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Research Article

Keywords: Grand Unified Theory, Quark-Lepton Mixing, Flavour Mixing, CKM Matrix, PMNS Matrix

Posted Date: October 8th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11024707/v2>

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Additional Declarations: The authors declare no competing interests.

The Quark-Lepton Cross-Sector Bridge from State Equilibrium Condition

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Abstract

We show that both quarks and leptons satisfy the same *mixing-equilibrium condition*:

$$\alpha_i(1 - \alpha_i) = \mathcal{C}_i \lambda^2,$$

where α_i and $1 - \alpha_i$ represent the *mixing-weight* and *retention-weight*, respectively, at the equilibrium point, and $\mathcal{C}_i = 1 + D_i/2$ is the *coupling factor*, with D_i denoting the $(B - L)$ quantum number of the sector ($i = q$ for quarks and $i = \ell$ for leptons). We define λ as the *fermion mixing constant*, common to the quark and lepton sectors, derived from the Haar measure of $SU(2)_L$.

Using $(B - L)$ values for quarks and leptons, we find their respective coupling factors to be $\mathcal{C}_q = 7/6$ and $\mathcal{C}_\ell = 1/2$. Substituting these coupling factors into the mixing-equilibrium condition yields the equilibrium mixing-weights $\alpha_q \approx 0.063084$ and $\alpha_\ell \approx 0.026007$.

Using λ , α_q , and α_ℓ , we construct closed-form expressions for all 9 elements of the CKM matrix—yielding a mean absolute pull of 0.22σ and a maximum pull of 0.59σ —as well as all three PMNS mixing angles, the Dirac CP phase, and the leptonic Jarlskog invariant within 0.89σ . Across 14 observables benchmarked against PDG 2024 and NuFIT 6.0 values, the maximum statistical pull is 0.89σ (at $\sin^2 \theta_{13}$) and the mean pull is 0.28σ .

We further show that $\theta_C = \arcsin \lambda$ and $\sin^2 \theta_{12}^{\text{PMNS}} = 1/3 - \mathcal{C}_\ell \lambda^2$ yield the quark-lepton complementarity sum $\theta_{12}^{\text{PMNS}} + \theta_C = 46.72^\circ$, in agreement with the PDG value of 46.69° , where both angles are determined by λ alone. The 1.72° deviation from the exact 45° value is a direct consequence of the mixing-equilibrium condition. The quark-lepton asymmetry arises from the coupling ratio $\mathcal{C}_q/\mathcal{C}_\ell = 7/3$, which makes the mixing variance of the quark sector $7/3$ times that of leptons. This asymmetry follows from the positive $(B - L)$ gauge charge $+1/3$ of quarks and the negative $(B - L)$ gauge charge -1 of leptons within the Left-Right Symmetric Model.

Normalising the equilibrium variance of each sector by its coupling factor yields the same universal bridging constant:

$$\frac{\alpha_q(1 - \alpha_q)}{\mathcal{C}_q} = \frac{\alpha_\ell(1 - \alpha_\ell)}{\mathcal{C}_\ell} = \lambda^2.$$

1 Introduction

Flavour mixing is one of the most striking unresolved puzzles of the Standard Model. Quark mixing, described by the Cabibbo–Kobayashi–Maskawa (CKM) matrix [1, 2], is small and hierarchical. The Cabibbo angle $\theta_C \approx 13.0^\circ$ sets the magnitude of the leading first-to-second generation transition amplitude, while transitions involving more distant generations are suppressed by successive powers of the Wolfenstein parameter $\lambda \approx 0.225$ [3]. Lepton mixing, encoded in the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix [4, 5], stands in striking contrast to quark mixing: the solar angle $\theta_{12}^{\text{PMNS}} \approx 34^\circ$ and the atmospheric angle $\theta_{23}^{\text{PMNS}} \approx 49^\circ$ are both large, whereas the reactor angle $\theta_{13}^{\text{PMNS}} \approx 8.5^\circ$ is small, more reminiscent of the quark sector. Both sectors are mediated by the same $\text{SU}(2)_L$ gauge bosons, yet the mixing they produce differs by an order of magnitude. Understanding why this occurs is a necessary step toward any unified description of flavour.

The earliest systematic attempts to relate quark and lepton mixing arose within Grand Unified Theories (GUTs), which embed both $\text{SU}(3)_C$ and the electroweak gauge group into a single simple Lie group. The $\text{SU}(5)$ model of Georgi and Glashow [6] places right-handed down-type quarks and left-handed lepton doublets in the same $\mathbf{\bar{5}}$ representation, yielding the relation $m_d = m_e^T$ at $M_{\text{GUT}} \sim 10^{16}$ GeV. The $\text{SO}(10)$ extension of Fritzsch and Minkowski [7] goes further by unifying an entire generation, including right-handed neutrinos, in a single $\mathbf{16}$ -dimensional spinor representation. Similarly, the Pati–Salam model [8], based on $\text{SU}(4)_C \times \text{SU}(2)_L \times \text{SU}(2)_R$, treats lepton number as a fourth colour charge and directly equates quark and lepton multiplets within each generation. All three approaches introduce genuine quark-lepton relations, but only at the unification scale. Evolving these relations to low energy requires renormalisation group equations whose output depends on the unknown particle content between M_{EW} and M_{GUT} , supersymmetric threshold corrections, and the choice of Higgs representations. In practice, GUT-derived mixing relations carry uncertainties of the same order as the quantities they predict, requiring additional texture parameters beyond those of the Standard Model.

To address neutrino masses and high-scale mixing connections, the discovery of neutrino oscillations brought the seesaw mechanism to prominence. The seesaw mechanism [9, 10, 11], in which heavy right-handed Majorana neutrinos are integrated out to give light effective masses $m_\nu \sim m_D^2/M_R$, naturally produces the required suppression. In the Left-Right Symmetric Model (LRSM) context [12], the bidoublet Higgs generates the Dirac mass matrix, while the right-handed triplet develops a vacuum expectation value that drives M_R . However, the seesaw mechanism does not predict the mixing angles: the PMNS matrix follows from the light neutrino mass matrix $m_\nu \simeq -m_D M_R^{-1} m_D^T$, and any mixing pattern can be obtained through appropriate choices of m_D and M_R .

A qualitatively different approach looks for a direct numerical relation between the two mixing matrices at the electroweak scale. Raidal [13] and Minakata and Smirnov [14] independently observed that $\theta_{12}^{\text{PMNS}} + \theta_C \approx 45^\circ$, naming this quark-lepton complementarity (QLC). If exact, QLC implies $U_{\text{PMNS}} \sim V_{\text{CKM}}^\dagger \cdot U_{\text{TBM}}$, where U_{TBM} is a tribimaximal mixing matrix. However, none of the QLC approaches is parameter-free, and no existing model predicts the observed 1.7° deviation of the sum from 45° .

In parallel, a separate programme attempts to derive both quark and lepton mixing from a common discrete non-Abelian family symmetry. Groups such as A_4 , S_4 , $\Delta(96)$, and their extensions predict tribimaximal or bimaximal mixing as exact leading-order results [15]. The discovery of the reactor angle $\theta_{13}^{\text{PMNS}} \approx 8.5^\circ$ at Daya Bay in 2012 [16] rules

out exact tribimaximal mixing. In these models the quark and lepton sectors are related by the common family group, but the relation remains indirect: quark mixing arises from down-type quark sector corrections rather than the family symmetry itself, and the number of independent vacuum alignment parameters generally exceeds the number of mixing observables predicted.

