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A STIMULATED ARRIVAL MULTI-SERVER MARKOVIAN QUEUE WITH DIFFERENTIATED VACATIONS

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Abstract

This paper investigates a multi-channel Markovian queueing system with stimulated arrivals and differentiated server vacations. The model is a multi-channel system in which the arrival pattern is modelled as a Poisson process with state and load-dependent rates, and service times are exponentially distributed. Servers remain active while customers are present, but switch to a primary vacation mode when the system has no customers. If no customers are found upon return, a secondary vacation begins; otherwise, service resumes immediately. Primary and secondary vacation times are independent and exponentially distributed. Arrival rates are adjusted by state-dependent encouragement factors during congestion, ensuring arrivals continue even when all servers are busy. The joint process of customer count and server state is modelled as Markov process in continuous time. The long run probabilities were calculated using the Probability Generating Function method. Both quantitative examples and performance measures are obtained, providing insights into realistic service systems under heterogeneous operating conditions.

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1.Introduction

Queueing theory is a technique for examining and enhancing procedures by locating and resolving delays-causing bottlenecks. In real-world congestion systems like telecommunications, service and distribution centres, manufacturing, and production systems, queueing theory has a significant applicability.

The concept of "stimulated arrivals" originated from a system's experience accompanying the start of deals and discounts by organisations. A recent contribution to the current understanding of consumer behaviour in queueing theory is stimulated arrivals. This facilitates the creation of policies by the organisations to attract new clients. Additionally, consumers continue to search for profitable offers and discounts from various organisations, which induces them to visit the particular organisation drawn by incentives may be more patient, less likely to balk or more likely remain longer in the system to fully utilize the offer. We refer to these enticed customers as stimulated arrivals.

Numerous studies have examined queueing systems characterized by stimulated arrivals and working breakdowns. In most analyses, it is inferred that the server terminates service entirely throughout the duration of an unplanned breakdown. Kalidass and Kasturi [8] investigated a breakdown policy that allows customer service to proceed at a diminished rate during periods when the system is partially defective. Deepa and Kalidass [4] provided a well-established formulation of the queueing model where servers experience outages and vacations. In their work, Deepa, Kalidass, and Vijayalakshmi [5] examined a capacity-constrained M/M/1 model that incorporates two types of server experiences downtime and operational breakdowns. By applying the direct search method, they performed a sensitivity analysis of the model, focusing on optimizing service charges by lowering the mean expenditure per unit time.

The stationary behaviour of single-server queueing systems with Markovian bulk arrivals has been investigated by Deepa, Kalidass [4] with subject to interruptions and malfunctions. They assessed the model's relevance to a cost optimization problem, aiming to optimize the mean customer count by applying a direct search strategy to reduce the average cost per unit hour.

A study by Vijayalakshmi, Kalidass, and Deepa [5] focused on Markovian queueing model with finite capacity incorporating two-phase service and system breakdowns. In this case, the server offers both the primary and secondary elective services, as well as cost optimisation through direct search to maximise service rates. Bhupender Kumar Som [10] proposed a cost model and carried out an economic evaluation of a Markovian single-channel queueing system under stimulated arrivals.

The multi-server queueing paradigm with idle intervals was first introduced by Levy and Yechiali [9]. By analysing the multichannel queueing model with asynchronous vacations, the predicted number of customers and the stationary distribution of busy servers were derived.

Shakir Majid and Manoharan [2] took into consideration the multichannel queueing system operating under synchronous working vacations, with both single and multiple working vacation policies. To show how different factors affect system performance metrics based on these two indices, they presented a comparative analysis of the models supported by numerical data. By studying impatient consumers in a multi-server working vacation queue, Manoharan and Majid [2] obtained clear expressions of performance measures and their stochastic decomposition. An $M^x/M/1$ queue with differentiated vacations was analyzed and considered by Vijaya.K and Deepa.B [6].

Vijayshree K.V., Janani.B, and Ambine.K [7] analyzes an $M/M/1$ queueing system featuring differentiated vacations and interruptions. The study likely investigates steady-state performance metrics, including queue length and waiting time., under the specific operational constraints, incorporating interrupted service and vacation periods. Yohapriyadharsini R.S. and Suvitha [1] considered a queue with heterogeneous arrivals and two working vacation policies under customer impatience and examined using PGF.

The structure of the paper is organized as follows. Section 2 presents the model in detail. Section 3 explains the steady-state analysis of the proposed model. The stability condition, reliability measures, and performance measures are derived in Sections 4 and 5. Sections 6 and 7 provide practical applications and illustrative numerical examples to highlight the effects of the server. Section 8 discusses a cost optimization problem, and Section 9 concludes with final remarks.

2.Model Description

- We consider a queueing scenario with multi-server queueing model (c servers) with Poisson arrivals and exponential service times that handle customers arriving and being served at random rates. The system specifically models situations where a server might partially fail but keep working and are also subject to unscheduled maintenance.
- Under normal conditions, the server operates at its standard service rate when customers are present in the queue. Once the system is empty, the server departs for a primary vacation of fixed duration.
- If the server returns from a primary vacation and finds no customers, it departs for a secondary vacation. After the secondary vacation, the server resumes service if customers are present; otherwise, it waits for new arrivals.
- Customer arrivals are governed by Poisson process with heterogenous arrival rate λ_i for $0 \leq i \leq c$, and $\lambda_i \beta_i$ for $i > c$, where

$$\lambda_i = \begin{cases} \lambda_0, & \text{arrival rate in normal state} \\ \lambda_1, & \text{arrival rate during primary vacation state} \\ \lambda_2, & \text{arrival rate during secondary vacation state} \end{cases}$$

$$\lambda_i \beta_i = \begin{cases} \lambda_0 \beta_0, & \text{arrival rate in normal state} \\ \lambda_1 \beta_1, & \text{arrival rate during primary vacation state} \\ \lambda_2 \beta_2, & \text{arrival rate during secondary vacation state} \end{cases}$$

- When customers arrive and find all servers busy, the system continues to receive additional arrivals depending on its current operational state.

Let: β_0 denote the arrival intensity modifier during the normal operational state.

β_1 denote the arrival intensity modifier during the primary vacation state.

β_2 denote the arrival intensity modifier during the secondary state.

These modifiers scale the base arrival rates λ_0 , λ_1 , and λ_2 respectively, resulting in effective arrival rates of $\lambda_0\beta_0$, $\lambda_1\beta_1$, and $\lambda_2\beta_2$ for states where the number of customers exceeds the number of servers ($i > c$).

- The service times of each server are assumed to be independent and identically distributed, following an exponential distribution with rate μ .
- The primary vacation rate γ_1 and secondary vacation rate γ_2 are modeled as independent and identically distributed exponential random variables.
- Let $X(t)$ be the number of customers in the system at time t , and $C(t)$ represents the state of server at time t , where

$$C(t) = \begin{cases} 0, & \text{the server is in the normal state at time } t \\ 1, & \text{the server is in the primary vacation state at time } t \\ 2, & \text{the server is in the secondary vacation state at time } t \end{cases}$$

Then $\{C(t), X(t), t \geq 0\}$ is considered a continuous time Markov Chain.

