural: chronological scheduling and channel admissibility still require separate checks.

**Proposition 6** (Set aggregation can destroy Eulericity). *Two strongly connected directed Eulerian snapshots with overlapping vertices can have a non-Eulerian set aggregate.*

*Proof.* Take

$$\begin{aligned} E_1 &= \{A \rightarrow B, B \rightarrow A, B \rightarrow C, C \rightarrow B\}, \\ E_2 &= \{A \rightarrow B, C \rightarrow A, B \rightarrow C, B \rightarrow D, D \rightarrow B\}. \end{aligned}$$

The first snapshot consists of two directed 2-cycles sharing  $B$ . In the second,  $A \rightarrow B \rightarrow C \rightarrow A$  is a directed 3-cycle and  $B \rightarrow D \rightarrow B$  a directed 2-cycle. Hence both snapshots are strongly connected and balanced. In the set union, repeated  $A \rightarrow B$  and  $B \rightarrow C$  are retained once. Vertex  $A$  has out-degree one and in-degree two, whereas  $C$  has out-degree two and in-degree one. The set aggregate is not Eulerian.  $\square$

## S5 Capability and duration boundary cases

**Proposition 7** (Living-state dependence). *The existence of a chronological single-payload schedule can change when only the intermediate node state changes.*

*Proof.* Let a type-D opportunity carry the payload from  $A$  to  $B$  in cell 1 and a second carry it from  $B$  to  $C$  in cell 3. The payload waits at  $B$  during cell 2. If  $B$  is Sleep in cell 2, the capability vector is  $(0, 0, 1)$ , so the payload is retained and the schedule is feasible once  $B$  becomes Active in cell 3. Replacing Sleep by Offline sets memory to zero and invalidates the wait. Independently, RxOff at the first completion invalidates reception, and TxOff at the second start invalidates transmission.  $\square$

**Proposition 8** (Discrete/continuous window and boundary dependence). *Under the unique-cell convention, an instantaneous opportunity on an interior cell boundary belongs only to the cell on its right, and a positive-duration opportunity may belong to a multi-cell window without belonging to either singleton.*

*Proof.* Let  $\zeta_1$  separate cells 1 and 2. Since  $C_1 = [\zeta_0, \zeta_1)$  and  $\zeta_1 \in C_2$ , an instantaneous type-D opportunity at  $\zeta_1$  belongs to singleton  $\{2\}$  and not to  $\{1\}$ . Next let a type-C opportunity begin in the interior of cell 1 and complete in the interior of cell 2. Its closed occupied interval is contained in  $H([1, 2])$  but in neither singleton span. The same conclusion holds when it completes exactly at  $\zeta_1$ : the completion point is owned by cell 2, so the opportunity is not contained in  $H(\{1\})$ . This is a conservative completion-within-window convention, not an active-overlap convention.  $\square$

## S6 Physical non-collapse under homogeneous teleportation-channel composition

**Proposition 9.** *Local quantum-admissible Eulericity need not imply Strong quantum-admissible Eulericity, even when the occurrence aggregate is structurally Eulerian and all consecutive nonzero-degree vertex sets coincide.*

*Proof.* Let  $\mathcal{T} = \{1, 2\}$  and  $V_1 = V_2 = \{A, B\}$ . At each time there is one occurrence of  $A \rightarrow B$  and one of  $B \rightarrow A$ , with all four opportunities contained in their respective atomic cells. Every snapshot is a directed Eulerian 2-cycle, and the snapshot-additive occurrence aggregate contains two parallel copies in each direction.

Model each successful opportunity by standard teleportation using the two-qubit Bell-diagonal isotropic resource

$$\rho(F_0) = F_0 |\Phi^+\rangle\langle\Phi^+| + \frac{1-F_0}{3} \sum_{\beta \neq \Phi^+} |\beta\rangle\langle\beta|,$$

where  $F_0$  is the fully entangled fraction with respect to  $|\Phi^+\rangle$ . Under the standard teleportation-channel correspondence, its conditional payload channel is depolarizing with  $q_0 = (4F_0 - 1)/3^2$ . Assume independent opportunity channels, unit joint success, and identity memory with zero memory cost during the positive wait between the two cell cycles. The memory therefore does not change the payload channel, and the opportunity-channel visibilities multiply, giving

$$F(m) = \frac{1}{4} + \frac{3}{4} q_0^m.$$

For the closed round trip we fix the ideal payload target to  $\mathcal{U} = \mathcal{I}$ ; hence  $F(m)$  is the entanglement fidelity of the composed payload channel relative to that target. This is a declared homogeneous model, not a statement about arbitrary channels.