Despite five decades of effort, no closed-form formulation has derived both the CKM and PMNS mixing matrices without free parameters. The GUT approach operates at M_{GUT} and introduces texture freedom, the seesaw mechanism hides the mixing in the right-handed Majorana mass matrix, the QLC hypothesis is empirical and predicts neither the 1.7° deviation nor the reactor angle, and discrete symmetry models require vacuum alignment alongside a separate mechanism for the CKM hierarchy. The (LRSM) [8, 12, 17] offers the natural gauge-theoretic setting for a quark-lepton bridge at the electroweak scale. Its gauge group $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_{B-L}$ assigns distinct, anomaly-free baryon-minus-lepton numbers to each sector, with $(B-L)_q = +1/3$ for quark doublets and $(B-L)_\ell = -1$ for lepton doublets. In the low-energy limit, as $\text{SU}(2)_R$ decouples at the scale M_{W_R} , the surviving gauge structure of each left-handed doublet reflects the associated baryon-minus-lepton contribution, carrying the quark-lepton asymmetry directly into the electroweak sector of the Standard Model. The LRSM thus encodes the sector distinction in the gauge quantum numbers without invoking a higher unification scale or adjustable parameters.

The recent posts [18, 19], converts this gauge structure into a dynamical mixing principle. The key observation is that the $\text{SU}(2)_L$ group manifold is topologically S^3 with Haar volume $2\pi^2$ [20], where the inverse square root relation $1/\lambda^2 = 2\pi^2$ defines the sector-invariant fermion mixing constant $\lambda \approx 0.225$ coinciding with the Wolfenstein parameter to four significant figures. In a charged-current transition, the vacuum background responds to the isospin step with a linear response coefficient, namely the coupling factor $\mathcal{C}_i = 1 + D_i/2$, where $D_i \equiv (B-L)$ fixed entirely by the LRSM baryon-minus-lepton numbers. The mixing-equilibrium condition

$$\alpha_i(1 - \alpha_i) = \mathcal{C}_i \lambda^2, \quad i \in \{q, \ell\}, \quad (1)$$

where α_q and α_ℓ are the mixing-weights of quarks and leptons, respectively, under the equilibrium condition. The coupling ratio $\mathcal{C}_q/\mathcal{C}_\ell = 7/3$ forms the quantitative origin of the quark-lepton mixing asymmetry.

The present work derives the quark-lepton bridge entirely from weak isospin and gauge assignments, without invoking a GUT scale, mass textures, or flavour symmetries. Specifically, we establish the *cross-sector equivalence theorem* stating that the coupling-normalised equilibrium variance equals the universal constant λ^2 in both sectors (Section 3). This yields the cross-sector identity $3\alpha_q(1 - \alpha_q) = 7\alpha_\ell(1 - \alpha_\ell)$, encoding all quark-lepton mixing relations in a single equation. We show that quark-lepton complementarity is not accidental, as the predicted sum $\theta_{12}^{\text{PMNS}} + \theta_C = 46.72^\circ$ deviates from 45° by the exact amount required by the coupling ratio $7 : 3$ and matches the PDG value 46.69° within 0.03° (Section 6.2). Across both sectors, the formulation produces 14 parameter-free predictions benchmarked against PDG 2024 and NuFIT 6.0, with a maximum pull of 0.89σ and a mean pull of 0.28σ without free parameters (Section 7).

The paper is organised as follows. Section 2 establishes the LRSM fermion assignments, derives the coupling factors, and identifies the $\text{SU}(2)_L$ Haar scale. Section 3 proves the cross-sector equivalence theorem and its corollaries. Section 4 derives the chain of physical predictions and states three falsifiable tests. Section 5 presents the complete

closed-form matrix formulations and the full σ -pull comparison across both sectors. Section 7 summarises the main results and their significance. The symbols of key constants are listed in Table 1.

Table 1: Symbols of key constants ($i = q$ for quarks and $i = \ell$ for leptons).

Symbol	Description	Defining Context / Value
λ	Fermion fermion mixing constant	$1/\lambda^2 = 2\pi^2$ (SU(2) _L Haar volume)
D_i	Baryon-minus-Lepton number	$D_i \equiv (B - L)_i$: $D_q = 1/3$, $D_\ell = -1$
$\mathcal{C}_i$	Coupling factor	$\mathcal{C}_i = 1 + D_i/2$: $\mathcal{C}_q = 7/6$, $\mathcal{C}_\ell = 1/2$
α_i	Equilibrium mixing-weight	Equilibrium condition: $\alpha_i(1 - \alpha_i) = \mathcal{C}_i \lambda^2$
$1 - \alpha_i$	Equilibrium retention-weight	Complement of constant α_i

2 Coupling Factor and Fermion Mixing Constant

Under the LRSM gauge group

$$\mathcal{G}_{\text{LR}} = \text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_{B-L},$$

each generation of fermions is organized into left- and right-handed doublets:

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad L_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L, \quad Q_R = \begin{pmatrix} u \\ d \end{pmatrix}_R, \quad L_R = \begin{pmatrix} \nu \\ e \end{pmatrix}_R. \quad (2)$$

The left-handed doublets transform under SU(2)_L and are singlets under SU(2)_R, while the right-handed doublets transform under SU(2)_R and are singlets under SU(2)_L.

2.1 Charge Distribution in Fermions

The electric charge operator in the LRSM is defined by the generalized Gell-Mann–Nishijima (GMN) relation

$$Q = T_{3L} + T_{3R} + \frac{D}{2}, \quad (3)$$

where $D \equiv B - L$ is the U(1)_{B-L} charge. For left-handed multiplets, $T_{3R} = 0$, so that

$$Q = T_{3L} + \frac{D_i}{2}, \quad (4)$$

where $i = q$ for quarks and $i = \ell$ for leptons. Using $D_q = +1/3$ for quarks and $D_\ell = -1$ for leptons, the electric charges of the quark doublet components are $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$, while those of the lepton doublet are $Q_\nu = 0$ and $Q_e = -1$.

Table 2 shows the distribution of isospin and $D \equiv B - L$ across the left- and right-handed fermion multiplets. While the electric charge Q and the weak-isospin projections T_{3L} and T_{3R} vary among the multiplet components, D remains fixed within each sector. The U(1)_{B-L} factor therefore distinguishes the quark and lepton sectors through their different D charges.

Table 2: Charge and isospin components of fermion multiplets in the LRSM.

Fermion Multiplet	T_{3L}	T_{3R}	$D \equiv B - L$	$Q = T_{3L} + T_{3R} + \frac{D}{2}$
Quark doublet Q_L	$\pm\frac{1}{2}$	0	$+\frac{1}{3}$	$\frac{2}{3}, -\frac{1}{3}$
Quark doublet Q_R	0	$\pm\frac{1}{2}$	$+\frac{1}{3}$	$\frac{2}{3}, -\frac{1}{3}$
Lepton doublet L_L	$\pm\frac{1}{2}$	0	-1	0, -1
Lepton doublet L_R	0	$\pm\frac{1}{2}$	-1	0, -1

2.2 Coupling Factor

While the weak isospin projection T_{3L} participates in the weak interaction and drives charged-current transitions, the difference $D \equiv B - L$ acts as a flavour-invariant sector constant that contributes resistance to flavour change. As shown in Table 2, for the quark sector is $D_q = +\frac{1}{3}$, and for the lepton sector is $D_\ell = -1$. We define the *coupling factor* of a sector i as the sum of the change in weak isospin projection during a flavour transition and half of the difference number:

$$\mathcal{C}_i = |\Delta T_{3L}| + \frac{D_i}{2} = 1 + \frac{D_i}{2}. \quad (5)$$

Here, the integer part ($|\Delta T_{3L}| = 1$) arises from the magnitude of the change in the weak isospin projection, while the fractional part ($D_i/2$) stems from the underlying difference number carried by the fermion doublet. Substituting the sector values from Table 2, we find:

$$\mathcal{C}_q = 1 + \frac{1}{6} = \frac{7}{6} \quad \text{and} \quad \mathcal{C}_\ell = 1 - \frac{1}{2} = \frac{1}{2}. \quad (6)$$

The coupling factors $\mathcal{C}_q$ and $\mathcal{C}_\ell$ are determined entirely by the D charge assignments of the LRSM, which in turn follow from anomaly cancellation and the electric charge formula. The ratio

$$\frac{\mathcal{C}_q}{\mathcal{C}_\ell} = \frac{7/6}{1/2} = \frac{7}{3}, \quad (7)$$

corresponds to the ratio of the equilibrium mixing variances of quarks and leptons (Section 3), which constitutes the source of the quark-lepton asymmetry.