Let $P_{i,n}(t) = P\{C(t) = i, X(t) = n\}$, $n \geq 0$, $i = 0, 1, 2$

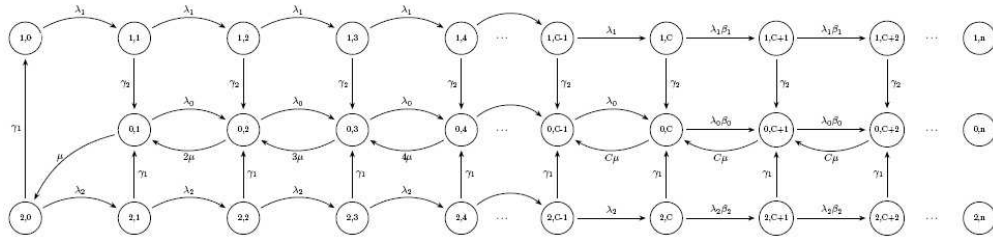


Figure 1. Transient State Diagram

3. Analysis of steady states

The system yields the following steady-state balance equations

$$\lambda_0 P_{0,1} + \mu P_{0,1} = \gamma_1 P_{2,1} + 2\mu P_{0,2} + \gamma_2 P_{1,1}, \quad n = 1 \quad (1)$$

$$(\lambda_0 + n\mu)P_{0,n} = \gamma_1 P_{2,n} + \lambda_0 P_{0,n-1} + (n+1)\mu P_{0,n+1} + \gamma_2 P_{1,n}, \quad 2 \leq n \leq c-1 \quad (2)$$

$$(\lambda_0\beta_0 + n\mu)P_{0,n} = \gamma_1 P_{2,n} + \lambda_0 P_{0,n-1} + n\mu P_{0,n+1} + \gamma_2 P_{1,n}, \quad n = c \quad (3)$$

$$(\lambda_0\beta_0 + c\mu)P_{0,n} = \gamma_1 P_{2,n} + \lambda_0\beta_0 P_{0,n-1} + c\mu P_{0,n+1} + \gamma_2 P_{1,n}, \quad n \geq c+1 \quad (4)$$

$$\lambda_1 P_{1,0} = \gamma_1 P_{2,0}, \quad n = 0 \quad (5)$$

$$(\lambda_1 + \gamma_2)P_{1,n} = \lambda_1 P_{1,n-1}, \quad 1 \leq n \leq c-1 \quad (6)$$

$$(\lambda_1\beta_1 + \gamma_2)P_{1,n} = \lambda_1 P_{1,n-1}, \quad n = c \quad (7)$$

$$(\lambda_1\beta_1 + \gamma_2)P_{1,n} = \lambda_1\beta_1 P_{1,n-1}, \quad n \geq c+1 \quad (8)$$

$$\gamma_1 P_{2,0} + \lambda_2 P_{2,0} = \mu P_{0,1}, \quad n = 0 \quad (9)$$

$$(\lambda_2 + \gamma_1)P_{2,n} = \lambda_2 P_{2,n-1}, \quad 1 \leq n \leq c-1 \quad (10)$$

$$(\lambda_2\beta_2 + \gamma_1)P_{2,n} = \lambda_2 P_{2,n-1}, \quad n = c \quad (11)$$

$$(\lambda_2\beta_2 + \gamma_1)P_{2,n} = \lambda_2\beta_2 P_{2,n-1}, \quad n \geq c+1 \quad (12)$$

The PGF is given by:

$$G_0(z) = \sum_{n=1}^{\infty} P_{0,n} z^n, \quad G_1(z) = \sum_{n=0}^{\infty} P_{1,n} z^n, \quad G_2(z) = \sum_{n=0}^{\infty} P_{2,n} z^n,$$

- $R_1(z) = \sum_{n=1}^{c-1} (n-c) P_{0,n} z^n$
- $R_2(z) = \sum_{n=1}^{c-1} P_{0,n} z^n$
- $R_3(z) = \sum_{n=0}^{c-1} (n-c) P_{1,n} z^n$
- $R_4(z) = \sum_{n=0}^{c-1} P_{1,n} z^n$
- $R_5(z) = \sum_{n=0}^{c-1} (n-c) P_{2,n} z^n$
- $R_6(z) = \sum_{n=0}^{c-1} P_{2,n} z^n$

Solving the earlier equations with generating function of a probability distribution we obtain:

- $\left(c\mu \left(1 - \frac{1}{z} \right) + (1-z)\lambda_0 \beta_0 \right) G_0(z) + R_2(z)(1-z)(\lambda_0 + \lambda_0 \beta_0) +$
 $\mu R_1(z) \left(1 - \frac{1}{z} \right) - \gamma_1 (G_2(z) - P_{2,0}) - \gamma_2 (G_1(z) - P_{1,0}) + \frac{\mu}{z} P_{0,1} = 0.$
 - $\lambda_1 (1-z) R_4(z) + \gamma_2 (G_1(z) - P_{1,0}) + \lambda_1 \beta_1 (1-z)(G_1(z) - R_4(z)) - \gamma_1 P_{2,0} = 0$
 - $\lambda_2 (1-z) R_6(z) + \gamma_1 G_2(z) + \lambda_2 \beta_2 (1-z)(G_2(z) - R_6(z)) - \mu P_{0,1} = 0$
- which in turn yields

- $G_0(z) = \frac{H(z)}{A_1(z)A_2(z) \left(c\mu \left(1 - \frac{1}{z} \right) + (1-z)\lambda_0 \beta_0 \right)}$
- $G_1(z) = \frac{\gamma_2 P_{1,0} + \gamma_1 P_{2,0} + \lambda_1 \beta_1 (1-z) R_4(z)}{\lambda_2 \beta_2 (1-z) + \gamma_1}$
- $G_2(z) = \frac{\mu P_{0,1} + \lambda_2 \beta_2 (1-z) R_6(z)}{\lambda_2 \beta_2 (1-z) + \gamma_1}$

$$\therefore G(z) = G_0(z) + G_1(z) + G_2(z)$$

where

- $A_1(z) = \lambda_1 \beta_1 (1-z) + \gamma_2$
- $A_2(z) = \lambda_2 \beta_2 (1-z) + \gamma_1$
- $H(z) = [(\lambda_0 \beta_0 - \lambda_0)(1-z)R_2(z) - \gamma_1 P_{2,0} - \gamma_2 P_{1,0} - \mu \left(1 - \frac{1}{z} \right) R_1(z)] A_1(z) A_2(z)$
 $+ \gamma_2 A_2(z) [\gamma_2 P_{1,0} + \gamma_1 P_{2,0} + (\lambda_1 \beta_1 - \lambda_1) (1-z) R_4(z)]$
 $+ \gamma_1 A_1(z) [\mu P_{0,1} + (\lambda_2 \beta_2 - \lambda_2) (1-z) R_6(z)]$
- $B_1 = \gamma_1 \gamma_2 (c\mu - \lambda_0 \beta_0)$

Letting $z = 1$, the equations reduce to the following form

- $G_0(1) = \frac{H(1)}{B_1}$
- $G_1(1) = \frac{\gamma_2 P_{1,0} + \gamma_1 P_{2,0}}{\gamma_2}$
- $G_2(1) = \frac{\mu P_{0,1}}{\gamma_1}$
- $R_1(1) = \sum_{n=1}^{c-1} (n-c) \frac{\lambda_0^{n-1}}{n! \mu^n} P_{0,n}$
- $R_2(1) = \sum_{n=1}^{c-1} \frac{\lambda_0^{n-1}}{n! \mu^n} P_{0,n}$
- $R_3(1) = \sum_{n=0}^{c-1} (n-c) \lambda_1^n P_{1,n}$
- $R_4(1) = \sum_{n=0}^{c-1} \lambda_1^n P_{1,n}$
- $R_5(z) = \sum_{n=0}^{c-1} (n-c) \lambda_2^n P_{2,n}$
- $R_6(z) = \sum_{n=0}^{c-1} \lambda_2^n P_{2,n}$