Set  $F_0 = 0.95 = 19/20$ , so  $q_0 = 14/15$ . A snapshot tour has two traversals and

$$F(2) = \frac{1}{4} + \frac{3}{4} \left( \frac{14}{15} \right)^2 = 0.903\bar{3}.$$

Every aggregate tour uses four occurrences and has

$$F(4) = \frac{1}{4} + \frac{3}{4} \left( \frac{14}{15} \right)^4 \approx 0.819126.$$

At  $\theta = 0.86$ , with non-binding success and resource constraints, both singleton tours pass and the four-occurrence tour fails. Thus the structural collapse does not imply a quantum collapse.  $\square$

## S7 Complexity statements

**Theorem 10** (Structural Hamiltonian hierarchy). *If  $\tau$  and all snapshots are explicitly represented, the Weak, Local, Strong, and Interval-Strong structural Hamiltonian decisions are NP-complete.*

*Proof.* A certificate lists one Hamiltonian cycle for every required window. There are at most  $O(\tau^2)$  windows and each cycle has polynomial size, so membership in NP follows. With  $\tau = 1$ , all four decisions coincide with directed Hamiltonian cycle, giving NP-hardness<sup>3</sup>.  $\square$

**Theorem 11** (Structural Euler hierarchy). *The four structural Eulerian robustness decisions are polynomial-time solvable under occurrence aggregation.*

*Proof.* For every required window, construct the occurrence multigraph and test degree balance and strong connectivity on its nonzero-degree vertices. These operations are linear in the window multigraph size, and there are at most  $O(\tau^2)$  windows<sup>1</sup>.  $\square$

**Proposition 12** (Operational boundary). *The polynomial structural Euler test does not supply a polynomial algorithm for all chronological Euler coverage problems. General undirected temporal Eulerian walk and trail decisions are NP-complete. A quantum-admissible Hamiltonian decision is NP-complete under polynomial-time physical evaluators, while a proposed quantum-admissible Euler schedule is polynomially verifiable under the same assumption.*

*Proof.* Marino and Silva establish NP-completeness for undirected temporal Eulerian walk and trail variants<sup>4</sup>. Their result retains temporal order and is not used as a hardness proof for every directed closed-tour formulation. For each of the four quantum Hamiltonian robustness decisions, membership in NP follows by verifying the required polynomial-size list of schedules. NP-hardness holds already at  $\tau = 1$ : represent every directed input edge by an identity-channel opportunity at the same interior instant, with unit success and zero cost, so a directed Hamiltonian cycle is chronological if and only if the static graph has one. For each of the four quantum Euler robustness decisions, a polynomial-size list of proposed schedules can likewise be verified in polynomial time, placing these decisions in NP without implying a polynomial search algorithm or NP-hardness.  $\square$

## S8 Analytical values and verification procedure

Under zero waiting and perfect swap visibility,

$$F(m) = \frac{1}{4} + \frac{3}{4} q_0^m, \quad q_0 = \frac{4F_0 - 1}{3}.$$

For  $\theta > 1/4$  and  $0 < q_0 < 1$ , admissible integer link counts satisfy

$$m \leq \left\lfloor \frac{\log((4\theta - 1)/3)}{\log q_0} \right\rfloor.$$

Both logarithms are negative. At  $\theta = 0.85$ , the deterministic values are

| $F_0$ | $q_0$     | maximum $m$ | $F(m)$    | $F(m + 1)$ |
|-------|-----------|-------------|-----------|------------|
| 0.90  | 0.8666667 | 1           | 0.9000000 | 0.8133333  |
| 0.95  | 0.9333333 | 3           | 0.8597778 | 0.8191259  |
| 0.98  | 0.9733333 | 8           | 0.8541629 | 0.8380519  |

The reproducibility code enumerates the small structural certificates, checks closed chronological adjacency and non-decreasing start/completion times, applies the state-capability map, verifies unique ownership of boundary instants and completion-within-window semantics, multiplies the Pauli unitary matrices, and evaluates the homogeneous teleportation-channel expressions. Automated tests fail if the Pauli instance is not Local and Strong but not Interval-Strong, or if the static three-cycle Euler witness is incorrectly accepted as a chronological whole-horizon tour. No random numbers or external data are used.

## References

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