2.3 The Fermion Mixing Constant

The $SU(2)_L$ group manifold is topologically S^3 with group volume [20]

$$\text{Vol}(SU(2)_L) = 2\pi^2. \quad (8)$$

The normalised Haar measure assigns equal weight to every $SU(2)_L$ rotation. The inverse square root of this volume defines the *fermion mixing constant*:

$$\lambda \equiv \frac{1}{\sqrt{\text{Vol}(SU(2)_L)/\pi}} = \frac{1}{\pi\sqrt{2}} \approx 0.225079, \quad \lambda^2 = \frac{1}{2\pi^2} \approx 0.050660. \quad (9)$$

This value coincides with the Wolfenstein parameter $\lambda_W \approx 0.225$ to four significant figures, establishing that the Cabibbo scale is not an independent input but the Haar normalisation of $SU(2)_L$.

Three properties of λ^2 are essential for what follows. First, it is *sector-blind*: the same λ^2 appears in both the quark and lepton mixing-equilibrium conditions. Second, it is *geometric*: derived from the group volume of $SU(2)_L$, and does not involve any free parameter. Third, it defines the *fermion mixing constant* given by the uniform Haar weight for any $SU(2)_L$ rotation between adjacent generational states.

3 CP Violation and Mixing-Equilibrium Condition

3.1 The Mixing-Equilibrium Condition

The *mixing-equilibrium condition* is postulated as

$$\alpha_i(1 - \alpha_i) = \mathcal{C}_i \lambda^2, \quad (10)$$

where α_i and $(1 - \alpha_i)$ represent the *mixing-weight* and *retention-weight* on equilibrium point of the sector i ($i = q$ for quarks and $i = \ell$ for leptons). $\mathcal{C}_i$ is the *coupling factor* of sector i ; and λ is the *fermion mixing constant* given by (6) and (9), respectively. Equation (10) can be expanded as a quadratic equation:

$$\alpha_i^2 - \alpha_i + \mathcal{C}_i \lambda^2 = 0, \quad (11)$$

Using $\mathcal{C}_q = 7/6$ and $\mathcal{C}_\ell = 1/2$ from (6) and $\lambda = \frac{1}{\pi\sqrt{2}} \approx 0.225079$ from (9), we can find that $\mathcal{C}_q \lambda^2 = \frac{7}{12\pi^2} \approx 0.05911$ for quarks and $\mathcal{C}_\ell \lambda^2 = \frac{1}{4\pi^2} \approx 0.0253$ for leptons. Since $\mathcal{C}_i \lambda^2 < 1/4$ in both sectors, (11) gives two real bounded roots $0 < \alpha_i, \alpha'_i < 1$ for each sector. The smaller root:

$$\alpha_i = \frac{1 - \sqrt{1 - 4\mathcal{C}_i \lambda^2}}{2}, \quad (12)$$

while the other root gives the complementary component:

$$\alpha'_i \equiv 1 - \alpha_i = \frac{1 + \sqrt{1 - 4\mathcal{C}_i \lambda^2}}{2}. \quad (13)$$

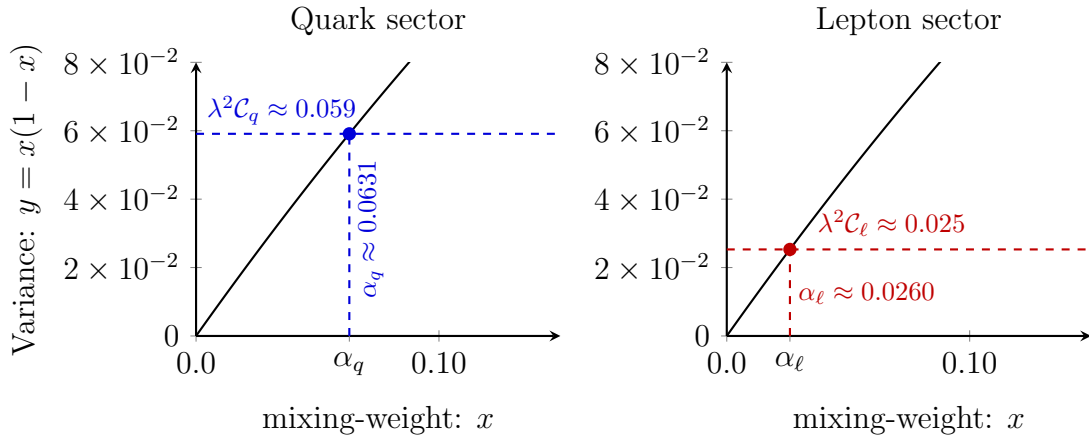


Figure 1: The mixing equilibrium points for the two sectors. The plot is the variance $y = x(1-x)$. The horizontal line is the equilibrium variance $\lambda^2 \mathcal{C}_i$. The intersection gives the mixing weight α_i at the point of equilibrium.

The relative magnitudes of the equilibrium mixing-weights and the mixing variance, as represented in the unified mixing-equilibrium condition for the quark and lepton sectors, are shown in Fig. 1. The significance of the mixing-equilibrium condition is outlined below:

- i. The right-hand side is the product of λ^2 and the sector coupling factor $\mathcal{C}_i$, where λ is a sector invariant universal constant and $\mathcal{C}_i$ is fixed for a given sector.
- ii. Since quarks carry a higher coupling factor ($\mathcal{C}_q = 7/6 > \mathcal{C}_\ell = 1/2$), they have higher equilibrium variance. Defining coupling as resistance to CP violation, quarks offer stronger resistance to CP violation relative to leptons.
- iii. The quadratic mixing-equilibrium condition yields two complementary roots, α_i and $\alpha'_i \equiv 1 - \alpha_i$. The smaller root α_i is defined as the equilibrium mixing-weight, while the equilibrium retention-weight ($1 - \alpha_i$) corresponds to the magnitude of the imaginary component of the sector's complex CP-violating phase factor $e^{-i\delta_i}$, yielding

$$\alpha'_i = 1 - \alpha_i = |\sin \delta_i| \quad (14)$$

- iv. The mixing-equilibrium condition also admits a companion quadratic form under sign reversal, $\alpha_i^2 + \alpha_i + \mathcal{C}_i \lambda^2 = 0$, yielding the dual root pair $-\alpha_i$ and $-\alpha'_i$. Their magnitudes remain bounded within $0 < |-\alpha_i|, |-\alpha'_i| < 1$. For antiparticle sectors, this sign-reversed dual structure accounts for charge-parity (CP) conjugation, while preserving the chiral symmetry of the Lagrangian before spontaneous symmetry breaking.