By applying the normalization condition, the equations are solved to obtain the results along with their vertical cuts we get,

$$G_0(1) + G_1(1) + G_2(1) = 1$$

we obtain the probability measures

- $P_{0,1} = \frac{\lambda_1 \gamma_1 \gamma_2 (\gamma_1 + \lambda_2) (c\mu - \lambda_0 \beta_0)}{[\mu \gamma_1^2 [\lambda_1^2 \beta_1 + \gamma_2 (c\mu - \lambda_0 \beta_0)] + \mu \gamma_2 \gamma_1^2 (\lambda_1 - \lambda_1 \beta_1) + \mu \lambda_1 \gamma_1 \gamma_2 [-\lambda_2 + (c\mu - \lambda_0 \beta_0)] + \lambda_1 \mu (c\mu - \lambda_0 \beta_0) (\gamma_1^2 + \gamma_2 \lambda_2) + \lambda_1 \lambda_2 \beta_2 \gamma_1 (\lambda_2 + \gamma_1)] - [\lambda_1 \gamma_1 \gamma_2 (\lambda_2 + \gamma_1)] [(\lambda_0 \beta_0 - \lambda_0) \sum_{n=1}^{c-1} \frac{\lambda_0^n}{n! \mu^n} + (\lambda_1 \beta_1 - \lambda_1) \frac{\gamma_2 \mu}{\lambda_1 (\lambda_2 + \gamma_1)} \sum_{n=0}^{c-1} \lambda_1^n + (\lambda_2 \beta_2 - \lambda_2) \frac{\mu}{(\lambda_2 + \gamma_1)} \sum_{n=0}^{c-1} \lambda_2^n - \mu \sum_{n=1}^{c-1} (n - c) \frac{\lambda_0^{n-1}}{n! \mu^n}]}$
- $P_{1,0} = \frac{\gamma_1 \mu}{\lambda_1 (\lambda_2 + \gamma_1)} P_{0,1}$
- $P_{2,0} = \frac{\mu}{(\lambda_2 + \gamma_1)} P_{0,1}$

3. Stability Condition

The model's stability criterion is determined by analysing the problem using a matrix geometric approach. For the process $\{C(t), X(t), t \geq 0\}$, the associated infinitesimal generator admits a block Jacobi matrix representation following lexicographical state order:

Here, $C_{00}, B_i, C_i (i = 1, 2, \dots), K_{00}, D_{00}$, and E_{00} are $m \times m$ matrices, where m denotes the number of environmental states ($m = 3$ in this study). The matrix K_{00} is the generator of the environmental process, and C_{00} is diagonal. The elements of the matrix Q are given below.

$$A_{00} = \begin{bmatrix} (-\lambda_1 + \gamma_1) & \gamma_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B_{00} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_i = \begin{bmatrix} i\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 1 \text{ to } c-2$$

$$B_i = \begin{bmatrix} (-\lambda_0 + i\mu) & 0 & 0 \\ \gamma_1 & -(\lambda_1 + \gamma_1) & 0 \\ \gamma_2 & 0 & -(\lambda_2 + \gamma_2) \end{bmatrix}, \quad i = 1 \text{ to } c-1$$

$$C_{00} = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad K_{00} = \begin{bmatrix} c\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = c-1 \text{ to } \infty$$

$$D_{00} = \begin{bmatrix} -(\lambda_0 \beta_0) + c\mu & 0 & 0 \\ \gamma_1 & -(\lambda_1 \beta_1 + \gamma_1) & 0 \\ \gamma_2 & 0 & -(\lambda_2 \beta_2 + \gamma_2) \end{bmatrix}$$

$$E_{00} = \begin{bmatrix} \lambda_0 \beta_0 & 0 & 0 \\ 0 & \lambda_1 \beta_1 & 0 \\ 0 & 0 & \lambda_2 \beta_2 \end{bmatrix}$$

A necessary and sufficient condition for the existence of the steady-state probability vector is $X E_{00} e < X K_{00} e$ Where X is the invariant probability of the matrix

$$D = C_1 + C_2 + \dots + B_1 + B_2 + \dots + C_{00} + K_{00} + D_{00} + E_{00}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ \gamma_1 & -\gamma_1 & 0 \\ \gamma_2 & 0 & -\gamma_2 \end{bmatrix}$$

From the calculations, the vector X is obtained

$$X = [1 \quad 0 \quad 0]$$

Substituting into the condition and simplifying algebraically gives the result

$$\rho = \frac{\lambda_0 \beta_0}{c\mu} < 1$$

4. Performance Measures

According to the distribution of the steady state, various system performance measures can be developed.

The expected system size is given by:

$$E[X] = G'_0(1) + G'_1(1) + G'_2(1)$$

$$= \left[\frac{\gamma_1 \gamma_2 (c\mu - \lambda_0 \beta_0) \Phi_3 - \Phi_1 \Phi_4}{(\gamma_1 \gamma_2 (c\mu - \lambda_0 \beta_0))^2} + \frac{\gamma_2 (-\lambda_1 \beta_1 - \lambda_1) R_4(1) - (-\lambda_1 \beta_1 - \lambda_1) R_4(1) (-\lambda_1 \beta_1)}{\gamma_2^2} + \frac{\gamma_1 (-\lambda_2 \beta_2 - \lambda_2) R_6(1) - \mu P_{0,1} (-\lambda_2 \beta_2)}{\gamma_1^2} \right]$$

where

$$\Phi_1 = -\gamma_1 \gamma_2 [(\lambda_0 \beta_0 - \lambda_0) R_2(1) + (\lambda_1 \beta_1 - \lambda_1) R_4(1) + (\lambda_2 \beta_2 - \lambda_2) R_6(1) - \mu R_1(1)] + \gamma_1^2 \lambda_1 \beta_1 P_{2,0} + \gamma_1 \gamma_2 \lambda_1 \beta_1 P_{1,0} + \mu \lambda_2 \beta_2 \gamma_2 P_{0,1} - \mu \gamma_1 \gamma_2 P_{0,1}$$

$$\begin{aligned} \Phi_3 = & 2\gamma_2 (\lambda_1 \beta_1 - \lambda_1) (\lambda_2 \beta_2 R_4(1) - \gamma_1 R_4'(1)) + 2\gamma_1 \lambda_1 \beta_1 (\lambda_2 \beta_2 - \lambda_2) (\lambda_1 \beta_1 R_6(1) - \gamma_2 R_6'(1)) \\ & - 2(\lambda_1 \beta_1 \gamma_2 + \lambda_2 \beta_2 \gamma_1) [(\gamma_2 + \lambda_1) P_{1,0} - \mu R_1(1) - (\lambda_0 \beta_0 - \lambda_0) R_2(1)] \\ & - 2\gamma_1 \gamma_2 (\lambda_0 \beta_0 - \lambda_0) R_2'(1) - 2\mu R_1(1) - \gamma_2 P_{1,0} - \gamma_1 P_{2,0} - 2\mu R_1'(1) - \\ & 2(\lambda_1 + \gamma_2) P_{1,0}, \end{aligned}$$

$$\Phi_4 = 2\lambda_0 \beta_0 \lambda_1 \beta_1 \gamma_1 + 2\lambda_0 \beta_0 \lambda_2 \beta_2 \gamma_2 - c\mu \gamma_1 (\lambda_1 \beta_1 + \lambda_2 \beta_2 - 2\gamma_2 - \frac{1}{\gamma_1})$$