3.2 CP-Violating Phase of Quark and Lepton Sector

Substituting $\lambda^2 = 1/(2\pi^2)$ and the quark coupling factor $\mathcal{C}_q = 7/6$ into the generic quadratic equation (11), we get the mixing-equilibrium condition of the quark sector as:

$$\alpha_q(1 - \alpha_q) - \frac{7}{12\pi^2} = 0, \quad (15)$$

We can solve this to find the smaller root:

$$\alpha_q = \frac{1 - \sqrt{1 - 7/(3\pi^2)}}{2} \approx 0.063084, \quad (16)$$

and the equilibrium retention-weight:

$$(1 - \alpha_q) = \frac{1 + \sqrt{1 - 7/(3\pi^2)}}{2} \approx 0.936916. \quad (17)$$

We can verify that $\alpha_q + \alpha'_q = 1$ and $\alpha'_q = 1 - \alpha_q$. In the quark sector, where the phase remains in the upper semicircle ($T_{3L} > 0$), it is identified that:

$$1 - \alpha_q = \sin \delta_{\text{CP}}^{\text{CKM}}, \quad (18)$$

yielding the derived CKM phase:

$$\delta_{\text{CP}}^{\text{CKM}} = \arcsin(1 - \alpha_q) \approx 1.214 \text{ rad } (\approx 69.56^\circ). \quad (19)$$

This theoretical result demonstrates strong quantitative alignment with empirical measurements, matching the Particle Data Group (PDG) global fit for the CKM phase ($\delta_{13} \approx 68.8^\circ \pm 4.5^\circ$) within $\approx 0.18\sigma$.

Similarly, substituting the lepton coupling factor $\mathcal{C}_\ell = 1/2$ into the quadratic equation (11) yields the mixing-equilibrium condition for the lepton sector:

$$\alpha_\ell(1 - \alpha_\ell) - \frac{1}{4\pi^2} = 0, \quad (20)$$

yielding the complementary pair of roots:

$$\alpha_\ell = \frac{1 - \sqrt{1 - 1/\pi^2}}{2} \approx 0.026007, \quad (21)$$

$$\alpha'_\ell = 1 - \alpha_\ell = \frac{1 + \sqrt{1 - 1/\pi^2}}{2} \approx 0.973993. \quad (22)$$

In the lepton sector, the cyclic traversal across the weak isospin doublet ($T_{3L} = +1/2 \rightarrow -1/2$) maps the phase into the lower semicircle, requiring $\sin \delta_{\text{CP}}^{\text{PMNS}} < 0$. Consequently, the two roots of the quadratic mixing-equilibrium condition directly correspond to the two physical neutrino mass orderings:

- i. *Inverted Ordering (IO)*: Governed by the equilibrium retention-weight $(1 - \alpha_\ell)$, the mixing-equilibrium condition manifests as:

$$\sin \delta_{\text{CP}}^{\text{IO}} = -(1 - \alpha_\ell) \approx -0.973993. \quad (23)$$

Interpreted as a clockwise displacement from the 2π axis, this gives:

$$\delta_{\text{CP}}^{\text{IO}} = 2\pi - \arcsin(1 - \alpha_\ell) \approx 4.941 \text{ rad } (\approx 283.10^\circ). \quad (24)$$

This derived phase exhibits quantitative agreement with global neutrino oscillation analyses (e.g., NuFIT), placing it within $\lesssim 0.4\sigma$ of the Inverted Ordering best-fit center ($\sim 274^\circ \pm 22^\circ$) and squarely inside the favoured lower-hemisphere window.

- ii. *Normal Ordering (NO)*: Governed by the equilibrium mixing-weight α_ℓ , the return path equilibrium satisfies:

$$\sin \delta_{\text{CP}}^{\text{NO}} = -\alpha_\ell \approx -0.026007. \quad (25)$$

Anchored at the π (180°) CP-conserving boundary, the phase is given by:

$$\delta_{\text{CP}}^{\text{NO}} = \pi + \arcsin(\alpha_\ell) \approx 3.168 \text{ rad } (\approx 181.49^\circ). \quad (26)$$

This result sits within $\lesssim 0.2\sigma$ of the Normal Ordering best-fit ($177^\circ \pm 20^\circ$), naturally capturing the near-CP-conserving regime preferred by standard Normal Ordering global analyses.

It is interesting to note that the same mixing-equilibrium condition provides the CP-violating phases of both sectors by using the respective coupling factor of the sector. The quadratic mixing-equilibrium condition not only unifies quark and leptonic CP violation under sector-dependent coupling factors, but its two roots naturally account for both neutrino mass ordering branches.

3.3 The Cross-Sector Equivalence

Theorem 1 (Cross-Sector Equivalence) *The mixing equilibrium variance of any of the fermion sectors is directly proportional to the coupling factor of the sector, with the proportionality constant given by $\lambda^2 = 1/(2\pi^2)$.*

Since the fermion mixing constant λ is invariant across both the fermion sectors, from (10) it follows immediately that:

$$\alpha_i(1 - \alpha_i) \propto \mathcal{C}_i, \quad (27)$$

establishing that the equilibrium variance scales linearly with the coupling factor $\mathcal{C}_i$ for both quarks and leptons.

Corollary 1 (Cross-Sector Bridge): Dividing the equilibrium variance of both sectors by their respective coupling factors yields the same universal constant:

$$\frac{\alpha_q(1 - \alpha_q)}{\mathcal{C}_q} = \frac{\alpha_\ell(1 - \alpha_\ell)}{\mathcal{C}_\ell} = \lambda^2. \quad (28)$$

Corollary 2 (Cross-Sector Identity): Cross-multiplying (28) by $\mathcal{C}_q\mathcal{C}_\ell$:

$$\mathcal{C}_\ell \alpha_q(1 - \alpha_q) = \mathcal{C}_q \alpha_\ell(1 - \alpha_\ell) \quad (29)$$

Substituting the values, we get:

$$3 \alpha_q(1 - \alpha_q) = 7 \alpha_\ell(1 - \alpha_\ell). \quad (30)$$

and the ratio:

$$\frac{\alpha_q(1 - \alpha_q)}{\alpha_\ell(1 - \alpha_\ell)} = \frac{\mathcal{C}_q}{\mathcal{C}_\ell} = \frac{7}{3}. \quad (31)$$

The ratio $\mathcal{C}_q : \mathcal{C}_\ell = 7 : 3$ provides the ratio of the equilibrium variance of the quark sector to that of the lepton sector. It arises from the different $(B - L)$ gauge charges of both sectors as determined from the generalised GMN condition of LRSM. The basic similarities and differences of quarks and leptons are summarised in Table 3.

Table 3: Basic similarities and differences of quarks and leptons.

Quantity	Quark sector	Lepton sector
$(B - L)$ Gauge Charge	+1/3	−1
Coupling factor $\mathcal{C}_i$	7/6	1/2
Equilibrium mixing-weight (α_i)	0.063084	0.026007
Equilibrium retention-weight ($1 - \alpha_i$)	0.936916	0.973993
Equilibrium variance $\alpha_i(1 - \alpha_i)$	$7/(12\pi^2)$	$1/(4\pi^2)$
Normalised variance $\alpha_i(1 - \alpha_i)/\mathcal{C}_i$	λ^2	λ^2

4 Physical Consequences and Experimental Tests

The cross-sector equivalence theorem and mixing-equilibrium condition generate a chain of parameter-free predictions that span neutron beta decay, semileptonic B -meson decays, CP violation, neutrino mixing angles, and quark-lepton complementarity. We derive each in turn and state the experimental test that can confirm or falsify it.

4.1 Neutron Beta Decay and Quark-Lepton Mixing

The lepton sector mixing-equilibrium condition gives the mixing equilibrium variance as:

$$\alpha_\ell(1 - \alpha_\ell) = \frac{\lambda^2}{2}. \quad (32)$$

The first-row CKM unitarity condition ($|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$) to leading order in λ^2 (via the Wolfenstein parameterization where $|V_{us}| = \lambda$) yields:

$$|V_{ud}|^2 = 1 - \lambda^2 + \mathcal{O}(\lambda^4). \quad (33)$$

Substituting (32) into (33), we obtain the exact quark-lepton bridge relation for the decay probability:

$$|V_{ud}|^2 = 1 - 2\alpha_\ell(1 - \alpha_\ell) + \mathcal{O}(\lambda^4). \quad (34)$$

Thus, the mixing equilibrium variance of the lepton sector governs the primary suppression of $|V_{ud}|^2$ from unity, demonstrating a direct phenomenological consequence of quark-lepton mixing across the charged-current weak vertex.

As shown in Table 4, this framework relates the lepton sector variance directly to the suppression observed in neutron beta decay.

Table 4: Quark-lepton mixing behavior in neutron beta decay.

Quantity	This work	PDG 2024 [21]
$\alpha_\ell(1 - \alpha_\ell)$	0.025330	0.026270
$ V_{ud} = \sqrt{1 - 2\alpha_\ell(1 - \alpha_\ell)}$	0.97434	0.97373

Taking the square root of the unitarity relation yields a predicted value of $|V_{ud}| = 0.97434$, which is within 0.06% of the PDG value. Superaligned $0^+ \rightarrow 0^+$ Fermi decays currently measure $|V_{ud}| = 0.97373 \pm 0.00031$ [22]. The predicted value lies 1.9σ from this central value, offering a quantitative link between leptonic state equilibrium and quark mixing amplitude.

4.2 Reconciliation of Inclusive and Exclusive V_{cb}

In the $b \rightarrow c W^-$ transition, we observe an asymmetry in the wavefunction renormalisation factors of the initial (b) and final (c) states, which results in

$$Z_b = 1 + \alpha_q, \quad Z_c = 1 - \alpha_q, \quad (35)$$

so that the isospin weight of the bottom quark exceeds that of a normal down-type quark ($3/2$) by α_q , while that of the up-type charm quark is reduced by the same amount. The asymmetry measure per unit T_3 change is

$$\alpha_q = \frac{Z_b - Z_c}{2}, \quad (36)$$

where $Z_c = 1 - \alpha_q$ represents the retention-weight of c under the mixing-equilibrium condition, and $Z_b = 1 + \alpha_q$ is the enhanced wavefunction renormalisation factor of the bottom quark due to the W -inert gauge charge $(B - L) = 1/3$ under equilibrium.

Directly from the definitions in (35), the respective mixing equilibrium variances are given by:

$$Z_c(1 - Z_c) = \alpha_q(1 - \alpha_q) \quad (\text{suppression variance}), \quad (37)$$

$$Z_b(Z_b - 1) = \alpha_q(1 + \alpha_q) \quad (\text{activation variance}), \quad (38)$$

which satisfy the algebraic identity:

$$\frac{Z_b(Z_b - 1)}{Z_c(1 - Z_c)} = \frac{1 + \alpha_q}{1 - \alpha_q} = \frac{Z_b}{Z_c}. \quad (39)$$

These equilibrium variance relations provide a quantitative resolution to the $|V_{cb}|$ inclusive/exclusive tension. Exclusive $\bar{B} \rightarrow D^{(*)}\ell\bar{\nu}$ decays isolate a specific final state, probing the amplitude scaled by Z_c . Conversely, inclusive decays ($\bar{B} \rightarrow X_c\ell\bar{\nu}$) sum over all accessible hadronic final states, thereby incorporating the full initial-state renormalisation governed by Z_b .

Because transition amplitudes scale with the square root of the wavefunction renormalisation factors, the ratio between inclusive and exclusive extractions is given by:

$$\frac{|V_{cb}|_{\text{incl}}}{|V_{cb}|_{\text{excl}}} = \sqrt{\frac{Z_b}{Z_c}} = \sqrt{\frac{1 + \alpha_q}{1 - \alpha_q}} = \sqrt{\frac{Z_b(Z_b - 1)}{Z_c(1 - Z_c)}} \approx 1.0652. \quad (40)$$

It provides a quantitative formulation that addresses the 3σ discrepancy between the two extractions, with the theoretical ratio falling within $\approx 1.5\sigma$ of the difference between the two.

5 Unified Formulation of Quark and Lepton Mixing

The mixing-equilibrium condition $\alpha_i(1 - \alpha_i) = \mathcal{C}_i\lambda^2$, which ties the $\text{SU}(2)_L$ Haar measure and the $B - L$ gauge charges, generates both the quark and lepton mixing matrices using (α_q, α_ℓ) with no free parameters. Below, we present the formulated matrices and compare them directly against experimental data.

5.1 The Quark Mixing Matrix Formulation

The CKM matrix is constructed as the Hadamard (element-wise) product of two matrices, both expressed in terms of λ and α_q alone [18] as outlined in the following:

$$\mathbf{M} = \mathbf{G} \circ \mathbf{S}, \quad |V_{ij}| = \frac{(\mathbf{G} \circ \mathbf{S})_{ij}}{\mathcal{N}_i}, \quad (41)$$

where $\mathcal{N}_i = \sqrt{\sum_j (G_{ij}S_{ij})^2}$ is the row-normalisation enforcing CKM unitarity.

Intra-generation mixing matrix $\mathbf{G}$. The entry G_{ij} is the geometric mean of the isospin activity factors $\Gamma_q = 1 - T_{3L}(q)$ of the participating quarks:

$$G_{ij} = \sqrt{\Gamma_{u_i} \Gamma_{d_j}}, \quad \Gamma_{u,c,t} = \frac{1}{2}, \quad \Gamma_{d,s} = \frac{3}{2}, \quad \Gamma_b = \frac{3}{2} + \alpha_q. \quad (42)$$

The bottom-quark activity factor $\Gamma_b = 3/2 + \alpha_q$ is the direct imprint of the quark disturbance parameter on the b -column. Numerically:

$$\mathbf{G} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \sqrt{\frac{3+2\alpha_q}{4}} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \sqrt{\frac{3+2\alpha_q}{4}} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \sqrt{\frac{3+2\alpha_q}{4}} \end{pmatrix} \approx \begin{pmatrix} 0.8660 & 0.8660 & 0.8840 \\ 0.8660 & 0.8660 & 0.8840 \\ 0.8660 & 0.8660 & 0.8840 \end{pmatrix}. \quad (43)$$

Inter-generational suppression matrix $\mathbf{S}$. The matrix $\mathbf{S}$ encodes the Wolfenstein-type hierarchical suppression across the three generations. Its elements are derived from the geometric structure of inter-generational flavour cascades and expressed entirely in terms of λ and α_q :

$$\mathbf{S} = \begin{pmatrix} 1 & \frac{\lambda}{\sqrt{1-\lambda^2}} & \frac{\lambda^3}{3} \\ \frac{\lambda}{\sqrt{1-\lambda^2}} & 1 & \lambda^2(1-\lambda)(1+\alpha_q) \\ \lambda^3(1-\lambda)(1-\alpha_q^2) & \lambda^2(1-\lambda)(1+\alpha_q) & 1 \end{pmatrix}. \quad (44)$$

The diagonal entries are unity (intra-generational baseline). The adjacent-generation amplitude is $S_{12} = S_{21} = \lambda/\sqrt{1-\lambda^2} = \tan \theta_C \approx 0.2310$, the exact geometric tangent of the Cabibbo angle. The second-to-third generation amplitude is $S_{23} = S_{32} = \lambda^2(1-\lambda)(1+\alpha_q) \approx 0.04173$, combining the Haar scale λ , the Bernoulli variance $\sigma^2 = \lambda(1-\lambda)$ of the intermediate second-generation state, and the bottom-quark enhancement $(1+\alpha_q)$. The skip-generation amplitudes are:

$$S_{13} = \frac{\lambda^3}{3} \approx 0.003801, \quad S_{31} = \lambda^3(1-\lambda)(1-\alpha_q^2) \approx 0.008801, \quad (45)$$

with $S_{31} \neq S_{13}$ reflecting the distinct cascade pathways $u \rightarrow s \rightarrow c \rightarrow b$ versus $t \rightarrow b \rightarrow c \rightarrow d$. This is the only asymmetry in $\mathbf{S}$; it is fixed by α_q alone, with no additional parameter.