- The steady-state probability that the server is occupied:

$$P_b = G_0(1) - P_{0,1}$$

- Expected queue length:

$$E(Q) = \sum_{n=c}^{\infty} (n - c) P_n$$

- Expected system time:

$$E(W_s) = \frac{E(X)}{\sum_{i=1}^3 (\lambda_i + \lambda_i \beta_i)}$$

- The fraction of time during which the server operates in the normal state

$$Q_0 = G_0(1)$$

- The fraction of time during which the server remains in the primary vacation state

$$Q_1 = G_1(1)$$

- The fraction of time during which the server remains in the secondary vacation state

$$Q_2 = G_2(1)$$

5. Practical Applications

1. In Healthcare systems, that is in hospitals serve different patient categories (emergency, outpatient, telemedicine), when there are fewer patients, hospitals encourage more

checkups or walk-ins (i.e., promotional health camps with offers). Doctors and nurses take differentiated vacations-short breaks when few patients, longer breaks during low demand hours. The assumptions regarding the parameters are as follows.

- λ_0 : Arrival rate of patients during regular hospital working hours.
- λ_1 : Arrival rate during short breaks or reduced service periods (primary vacation).
- λ_2 : Arrival rate during extended low-demand hours such as night shifts or off-peak periods (secondary vacation).
- β_0 : Patient arrival encouragement during congestion, representing health camps, appointment discounts, or walk-in promotions.
- β_1 : Encouragement factor during short staff breaks, indicating limited but ongoing patient intake.
- β_2 : Encouragement factor during longer breaks, reflecting restricted emergency-only arrivals.
- μ : Service rate of doctors or nurses, corresponding to the rate of patient diagnosis or treatment.
- γ_1 : Rate at which medical staff return from short breaks.
- γ_2 : Rate at which staff return from extended breaks or low-demand duty periods.

6. Numerical Calculations

Here a few illustrative instances in this part to highlight the qualitative features of the queuing model that is being studied. We examine 10 cases that have been assessed, four of which are based on reliability metrics. We make $\lambda_0 = 25$, $\lambda_1 = 8$, $\lambda_2 = 5$, $\beta_1 = 1$, $\beta_2 = 0.5$, $\beta_0 = 1$, $\gamma_1 = 2$, $\gamma_2 = 0.2$, $c = 3$.

Example 7.1 From Figure 2: Availability decreases steadily as the service rate μ increases across all arrival rates. Systems with higher λ_0 values consistently achieve greater availability compared to lower ones. This demonstrates that both μ and λ_0 jointly determine system performance with λ_0 .

Example 7.2 From Figure 3: Non-availability rises as the parameter γ_1 increases, showing a clear positive correlation. Larger λ_0 values correspond to higher non-availability levels across the range of γ_1 . The figure highlights how increasing γ_1 and λ_0 together reduce system reliability.

Example 7.3 From Figure 4: The probability measure $G_0(1)$ is influenced by the input rate λ_0 and the processing rate μ . For a fixed processing rate μ , the probability $G_0(1)$ declines as the input rate λ_0 increases. Conversely, for a constant input rate λ_0 , the probability $G_0(1)$ increases with the service rate μ .

Example 7.4 From Figure 5: The probability measure $G_1(1)$ is influenced by the inflow rate λ_1 and the parameter γ_1 . For a fixed value of γ_1 , the probability measure $G_1(1)$ declines as the inflow rate λ_1 increases. Conversely, for a constant inflow rate λ_1 , the probability $G_1(1)$ increases with higher values of γ_1 .

Example 7.5 From Figure 6: The probability measure $G_2(1)$ is influenced by the variable λ_2 and the parameter γ_2 . For a fixed value of γ_2 , the probability measure $G_2(1)$ declines as λ_2 rises. Conversely, for a constant value of λ_2 , the probability $G_2(1)$ rises with higher values of γ_2 .

Example 7.6 From Figure 7: The probability measure $P_{0,1}$ is influenced by the input rate λ_0 and the processing rate μ . For a fixed service capacity μ , the probability $P_{0,1}$ rises as the arrival rate λ_0 increases. Conversely, for a constant input rate λ_0 , the probability $P_{0,1}$ declines with the service rate μ .

Example 7.7 From Figure 8: The probability measure $P_{1,0}$ is influenced by the input rate λ_1 and the primary vacation rate γ_1 . For a fixed vacation rate γ_1 , the probability $P_{1,0}$

declines as the input rate λ_1 increases. Conversely, for a constant input rate λ_1 , the probability $P_{1,0}$ rises with the vacation rate γ_1 .

Example 7.8 From Figure 9: The probability measure $P_{2,0}$ is influenced by the inflow rate λ_2 and the secondary vacation rate γ_2 . For a fixed vacation rate γ_2 , the probability $P_{2,0}$ declines as the input rate λ_2 increases. Conversely, for a constant input rate λ_2 , the probability $P_{2,0}$ rises with the vacation rate γ_2 .

Example 7.9 From Figure 10: The expected value $E(X)$ is influenced by the input rate λ_1 and the service rate μ . For a fixed service capacity μ , the expected value $E(X)$ declines as the inflow rate λ_0 increases. Conversely, for a constant inflow rate λ_0 , the expected value $E(X)$ rises with the processing rate μ .

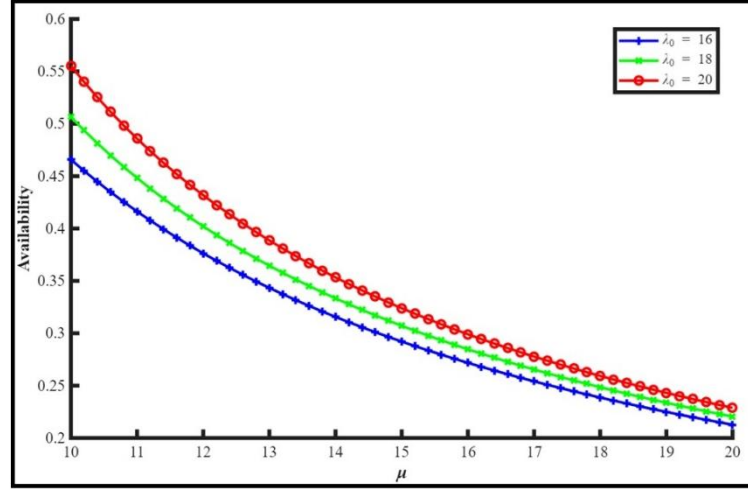


Figure 2. Availability rate with respect to λ_0 and μ .

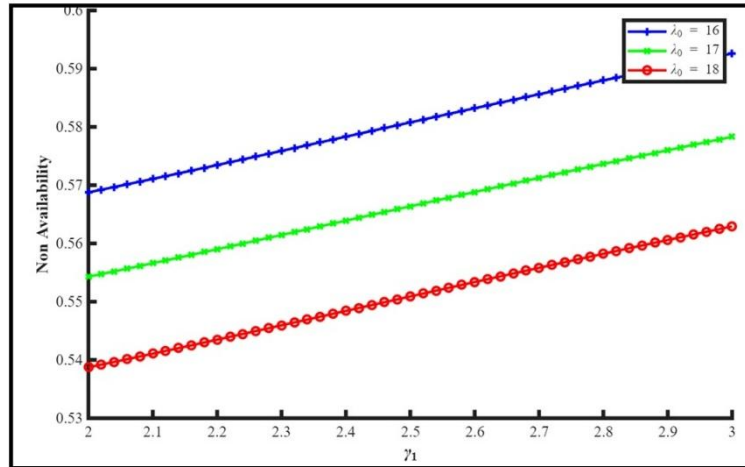


Figure 3. NonAvailability rate with respect to λ_0 and γ_1 .

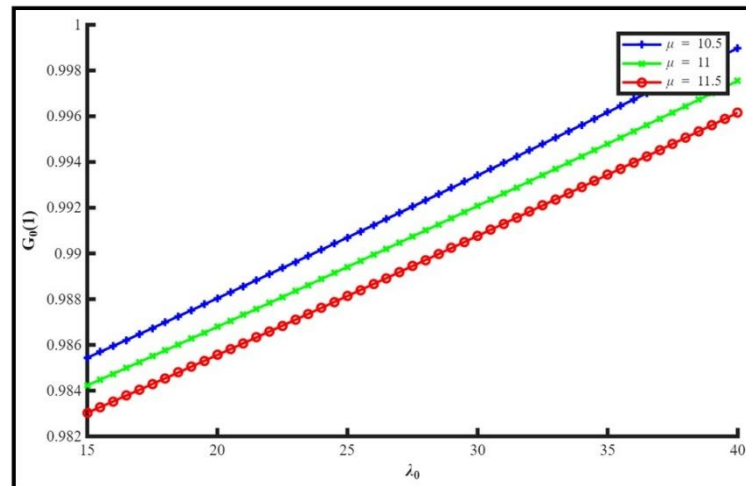


Figure 4. The impact of λ_0 and μ on $G_0(1)$.

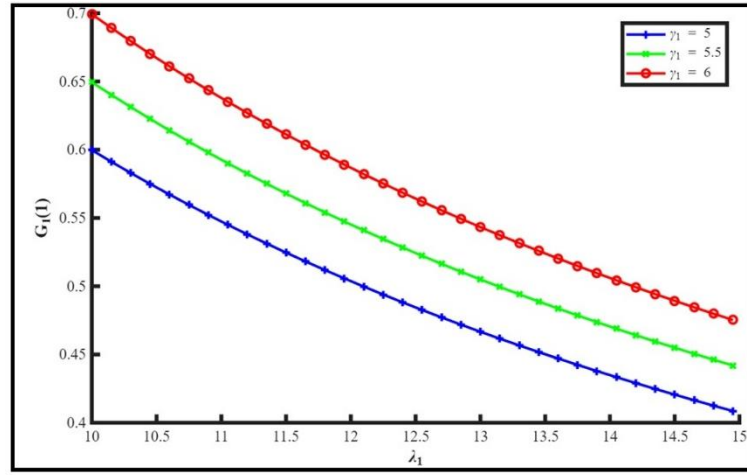


Figure 5. The impact of λ_1 and γ_1 on $G_1(1)$.

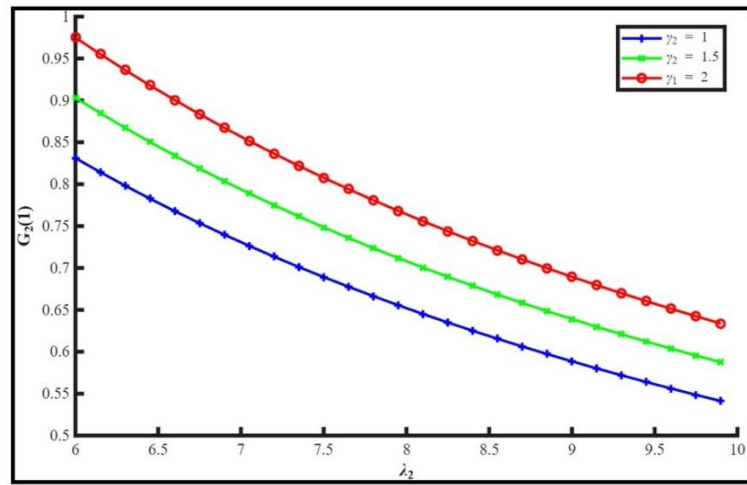


Figure 6. The impact of λ_2 and γ_2 on $G_2(1)$.

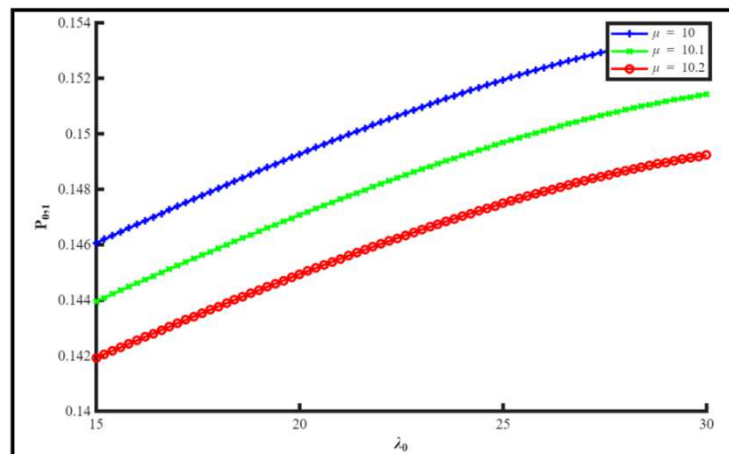


Figure 7. The impact of λ_0 and μ on $P_{0,1}$.

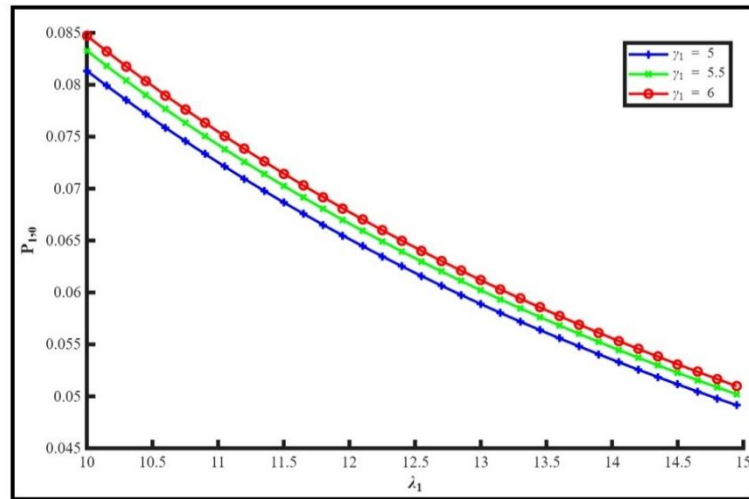


Figure 8. The impact of λ_1 and γ_1 on $P_{1,0}$.