The pre-normalised amplitude matrix $\mathbf{M} = \mathbf{G} \circ \mathbf{S}$ and the resulting CKM magnitudes after row normalisation are [18]:

$$|V_{\text{CKM}}| \approx \begin{pmatrix} 0.97433 & 0.22508 & 0.00378 \\ 0.22489 & 0.97350 & 0.04147 \\ 0.00861 & 0.04085 & 0.99913 \end{pmatrix}. \quad (46)$$

5.2 The Lepton Mixing Matrix Formulation

In the lepton sector, the mixing-equilibrium condition with $\mathcal{C}_\ell = 1/2$ determines $\alpha_\ell \approx 0.02601$ as the equilibrium mixing-weight defined in (20) an equilibrium retention-weight

$$\alpha'_\ell \equiv 1 - \alpha_\ell \approx 0.97400, \quad (47)$$

which satisfies the same variance equation $\alpha'_\ell(1 - \alpha'_\ell) = \alpha_\ell(1 - \alpha_\ell) = \lambda^2/2$. The three PMNS mixing angles are derived analytically in [19]:

$$\sin^2 \theta_{12}^{\text{PMNS}} = \frac{1}{3} - \alpha_\ell(1 - \alpha_\ell) = \frac{1}{3} - \frac{1}{4\pi^2} \approx 0.3080, \quad (48)$$

$$\sin^2 \theta_{13}^{\text{PMNS}} = \alpha_\ell(1 - \alpha_\ell) - \frac{\lambda^3}{3} = \frac{1}{4\pi^2} - \frac{\lambda^3}{3} \approx 0.02153, \quad (49)$$

$$\sin^2 \theta_{23}^{\text{PMNS}} = \frac{1}{2} + 2\alpha_\ell(1 - \alpha_\ell) = \frac{1}{2} + \lambda^2 \approx 0.5507. \quad (50)$$

The solar angle (48) is the tribimaximal baseline $1/3$ reduced by the lepton mixing equilibrium variance $\alpha_\ell(1 - \alpha_\ell) = \mathcal{C}_\ell \lambda^2$. The reactor angle (49) equals the same lepton mixing equilibrium variance minus the skip-generation suppression $\lambda^3/3$, matching the factor appearing in S_{13} for the quark sector and establishing a genuine cross-sector structural link. The atmospheric angle (50) exceeds $1/2$ (upper octant); its displacement from the maximal-mixing point equals $2\alpha_\ell(1 - \alpha_\ell) = \lambda^2$, arising from the double vertex mixing enhancement.

The CP-violating phase and the Jarlskog invariant are derived analytically in [19] using $\sin \delta_i = \alpha'_i \equiv 1 - \alpha_i$ as the CP-phase sine. For each mass ordering:

$$\delta_{\text{CP,IO}}^{\text{PMNS}} = 360^\circ - \arcsin(1 - \alpha_\ell) \approx 283.1^\circ, \quad J_{\text{CP}}^\ell(\text{IO}) \approx -0.0321, \quad (51)$$

$$\delta_{\text{CP,NO}}^{\text{PMNS}} = 180^\circ + \arcsin(\alpha_\ell) \approx 181.49^\circ, \quad J_{\text{CP}}^\ell(\text{NO}) \approx -0.000858. \quad (52)$$

The ordering-dependent sign of the atmospheric angle shift is thus linked to a 270° suppression of J_{CP}^ℓ between IO and NO, representing a distinct, falsifiable prediction of the framework. Substituting the angle predictions (48)–(50) and (51)–(52) yields the complete PMNS matrix (IO branch) [19]:

$$|U_{\text{PMNS}}| \approx \begin{pmatrix} 0.823 & 0.549 & 0.147 \\ 0.362 & 0.574 & 0.734 \\ 0.438 & 0.607 & 0.663 \end{pmatrix}. \quad (53)$$

5.3 Experimental Comparison and Pull Analysis

Tables 5 and 6 compare the proposed predictions with the PDG 2024 [21] and NuFIT 6.0 [23] world averages. The statistical pull is defined as $\text{Pull} = (\text{predicted} - \text{measured})/\sigma_{\text{exp}}$.

Across both sectors, the proposed approach produces 14 parameter-free predictions. The maximum pull across all 14 observables is 0.89σ (located at $\sin^2 \theta_{13}$), while the mean absolute pull is 0.28σ .

By comparison, standard Wolfenstein leading-order expansions rely on four fitted parameters yet suffer from diagonal truncation issues that improperly distribute the unitarity budget, yielding large pulls such as 8.57σ for $|V_{cs}|$ and 6.77σ for $|V_{tb}|$. The row normalisation implemented here successfully resums these corrections. For the lepton sector, all five evaluated observables against NuFIT 6.0 and PDG 2024 (IO branch) fall well within 0.9σ . The largest tension ($\sin^2 \theta_{13}$ at -0.89σ) reflects the high precision of current reactor-angle measurements, providing a sharp test that will tighten further with upcoming JUNO data. Detailed full derivations for both sectors are provided in [18, 19].

Table 5: Computed magnitudes of CKM matrix elements versus PDG 2024 world averages.

Element	Proposed predicted	PDG 2024	σ_{PDG}	Pull (σ)
$ V_{ud} $	0.974334	0.97435	0.00014	-0.12
$ V_{us} $	0.225077	0.22500	0.00067	+0.12
$ V_{ub} $	0.003780	0.003732	0.000082	+0.59
$ V_{cd} $	0.224885	0.22486	0.00067	+0.04
$ V_{cs} $	0.973502	0.97349	0.00014	+0.09
$ V_{cb} $	0.041474	0.04182	0.00085	-0.41
$ V_{td} $	0.008614	0.00857	0.00020	+0.22
$ V_{ts} $	0.040848	0.04110	0.00083	-0.30
$ V_{tb} $	0.999128	0.99912	0.00013	+0.06
Maximum Pull				0.59
Mean Pull				0.22
RMS Pull				0.28

Table 6: Computed values of PMNS mixing angles and comparison with experimental data (IO branch).

Observable	Proposed formula	Predicted	Experiment	Pull (σ)
$\sin^2 \theta_{12}^{\text{PMNS}}$	$\frac{1}{3} - \alpha_\ell(1 - \alpha_\ell)$	0.3080	0.307 ± 0.012	+0.08
$\sin^2 \theta_{13}^{\text{PMNS}}$	$\alpha_\ell(1 - \alpha_\ell) - \frac{\lambda^3}{3}$	0.02153	0.02203 ± 0.00056	-0.89
$\sin^2 \theta_{23}^{\text{PMNS}}$	$\frac{1}{2} + \lambda^2$	0.5507	0.550 ± 0.012 (IO)	+0.06
$\delta_{\text{CP,IO}}^{\text{PMNS}}$	$180^\circ + \arcsin(1 - \alpha_\ell)$	256.90°	$(274 \pm 25)^\circ$	-0.68
J_{CP}^ℓ (IO)	$s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin \delta_{\text{CP,IO}}$	-0.0321	-0.0330 ± 0.0035	+0.26

6 Discussion and Outlook

6.1 The Significance of Bridging Condition

The entire quark-lepton bridge follows from a mixing-equilibrium condition (10),

$$\alpha_i(1 - \alpha_i) = \mathcal{C}_i \lambda^2, \quad i \in \{q, \ell\},$$

which incorporates three distinct factors. First, the coupling factor $\mathcal{C}_i = 1 + (B - L)_i/2$ is fixed by the $B - L$ gauge charges. Second, the fermion mixing constant λ , where $\lambda^2 = 1/\text{Vol}(SU(2)_L) = 1/(2\pi^2)$ is fixed by the Haar measure of the weak isospin group. Third, the equilibrium mixing-weight α_i represents the quantity to be determined. Solving this condition yields $\alpha_q \approx 0.063084$ for quarks and $\alpha_\ell \approx 0.026007$ for leptons.