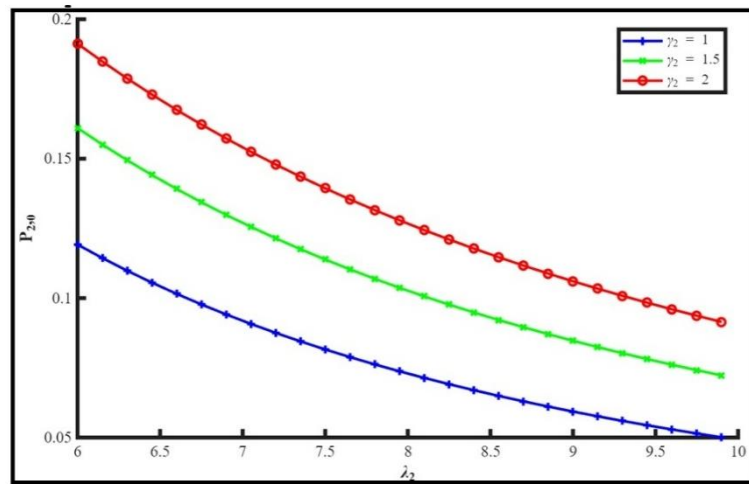


Figure 9. The impact of λ_2 and γ_2 on $P_{2,0}$.

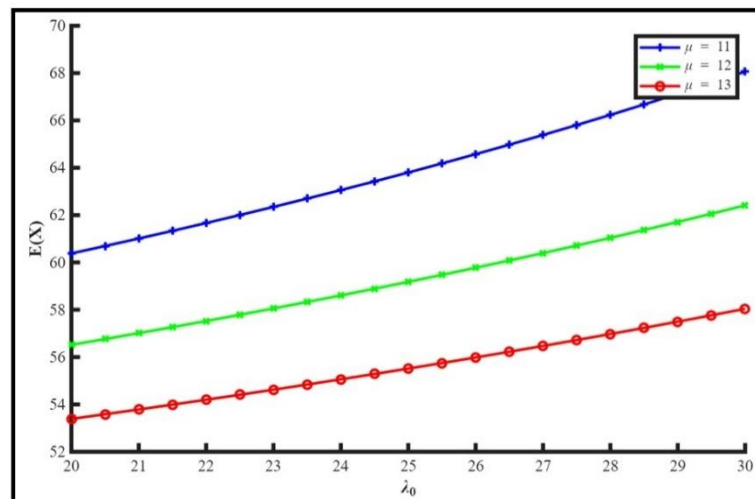


Figure 10. The impact of λ_0 and μ on $E(X)$.

7. The optimisation of the cost model

This section examines the cumulative expected cost function of the model specified above using the decision parameter μ . This study aims to identify the optimal parameters associated with the continuous random variables, such as $F(\mu^*)$, with the aim of minimizing the cost.

Cost Function

The cost components are specified as follows.

C_0 : The cost incurred per unit period during server busy time.

C_1 : The cost incurred per unit period when the server is in its primary vacation state.

C_2 : The cost incurred per unit period when the server is in its secondary vacation state.

C_3 : Fixed operational cost incurred per unit time by the server.

C_4 : The cost incurred per unit time by a customer waiting in the system.

Based on the definitions of the cost components presented above, the expected cost rate is formulated as

$$F(\mu^*) = C_0 G'_0(1) + C_1 G'_1(1) + C_2 G'_2(1) + C_3 \mu + C_4 E(W_s)$$

The problem of minimizing the cost is expressed as

$$F(\mu^*) = \text{Minimize } F(\mu)$$

The complexity of $E(X)$ and $E(W_s)$ increases the difficulty of the cost function in the equation. Analytical expressions for the most efficient processing rates at the least anticipated cost are not attainable. Therefore, the optimal processing rate μ^* was computed numerically using a direct search algorithm.

Direct optimization method:

The assigned cost parameters are: $C_0 = 150$, $C_1 = 270$, $C_2 = 380$, $C_3 = 250$, $C_4 = 270$. To determine the optimal μ , the direct search method is employed to present numerical examples. we specify $\lambda_0 = 25$, $\lambda_1 = 8$, $\lambda_2 = 5$, $\beta_1 = 0.3$, $\beta_2 = 3$, $\beta_0 = 0.5$, $\gamma_1 = 2$, $\gamma_2 = 0.2$, $c = 3$.

From Figure 11, the cost function $F(\mu)$ exhibits a convex, unimodal behaviour with a single relative minimum for each fixed value of λ_0 . As μ increases from 11 to 15, the cost initially decreases, reaches a minimum, and then begins to rise, confirming the existence of an optimal service rate μ for each λ_0 .

Table 1 summarizes the lowest expected cost $F(\mu)$ for different values of $\lambda_0 = 24, 25$ and 26 . The study reveals that the smallest estimated cost of **Rs. 2377236.11** is obtained at $\mu = 13.2$ for $\lambda_0 = 24$, **Rs. 2407164.31** is obtained at $\mu = 13.4$ for $\lambda_0 = 25$, and **Rs. 2436852.27** is obtained at $\mu = 13.6$ for $\lambda_0 = 26$.

With μ ranging from 11 to 15 and while keeping λ_0 fixed, Table 1 displays the minimum projected cost $F(\mu)$. It is observed that for fixed values of μ , the total cost first falls and then begins to rise with the grant of λ_0 . By expanding the customers typical service domain, the convexity of the cost function in λ_0 indicates a rise in the optimal cost.

Figure 11 thus confirms that the overall cost function initially declines and subsequently rises, exhibiting a convex shape and validating the existence of optimal service rates for each λ_0 .

From Figure 12, the cost function $F(\mu)$ demonstrates a convex and unimodal structure, where the cost initially decreases with increasing μ , reaches a minimum, and then rises again. This behaviour confirms the existence of an optimal service rate μ for each fixed value of λ_1 .

Table 2 presents the lowest expected cost $F(\mu)$ for different values of $\lambda_1 = 8.2, 8.4$, and 8.6 . The analysis reveals that the lowest anticipated cost of **Rs. 2546575.4** is obtained at $\mu = 13.4$ for $\lambda_1 = 8.2$, **Rs. 2689107.45** is attained at $\mu = 13.4$ for $\lambda_1 = 8.4$, and **Rs. 2833251.05** is reached at $\mu = 13.2$ for $\lambda_1 = 8.6$.

From Figure 13, the cost function $F(\mu)$ displays a convex and unimodal trend, where the cost initially decreases with increasing μ , reaches a minimum, and then begins to rise. This confirms the existence of an optimal service rate μ for each fixed value of λ_2 .

Table 3 summarizes the lowest expected cost $F(\mu)$ for different values of $\lambda_2 = 4.95, 5.00$ and 5.05 . The analysis reveals that the lowest anticipated cost of **Rs. 2273340.7** is reached at $\mu = 13.2$ for $\lambda_2 = 4.95$, **Rs. 2407164.31** is attained at $\mu = 13.4$ for $\lambda_2 = 5.00$ and **Rs. 2546345.44** is obtained at $\mu = 13.8$ for $\lambda_2 = 5.05$.

From Figure 14, the cost function $F(\mu)$ exhibits a convex and unimodal pattern, where the cost decreases with increasing μ until reaching a minimum, and then begins to rise. This confirms the presence of an optimal service rate μ for each fixed value of γ_1 .

Table 4 presents the lowest expected cost $F(\mu)$ for different values of $\gamma_1 = 1.95, 2.05$ and 2.15 . The analysis shows that the lowest anticipated cost of **Rs. 2357572.75** is attained at $\mu = 13.6$ for $\gamma_1 = 1.95$, **Rs. 2407164.31** is reached at $\mu = 13.4$ for $\gamma_1 = 2.05$ and **Rs. 2454475.89** is obtained at $\mu = 13.2$ for $\gamma_1 = 2.15$.

It is observed that, for fixed values of μ , the total cost initially declines and then rises with respect to γ_1 . The convexity of the cost function in γ_1 indicates a rise in the optimal cost as the service intensity expands. Figure 14 thus illustrates the unimodal behaviour of the cost function and supports the optimization of service parameters.

From Figure 15, the cost function $F(\mu)$ exhibits a convex and unimodal behaviour, where the cost decreases with increasing μ until reaching a minimum, and then begins to rise. This confirms the existence of an optimal service rate μ for each fixed value of γ_2 .

Table 5 presents the lowest expected cost $F(\mu)$ for different values of $\gamma_2 = 0.210, 0.215$, and 0.220 . The analysis reveals that the lowest anticipated cost of **Rs. 1852657.8** is reached at $\mu = 13.2$ for $\gamma_2 = 0.210$, **Rs. 1906646.44** is obtained at $\mu = 13.4$ for $\gamma_2 = 0.215$, and **Rs. 1960038.69** is attained at $\mu = 13.6$ for $\gamma_2 = 0.220$. It is observed that unimodal cost function in γ_2 supports optimization of service parameters.

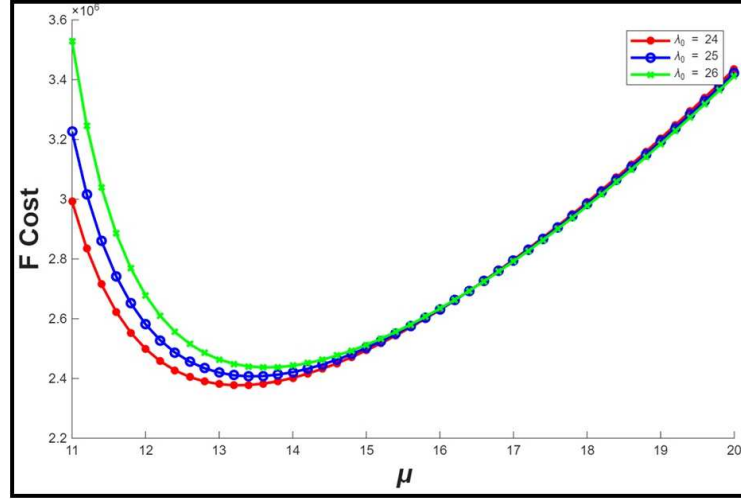


Figure 11. The influence of μ on total cost for varying λ_0 .

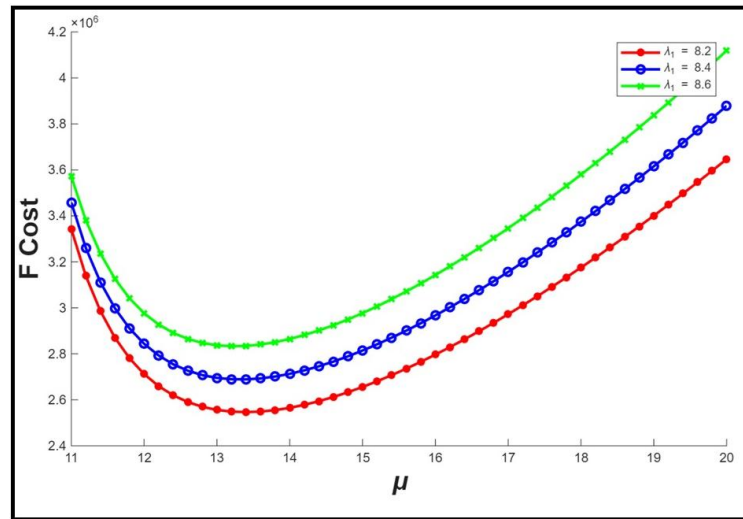


Figure 12. The influence of μ on total cost for varying λ_1 .

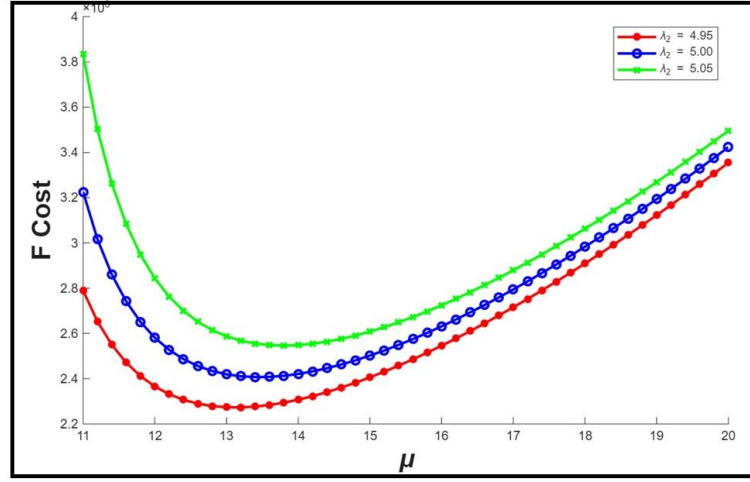


Figure 13. The influence of μ on total cost for varying λ_2 .

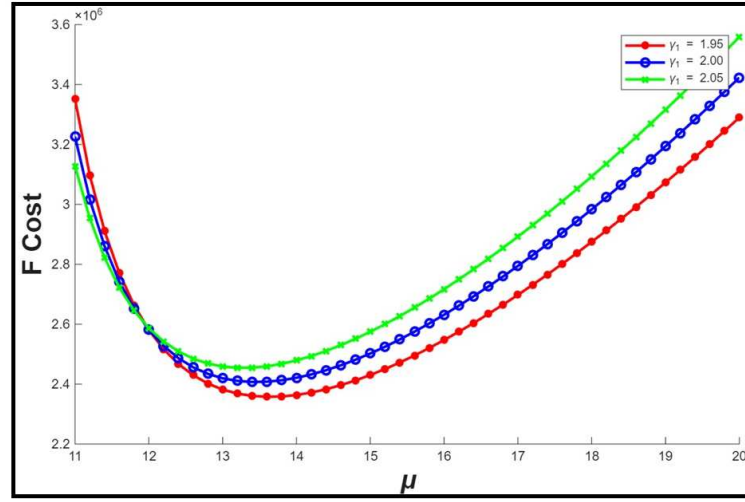


Figure 14. The influence of μ on total cost for varying γ_1 .

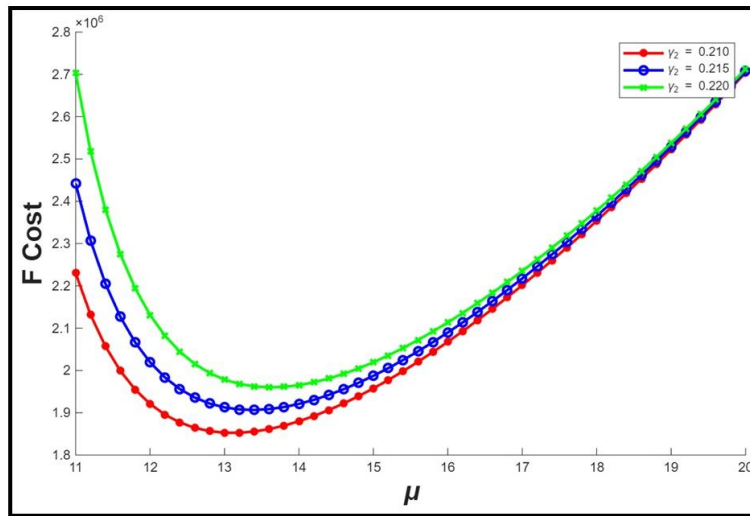


Figure 15. The influence of μ on total cost for varying γ_2 .