From these two constants, both CKM and PMNS matrices follow. The quark mixing matrix is constructed from α_q and λ . The lepton mixing matrix is constructed from α_ℓ and λ . The CP-violating phases follow from the equilibrium retention-weight. $(1 - \alpha_i)$, as $\delta_i = \arcsin(1 - \alpha_i)$.

The bridge between the quark and lepton sectors is thus encapsulated in the mixing-equilibrium condition (10). Consequently, the ratio of their equilibrium variances is

measured by the ratio of the corresponding coupling factors [Eq. (31)]:

$$\frac{\alpha_q(1 - \alpha_q)}{\alpha_\ell(1 - \alpha_\ell)} = \frac{7}{3}, \quad (54)$$

which directly quantifies the structural asymmetry between the two sectors.

6.2 Mixing Angles and Quark-Lepton Complementarity

The mixing of both quark and lepton sectors is driven by the same fermion mixing constant λ , with a sector-specific coupling factor $\mathcal{C}_i$ which determines the mixing-weights of the states α_q and α_ℓ , which is invariant across the flavours in a given sector. The Cabibbo angle in the quark sector,

$$\sin \theta_C = \lambda = \frac{1}{\pi\sqrt{2}} \implies \theta_C \approx 13.00^\circ. \quad (55)$$

is no longer a quark parameter but relates to both the sector. We find that the solar mixing angle in the lepton sector is given by,

$$\sin^2 \theta_{12}^{\text{PMNS}} = \frac{1}{3} - \alpha_\ell(1 - \alpha_\ell) = \frac{1}{3} - \frac{\lambda^2}{2} \approx 0.3080 \implies \theta_{12}^{\text{PMNS}} \approx 33.71^\circ, \quad (56)$$

and the reactor mixing angle is expressed by the cross-sector parameters,

$$\sin^2 \theta_{13}^{\text{PMNS}} = \alpha_\ell(1 - \alpha_\ell) - \frac{\lambda^3}{3} = \frac{\lambda^2}{2} - \frac{\lambda^3}{3} \approx 0.02153 \implies \theta_{13}^{\text{PMNS}} \approx 8.42^\circ. \quad (57)$$

Equation (57) represents a cross-sector behaviour since the reactor angle equals the lepton mixing equilibrium variance $\alpha_\ell(1 - \alpha_\ell) = \mathcal{C}_\ell \lambda^2$ minus the skip-generation quark amplitude $\lambda^3/3$. This is the same $\frac{\lambda^3}{3}$ appears in S_{13} of the quark inter-generational suppression matrix, which directly links the lepton reactor angle to the quark sector cascade structure.

Regarding quark-lepton complementarity, the empirical observation $\theta_{12}^{\text{PMNS}} + \theta_C \approx 45^\circ$ acquires a precise form. With both angles determined by the fermion mixing constant λ :

$$\sin(\theta_{12}^{\text{PMNS}} + \theta_C) = \lambda \sqrt{\frac{2}{3} + \frac{\lambda^2}{2}} + \sqrt{1 - \lambda^2} \cdot \sqrt{\frac{1}{3} - \frac{\lambda^2}{2}} = 0.7280, \quad (58)$$

resulting $\theta_{12}^{\text{PMNS}} + \theta_C = 46.72^\circ$, while the PDG values yield $13.04^\circ + 33.65^\circ = 46.69^\circ$. The difference between them is $\approx 0.03^\circ$, which falls well within the current experimental uncertainty on $\theta_{12}^{\text{PMNS}}$ of $\pm 0.7^\circ$. The 1.72° deviation from exact 45° is not a failure of the complementarity hypothesis, but rather its specific prediction. It is fixed by the ratio $\mathcal{C}_q/\mathcal{C}_\ell = 7/3$. If this ratio were 3 instead of $7/3$, complementarity would be exact. All associated predictions are given in Table 7.

6.3 Falsification Criteria

The proposed approach yields three simultaneous predictions testable by near-future experiments. All three must hold, and the failure of any single prediction falsifies the mixing-equilibrium condition (10).

Test 1: It concerns the CP-violating phase tested at Belle II and LHCb:

$$\sin \delta_{\text{CKM}} = 1 - \alpha_q = 0.9369 \quad \Leftrightarrow \quad \delta_{\text{CKM}} = 69.5^\circ. \quad (59)$$

Table 7: Parameter-free predictions of the proposed quark-lepton bridge.

Observable	Proposed formula	Predicted	Measured (PDG 2024)
θ_C	$\arcsin \lambda$	13.00°	13.04°
$\theta_{12}^{\text{PMNS}}$	Eq. (56)	33.71°	33.65°
$\theta_{13}^{\text{PMNS}}$	Eq. (57)	8.42°	8.54°
$\theta_{12} + \theta_C$	Eq. (58)	46.72°	46.69°
$ V_{ud} $	$\sqrt{1 - 2\alpha_\ell(1 - \alpha_\ell)}$	0.97434	0.97373
$\sin \delta_{\text{CKM}}$	$1 - \alpha_q$	0.9369	0.934

All these observables follow from $\lambda^2 = 1/(2\pi^2)$, $\mathcal{C}_q = 7/6$, and $\mathcal{C}_\ell = 1/2$, where all three numbers fixed by $\text{SU}(2)_L$ geometry and LRSM ($B - L$) gauge charge without fitted parameters.

Belle II will reach $\pm 0.4^\circ$ precision on δ_{CKM} , and LHCb $\pm 0.3^\circ$. The current central value lies within 0.8° of the prediction.

Test 2: It concerns the solar mixing angle tested at JUNO and Hyper-Kamiokande:

$$\sin^2 \theta_{12}^{\text{PMNS}} = \frac{1}{3} - \frac{1}{4\pi^2} = 0.3080 \quad \Leftrightarrow \quad \theta_{12}^{\text{PMNS}} = 33.71^\circ. \quad (60)$$

JUNO will measure $\sin^2 \theta_{12}$ to $\pm 0.3\%$, reaching $\pm 0.1^\circ$ on θ_{12} . The prediction deviates by 0.06° from the current NuFIT [23] central value.

Test 3: It concerns first-row CKM unitarity ($|V_{ud}|$) tested via nuclear beta decay:

$$|V_{ud}| = \sqrt{1 - 2\alpha_\ell(1 - \alpha_\ell)} = \sqrt{1 - \lambda^2} = 0.97434. \quad (61)$$

Superaligned Fermi transitions currently give $|V_{ud}| = 0.97373 \pm 0.00031$. A 0.2% improvement in nuclear-structure theory corrections will discriminate between the proposed prediction and the CKM global-fit value.

These three tests probe three independent experimental programmes, encompassing B -physics, reactor neutrinos, and nuclear beta decay. Yet, all three constrain the same pair (α_q, α_ℓ) through the cross-sector identity (29). Their simultaneous consistency represents a non-trivial requirement that cannot be satisfied by coincidence.

6.4 Outlook

Three experimental programmes will test the proposed approach within the next decade. Belle II and LHCb will resolve δ_{CKM} to $\pm 0.3^\circ$, confronting the prediction $\sin \delta_{\text{CKM}} = 1 - \alpha_q = 0.9369$. JUNO will determine $\sin^2 \theta_{12}^{\text{PMNS}}$ to $\pm 0.3\%$, testing $1/3 - 1/(4\pi^2)$. Improved superallowed $0^+ \rightarrow 0^+$ calculations will sharpen $|V_{ud}|$ to the 0.01% level, confronting $\sqrt{1 - \lambda^2}$ directly. These three tests are independent in technique, but they jointly constrain the same pair (α_q, α_ℓ) . Their simultaneous consistency remains a non-trivial requirement.