The sensitivity analysis presented in following Tables 1–5 examines the effect of the service rate μ on the system performance measures $P_{0,1}$, $E(W_s)$, and the total cost function under different parameter settings of λ_0 , λ_1 , λ_2 , γ_1 , and γ_2 . Across all tables, as the service rate μ increases, the probability $P_{0,1}$ consistently decreases, indicating a reduction in the likelihood of the system being in that state due to improved service capability. The expected

waiting time $E(W_s)$ generally decreases or stabilizes as the service rate increases, reflecting enhanced system efficiency and reduced congestion. The total cost function initially decreases with increasing μ , reaches a minimum around $\mu \approx 13.0$ – 13.2 , and then begins to increase, demonstrating a typical convex cost behavior. This pattern is observed for all variations of the arrival rates ($\lambda_0, \lambda_1, \lambda_2$) and other system parameters (γ_1, γ_2). Higher values of arrival rates tend to increase the overall system cost due to greater demand on the system, while changes in γ_1 and γ_2 influence the waiting time and cost levels but do not significantly alter the optimal region of μ . Overall, the results indicate that an optimal service rate exists around $\mu \approx 13.2$, where the balance between service efficiency and operational cost minimizes the total system cost.

Table 1. Impact of μ on the cost function at varying λ_0

μ	$\lambda_0 = 24$			$\lambda_0 = 25$			$\lambda_0 = 26$		
	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost
12.8	0.1202	0.7563	2389846.57	0.1245	0.7265	2434304.51	0.1293	0.6958	2485182.97
13	0.114	0.7552	2380954.98	0.1177	0.7268	2419513.56	0.1219	0.6977	2463479.57
13.2	0.1084	0.7538	2377236.1	0.1117	0.7265	2410757.22	0.1154	0.6988	2448878.78
13.4	0.1034	0.7521	2377987.23	0.1063	0.7259	2407164.3	0.1095	0.6994	2440290.66
13.6	0.0988	0.7503	2382639.48	0.1014	0.725	2408037.05	0.1043	0.6996	2436852.3
13.8	0.0947	0.7484	2390727.44	0.097	0.7239	2412809.92	0.0996	0.6993	2437871.53
14	0.0909	0.7463	2401866.61	0.093	0.7226	2421019.77	0.0952	0.6988	2442786.87

Table 2: Impact of μ on the cost function at varying λ_1

μ	$\lambda_1 = 8.2$			$\lambda_1 = 8.4$			$\lambda_1 = 8.6$		
	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost
13	0.116	0.878	2556130.13	0.115	1.027	2695429.21	0.113	1.174	2837252.67
13.2	0.11	0.871	2548729.62	0.109	1.013	2689611.23	0.107	1.153	2833251.1
13.4	0.105	0.864	2546575.4	0.104	1	2689107.5	0.102	1.134	2834616.61
13.6	0.1	0.857	2548971.25	0.099	0.987	2693225.91	0.098	1.116	2840664.25
13.8	0.096	0.851	2555351.32	0.095	0.976	2701402.61	0.094	1.1	2850833.97

Table 3. Impact of μ on the cost function at varying λ_2 .

μ	$\lambda_2 = 4.95$			$\lambda_2 = 5.00$			$\lambda_2 = 5.05$		
	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost
13	0.11	0.78	2273874.9	0.118	0.727	2419513.56	0.127	0.664	2587619.09
13.2	0.105	0.777	2273340.7	0.112	0.727	2410757.22	0.12	0.668	2568222.26
13.4	0.1	0.774	2276872.33	0.106	0.726	2407164.3	0.113	0.671	2555497.25
13.6	0.096	0.77	2283964.19	0.101	0.725	2408037.05	0.108	0.673	2548465.74
13.8	0.092	0.767	2294201.67	0.097	0.724	2412809.92	0.103	0.675	2546345.4
14	0.088	0.764	2307241.56	0.093	0.723	2421019.77	0.098	0.676	2548503.35
13	0.11	0.78	2273874.9	0.118	0.727	2419513.56	0.127	0.664	2587619.09

Table 4. Impact of μ on the cost function at varying γ_1 .

μ	$\gamma_1 = 1.95$			$\gamma_1 = 2.00$			$\gamma_1 = 2.05$		
	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost	$P_{0,0}$	$E(W_s)$	F-cost
12.6	0.1386	0.669	2429801.44	0.1321	0.726	2456232.01	0.1262	0.776	2484955.91
12.8	0.1302	0.673	2401845.02	0.1245	0.727	2434304.51	0.1192	0.774	2468632.46
13	0.1229	0.676	2381789.38	0.1177	0.727	2419513.56	0.113	0.772	2458791.43

13.2	0.1164	0.679	2368358	0.1117	0.727	2410757.22	0.1074	0.77	2454475.9
13.4	0.1105	0.68	2360550.71	0.1063	0.726	2407164.3	0.1024	0.768	2454923.11
13.6	0.1053	0.681	2357572.8	0.1014	0.725	2408037.05	0.0979	0.765	2459517.62
13.8	0.1005	0.681	2358784.6	0.097	0.724	2412809.92	0.0937	0.763	2467757.32

Table 5. Impact of μ on the cost function at varying γ_2

μ	$\gamma_2 = 0.210$			$\gamma_2 = 0.215$			$\gamma_2 = 0.220$		
	$P_{0,0}$	E(Ws)	F-cost	$P_{0,0}$	E(Ws)	F-cost	$P_{0,0}$	E(Ws)	F-cost
12.6	0.124	0.844	1864308.03	0.133	0.843	1935980.44	0.142	0.845	2015461.69
12.8	0.117	0.824	1856467.01	0.125	0.822	1921818.51	0.134	0.821	1993881.61
13	0.112	0.807	1852773.79	0.119	0.803	1912675.94	0.127	0.8	1978410.6
13.2	0.106	0.79	1852657.8	0.113	0.785	1907815.02	0.12	0.781	1968091.56
13.4	0.102	0.775	1855658.1	0.108	0.769	1906646.4	0.114	0.764	1962169.02
13.6	0.097	0.762	1861398.31	0.103	0.754	1908693.74	0.109	0.748	1960038.7
13.8	0.094	0.749	1869568.01	0.099	0.741	1913567.5	0.104	0.733	1961211.43

8. Conclusion

This paper studies an M/M/C queueing model with stimulated arrivals and differentiated vacations. The probability-generating function method is employed to derive the model's steady state and steady-state probabilities. Furthermore, various system performance measures are analysed numerically. A cost function is constructed to determine the optimal (minimum) cost, and the corresponding optimal service rate is obtained using the direct search method.

The preceding queueing model's cost function and numerical examples are shown using MATLAB software. Future work may include extending the model to incorporate customer impatience, priority classes, or time-varying arrival and service rates.

Author Contributions: B. Deepa contributed to the conceptualization, formulation of the queueing model, mathematical analysis, and preparation of the manuscript. A. Kalaivani contributed to the mathematical derivations, analysis, and interpretation of the results. K. Vijaya contributed to the validation of the model, computational analysis, and review of the manuscript. All authors read and approved the final manuscript.

Competing Interests

The authors declare that they have no competing interests.

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Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed

Ethics and Consent to Participate declarations: Not applicable.

Consent to Publish declaration: Not applicable.

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