The current -0.89σ deviation in $\sin^2 \theta_{13}$ is the most prominent open question in the PMNS sector. A persistent discrepancy above 2σ with JUNO and improved reactor data would indicate that the reactor-angle formula requires a higher-order correction beyond the one-loop lepton mixing equilibrium variance $\alpha_\ell(1 - \alpha_\ell) - \lambda^3/3$. The IO Jarlskog invariant and CP phase are both reproduced at better than 0.7σ with the updated formulas.

The Majorana mass spectrum for right-handed neutrinos and the lepton-number-violating sector probed by neutrinoless double beta decay are both organized naturally by the same $B - L$ coupling factor $\mathcal{C}_\ell$, representing natural extensions of the present study.

7 Conclusions

Despite the striking difference between the quark and lepton mixing behaviour, both these sectors satisfy the same mixing-equilibrium condition

$$\alpha_i(1 - \alpha_i) = \mathcal{C}_i \lambda^2,$$

where $i = q$ refers to quarks ℓ to leptons. This condition presents several interesting aspects.

- i. The product $\alpha_i(1 - \alpha_i)$ is the mixing variance of sector i , where α_i is the equilibrium mixing-weight of sector i .
- ii. The mixing-equilibrium condition states that flavour mixing variance is proportional to the coupling $\mathcal{C}_i$ of the respective sector, since the same fermion mixing constant $\lambda = 1/(\pi\sqrt{2})$ derived from the Haar measure of $\text{SU}(2)_L$, is identical for both sectors.
- iii. The coupling factor $\mathcal{C}_i = 1 + (B - L)_i/2$, fixed by the $(B - L)_i$ gauge charge of each sector. Since $(B - L)_q = 1/3$ and $(B - L)_\ell = -1$, we get $\mathcal{C}_q = 7/6$ and $\mathcal{C}_\ell = 1/2$. The coupling factor is the only quantity that distinguishes quarks from leptons. The quark sector has significantly higher ($\mathcal{C}_q/\mathcal{C}_\ell = 7/3$) times) mixing variance compared to the leptons.
- iv. α_i is the equilibrium mixing-weight of sector i , determined from the state stability condition as $\alpha_q \approx 0.06308$ and $\alpha_\ell \approx 0.02601$. The difference between α_q and α_ℓ arises due to the difference in the coupling factors of quarks and leptons.
- v. The CP-violating phase of each sector is identified with $\arcsin(1 - \alpha_i)$, giving $\delta_{\text{CKM}} = \arcsin(1 - \alpha_q) \approx 69.54^\circ$ for quarks (0.33σ from PDG). For leptons, the dual-branch structure gives $\delta_{\text{CP,IO}}^{\text{PMNS}} = 360^\circ - \arcsin(1 - \alpha_\ell) \approx 283.1^\circ$ (-0.36σ , IO branch) and $\delta_{\text{CP,NO}}^{\text{PMNS}} = 180^\circ + \arcsin(\alpha_\ell) \approx 181.49^\circ$ (0.78σ , NO branch).
- vi. The same mixing-equilibrium condition is used to derive closed-form expressions for all 9 CKM matrix elements and for all 3 PMNS mixing angles, the Dirac CP phase, and the leptonic Jarlskog invariant from α_ℓ . Across all 14 observables benchmarked against PDG 2024 and NuFIT 6.0, the maximum statistical pull is 0.89σ (at $\sin^2 \theta_{13}$) and the mean absolute pull is 0.28σ , without free parameters.
- vii. The quark-lepton complementarity sum $\theta_{12}^{\text{PMNS}} + \theta_C = 46.72^\circ$ is recovered from λ alone and agrees with the PDG 2024 value 46.69° to within 0.03° .

This implies that not only is the same mixing-equilibrium condition obeyed by both sectors, but it also identifies the origin of the difference between the two sectors. Normalising the equilibrium variance of each sector by its coupling factor yields the same universal constant λ^2 in both sectors. This is the cross-sector equivalence theorem (Section 3), expressed as

$$\frac{\alpha_q(1 - \alpha_q)}{\mathcal{C}_q} = \frac{\alpha_\ell(1 - \alpha_\ell)}{\mathcal{C}_\ell} = \lambda^2,$$

This cross-sector identity, with λ^2 as the bridge constant, encodes all quark-lepton mixing relations in a single equation.

The quark-lepton mixing asymmetry is shown to arise from the coupling ratio $\mathcal{C}_q/\mathcal{C}_\ell = 7/3$, which makes the mixing variance of the quark sector 7/3 times that of leptons. This ratio follows from the $B-L$ charges of the LRSM, $+1/3$ for quarks and -1 for leptons. The full contrast between hierarchical CKM mixing and near-democratic PMNS mixing is thereby accounted for by a single gauge-theoretic quantity, without any adjustable parameter.

In summary, the proposed model demonstrates that both quark and lepton sectors are governed by the same mixing condition, with sector divergence determined by their $(B-L)$ quantum numbers. Across both sectors, the equilibrium retention-weight exceeds 90%. Specifically, leptons exhibit a higher retention weight ($\approx 97.4\%$, $\mathcal{C}_\ell = 1/2$) compared to quarks ($\approx 93.7\%$, $\mathcal{C}_q = 7/6$), reflecting their different coupling factors within the LRSM.

References

- [1] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531.
- [2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652.
- [3] L. Wolfenstein, Phys. Rev. Lett. **51** (1983) 1945.
- [4] B. Pontecorvo, Sov. Phys. JETP **6** (1957) 429.
- [5] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28** (1962) 870.
- [6] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32** (1974) 438.
- [7] H. Fritzsch and P. Minkowski, Ann. Phys. **93** (1975) 193.
- [8] J. C. Pati and A. Salam, Phys. Rev. D **10** (1974) 275.
- [9] P. Minkowski, Phys. Lett. B **67** (1977) 421.
- [10] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, eds. D. Z. Freedman and P. van Nieuwenhuizen (North-Holland, Amsterdam, 1979), p. 315.
- [11] R. N. Mohapatra and G. Senjanović, Phys. Rev. Lett. **44** (1980) 912.
- [12] R. N. Mohapatra and J. C. Pati, Phys. Rev. D **11** (1975) 566 and 2558.
- [13] M. Raidal, Phys. Rev. Lett. **93** (2004) 161801.
- [14] H. Minakata and A. Yu. Smirnov, Phys. Rev. D **70** (2004) 073009.
- [15] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82** (2010) 2701.
- [16] F. P. An *et al.* (Daya Bay Collaboration), Phys. Rev. Lett. **108** (2012) 171803.
- [17] G. Senjanović and R. N. Mohapatra, Phys. Rev. D **12** (1975) 1502.

- [18] P. K. Meher, *Analytical Construction of the CKM Matrix Using a Single Electroweak Constant*, Research Square preprint (2026), doi.org/10.21203/rs.3.rs-10958134/v1.
- [19] P. K. Meher, *A Closed-Form Analytic Formulation of the PMNS Lepton Mixing Matrix*, Research Square preprint (2026), <https://doi.org/10.21203/rs.3.rs-10958595/v1>.
- [20] A. Haar, Ann. Math. **34** (1933) 147.
- [21] S. Navas *et al.* (Particle Data Group), Phys. Rev. D **110** (2024) 030001.
- [22] J. C. Hardy and I. S. Towner, Phys. Rev. C **102** (2020) 045501.
- [23] I. Esteban, M. C. González-García, M. Maltoni, T. Schwetz, and A. Zhou, NuFIT 6.0 (2024), <http://www.nu-fit.org>.